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MAXIMAL SOLVABLE SUBGROUPS OF ODD INDEX IN SYMMETRIC GROUPS

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Dedicated to Yu. L. Ershov
on the occasion of his 80th birthday

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Maximal solvable subgroups of odd index in symmetric groups are classified up to conjugation.

INTRODUCTION

We study maximal solvable subgroups of odd index in symmetric groups. Interest in solvable subgroups of symmetric groups has its origins in group theory, Galois theory, and issues in solvability of algebraic equations in radicals (see Galois' Memoir [1], Jordan's Commentary [2] and especially his Treatise on substitutions [3]). In the frames of the approach adopted in group theory, it is natural to attempt to find maximal solvable subgroups. In dealing with the general case, in turn, we usually succeed in reducing it to one where a group is primitive.

Every primitive maximal solvable subgroup of the symmetric group Sym_n should be contained in an affine group. That is, every such subgroup must be an extension of a regular elementary Abelian group, whose order p^α coincides with the degree n of the symmetric group, by a maximal

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solvable subgroup of the group $\mathrm{GL}_\alpha(p)$. Much progress in the study of such subgroups has been made due to Suprunenko's theorem on primitive solvable subgroups of linear groups [4, Chap. 5]. Unfortunately, even that theorem does not make it possible to classify maximal solvable subgroups of symmetric groups up to conjugation. In the literature, therefore, more narrow classes of maximal solvable subgroups of symmetric groups are examined.

Thus, Mann in [5] showed that all solvable subgroups of greatest order are conjugate in Sym_n, and studied their properties. Maximal subgroups of odd index in simple and almost simple (in particular, in symmetric) groups have been described by Kantor [6], Liebeck and Saxl [7], and Maslova [8-12]. Therefore, it seems natural to concentrate efforts on examining maximal solvable subgroups of odd index. Such subgroups exist since a Sylow 2-subgroup is solvable, has odd index, and is always contained in some (and, up to conjugation, in any) maximal solvable subgroup of odd index.

In [13, 14], we came up with the classification program—which goes back to Wielandt's report [14]—, for so-called submaximal $\mathfrak{X}$-subgroups of odd index in finite simple groups for any complete (i.e., closed under taking subgroups, homomorphic images, and extensions) class $\mathfrak{X}$ of finite groups containing a group of even order. (The implementation of that program would allow us to find maximal $\mathfrak{X}$-subgroups of odd index in an arbitrary finite group.) An important particular case of such a complete class is the class of solvable groups, and, as follows from [13, Prop. 7], finding submaximal solvable subgroups of odd index in alternating groups requires a knowledge of all maximal solvable subgroups of odd index in symmetric groups. Aschbacher in [16] noted that every maximal solvable subgroup H in Sym_n, whose index $|\mathrm{Sym}_n : H|$ is odd, should be a $\{2,3\}$-group. The objective of the present paper is to propose a relatively simple arithmetic-combinatorial parametrization of conjugacy classes of such subgroups H.

We parametrize maximal solvable subgroups of odd index in Sym_n by so-called admissible diagrams associated with a number n.

Let sequences $\overline{a_k a_{k-1} \dots a_0}$ and $\overline{b_k b_{k-1} \dots b_0}$ be given in which every element is equal to 0, 1, or 3. We say that the sequence $\overline{b_k b_{k-1} \dots b_0}$ is obtained from $\overline{a_k a_{k-1} \dots a_0}$ *by elementary replacement* if there exists a number $i \geqslant 1$ such that $a_i = a_{i-1} = 1$, $b_i = 0$, $b_{i-1} = 3$, and $a_j = b_j$ for $j \neq i, (i-1)$. In other words, an elementary replacement is the replacement of two consecutive 1's in the sequence with 0 and 3. If $\overline{b_k b_{k-1} \dots b_0}$ is obtained from $\overline{a_k a_{k-1} \dots a_0}$ by elementary replacement, then, in view of $2^{i+1} + 2^i = 3 \cdot 2^i$,

$$\sum_{i=0}^{k} a_i \cdot 2^i = \sum_{i=0}^{k} b_i \cdot 2^i.$$

We define an extended binary representation of a natural number n. First, consider the binary decomposition

$$n = \overline{a_k a_{k-1} \dots a_0} = \sum_{i=0}^{\infty} a_i \cdot 2^i, \text{ where } a_i \in \{0,1\},\ a_i = 0 \text{ for } i > k, \text{ and } a_k = 1.$$

Clearly, $k = [\log_2 n]$.

An *extended* (*binary*) *representation of a number* n is the decomposition

$$n = \overline{b_k b_{k-1} \dots b_0} = \sum_{i=0}^{\infty} b_i \cdot 2^i \tag{1}$$

such that $b_i \in \{0, 1, 3\}$, $b_i = 0$ for $i > k$, and the representation $\overline{b_k b_{k-1} \dots b_0}$ is obtained from $\overline{a_k a_{k-1} \dots a_0}$ by a series of elementary replacements. Note that a given number has only finitely many extended binary representations, and finding them is a simple matter. For instance, the number $n = 22 = 2^4 + 2^2 + 2^1$ has exactly two extended binary representations: $\overline{10110}$ and $\overline{10030}$.

With each extended representation $\overline{b_k b_{k-1} \dots b_0}$ of n we associate a *template of a diagram* using the following rule. The template consists of $k+1$ rows, where $k = [\log_2 n]$, and the numbering starts from 0, from the bottom upward. The length of an ith row equals i (the zero row is empty). All rows are left-aligned. To the right of the ith row we put the digit b_i from the extended representation $\overline{b_k b_{k-1} \dots b_0}$.

Diagrams corresponding to the extended binary representation

$$n = \overline{b_k b_{k-1} \dots b_0},$$

are defined as follows.

(a) If a diagram template for such a representation has 0 opposite a row, then we delete the row.

(b) A row with number i opposite which there is a nonzero digit b_i is cut into bands of lengths 1 or 2, so that no two rows of length 1 would be consecutive. Such a partition of a row corresponds to the ordered partition $(l_1, \dots, l_t)$ of a number $i = l_1 + \dots + l_t$, where $l_j \in \{1, 2\}$, and if $l_j = 1$ for some $j < t$ then $l_{j+1} = 2$. We assume that bands of lengths $l_1, \dots, l_t$ are arranged in the specified order, from left to right. Notice that

$$2^i = 2^{l_1 + \dots + l_t} = 2^{l_1} \dots 2^{l_t}. \tag{2}$$

A template whose rows are partitioned in the above-indicated way is called a *diagram representing a number* n. A diagram is said to be *inadmissible* if it contains two consecutive rows opposite which is digit 1, and a larger row is obtained from a smaller one by adding on the right bands of length 1. Otherwise, a diagram is said to be *admissible*. The fact that $\mathscr{D}$ is an admissible diagram for a number n is written thus:

$$n \Vdash \mathscr{D} \text{ or } \mathscr{D} \dashv\vert\, n.$$

Note that not every extended binary representation will comply with an admissible diagram. For example, if an extended binary representation of n ends with two ones, i.e., $n = \dots + 1 \cdot 2^1 + 1 \cdot 2^0$, then a row with number 1 in any diagram complying with the extended binary representation is obtained from an (empty) row with number 0 by pasting a band of length 1, and such a diagram will always be inadmissible.

Let $\mathscr{D} \dashv\!\| \, n$ be an admissible diagram complying with the extended representation of a number $n = \overline{b_k b_{k-1} \dots b_0}$, and let the ith row of the diagram corresponding to a nonzero digit b_i be partitioned into bands whose lengths form an ordered tuple $(l_1, \dots, l_t)$. With the given diagram we associate a subgroup in Sym_n using the following rule. An ith row is assigned the wreath product

$$\mathrm{Sym}_{2^{l_1}} \wr \dots \wr \mathrm{Sym}_{2^{l_t}} \wr \mathrm{Sym}_{b_i},$$

which is treated as a transitive subgroup of $\mathrm{Sym}_{b_i \cdot 2^i}$ and is defined in it uniquely up to conjugation (see equality (2)). The entire diagram is assigned the direct wreath product (corresponding to its rows)[1]

$$\prod_{i=0}^{k} \mathrm{Sym}_{2^{l_1}} \wr \dots \wr \mathrm{Sym}_{2^{l_t}} \wr \mathrm{Sym}_{b_i}, \text{ where } k = [\log_2 n].$$

In view of equality (1), we can naturally identify this product with a subgroup in Sym_n and denote it by $S_{\mathscr{D}}$. The subgroup $S_{\mathscr{D}}$ is defined uniquely up to conjugation in Sym_n.

The main result of the present paper is describing maximal solvable subgroups of odd index in symmetric groups. Namely, the following holds:

THEOREM. Let $G = \mathrm{Sym}_n$. Then the map

$$\mathscr{D} \mapsto S_{\mathscr{D}}^G$$

yields a bijection between a set of admissible diagrams $\mathscr{D} \dashv\!\| \, n$ and a set of conjugacy classes of maximal solvable subgroups of odd index in Sym_n.

As an illustration we list all (up to conjugation) maximal solvable subgroups of odd index in Sym_{15}. We start with a canonical binary representation of the number 15:

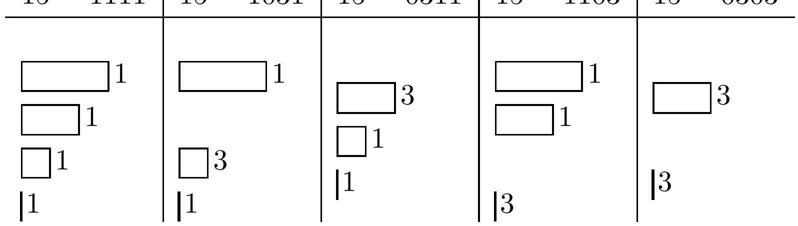
$$15 = \overline{1111}.$$

By elementary replacements we obtain all extended binary representations of 15:

$$\overline{1111},\ \overline{1031},\ \overline{0311},\ \overline{1103}, \overline{0303}.$$

Now we are in a position to create diagram templates for extended representations of the number $15 = 2^3 + 2^2 + 2^1 + 2^0$:

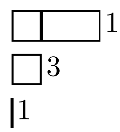
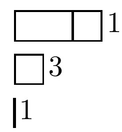
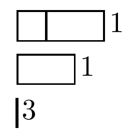
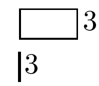
$15 = \overline{1111}$	$15 = \overline{1031}$	$15 = \overline{0311}$	$15 = \overline{1103}$	$15 = \overline{0303}$



[1] Rows opposite which we wrote 0 and then removed them from the template are assigned the wreath product equal to $\mathrm{Sym}_0 = 1$ by definition; i.e., the rows make no contribution to the direct product associated with the diagram.

The templates show that no admissible diagram will correspond to the extended representations $\overline{1111}$ and $\overline{0311}$.

Our present goal is to obtain all admissible diagrams and their corresponding subgroups (the group Sym_1 is trivial and so omitted):

$\overline{1111}$	$\overline{1031}$		$\overline{0311}$	$\overline{1103}$	$\overline{0303}$
$\varnothing$			$\varnothing$		
	$\mathrm{Sym}_2 \wr \mathrm{Sym}_4 \times \mathrm{Sym}_2 \wr \mathrm{Sym}_3$	$\mathrm{Sym}_4 \wr \mathrm{Sym}_2 \times \mathrm{Sym}_2 \wr \mathrm{Sym}_3$		$\mathrm{Sym}_2 \wr \mathrm{Sym}_4 \times \mathrm{Sym}_4 \times \mathrm{Sym}_3$	$\mathrm{Sym}_4 \wr \mathrm{Sym}_3 \times \mathrm{Sym}_3$

Thus the group Sym_{15} has four classes of conjugate maximal solvable subgroups of odd index:

$\mathrm{Sym}_2 \wr \mathrm{Sym}_4 \times \mathrm{Sym}_2 \wr \mathrm{Sym}_3$ with a fixed point;

$\mathrm{Sym}_4 \wr \mathrm{Sym}_2 \times \mathrm{Sym}_2 \wr \mathrm{Sym}_3$ with a fixed point;

$\mathrm{Sym}_2 \wr \mathrm{Sym}_4 \times \mathrm{Sym}_4 \times \mathrm{Sym}_3$ without fixed points;

$\mathrm{Sym}_4 \wr \mathrm{Sym}_3 \times \mathrm{Sym}_3$ without fixed points.

Here, the presence or absence of fixed points is understood in the sense that the listed groups, when treated as subgroups of Sym_{15}, act naturally.

1. PRELIMINARY RESULTS

Let G be a subgroup of $\mathrm{Sym}\,\Omega$ and let $\Delta \subseteq \Omega$. Following [17], we denote by $G_{\{\Delta\}}$ the stabilizer of a subset Δ in G, i.e., $G_{\{\Delta\}} = \{g \in G \mid \Delta^g = \Delta\}$, and by $G_{(\Delta)}$ the pointwise stabilizer of Δ, i.e., $G_{(\Delta)} = \{g \in G \mid \delta^g = \delta \text{ for all } \delta \in \Delta\}$. Clearly, the group $G_{\{\Delta\}}$ acts on the set Δ, i.e., the homomorphism $g \mapsto g^\Delta$ from $G_{\{\Delta\}}$ to $\mathrm{Sym}\,\Delta$ is defined, and its kernel coincides with $G_{(\Delta)}$.

LEMMA 1. Let G be a finite group, $N \trianglelefteq G$, and H be a subgroup of odd index in the group G. Then $|N : (H \cap N)|$ and $|G/N : HN/N|$ are odd indices.

Proof. We write the following chain of equalities:

$$
\begin{aligned}
|G:H| &= |G:HN||HN:H| \\
&= |G/N : HN/N|\frac{|H||N|}{|H \cap N||H|} \\
&= |G/N : HN/N||N : (H \cap N)|.
\end{aligned}
$$

The left part is odd, and so therefore are both factors in the right part. □

LEMMA 2 [18, Thm. 13.3]. Let H be a primitive subgroup of Sym_n containing a transposition. Then $H = \mathrm{Sym}_n$.

Let two natural numbers m and n be given. We introduce a relation $\preccurlyeq$ on a set of natural numbers. Following [8], for the natural numbers m and n, we write $m \preccurlyeq n$ if the binary representation of m is obtained from the binary representation of n by replacing some ones with zeros. In other words, if

$$n = \sum_{i=0}^{\infty} a_i \cdot 2^i \text{ and } m = \sum_{i=0}^{\infty} b_i \cdot 2^i, \quad \text{where } a_i, b_i \in \{0, 1\},$$

and almost all a_i and b_i are equal to zero, then $m \preccurlyeq n$ iff $b_i \leqslant a_i$ for all $i \geqslant 0$. It is easy to see that the relation $\preccurlyeq$ is a partial order on the set of natural numbers, and more exactly, a suborder of the natural order.

LEMMA 3 [10, 19]. Let $m \leqslant n$. Then the index of a subgroup $\mathrm{Sym}_m \times \mathrm{Sym}_{n-m}$ in Sym_n is equal to C_n^m. This index is odd if and only if $m \preccurlyeq n$.

The **proof** is given in a shorter and independent form. We have

$$|\mathrm{Sym}_n : (\mathrm{Sym}_m \times \mathrm{Sym}_{n-m})| = \frac{|\mathrm{Sym}_n|}{|\mathrm{Sym}_m||\mathrm{Sym}_{n-m}|} = \frac{n!}{m!(n-m)!} = \mathrm{C}_n^m.$$

Let $n = 2^{l_1} + \cdots + 2^{l_t}$, where $l_1 > \cdots > l_t \geqslant 0$. Consider a field F of characteristic 2 and a polynomial $(1+x)^n$ over the field F. Since $f \mapsto f^2$ is an endomorphism of the polynomial ring $F[x]$, we have

$$(1+x)^n = \prod_{i=1}^{t} (1+x)^{2^{l_i}} = \prod_{i=1}^{t} (1+x^{2^{l_i}}).$$

Taking into account $l_1 > \cdots > l_t$ and removing the brackets in the left part, we obtain

$$(1+x)^n = \sum_{m \preccurlyeq n} x^m.$$

Keeping in mind that F is a field of characteristic 2 and applying the binomial formula, we have

$$\mathrm{C}_n^m \equiv \begin{cases} 1 \pmod 2 & \text{if } m \preccurlyeq n, \\ 0 \pmod 2 & \text{if } m \not\preccurlyeq n. \end{cases} \quad \square$$

LEMMA 4. (1) Let m and k be natural numbers and let $k > 1$. The index $|\mathrm{Sym}_{mk} : (\mathrm{Sym}_m \wr \mathrm{Sym}_k)|$ is odd if and only if m is the degree of two.

(2) Let H be a transitive imprimitive subgroup of odd index in Sym_n and Δ be a block for H such that $|\Delta| < n$. Then $|\Delta|$ is the degree of two.

Proof. (1) See [19, Lemma 2].

(2) Let $|\Delta| = m$ and let a system of imprimitivity containing Δ consist of k blocks. Then $n = mk$ and the full stabilizer of this imprimitivity system in Sym_n is equal to $\mathrm{Sym}_m \wr \mathrm{Sym}_k$. Since

H is contained in $\mathrm{Sym}_m \wr \mathrm{Sym}_k$ and has odd index in Sym_n, the index $|\mathrm{Sym}_n : (\mathrm{Sym}_m \wr \mathrm{Sym}_k)|$ is also odd, and by (1), $|\Delta|$ is the degree of two. □

LEMMA 5 [20, Thm. 13.1.1, Ex. 3.4.2]. The group Sym_n is solvable if and only if $n \leqslant 4$.

LEMMA 6. If Sym_n contains a transitive solvable subgroup of odd index, then either $n = 2^t$ or $n = 2^t \cdot 3$ for some nonnegative integer t.

Proof. Let H be a transitive solvable subgroup of odd index in Sym_n. We use induction on n. There are two cases to consider—one where H is primitive and one where H is imprimitive.

Assume that H is primitive. Then $H = \mathrm{Sym}_n$ by Lemma 2 and $n \in \{1, 2, 3, 4\}$ by Lemma 5.

Now suppose that H is imprimitive and possesses an imprimitivity system consisting of k blocks of cardinality m for some m and k such that $n = mk$ and $m, k > 1$. Then

$$H \leqslant \mathrm{Sym}_m \wr \mathrm{Sym}_k.$$

Since H has odd index in Sym_n, the index $|\mathrm{Sym}_n : (\mathrm{Sym}_m \wr \mathrm{Sym}_k)|$ is also odd, and by Lemma 4, m is the degree of two. To complete the proof, we need only show the following:

($*$) either $k = 2^s$ or $k = 2^s \cdot 3$.

Since H acts on a set of blocks transitively, its image $\overline{H}$ under the natural epimorphism $\mathrm{Sym}_m \wr \mathrm{Sym}_k \to \mathrm{Sym}_k$ is a transitive solvable subgroup of odd index in Sym_k. Statement ($*$) is valid by the inductive assumption. □

LEMMA 7. Let H be a transitive imprimitive solvable subgroup of odd index in Sym_n and Δ be a nontrivial block of minimal size for the group H. Then $|\Delta| \in \{2, 4\}$.

Proof. According to Lemma 4, the number $|\Delta|$ is the degree of two, and it suffices to state that $|\Delta| \leqslant 4$. We show that the group $\mathrm{Sym}\,\Delta$ contains a solvable primitive subgroup of odd index and coincides with it by virtue of Lemma 2 and Sylow's theorem. In particular, $\mathrm{Sym}\,\Delta$ is solvable and $|\Delta| \leqslant 4$ in view of Lemma 5.

It follows from [17, 1.5.6] that $H_{\{\Delta\}}$ acts transitively on Δ. In view of $|\Delta|$ being minimal and according to [17, 1.5.10], the action of $H_{\{\Delta\}}$ on Δ is primitive. Let $\Delta_1 = \Delta, \Delta_2, \ldots, \Delta_m$ be an imprimitivity system containing a block Δ. Suppose that M is the full stabilizer of this system of imprimitivity in Sym_n. Then

$$M = \mathrm{Sym}\,\Delta \wr \mathrm{Sym}_m = (\mathrm{Sym}\,\Delta_1 \times \mathrm{Sym}\,\Delta_2 \times \cdots \times \mathrm{Sym}\,\Delta_m) \rtimes \mathrm{Sym}_m.$$

Denote by B a basis for the above wreath product, i.e., the subgroup

$$B = \mathrm{Sym}\,\Delta_1 \times \mathrm{Sym}\,\Delta_2 \times \cdots \times \mathrm{Sym}\,\Delta_m.$$

Consider the stabilizer $M_{\{\Delta\}}$ of a block $\Delta = \Delta_1$ in M. Clearly, $B \leqslant M_{\{\Delta\}}$ and $H_{\{\Delta\}} = H \cap M_{\{\Delta\}}$. The group $M_{\{\Delta\}}$ acts on the set Δ, thereby defining the homomorphism

$$\rho : g \mapsto g^{\Delta}$$

from $M_{\{\Delta\}}$ to $\operatorname{Sym}\Delta = \operatorname{Sym}\Delta_1$, and the kernel of this homomorphism coincides with $M_{(\Delta)}$. It is easy to see that

$$M_{\{\Delta\}} = \operatorname{Sym}\Delta_1 \times (\operatorname{Sym}\Delta_2 \times \cdots \times \operatorname{Sym}\Delta_m) \rtimes \operatorname{Sym}_{m-1} = \operatorname{Sym}\Delta \times M_{(\Delta)},$$

and the restriction of ρ to the factor $\operatorname{Sym}\Delta$ is the identity mapping

$$\operatorname{Sym}\Delta \to \operatorname{Sym}\Delta.$$

We show that $H^{\rho}_{\{\Delta\}}$ is a primitive solvable subgroup of odd index in $\operatorname{Sym}\Delta$. The group $H^{\rho}_{\{\Delta\}}$ is a homomorphic image of a subgroup $H_{\{\Delta\}}$ of a solvable group H and is therefore solvable. The primitivity of $H^{\rho}_{\{\Delta\}}$ follows from the primitivity of the action of $H_{\{\Delta\}}$ on Δ.

We have $B \leqslant M_{\{\Delta\}}$, and so $B \cap H \leqslant M_{\{\Delta\}} \cap H = H_{\{\Delta\}}$. Since the restriction of ρ to $\operatorname{Sym}\Delta$ is an identity mapping, the following relations hold:

$$\operatorname{Sym}\Delta \cap H = (\operatorname{Sym}\Delta \cap H)^{\rho} \leqslant (B \cap H)^{\rho} \leqslant H^{\rho}_{\{\Delta\}} \leqslant \operatorname{Sym}\Delta. \tag{3}$$

Appealing to Lemma 1, we conclude that $H \cap B$ is a subgroup of odd index in B and $H \cap \operatorname{Sym}\Delta$ is one in $\operatorname{Sym}\Delta$. In view of (3), $H^{\rho}_{\{\Delta\}}$ is a solvable primitive subgroup of odd index in $\operatorname{Sym}\Delta$. □

LEMMA 8 [21, Chap. 1, Lemma 15.4]. Let G, H, and K be permutation groups of degrees m, n, and k respectively. Then, for the naturally defined subgroups in Sym_{nmk}, the following equalities hold:

$$(G \wr H) \wr K = G \wr (H \wr K).$$

LEMMA 9. Let K be a maximal solvable subgroup of odd index in Sym_n, and let K be transitive. Then, up to conjugation, $K = S_{\mathscr{D}}$ for some diagram $\mathscr{D} \models n$, and the diagram $\mathscr{D}$ corresponds to the extended representation of a number n containing exactly one significant digit.

Proof. If the subgroup K is primitive, then $K = \operatorname{Sym}_n$ and $n \leqslant 4$ by Lemmas 2 and 5. In this case $K = S_{\mathscr{D}}$ holds, where

$$\mathscr{D} = \begin{cases} \mid 1 & \text{for } n = 1, \\ \square\, 1 & \text{for } n = 2, \\ \square\, 3 & \text{for } n = 3, \\ \boxed{}\, 1 & \text{for } n = 4. \end{cases}$$

Now let K be imprimitive. Lemma 6 implies that $n = 2^i \cdot 3^j$, where $i \geqslant 0$ and $j \in \{0, 1\}$. Let $\Delta_1, \ldots, \Delta_m$ be an imprimitivity system of K having nontrivial blocks of minimal size and let Δ be one of those blocks. Then $|\Delta| = 2^{l_0}$, where $l_0 \in \{1, 2\}$ by Lemma 7, $K \leqslant \operatorname{Sym}\Delta \wr \operatorname{Sym}_m$, and $m = n/2^{l_0} = 2^{i-l_0} \cdot 3^j$. We denote by B a basis for the wreath product $W = \operatorname{Sym}\Delta \wr \operatorname{Sym}_m$ and consider a natural epimorphism $\bar{\ } : W \to \operatorname{Sym}_m$. The subgroup

$$B = \operatorname{Sym}\Delta_1 \times \cdots \times \operatorname{Sym}\Delta_m$$

is solvable, and so therefore is BK. Hence $B \leqslant K$ by the maximality of a solvable subgroup K. Then the image $\overline{K}$ is a solvable transitive subgroup in Sym_m. Since $B \leqslant K$, the subgroup $\overline{K}$ is a maximal solvable subgroup in Sym_m.

By the inductive assumption, there exists an admissible diagram $\overline{\mathscr{D}} \models m$ such that $\overline{K} = S_{\overline{\mathscr{D}}}$ and $\overline{\mathscr{D}}$ corresponds to the extended binary representation of a number m containing exactly one significant digit. Therefore, the diagram $\overline{\mathscr{D}}$ consists of one row. Clearly,

$$K = \mathrm{Sym}\,\Delta \wr \overline{K} = \mathrm{Sym}_{2^{l_0}} \wr S_{\overline{\mathscr{D}}}.$$

Let the sole row of $\overline{\mathscr{D}}$ be partitioned into bands of lengths $l_1, \ldots, l_t$, counting from left to right, with the digit $3^j \in \{1, 3\}$ opposite it. Then

$$\begin{aligned} \overline{K} &= S_{\overline{\mathscr{D}}} = \mathrm{Sym}_{2^{l_1}} \wr \cdots \wr \mathrm{Sym}_{2^{l_t}} \wr \mathrm{Sym}_{3^j}, \\ K &= \mathrm{Sym}_{2^{l_0}} \wr S_{\overline{\mathscr{D}}} = \mathrm{Sym}_{2^{l_0}} \wr \mathrm{Sym}_{2^{l_1}} \wr \cdots \wr \mathrm{Sym}_{2^{l_t}} \wr \mathrm{Sym}_{3^j}. \end{aligned}$$

Consider a diagram $\mathscr{D}$ obtained by attaching on the left a band of length l_0 to the sole row of $\overline{\mathscr{D}}$. To complete the proof, it suffices to show that $\mathscr{D}$ is admissible. In other words, if $l_0 = 1$, then the length l_1 of the leftmost band in $\overline{\mathscr{D}}$ equals 2.

Suppose that $l_0 = l_1 = 1$. Then $K = \mathrm{Sym}_{2^{l_0}} \wr \mathrm{Sym}_{2^{l_1}} \wr \mathrm{Sym}_{2^{l_2}} \wr \cdots \wr \mathrm{Sym}_{2^{l_t}} \wr \mathrm{Sym}_{3^j}$ is strictly contained in the solvable group $\mathrm{Sym}_4 \wr \mathrm{Sym}_{2^{l_2}} \wr \cdots \wr \mathrm{Sym}_{2^{l_t}} \wr \mathrm{Sym}_{3^j}$, which contradicts the maximality. □

If the diagram $\mathscr{D}$ is associated with a number $2^i \cdot 3^j$, where $i \geqslant 0$ and $j \in \{0, 1\}$, and consists of a single row cut into bands of lengths $l_1, \ldots, l_t$, then the group $S_{\mathscr{D}} = \mathrm{Sym}_{2^{l_1}} \wr \cdots \wr \mathrm{Sym}_{2^{l_t}} \wr \mathrm{Sym}_{3^j}$ possesses a natural imprimitivity system $\Gamma_1, \ldots, \Gamma_s$, where $s = 2^{l_2+\cdots+l_t} \cdot 3^j$ and $\Gamma_1, \ldots, \Gamma_s$ are blocks of size 2^{l_1}. Since $S_{\mathscr{D}}$ contains as a subgroup the full symmetric group $\mathrm{Sym}\,(\Gamma_k)$ of each of the blocks, the given system of imprimitivity will be unrefinable [17, 1.5.10].

LEMMA 10. Let an admissible diagram $\mathscr{D}$ be associated with a number $n = 2^i \cdot 3^j$, where $i \geqslant 0$ and $j = 0, 1$, and let it consist of one row. Consider a group $H = S_{\mathscr{D}}$ and a natural imprimitivity system $\Gamma_1, \ldots, \Gamma_s$ of the group H chosen as described above. Then each block of any nontrivial system of imprimitivity of H will be the union of some blocks $\Gamma_1, \ldots, \Gamma_s$.

Proof. Consider an arbitrary nontrivial imprimitivity system $\Delta_1, \ldots, \Delta_t$ of H. Let

$$\Gamma \in \{\Gamma_1, \ldots, \Gamma_s\} \quad \text{and} \quad \Delta \in \{\Delta_1, \ldots, \Delta_t\}.$$

By the definition of a subgroup $H = S_{\mathscr{D}}$, the inclusion $\mathrm{Sym}\,\Gamma \leqslant H$ holds. It suffices to show that $\Gamma \leqslant \Delta$ if $\Gamma \cap \Delta \neq \varnothing$.

Assume that $\Gamma \cap \Delta \neq \varnothing$ and $\Gamma \nleqslant \Delta$, i.e., there exist $\alpha \in \Gamma \cap \Delta$ and $\beta \in \Gamma \backslash \Delta$. Notice that $\Delta \nleqslant \Gamma$ since Γ is a nontrivial minimal block for the group H. Therefore, there exists $\delta \in \Delta \backslash \Gamma$. Consider a transposition $(\alpha\beta)$. We have $(\alpha\beta) \in \mathrm{Sym}\,\Gamma \leqslant H$. This yields $\delta = \delta^{(\alpha\beta)} \in \Delta^{(\alpha\beta)} \cap \Delta$. Consequently, $\Delta^{(\alpha\beta)} = \Delta$. In particular, $\alpha^{(\alpha\beta)} \in \Delta^{(\alpha\beta)} = \Delta$. On the other hand, $\alpha^{(\alpha\beta)} = \beta$ and $\beta \in \Gamma \backslash \Delta$. Contradiction. □

2. MAIN RESULT

Proof of the theorem. It suffices to verify the following statements:

(1) if H is a maximal solvable subgroup of odd index in Sym_n, then H coincides with some subgroup $S_{\mathscr{D}}$, where $\mathscr{D} \models n$ is an admissible diagram;

(2) every such subgroup $S_{\mathscr{D}}$ is a maximal solvable subgroup of Sym_n and its index in Sym_n is odd.

We start with statement (1). Let H be a maximal solvable subgroup of Sym_n, where $n > 1$, and let $|\operatorname{Sym}_n : H|$ be an odd number. We show that $H = S_{\mathscr{D}}$ for some admissible diagram $\mathscr{D} \models n$.

First suppose that H is primitive. By Lemma 2, the subgroup H coincides with Sym_n and $n \in \{2, 3, 4\}$. Thus $H = S_{\mathscr{D}}$, where

$$\mathscr{D} = \begin{cases} |\,1 & \text{for } n = 1, \\ \square\,1 & \text{for } n = 2, \\ \square\,3 & \text{for } n = 3, \\ \boxed{}\,1 & \text{for } n = 4, \end{cases}$$

in which case the theorem is true.

Now assume that H is transitive but imprimitive. Choose a nontrivial block Δ of minimal size. Then $|\Delta| = 2^l \in \{2, 4\}$ by Lemma 7, and $H \leqslant \operatorname{Sym}\Delta \wr \operatorname{Sym}_m$, where $m = n/|\Delta|$. Let $\overline{}$: $\operatorname{Sym}\Delta \wr \operatorname{Sym}_m \to \operatorname{Sym}_m$ be a natural epimorphism. Note that the kernel $B \cong \underbrace{\operatorname{Sym}\Delta \times \cdots \times \operatorname{Sym}\Delta}_{m \text{ times}}$ of this epimorphism is solvable. We argue to state that $\overline{H}$ is a maximal solvable subgroup of odd index in Sym_m. By virtue of Lemma 1, it suffices to show that $\overline{K} = \overline{H}$ if $\overline{H} \leqslant \overline{K}$ for some solvable subgroup $\overline{K} \leqslant \operatorname{Sym}_m$. Let K be the complete preimage of $\overline{K}$ in $\operatorname{Sym}\Delta \wr \operatorname{Sym}_m$. Then K is an extension of a solvable group B by a solvable group $\overline{K}$. Consequently, K is solvable and contains a maximal solvable subgroup H. Hence $K = H$ and $\overline{K} = \overline{H}$.

By the inductive assumption, $\overline{H} = S_{\overline{\mathscr{D}}}$ for some $\overline{\mathscr{D}} \Vdash m$. The group H is transitive and the subgroup $\overline{H}$ of Sym_m is also transitive; so the diagram $\overline{D}$ consists of one row. It follows from the definition that $H \leqslant B \cdot S_{\overline{\mathscr{D}}} \simeq \operatorname{Sym}\Delta \wr S_{\overline{\mathscr{D}}}$. The solvability of $\operatorname{Sym}\Delta \wr S_{\overline{\mathscr{D}}}$ implies that $H = \operatorname{Sym}\Delta \wr S_{\overline{\mathscr{D}}} = S_{\mathscr{D}}$, where $\mathscr{D}$ is a diagram obtained from the one-row diagram $\overline{\mathscr{D}}$ by attaching on the left a band of length l, where $2^l = |\Delta|$.

We show that the diagram $\mathscr{D}$ does not contain two consecutive bands of length 1. Assume the contrary. The diagram $\overline{\mathscr{D}}$ is admissible, and so we may suppose that $l = 1$ and the leftmost band also has length 1; i.e., $S_{\overline{\mathscr{D}}} = \operatorname{Sym}_2 \wr S_{\widetilde{\mathscr{D}}}$ for some diagram $\widetilde{\mathscr{D}} \Vdash m/2$. By Lemma 8,

$$H = \operatorname{Sym}_2 \wr S_{\overline{\mathscr{D}}} = \operatorname{Sym}_2 \wr (\operatorname{Sym}_2 \wr S_{\widetilde{\mathscr{D}}}) = (\operatorname{Sym}_2 \wr \operatorname{Sym}_2) \wr S_{\widetilde{\mathscr{D}}} < \operatorname{Sym}_4 \wr S_{\widetilde{\mathscr{D}}},$$

which clashes with H being a maximal solvable subgroup.

Now let H be intransitive, $\mathrm{O}_1, \ldots, \mathrm{O}_s$ be all of its orbits, and $n_i = |\mathrm{O}_i|$. Without loss of generality, we assume that

$$n_1 \leqslant n_2 \leqslant \cdots \leqslant n_s.$$

Then $H \leqslant \mathrm{Sym}\,\mathrm{O}_1 \times \cdots \times \mathrm{Sym}\,\mathrm{O}_s$. Since H is a maximal solvable subgroup, H is generated by its projections H_i onto $\mathrm{Sym}\,\mathrm{O}_i$, of which each is a maximal solvable subgroup of odd index in $\mathrm{Sym}\,\mathrm{O}_i$ and is transitive on O_i. Furthermore, the subgroup $\mathrm{Sym}\,\mathrm{O}_1 \times \cdots \times \mathrm{Sym}\,\mathrm{O}_s$ contains a Sylow 2-subgroup of the group Sym_n. Therefore, if A and B are two unions of some orbits O_i such that $\mathrm{A} \subseteq \mathrm{B}$, then $\mathrm{Sym}\,(\mathrm{A}) \times \mathrm{Sym}\,(\mathrm{B}\backslash\mathrm{A})$ has odd index in $\mathrm{Sym}\,(\mathrm{B})$, and by Lemma 3, $|\mathrm{A}| \preccurlyeq |\mathrm{B}|$. This means that for any subtuple $n_{i_1}, \ldots, n_{i_t}$ of the tuple of numbers $n_1, \ldots, n_s$, the binary representation of a number $n_{i_1} + \cdots + n_{i_t}$ (e.g., a binary representation of the numbers $n_1, \ldots, n_s$ themselves) is obtained from the binary representation of a number n by replacing some ones with zeros. If we take one of the orbits O_{i_1} to be A and take its union with another orbit O_{i_2} to be B we see that ones may hold only different positions in the binary representation of numbers n_{i_1} and n_{i_2} for $i_1 \neq i_2$. By Lemma 6, either $n_i = 2^j$ or $n_i = 2^j \cdot 3 = 2^{j+1} + 2^j$. Replacing each summand in the equality $n = \sum n_i$ either by 2^j or by $2^j \cdot 3$ yields an extended binary representation of n.

Every group H_i complies with an admissible one-row diagram $\mathscr{D}_i \Vdash n_i$, and the rows of such diagrams are pairwise different in length. Pasting these, we obtain a diagram $\mathscr{D}$ associated with a number n. We show that $\mathscr{D}$ is admissible. Suppose the contrary. Then the diagram $\mathscr{D}$ has two neighboring rows corresponding to diagrams $\mathscr{D}_i$ and $\mathscr{D}_{i+1}$, in which case $\mathscr{D}_{i+1}$ is obtained from $\mathscr{D}_i$ by attaching a band of length 1. Therefore, $H_{i+1} = H_i \wr \mathrm{Sym}_2$ and $H_i \times H_{i+1} < H_i \wr \mathrm{Sym}_3$. Consequently,

$$H = H_1 \times \cdots \times H_s < H_1 \times \cdots \times (H_i \wr \mathrm{Sym}_3) \times H_{i+2} \times \cdots \times H_s \leqslant \mathrm{Sym}_n,$$

which contradicts the maximality of H.

Now we consider statement (2). Let $H = S_{\mathscr{D}}$ for some admissible diagram $\mathscr{D} \models n$. We show that H is a maximal solvable subgroup of odd index in Sym_n. The definition of a group $S_{\mathscr{D}}$ implies that H is solvable. It is straightforward to verify that H has odd index in Sym_n. It remains to show that if $H \leqslant M$ for a maximal solvable subgroup M, then $H = M$.

The following lemma yields the desired statement for a particular case.

LEMMA 11. Let an admissible diagram $\mathscr{D}$ be associated with a number $n = 2^i \cdot 3^j$, where $i \geqslant 0$ and $j = 0, 1$. Then the group $S_{\mathscr{D}}$ corresponding to the diagram $\mathscr{D}$ is a maximal solvable subgroup of odd index in Sym_n.

Proof. Clearly, the diagram has either one or two rows. Furthermore, $S_{\mathscr{D}} = \mathrm{Sym}_n$ for $n \leqslant 4$, and so below we assume that $n > 4$.

Consider the case where $\mathscr{D}$ consists of one row. Then $H = S_{\mathscr{D}}$ is a transitive group. Assume that $H \leqslant M$, where M is a maximal solvable subgroup of odd index in Sym_n. In view of the above, $M = S_{\mathscr{E}}$ for some admissible diagram $\mathscr{E} \models n$. Since M is also transitive, $\mathscr{E}$ consists of one row. The condition $n > 4$ implies that M is imprimitive. Let the leftmost bands in $\mathscr{D}$ and $\mathscr{E}$ have lengths k_1 and l_1 respectively. Fix natural imprimitivity systems $\Gamma_1, \ldots, \Gamma_s$ and $\Delta_1, \ldots, \Delta_t$ for groups H and M, respectively, such that $|\Gamma_i| = 2^{k_1}$ and $|\Delta_i| = 2^{l_1}$. Then $k_1, l_1 \in \{1, 2\}$. The system $\Delta_1, \ldots, \Delta_t$ will also be an imprimitivity system for the group H. We show that the two imprimitivity systems coincide. Choose $\Gamma \in \{\Gamma_1, \ldots, \Gamma_s\}$ and $\Delta \in \{\Delta_1, \ldots, \Delta_t\}$ such that $\Gamma \cap \Delta \neq \varnothing$. By Lemma 10, $\Gamma \subseteq \Delta$.

By construction, the subgroup H is contained in $\operatorname{Sym}\Gamma \wr \operatorname{Sym}_s$, and the basis for the wreath product is contained in H. Consider a homomorphism $^{-} : \operatorname{Sym}\Gamma \wr \operatorname{Sym}_s \to \operatorname{Sym}_s$. The image of H under this homomorphism coincides with a group $S_{\overline{\mathscr{D}}}$ complying with a diagram $\overline{\mathscr{D}}$ which is obtained from $\mathscr{D}$ by removing the leftmost band and is therefore admissible. By the inductive assumption, $\overline{H}$ is a maximal solvable subgroup of odd index in Sym_s. The group $\overline{H}$ acts on the set $\Gamma_1, \ldots, \Gamma_s$.

If $\Delta = \Gamma$, then $M \leqslant \operatorname{Sym}\Gamma \wr \operatorname{Sym}_s$ and $\overline{M} = \overline{H}$, and hence $M = H$.

Now let $\Delta \neq \Gamma$. Then $|\Delta| = 4$, $|\Gamma| = 2$, and each block Δ_i is the union of two blocks in the system $\Gamma_1, \ldots, \Gamma_s$. An imprimitivity system $\Delta_1, \ldots, \Delta_t$ of H defines an imprimitivity system $\overline{\Delta_1}, \ldots, \overline{\Delta_t}$ of $\overline{H}$ on a set $\{\Gamma_1, \ldots, \Gamma_s\}$ via the following rule: Γ_i and Γ_j belong to one block $\overline{\Delta}$ iff Γ_i and Γ_j are contained in one block Δ. By Lemma 10, we have $|\overline{\Delta}| = 2^{k_2}$, where k_2 is the size of a left band in $\overline{\mathscr{D}}$. Hence $k_2 = 1$ and the diagram $\mathscr{D}$ contains two consecutive bands of length 1. Contradiction.

By induction, we conclude that $\mathscr{D} = \mathscr{E}$.

Consider the second case where $\mathscr{D}$ consists of two neighboring rows with 1 opposite each of these. A subgroup H is intransitive and has two orbits on a set Ω. Denote them by O_1 and O_2. To be specific, we assume that $|\mathrm{O}_1| \geqslant |\mathrm{O}_2|$. By the definition of a group $S_{\mathscr{D}}$, $|\mathrm{O}_1| = 2|\mathrm{O}_2|$ and both of the numbers are the degrees of two. Let $\mathscr{D}_1$ and $\mathscr{D}_2$ be one-row diagrams complying with orbits O_1 and O_2, respectively, with $H = S_{\mathscr{D}_1} \times S_{\mathscr{D}_2}$. As above, consider a diagram $\mathscr{E}$ such that $M = S_{\mathscr{E}}$.

If M is not transitive, then M has two orbits of the same length as that of H. The above argument implies that $H = M$.

Suppose now that M is transitive. Let $\Delta_1, \ldots, \Delta_s$ be its unrefinable imprimitivity system with blocks of size $2^{l_1} \in \{2, 4\}$. The group H acts on the set of blocks $\{\Delta_1, \ldots, \Delta_s\}$.

Assume that there exists a block $\Delta \in \{\Delta_1, \ldots, \Delta_s\}$ for which $\Delta \cap \mathrm{O}_1 \neq \varnothing$ and $\Delta \cap \mathrm{O}_2 \neq \varnothing$. Then, for any block $\Delta' \in \{\Delta_1, \ldots, \Delta_s\}$, we have $|\Delta' \cap \mathrm{O}_i| = |\Delta \cap \mathrm{O}_i|$, where $i = 1, 2$. Indeed, let O_i be the orbit for which $\Delta' \cap \mathrm{O}_i \neq \varnothing$, and let $\alpha \in \Delta' \cap \mathrm{O}_i$. Choose $\beta \in \Delta \cap \mathrm{O}_i$ arbitrarily and let $x \in H$ be an element such that $\alpha = \beta^x$. Then $\Delta' = \Delta^x$ since $x \in M$. Furthermore, $\mathrm{O}_i^x = \mathrm{O}_i$. Consequently,

$$\Delta' \cap \mathrm{O}_i = \Delta^x \cap \mathrm{O}_i^x = (\Delta \cap \mathrm{O}_i)^x$$

and $|\Delta' \cap \mathrm{O}_i| = |\Delta \cap \mathrm{O}_i|$. Also

$$|\Delta' \cap \mathrm{O}_{3-i}| = |\Delta'| - |\Delta' \cap \mathrm{O}_i| = |\Delta| - |\Delta \cap \mathrm{O}_i| = |\Delta \cap \mathrm{O}_{3-i}|.$$

Now, for $i = 1, 2$, the following equalities hold:

$$|\mathrm{O}_i| = s|\Delta \cap \mathrm{O}_i|. \tag{4}$$

Then numbers $|\Delta \cap \mathrm{O}_i|$ are the degrees of two. Since the binary representation is unique, the degree of two cannot be the sum of two different degrees of two. From the equality

$$|\Delta| = |\Delta \cap \mathrm{O}_1| + |\Delta \cap \mathrm{O}_2|$$

and the fact that $|\Delta|$ is also the degree of two, we now derive

$$|\Delta \cap \mathrm{O}_1| = |\Delta \cap \mathrm{O}_2|. \tag{5}$$

Equalities (4) and (5) imply $|\mathrm{O}_1| = |\mathrm{O}_2|$, which clashes with $|\mathrm{O}_1| = 2|\mathrm{O}_2|$.

Hence, for any block $\Delta \in \{\Delta_1, \ldots, \Delta_s\}$, either $\Delta \subseteq \mathrm{O}_1$ or $\Delta \subseteq \mathrm{O}_2$. Arguing as in the case where $\mathscr{D}$ is a diagram consisting of one band, we make sure that the lengths of the leftmost bands in $\mathscr{D}_1$ and $\mathscr{D}_2$ are equal to l_1, where $|\Delta| = 2^{l_1}$. Moreover, the corresponding imprimitivity systems for the groups $S_{\mathscr{D}_1}$ and $S_{\mathscr{D}_2}$ on the sets O_1 and O_2 are composed of blocks $\Delta_1, \ldots, \Delta_s$ contained in these sets. Since H stabilizes the system $\Delta_1, \ldots, \Delta_s$, we have $H \leqslant \mathrm{Sym}\,\Delta \wr \mathrm{Sym}_s$. As above, let $\overline{} : \mathrm{Sym}\,\Gamma \wr \mathrm{Sym}_s \to \mathrm{Sym}_s$ be a natural epimorphism. Then $\overline{H} = S_{\overline{\mathscr{D}_1}} \times S_{\overline{\mathscr{D}_2}} = S_{\overline{\mathscr{D}}}$, where the diagram $\overline{\mathscr{D}}$ is obtained from $\mathscr{D}$ by removing the leftmost bands of length l_1 in the rows $\mathscr{D}_1$ and $\mathscr{D}_2$ and consists of bands $\overline{\mathscr{D}}_1$ and $\overline{\mathscr{D}}_2$.

By the inductive assumption, $\overline{H}$ is a maximal solvable subgroup of odd index in Sym_s, and its complete preimage H is a maximal solvable subgroup of odd index in $\mathrm{Sym}\,\Delta \wr \mathrm{Sym}_s$. On the other hand, $M = S_{\mathscr{E}} \leqslant \mathrm{Sym}\,\Delta \wr \mathrm{Sym}_s$, and so $H = S_{\mathscr{E}} = M$. □

Now we show that $H = S_{\mathscr{D}}$ is maximal as a solvable subgroup in the general case. We use induction on n. The *degree of a row* in a diagram $\mathscr{D}$ is the number $b \cdot 2^l$, where l is the length of the row and b is the corresponding digit in the extended representation, which is opposite that row in the diagram. Thus the degrees of rows are cardinalities of the orbits of the group $S_{\mathscr{D}}$.

If M is transitive, then, by Lemma 6, $n = 2^l \cdot 3^j$, where $j \in \{0, 1\}$. In view of Lemma 11, $H = M$.

Let M be intransitive. Then H is also intransitive, and the orbits of M are the unions of orbits of H, which, in turn, correspond to rows of $\mathscr{D}$. Furthermore, $M \leqslant \mathrm{Sym}_m \times \mathrm{Sym}_{n-m}$ for some $m \leqslant n$, and the rows of $\mathscr{D}$ can be divided into two disjoint subsets $\mathscr{D}_1$ and $\mathscr{D}_2$, so that m and $n - m$ will be the sums of the degrees of rows in these sets. Denote by $\mathscr{D}_1$ and $\mathscr{D}_2$ the diagrams composed of the rows of the two sets. Clearly, $\mathscr{D}_1 \models m$ and $\mathscr{D}_2 \models n - m$.

The definition of $S_{\mathscr{D}}$ implies that

$$H = S_{\mathscr{D}} = S_{\mathscr{D}_1} \times S_{\mathscr{D}_2} \leqslant \mathrm{Sym}_m \times \mathrm{Sym}_{n-m},$$

and by the inductive assumption, $S_{\mathscr{D}_1}$ and $S_{\mathscr{D}_2}$ are maximal solvable subgroups of odd index in Sym_m and Sym_{n-m}. Let M_1 and M_2 be projections of the subgroup M onto Sym_m and Sym_{n-m}

respectively. Then

$$S_{\mathscr{D}_i} \leqslant M_i,$$

and hence $M_i = S_{\mathscr{D}_i}$. Thus

$$H \leqslant M \leqslant M_1 \times M_2 = S_{\mathscr{D}_1} \times S_{\mathscr{D}_2} = H,$$

and so $M = H$. $\square$

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