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NEIGHBORHOODS AND ISOLATED POINTS IN SPACES OF FUNCTIONAL CLONES ON SETS

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In a previous paper, on a collection F_A of functional clones on a set A, we introduced a natural metric d turning it into a topological (metric) space $\mathfrak{F}_A = \langle F_A; d\rangle$. In this paper, we describe the structure of neighborhoods of clones in spaces $\mathfrak{F}_A$ and establish a number of consequences of this result.

A collection F_A of functional clones on a set A is conventionally treated as a lattice $L_A = \langle F_A; \wedge, \vee\rangle$ with respect to a relation $\subseteq$ on these clones. To date, significant information has been accumulated concerning such lattices, their cardinalities, atoms, coatoms, sublattices, intervals, unrefinable chains, unsolvability of lattice identities on them, etc. A review of basic results on functional clones and their lattices can be found in [1, 2].

In [3], we came up with another approach to studying a collection of functional clones on a set A. Namely, on F_A, we introduced a natural metric d turning it into a topological (metric) space $\mathfrak{F}_A = \langle F_A; d\rangle$ with respect to which the operations $\wedge$ and $\vee$ are continuous. In [3, 4], the following properties of spaces $\mathfrak{F}_A$ were proved: (1) the spaces $\mathfrak{F}_A$ are complete; (2) $\mathfrak{F}_A$ is a compact space iff A is a finite set; (3) for any $B \subseteq A$, a space $\mathfrak{F}_B$ is isomorphically embedded in a space $\mathfrak{F}_A$ and is a topological retract of the space $\mathfrak{F}_A$; (4) for any not more than three-element set A, there exists a sublattice L of L_A forming a perfect subset of the space F_A which is homeomorphic to the Cantor discontinuum.

In the present paper, we describe the structure of neighborhoods of clones in spaces $\mathfrak{F}_A$ and establish a number of consequences of this result.

First we recall the definition of a metric d on a set F_A and some relevant notions.

Definition 1. For any natural n, by an *n-fragment* $\mathfrak{F}^{(n)}$ of a clone $\mathfrak{F}$ we mean the collection of all functions in $\mathfrak{F}$ depending on at most n arguments.

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Notice that $\mathfrak{F}^{(n)} \subseteq \mathfrak{F}^{(m)}$ for $n \leq m$ and $\mathfrak{F} = \bigcup\limits_{n\in\omega} \mathfrak{F}^{(n)}$. For any $\mathfrak{F}_1, \mathfrak{F}_2 \in F_A$, put

$$d(\mathfrak{F}_1, \mathfrak{F}_2) = \begin{cases} \dfrac{1}{\min\{n \in \omega \mid \mathfrak{F}_1^{(n)} \neq \mathfrak{F}_2^{(n)}\}} & \text{if } \mathfrak{F}_1 \neq \mathfrak{F}_2, \\ 0 & \text{if } \mathfrak{F}_1 = \mathfrak{F}_2. \end{cases}$$

Furthermore, for any collection S of functions on a set A, by $\langle S \rangle$ we denote the functional clone on A generated by the collection S.

Definition 2. A clone $\mathfrak{F}$ is said to be *boundedly generated* if $\mathfrak{F} = \langle \mathfrak{F}^{(n)} \rangle$ for some natural n.

The collection of all boundedly generated clones on A forms a sublattice L'_A of L_A having cardinality less than that of L_A but inheriting most of the properties of the lattice L_A (see [5]).

For any natural n, the $\frac{1}{n}$-neighborhood $D_{\frac{1}{n}}(\mathfrak{F}) = \{\mathfrak{F}' \mid d(\mathfrak{F}', \mathfrak{F}) < \frac{1}{n}\}$ of a clone $\mathfrak{F}$ has the form $\{\mathfrak{F}' \in F_A \mid \mathfrak{F}'^{(n)} = \mathfrak{F}^{(n)}\}$ and is, obviously, a convex sublattice of L_A with a least element—a boundedly generated clone $\mathfrak{F}^n_{\min} = \langle \mathfrak{F}^{(n)} \rangle$—and a greatest element $\mathfrak{F}^n_{\max}$.

Definition 3. For any clone $\mathfrak{F}$ on A and for arbitrary natural $n < m$, a function $g(x_1, \ldots, x_m)$ on A is said to be *n-reducible to a fragment* $\mathfrak{F}^{(n)}$ of the clone $\mathfrak{F}$ ($g \in n\text{-Red}\,\mathfrak{F}$) if, in identifying some variables from $x_1, \ldots, x_m$ with at most n pairwise distinct variables in the function $g(x_1, \ldots, x_m)$, we will obtain functions in $\mathfrak{F}^{(n)}$.

Note that the above relation is transitive, i.e., if $g(x_1, \ldots, x_m)$ is n-reducible to $\mathfrak{F}^{(n)}$, and all functions in $\mathfrak{F}^{(n)}$ are k-reducible to a fragment $\mathfrak{F}_1^{(k)}$ of some clone $\mathfrak{F}_1$ (here $k < n$), then g is itself k-reducible to $\mathfrak{F}_1^{(k)}$.

Superposition of any functions n-reducible to a fragment $\mathfrak{F}^{(n)}$ on A is itself n-reducible, and each of the selector functions $e_i^m(x_1, \ldots, x_m) = x_i$ is n-reducible to $\mathfrak{F}^{(n)}$ for any clone $\mathfrak{F}$. Thus the collection $n\text{-Red}\,\mathfrak{F}$ is a functional clone on A such that $(n\text{-Red}\,\mathfrak{F})^{(n)} = \mathfrak{F}^{(n)}$, while the clone $n\text{-Red}\,\mathfrak{F}$ is a maximal clone with an n-fragment equal to $\mathfrak{F}^{(n)}$. In other words, $n\text{-Red}\,\mathfrak{F} = \mathfrak{F}^n_{\max}$.

We have $n\text{-Red}\,\mathfrak{F} = \bigcup\limits_{m \geq n} (n\text{-Red}\,\mathfrak{F})^{(m)}$. In order to construct $n\text{-Red}\,\mathfrak{F}$, therefore, we need only point out the chain of expansions

$$\mathfrak{F}^{(n)} = (n\text{-Red}\,\mathfrak{F})^{(n)} \subseteq (n\text{-Red}\,\mathfrak{F})^{(n+1)} \subseteq \ldots \subseteq (n\text{-Red}\,\mathfrak{F})^{(m)} \subseteq (n\text{-Red}\,\mathfrak{F})^{(m+1)} \subseteq \ldots,$$

and then take $n\text{-Red}\,\mathfrak{F}$ to be their union.

Thus suppose that $m \geq n$ and that an m-fragment $(n\text{-Red}\,\mathfrak{F})^{(m)}$ of the clone $n\text{-Red}\,\mathfrak{F}$ has been constructed. Then $(n\text{-Red}\,\mathfrak{F})^{m+1}$ is the collection of all $(m+1)$-ary functions m-reducible to the constructed fragment $(n\text{-Red}\,\mathfrak{F})^{(m)}$.

($*$) For any i and any j such that $1 \leq i < j \leq m+1$, we denote by $\Phi_{ij}(x_1, \ldots, x_{m+1})$ the formula $x_i = x_j$, and by $\Phi(x_1, \ldots, x_{m+1})$ the formula $\bigwedge\limits_{1 \leq i < j \leq m+1} x_i \neq x_j$. As usual, for $1 \leq j \leq m+1$ and for a sequence $a_1, \ldots, a_{m+1}$, we write $a_1, \ldots, \widehat{a_j}, \ldots, a_{m+1}$ to denote this sequence with an omitted element a_j.

Definition 4. A set consisting of m-ary functions $S_{ij}(x_1, \ldots, x_m)$ on A (for $1 \le i < j \le m+1$) is said to be *consistent* if, for any i, j, i_1, j_1 such that $1 \le i < j \le m+1$ and $1 \le i_1 < j_1 \le m+1$ and for arbitrary $a_1, \ldots, a_{m+1}$ in A for which $a_i = a_j$ and $a_{i_1} = a_{j_1}$, the following equalities hold:

$$S_{ij}(a_1, \ldots, \widehat{a_j}, \ldots, a_{m+1}) = S_{i_1 j_1}(a_1, \ldots, \widehat{a_{j_1}}, \ldots, a_{m+1}).$$

Definition 5. An $(m+1)$-ary function $g(x_1, \ldots, x_{m+1})$ on a set A is said to be $(n\text{-Red}\,\mathfrak{F})^{(m)}$-*conservatively conditional* if, for some consistent set of $(n\text{-Red}\,\mathfrak{F})^{(m)}$-functions $S_{ij}(x_1, \ldots, x_m)$ (with $1 \le i < j \le m+1$), the following holds: for any $a_1, \ldots, a_{m+1}$ in A, if some of the formulas $\Phi_{ij}(a_1, \ldots, a_{m+1})$ is true, then

$$g(a_1, \ldots, a_{m+1}) = S_{ij}(a_1, \ldots, \widehat{a_j}, \ldots, a_{m+1}).$$

Thus every $(n\text{-Red}\,\mathfrak{F})^{(m)}$-conservatively conditional function $g(x_1, \ldots, x_{m+1})$ is uniquely defined by some (any) $(m+1)$-ary function $h(x_1, \ldots, x_{m+1})$ on A, some (any) consistent set of functions $S_{ij}(x_1, \ldots, x_m)$ in $(n\text{-Red}\,\mathfrak{F})^{(m)}$, and the following condition: for any $a_1, \ldots, a_{m+1}$ in A, if $\Phi(a_1, \ldots, a_{m+1})$ holds, then $g(a_1, \ldots, a_{m+1}) = h(a_1, \ldots, a_{m+1})$, and the equality $g(a_1, \ldots, a_{m+1}) = S_{ij}(a_1, \ldots, \widehat{a_j}, \ldots, a_{m+1})$ holds whenever some formula $\Phi_{ij}(a_1, \ldots, a_{m+1})$ is true.

Obviously, every $(n\text{-Red}\,\mathfrak{F})^{(m)}$-conservatively conditional function on A will be m-reducible to a fragment $(n\text{-Red}\,\mathfrak{F})^{(m)}$, and it will be n-reducible to a fragment $(n\text{-Red}\,\mathfrak{F})^{(n)}$ by the transitivity of an n-reduction relation and by the inductive assumption on $(n\text{-Red}\,\mathfrak{F})^{(m)}$. The converse is also true: all $(m+1)$-ary functions m-reducible to $(n\text{-Red}\,\mathfrak{F})^{(m)}$ on A will be $(n\text{-Red}\,\mathfrak{F})^{(m)}$-conservatively conditional. Thus we arrive at

THEOREM 1. For any clone $\mathfrak{F}$ on a set A and any natural n, the $\frac{1}{n}$-neighborhood of $\mathfrak{F}$ in a space $\mathfrak{F}_A$ is a convex sublattice of the lattice L_A of all clones on A with a least element—a boundedly generated clone $\mathfrak{F}^n_{\text{min}} = \langle \mathfrak{F}^{(n)} \rangle$—and a greatest element $\mathfrak{F}^n_{\text{max}} = n\text{-Red}\,\mathfrak{F}$, which is the union of the chain of m-fragments of clones ($m \ge n$):

$$\mathfrak{F}^{(n)} \subseteq (n\text{-Red}\,\mathfrak{F})^{(n+1)} \subseteq \ldots \subseteq (n\text{-Red}\,\mathfrak{F})^{(m)} \subseteq (n\text{-Red}\,\mathfrak{F})^{(m+1)} \subseteq \ldots,$$

where $(n\text{-Red}\,\mathfrak{F})^{(m+1)}$ consists of all $(n\text{-Red}\,\mathfrak{F})^{(m)}$-conservatively conditional functions.

Notice that the same clone $\mathfrak{F}^n_{\text{max}} = n\text{-Red}\,\mathfrak{F}$ of functions n-reducible to a fragment $\mathfrak{F}^{(n)}$ of a clone $\mathfrak{F}$ can be defined, avoiding an inductive construction of fragments $(n\text{-Red}\,\mathfrak{F})^{(m)}$ for $m \ge n$, as follows. We fix a natural number m greater than or equal to n. Let $\{S^m_k \mid k \in K\}$ be the collection of all partitions of a set $\{1, \ldots, m\}$ into n nonempty disjoint subsets $S^m_k = \{S^m_{kj} \mid 1 \le j \le n\}$ for $k \in K$.

$(**)$ By $\Phi^m_k(x_1, \ldots, x_m)$ we denote a condition (the conjunction of equalities between variables x_i for $1 \le i \le m$) of the form

$$\Phi^m_k(x_1, \ldots, x_m) = \bigwedge_{j=1}^{n} \left[\bigwedge_{i_1, i_2 \in S^m_{kj}} x_{i_1} = x_{i_2} \right].$$

A collection $\{F_k(x_1,\ldots,x_n) \mid k \in K\}$ of n-ary function on a set A is said to be *n-consistent* if, for any $k_1, k_2 \in K$ and any $a_1,\ldots,a_m$ in A, the fact that $\Phi_{k_1}(a_1,\ldots,a_m)$ and $\Phi_{k_2}(a_1,\ldots,a_m)$ are true on a tuple $\langle a_1,\ldots,a_m\rangle$ implies that $F_{k_1}(a_{i_1},\ldots,a_{i_n}) = F_{k_2}(a_{i_1},\ldots,a_{i_n})$, where $i_j \in S^m_{k_1 j} \bigcap S^m_{k_2\pi(j)}$; here π is a permutation on $\{1,\ldots,n\}$.

Definition 6. A function $F(x_1,\ldots,x_m)$ on a set A is said to be $\mathfrak{F}^{(n)}$-*conservatively conditional* if, for some n-consistent collection of functions F_k $(k \in K)$ from $\mathfrak{F}^{(n)}$ and some m-ary function $F'(x_1,\ldots,x_m)$, the following condition holds:

for all $a_1,\ldots,a_m$ in A, the truth of a condition $\Phi^m_k(a_1,\ldots,a_m)$ on a tuple $\langle a_1,\ldots,a_m\rangle$ (where $k \in K$) implies that $F(a_1,\ldots,a_m) = F_k(a_{i_1},\ldots,a_{i_n})$, where $i_j \in S^m_{kj}$, and if none of the conditions $\Phi^m_k(a_1,\ldots,a_m)$, where $k \in K$, holds on $\langle a_1,\ldots,a_m\rangle$, then $F(a_1,\ldots,a_m) = F'(a_1,\ldots,a_m)$.

Obviously, a collection of $\mathfrak{F}^{(n)}$-conservatively conditional functions coincides with a clone $\mathfrak{F}^n_{\max} = n\text{-Red}\,\mathfrak{F}$.

Below are some corollaries to Theorem 1. By the definition of $\mathfrak{F}^{(m)}$-conservatively conditional functions for arbitrary $\mathfrak{F} \in F_A$ and $m \in \omega$, we have

COROLLARY 1. Let $\mathfrak{F}$ be a clone on a set A and n a natural number. For any m-ary function $h(x_1,\ldots,x_m)$ on A, $m > n$, the clone $\mathfrak{F}^n_{\max}$ contains some m-ary function whose values coincide with those of a function h on any tuple $\langle a_1,\ldots,a_m\rangle$ including at least $(n+1)$ pairwise distinct elements of A.

We also point out some properties of being isolated for points in a space $\mathfrak{F}_A$, i.e., clones $\mathfrak{F}$ on A satisfying the equality $\mathfrak{F}^n_{\min} = \mathfrak{F}^n_{\max}$ for a natural number n.

COROLLARY 2. If a clone $\mathfrak{F}$ is an isolated point in $\mathfrak{F}_A$, i.e., $D_{\frac{1}{n}}(\mathfrak{F}) = \{\mathfrak{F}\}$ for some n, then a boundedly generated clone $\langle\mathfrak{F}^{(n)}\rangle$ for some $n \in \omega$ satisfies the following condition: for every $m > n$ and every m-ary function $h(x_1,\ldots,x_m)$ of a set A, the clone $\langle\mathfrak{F}^{(n)}\rangle$ contains an m-ary function whose values coincide with those of a function h on a tuple $\langle a_1,\ldots,a_m\rangle$ including at least $(n+1)$ pairwise distinct elements of A.

Denote by $T(\mathfrak{A})$ the clone of all termal functions of a universal algebra $\mathfrak{A} = \langle A;\sigma\rangle$. Notice that for any functional clone $\mathfrak{F}$ on a set A, $\mathfrak{F} = T(\mathfrak{A}_{\mathfrak{F}})$, where $\mathfrak{A}_{\mathfrak{F}} = \langle A;\mathfrak{F}\rangle$ is a universal algebra with universe A and all functions in $\mathfrak{F}$ as signature functions.

Denote by $CT(\mathfrak{A})$ the clone of all conditionally termal [6] functions of an algebra $\mathfrak{A}$. Recall that $CT(\mathfrak{A})$ contains all possible functions defined thus:

$$f(x_1,\ldots,x_n) = \begin{cases} g_1(x_1,\ldots,x_n) & \text{if } \Phi_1(x_1,\ldots,x_n), \\ \ldots & \\ g_k(x_1,\ldots,x_n) & \text{if } \Phi_k(x_1,\ldots,x_n), \end{cases}$$

where $g_i(x_1,\ldots,x_n) \in T(\mathfrak{A})$ and $\Phi_i(x_1,\ldots,x_n)$ are quantifier-free elementary formulas of a signature σ, in which case $\bigvee\limits_{i=1}^{k} \Phi_i(x_1,\ldots,x_n)$ is an identically true formula in σ, and for $1 \le i \ne j \le k$, $\Phi_i(x_1,\ldots,x_n) \wedge \Phi_j(x_1,\ldots,x_n)$ are unsatisfiable formulas.

Recall that a discriminator function $d_A(x,y,z)$ is defined on a set A as follows:

$$d_A(x,y,z) = \begin{cases} x & \text{if } x \neq y, \\ z & \text{if } x = y. \end{cases}$$

In this case, for any algebra $\mathfrak{A} = \langle A; \sigma \rangle$, we have $CT(\mathfrak{A}) = T(\mathfrak{A}_d)$, where $\mathfrak{A}_d = \langle A; \sigma, d_A \rangle$ is an enrichment of $\mathfrak{A}$ obtained by adding d_A to its signature functions.

Now we consider the following version of the definition of $\mathfrak{F}^{(n)}$-conservatively conditional functions for the case where $n \geq 3$ and $d_A \in \mathfrak{F}$. Let $m \geq n$ and $\{S_k^m \mid k \leq K\}$ be the collection of all partitions of a set $\{1, \ldots, m\}$ into n nonempty disjoint subsets $S_k^m = \{S_{k,j}^m \mid 1 \leq j \leq n\}$ for $k \leq K$.

(***) By $\psi_k^m(x_1, \ldots, x_m)$ for $1 \leq k \leq K$ we denote a condition (the conjunction of equalities and inequalities between variables x_i for $1 \leq i \leq m$) of the form

$$\psi_k^m(x_1, \ldots, x_m) = \bigwedge_{j=1}^{n} \left[\bigwedge_{i_1, i_2 \in S_{k,j}^m} x_{i_1} = x_{i_2} \right] \wedge \bigwedge_{j_1 \neq j_2 \leq n} \left[\bigwedge_{l_1 \in S_{kj_1}^m, l_2 \in S_{kj_2}^m} x_{l_1} \neq x_{l_2} \right].$$

Note that for any clone $\mathfrak{F}$ on a set A and any n, every $\mathfrak{F}^{(n)}$-conservatively conditional function $F(x_1, \ldots, x_m)$ for $m \geq n$ can be defined by some set of functions $G_k(x_1, \ldots, x_n)$, where $k \leq K$, in $\mathfrak{F}^{(n)}$ and some (any) function $F'(x_1, \ldots, x_m)$ on A as the following conditionally termal function of an algebra $\mathfrak{A}' = \langle A; \mathfrak{F}^{(n)}, F'(x_1, \ldots, x_m) \rangle$:

$$F(x_1, \ldots, x_m) = \begin{cases} G_1(x_{i_1}^1, \ldots, x_{i_n}^1) & \text{if } \psi_1(x_1, \ldots, x_m), \\ \ldots & \\ G_k(x_{i_1}^k, \ldots, x_{i_n}^k) & \text{if } \psi_k(x_1, \ldots, x_m), \\ \ldots & \\ G_K(x_{i_1}^K, \ldots, x_{i_n}^K) & \text{if } \psi_K(x_1, \ldots, x_m), \\ F'(x_1, \ldots, x_m) & \text{if the number of pairwise not equal} \\ & \text{values of variables } x_1, \ldots, x_m \\ & \text{is greater than } n. \end{cases}$$

Here $k \leq K$ and $x_{i_1}^k, \ldots, x_{i_n}^k$ are representatives of variables in the set $\{x_1, \ldots, x_m\}$ such that $i_1, \ldots, i_n$ occur in pairwise distinct sets in the partition S_k^m. Now it is no longer necessary to take care of the consistency of a set of functions G_k due to choosing conditions ψ_k disjunctive.

Thus the function $F(x_1, \ldots, x_m)$ occurs in the clone $\langle \mathfrak{F}^{(n)} \cup \{F', d_A\} \rangle$. This, together with Theorem 1, implies the following sufficient conditions for a clone $\mathfrak{F}$ to be an isolated point in a space $\mathfrak{F}_A$.

COROLLARY 3. Suppose that a clone $\mathfrak{F}$ includes a discriminator function, and for any $m > n \geq 3$ and any m-ary function $h(x_1, \ldots, x_m)$ on a set A, the clone $\langle \mathfrak{F}^{(n)} \rangle$ contains an m-ary function $h'(x_1, \ldots, x_m)$ whose values on any tuple $\langle a_1, \ldots, a_m \rangle$ of elements in A including at least $n+1$ pairwise distinct elements coincide with those of a function h on a same tuple. Then the clone $\mathfrak{F}$ is an isolated point in the space $\mathfrak{F}_A$.

A functional clone on a set A is called a *discriminator* clone if it contains a function d_A.

THEOREM 2. For any set A and any discriminator clone $\mathfrak{F}$ on A, every neighborhood of a point $\mathfrak{F}$ in a space $\mathfrak{F}_A$ contains an isolated point of this space.

Proof. Let $\mathfrak{F}$ be a discriminator clone on a set A. It suffices to show that every $\frac{1}{n}$-neighborhood of $\mathfrak{F}$ for $n \geq 3$ includes a clone $\mathfrak{F}'$ satisfying the conditions of Corollary 3, i.e., for any function $h(x_1, \ldots, x_m)$ on A, $m \geq n$, the clone $\mathfrak{F}'$ includes a function $h'(x_1, \ldots, x_m)$ whose values coincide with those of a function h on any tuple of elements of A containing at least $n+1$ pairwise distinct elements.

There are two cases to consider: (1) A is finite and (2) A is infinite.

(1) As is well known [7], a binary Webb function $v(x_1, x_2) = \max(x_1, x_2) + 1 \pmod{l}$, if $A = \{0, \ldots, l-1\}$, forms a complete system of functions on A. If the clone $\mathfrak{F}''$ on A is generated by the function

$$g_v(x_1, \ldots, x_{n+1}, y_1, y_2) = \begin{cases} y_1 & \text{if not all values of variables} \\ & x_1, \ldots, x_{n+1} \text{ are pairwise distinct,} \\ v(y_1, y_2) & \text{otherwise,} \end{cases}$$

then for any function $h(x_1, \ldots, x_m)$, $m \geq n+1$, on A there is a function $h'(x_1, \ldots, x_m)$ in $\mathfrak{F}''$ whose values coincide with those of a function h on any tuple of elements in A including at least $n+1$ pairwise distinct elements. By virtue of Corollary 3, the clone $\mathfrak{F}' = \langle \mathfrak{F}^{(n)} \cup \{g_v\} \rangle$ is isolated and $\mathfrak{F}'^{(n)} = \mathfrak{F}^{(n)}$; i.e., $\mathfrak{F}'$ occurs in the $\frac{1}{n}$-neighborhood of $\mathfrak{F}$, while its $\frac{1}{n+3}$-neighborhood consists of a single point—$\mathfrak{F}'$.

(2) Let $c(x_1, x_2)$ be a one-to-one mapping of A^2 to A and let R be the collection of all unary functions on A. Then the collection $R \cup \{c\}$ is a complete system of functions on A.

Let a clone $\mathfrak{F}''$ be generated by functions g_c and g_v, $r \in R$, where

$$g_c(x_1, \ldots, x_{n+1}, y_1, y_2) = \begin{cases} y_1 & \text{if not all values of variables} \\ & x_1, \ldots, x_{n+1} \text{ are pairwise distinct,} \\ c(y_1, y_2) & \text{otherwise;} \end{cases}$$

$$g_r(x_1, \ldots, x_{n+1}, y_1) = \begin{cases} y_1 & \text{if not all values of variables} \\ & x_1, \ldots, x_{n+1} \text{ are pairwise distinct,} \\ r(y_1) & \text{otherwise.} \end{cases}$$

Then for any function $h(x_1, \ldots, x_m)$, $m \geq n+1$, on A there is a function $h'(x_1, \ldots, x_m)$ in $\mathfrak{F}''$ whose values coincide with those of a function h on any tuple of elements in A including at least $n+1$ pairwise distinct elements. In view of Corollary 3, the clone $\mathfrak{F}' = \langle \mathfrak{F}^{(n)} \cup \{g_c, g_r \mid r \in R\} \rangle$ is an isolated point of the space $\mathfrak{F}_A$, with $\mathfrak{F}'^{(n)} = \mathfrak{F}^{(n)}$; i.e., $\mathfrak{F}'$ occurs in the $\frac{1}{n}$-neighborhood of a clone $\mathfrak{F}$, while the $\frac{1}{n+3}$-neighborhood of a clone $\mathfrak{F}'$ consists of a single point—$\mathfrak{F}'$. The theorem is proved.

Thus, for any set A, a collection of isolated points of a space $\mathfrak{F}_A$ is everywhere dense in a set of discriminator points of the space.

Of interest is the question whether a collection of isolated points of a space $\mathfrak{F}_A$ is everywhere dense in the space for any sets A. The answer is 'yes' for at least a two-element set A: in this case a countable space $\mathfrak{F}_A$ has only eight limit points.

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