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MULTI-AGENT TEMPORAL NONTRANSITIVE LINEAR LOGICS AND THE ADMISSIBILITY PROBLEM

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We study an extension of temporal logic, a multi-agent logic on models with nontransitive linear time (which is, in a sense, also an extension of interval logic). The proposed relational models admit lacunas in admissibility relations among agents: information accessible for one agent may be inaccessible for others. A logical language uses temporary operators 'until' and 'next' (for each of the agents), via which we can introduce modal operations 'possible' and 'necessary.' The main problem under study for the logic introduced is the recognition problem for admissibility of inference rules. Previously, this problem was dealt with for a logic in which transitivity intervals have a fixed uniform length. Here the uniformity of length is not assumed, and the logic is extended by individual temporal operators for different agents. An algorithm is found which decides the admissibility problem in a given logic, i.e., it recognizes admissible inference rules.

INTRODUCTION

Temporal logic is a branch of modern nonclassical logics within which models are constructed to analyze propositions whose truth values vary over time. The advent of temporal logic goes back to the early 1950s, to A. N. Prior's works. Since then, it has been (and still is) an active field of research in mathematical logic, computer science, and artificial intelligence (see [1-3]). An

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important particular case is the linear temporal logic (LTL) which has been employed in analyzing computation protocols and correctness verification. The use of mathematical logic in information sciences involves the employment of the machinery of nonclassical logic in analyzing correctness.

For example, multi-agent logics applying modalities interpreted via agent relations of accessibility for model verification were used in studying the interaction and autonomy of agents (see [4-12]).

Modeling interactions between agents as a dual representation of common knowledge (information) was offered in [12]. The very conception of common knowledge for agents was proposed and thoroughly studied in [13] in which agent relations of knowledge are used as (S5-like) modalities. Properties of interval linear logics were examined in [14]. Knowledge, as a general conception, was based on a multi-agent approach, since personal knowledge may generate common knowledge only through interactions among agents. A similar simulation in terms of symbolic logic can be traced back to the late 1950s. [15] was probably the first work of a book size which proposed to employ modalities to describe mathematical semantics for the concept of knowledge. Temporal ontology and arguments that include temporal components were taken up in [16]. The technique of formal automata for solving the satisfiability problem in linear temporal logic was developed in [17, 18]. Central to research in nonclassical logics are algorithmic problems such as decidability and recognizability [14, 19].

Recently, I have undertaken a study into nontransitive temporal logics that are close to an interval temporal logic [20-23]. The satisfiability and decidability problems for such logics were solved. In particular, a solution to the admissibility problem for a nontransitive temporal logic was found, but only in the case where all transitivity intervals in models have a fixed uniform length [23]. The present paper gives a solution to the admissibility problem in the general case where all transitivity intervals have an arbitrary nonuniform length bounded by a given fixed number. Furthermore, the models may have new expanded properties: (1) different accessibility operations are allowed for different agents; (2) accessibility operations can have lacunas in basic frames. The latter approach for standard models was considered in [24]. We point out a decision algorithm for the admissibility problem in a given logic—namely, it recognizes admissible rules of inference.

1. INTERVAL LINEAR MULTI-AGENT LOGIC

First we describe the semantics and language for the logical system proposed. Models are not transitive linear structures; they are based on composite interval frames. The assumption that all computational flows are linear and potentially infinite is too strong. In fact, all resources are always bounded: these can be rather rich, but with a certain hypothetical upper bound. Therefore, as basic semantics we choose the following frames.

In what follows, N, as is common in the mathematical notation, denotes the set of natural numbers. We fix some $In \subset N$, an infinite set of time indices. For every $i \in In$, by i(next) we denote a least number greater than i in In; $In(i) := [i, i(\text{next})]$. Thus $N = \bigcup_{i \in In} In(i)$. We consider $In(i)$ as transitive time-intervals.

A *temporary interval linear multi-agent k-frame* is the structure

$$\mathcal{F} := \left\langle \bigcup_{i \in In} In(i), R_1, \ldots, R_k, \text{Next} \right\rangle,$$

where any R_j is a binary relation, which is some bound for the standard linear order $\leq$ in the interval $I(n)$, i.e.,

$$\forall i \in In \ \forall j \ R_j \subseteq (In(i) \times In(i)) \cap (\leq).$$

Moreover, different transitive intervals are inaccessible with respect to R_j, i.e., $\forall x \in In(i) \ \forall y \notin In(i) \ \neg(xR_jy)$. Furthermore, Next is a standard binary relation, the next natural number. We suppose that m is a fixed natural number and $\forall i \in In \ (i(\text{next}) - i) \leq m$; m is called an *upper bound for transitivity of a frame* (the frames themselves are naturally not transitive).

A model $\mathcal{M}$ on $\mathcal{F}$ is $\mathcal{F}$ with a valuation V for some set $Prop$ of propositional variables (i.e, $\forall p \in Prop \ V(p) \subseteq \bigcup_{i \in In} In(i)$). We write

$$\mathcal{M} := \left\langle \bigcup_{i \in In} In(i), R_1, \ldots, R_k, \text{Next}, V \right\rangle.$$

The universe of $\mathcal{M}$ is merely the set N of all natural numbers; we denote it by $|\mathcal{M}|$. For brevity, we write $a \in \mathcal{M}$ in place of $a \in |\mathcal{M}|$. If $a \in \mathcal{M}$ and $a \in V(p)$, then we write $(\mathcal{M}, a) \Vdash_V p$ and say that *p is true at a under a valuation V*. A logical language is introduced as follows.

Definition 1. The *set of all formulas* contains the set $Prop$ of all propositional variables and is closed under taking the Boolean operations $\wedge, \vee, \neg, \rightarrow$, the unary operation N (next), and the binary operations U_j (until), $j \in [1, k]$ (for each agent j).

Definition 2. For any model $\mathcal{M}$, *truth values* can be extended from propositional variables to all formulas as follows:

$$\begin{aligned}
\forall p \in Prop \ (\mathcal{M}, a) \Vdash_V p \ &\Leftrightarrow \ a \in V(p); \\
(\mathcal{M}, a) \Vdash_V (\varphi \wedge \psi) \ &\Leftrightarrow \ (\mathcal{M}, a) \Vdash_V \varphi \wedge (\mathcal{M}, a) \Vdash_V \psi; \\
(\mathcal{M}, a) \Vdash_V (\varphi \vee \psi) \ &\Leftrightarrow \ (\mathcal{M}, a) \Vdash_V \varphi \vee (\mathcal{M}, a) \Vdash_V \psi; \\
(\mathcal{M}, a) \Vdash_V (\varphi \rightarrow \psi) \ &\Leftrightarrow \ (\mathcal{M}, a) \nVdash_V \varphi \vee (\mathcal{M}, a) \Vdash_V \psi; \\
(\mathcal{M}, a) \Vdash_V \neg\varphi \ &\Leftrightarrow \ \neg[(\mathcal{M}, a) \Vdash_V \varphi]; \\
(\mathcal{M}, a) \Vdash_V N\varphi \ &\Leftrightarrow \ \forall b \ [(a \,\text{Next}\, b) \Rightarrow (\mathcal{M}, b) \Vdash_V \varphi]; \\
(\mathcal{M}, a) \Vdash_V (\varphi U_j \psi) \ &\Leftrightarrow \ \exists b \ (aR_jb)((\mathcal{M}, b) \Vdash_V \psi) \\
&\qquad \wedge \forall c \ [(aR_jc < b) \Rightarrow (\mathcal{M}, c) \Vdash_V \varphi].
\end{aligned}$$

Thus every operator U_j (until) works in the usual manner, but it has an upper bound for transitivity of the local part $[i, \text{next}(i)]$. This agrees well with ordinary intuition about computational procedures and computational flows—decisions (states satisfying a formula) should, if any, be reached before the end of computation of the current local computational flow.

Standard derivative logical operations can be specified via those postulated above. Modal operations $\Box_j$ (necessary for an agent j) and $\Diamond_j$ (possible for an agent j) are defined thus: $\Diamond_i p := \top U_i p$ and $\Box_i p := \neg\Diamond_i\neg p$. It is easy to verify that

$$(\mathcal{M}, a) \Vdash_V \Diamond_j \varphi \Leftrightarrow \exists b \in N\ [(aR_j b) \wedge (\mathcal{M}, b) \Vdash_V \varphi];$$
$$(\mathcal{M}, a) \Vdash_V \Box_j \varphi \Leftrightarrow \forall b \in N\ [(aR_j b) \Rightarrow (\mathcal{M}, b) \Vdash_V \varphi].$$

For example, suppose $(\mathcal{M}, a) \Vdash_V [\Box_1 p \to \Box_2 \neg p] \wedge [\Box_2 p \to \Box_1 \neg p]$. The truth of this formula expresses the property of agents 1 and 2 *to be in opposition in relation to truth of indisputable facts*: in the future, the agents always have opposite opinions—if one assumes that a fact is true, then the other thinks that it is false.

Definition 3. A *logic* $L(m, \max)$ is the set of all formulas true in all models with maximal transitivity boundary m under all valuations.

Recall that the temporal degree of a formula φ is the maximum number of nested occurrences of temporal operations in the formula. A formal definition is introduced inductively: the temporal degree of propositional variables equals zero, i.e., $td(p) := 0$; the temporal degree of a formula whose basic operation is Boolean is the maximum temporal degree of its components, i.e., for $\varphi := \varphi_1 \star \varphi_2$, where $\star$ is a binary logical operation, we put $td(\varphi) := \max\{td(\varphi_1), td(\varphi_2)\}$ and $td(\neg\varphi) := td(\varphi)$; for $\varphi := \varphi_1 U \varphi_2$, we put $td(\varphi) := \max\{td(\varphi_1), td(\varphi_2)\} + 1$, and $td(N\varphi) := td(\varphi) + 1$.

The fact that in our models, the truth of a formula of temporal degree n depends only on truth of propositional variables in succeeding n transitivity intervals is trivial (easily verifiable by induction on the temporal degree of formulas). Therefore, we have

LEMMA 4. For any m, the logic $L(m, \max)$ is decidable.

The question whether the admissibility problem is decidable in such logics is nontrivial. For the case where all transitivity intervals $In(i)$ in models have length m, it was decided in [23]. In this paper, we want to lift this restriction and decide the admissibility problem in the general case for a more expressive—multiagent—logic.

Definition 5. A rule

$$r := \varphi_1(x_1, \ldots, x_n), \ldots, \varphi_m(x_1, \ldots, x_n)/\psi(x_1, \ldots, x_n)$$

is admissible in a logic L if, for all formulas $\alpha_1, \ldots, \alpha_n$,

$$\left[\left(\bigwedge_{1 \le i \le m} \varphi_i(\alpha_1, \ldots, \alpha_n)\right) \in L\right] \Longrightarrow [\psi(\alpha_1, \ldots, \alpha_n) \in L\right].$$

Definition 6. A rule

$$r := \varphi_1(x_1, \ldots, x_n), \ldots, \varphi_m(x_1, \ldots, x_n)/\psi(x_1, \ldots, x_n)$$

is valid on a frame $\mathcal{F}$ (written $\mathcal{F} \Vdash r$) if, for all valuations V of variables from r in $\mathcal{F}$, whenever all premises of r are true at all states in $\mathcal{F}$ under V, the conclusion of r is also true at all states under V.

We can transform any formula φ to a rule $x \to x/\varphi$, and φ is a *theorem of the logic* $L(m, \max)$ (i.e., $\varphi \in L(m, \max)$) iff the rule $(x \to x/\varphi)$ is valid on any frame $\mathcal{F}$. It might be helpful to use rules in special uniform form without formulas of temporal degree higher than 1.

Definition 7. A rule $\mathbf{r}$ is in *reduced normal form* if $\mathbf{r} = \varepsilon/x_1$, where

$$\varepsilon := \bigvee_{1 \le j \le l} \left[\bigwedge_{1 \le i \le n} x_i^{t(j,i,0)} \wedge \bigwedge_{1 \le i \le n} (Nx_i)^{t(j,i,1)} \wedge \bigwedge_{a \in [1,k], 1 \le i, i_1 \le n} (x_i \, U_a x_{i_1})^{t(j,i,i_1,a,2)} \right],$$

$t(j, i, 0), t(j, i, 1), t(j, i, i_1, a, 2) \in \{0, 1\}$, and $\alpha^0 := \alpha$, $\alpha^1 := \neg\alpha$ for any formula α used above.

Definition 8. A rule in reduced normal form $\mathbf{r_{nf}}$ is called the *reduced normal form of a rule* $\mathbf{r}$ in a logic $L(m, \max)$ if these rules are equivalent with respect to admissibility in $L(m, \max)$ and with respect to validity in any frame for $L(m, \max)$.

Naturally, we confine ourselves to treating a rule with a single premise since every rule $\alpha_1, \ldots, \alpha_n/\beta$ is equivalent with respect to validity and admissibility to a rule $\alpha_1 \wedge \cdots \wedge \alpha_n/\beta$.

THEOREM 9. There exists a time-exponential algorithm which, given any rule $\mathbf{r}$, constructs its reduced form $\mathbf{r_{nf}}$ (for $L(m, \max)$).

Proof. We only give a scheme, which is similar to one in [23] and was applied in some of my earlier works. Let an inference rule $\mathbf{r} = \alpha/\beta$ be given. By $\mathrm{Sub}\,(\mathbf{r})$ we denote the set of all subformulas of formulas in $\mathbf{r}$. Introduce a set $Z = \{z_\gamma \mid \gamma \in \mathrm{Sub}\,(r)\}$ of variables which do not occur in $\mathbf{r}$ and define a rule (in intermediate form) as follows:

$$\mathbf{r_{if}} = z_\alpha \wedge \bigwedge_{\gamma \in \mathrm{Sub}\,(\mathbf{r}) \setminus \mathrm{Var}\,(\mathbf{r})} (z_\gamma \leftrightarrow \gamma^\sharp)/z_\beta,$$

where

$$\gamma^\sharp = \begin{cases} z_\delta * z_\epsilon & \text{if } \gamma = \delta * \epsilon \text{ for } * \in \{\wedge, \vee, \to, U_j\}; \\ *z_\delta & \text{if } \gamma = *\delta \text{ for } * \in \{\neg, N\}. \end{cases}$$

The rules $\mathbf{r}$ and $\mathbf{r_{if}}$ are both valid or refutable on any frames for $L(m, \max)$ and are equivalent with respect to admissibility. We start with validity. In fact, let $\mathcal{M}$ be a model for $L(m, \max)$ based on a frame $\mathcal{F}$ with valuation V, for which $\mathcal{M} \nVdash_V \mathbf{r}$. Then $\mathcal{M} \Vdash_V \alpha$ and there exists a state $a \in \mathcal{F}$ such that $(\mathcal{M}, a) \nVdash_V \beta$. Choose a valuation $V_1 : Z \to 2^N$, where $V_1(z_\gamma) := V(\gamma)$. Using induction on the length of formulas, we can readily see that $\mathcal{M} \Vdash_{V_1} z_\alpha \wedge \bigwedge\{z_\gamma \leftrightarrow \gamma^\sharp \mid \gamma \in \mathrm{Sub}\,(\mathbf{r}) \setminus \mathrm{Var}\,(\mathbf{r})\}$ and $(\mathcal{M}, w) \nVdash_{V_1} z_\beta$.

On the other hand, assume that $\mathcal{M}$ is a model for $L(m, \max)$ based on a frame $\mathcal{F}$ with valuation $V_1 : Z \to 2^N$ such that $\mathcal{M} \Vdash_{V_1} z_\alpha \wedge \bigwedge\{z_\gamma \leftrightarrow \gamma^\sharp \mid \gamma \in \mathrm{Sub}\,(\mathbf{r}) \setminus \mathrm{Var}\,(\mathbf{r})\}$ and $\exists w \in \mathcal{F}\ (\mathcal{M}, w) \nVdash_{V_1} z_\beta$.

We define $V : \mathrm{Var}\,(\mathbf{r}) \to 2^N$ by the rule $V(x_i) := V_1(z_{x_i})$. If we use induction on the length of formulas we obtain $V(\gamma) = V_1(z_\gamma)$ for all $\gamma \in \mathrm{Sub}\,(r)$. Consequently, $\mathcal{M} \Vdash_V \alpha$ and $(\mathcal{M}, w) \nVdash_V \beta$; so $\mathcal{M} \nVdash_V \mathbf{r}$. The same scheme applies in proving equivalence with respect to admissibility.

We transform the premise of $\mathbf{r_{if}}$ to the perfect disjunctive normal form constructed on formulas like x_i, Nx_i, and $x_i\, U_l x_j$. As is known, such a construction requires one-exponential time in the number of all formulas x_i, Nx_i, and $x_i\, U_l x_j$, hence in the number of subformulas of the initial rule and thereby in its length. □

The reduced normal forms obtained by using the algorithm given in the proof of this theorem are defined uniquely. For any $x, y \in |\mathcal{F}|$, where $x < y$, we assume that the *distance* between x and y

$$\mathrm{dist}\,(x, y) = y - x$$

is the length of a chain of states leading from x to y.

Definition 10. Given a model $\mathcal{M} = \langle \mathcal{F}, V\rangle$ and a new valuation V_s of variables in some set S on a frame $\mathcal{F}$, we say that V_s is *first-order definable* (*definable*) in $\mathcal{M}$ if there exist formulas β_i such that

$$\forall x_i \in S\ V_s(x_i) = V(\beta_i).$$

LEMMA 11. If the rule $\mathbf{r_{nf}} = \bigvee_{1\le j\le l} \varphi_j / x_1$ in normal reduced form is inadmissible in $L(m, \max)$, then there exist a frame $\mathcal{F}_1 = \left\langle \bigcup_{i\in In} In(i), R_1, \ldots, R_k, \mathrm{Next}\right\rangle$ with valuation V_1 of variables in $\mathbf{r_{nf}}$ and some $w_s \in In$ such that $\bigcup_{i\in In} In(i) = [0, w_s] \cup \bigcup_{i\in In, i\ge w_s} In(i)$, and the following conditions hold:

(i) every variable in the rule $\mathbf{r_{nf}}$ has the same truth value for V_1 at all states in $\bigcup_{i\in In, i\ge w_s} In(i)$;

(ii) there exists j_0 such that $(\mathcal{F}_1, n) \Vdash_{V_1} \varphi_{j_0}$ for any $n \in \bigcup_{i\in In, i\ge w_s} In(i)$;

(iii) for every $n \in N$, there exists j such that $(\mathcal{F}_1, n) \Vdash_{V_1} \varphi_j$ (we denote such unique φ_j by $\theta(n)$);

(iv) $(\mathcal{F}_1, 0) \nVdash_{V_1} x_1$.

Proof. Let $\mathbf{r_{nf}} = \phi / x_1$, where

$$\phi := \bigvee_{1\le j\le l}\left[\bigwedge_{1\le i\le n} x_i^{t(j,i,0)} \wedge \bigwedge_{1\le i\le n} (Nx_i)^{t(j,i,1)} \wedge \bigwedge_{1\le i, 1\le i_1, i\ne i_1, g\in[1,k]} (x_i\, U_g x_{i_1})^{t(j,i,i_1,g,2)}\right]$$

and the map $x_i \to \varepsilon(x_i)$ is a substitution of formulas $\varepsilon(x_i)$ for the variables x_i in $\mathbf{r_{nf}}$. Moreover, after extending this substitution to the formulas, we have

$$\varepsilon(\phi) \in L(m, \max) \text{ and } \varepsilon(x_1) \notin L(m, \max).$$

Define

$$\varphi_j := \bigwedge_{1\le i\le n} \left[x_i^{t(j,i,0)} \wedge \bigwedge_{1\le i\le n} (Nx_i)^{t(j,i,1)} \wedge \bigwedge_{1\le i, 1\le i_1, i\ne i_1, g\in[1,k]} (x_i\, U_g x_{i_1})^{t(j,i,i_1,g,2)} \right].$$

In the family of frames defining a logic $L(m, \max)$ with valuation V, there then exists a frame

$$\mathcal{F} := \left\langle \bigcup_{i\in In} In(i), R_1, \dots, R_k, \text{Next} \right\rangle$$

such that

$$(\mathcal{F}, b) \nVdash_V \varepsilon(x_1)$$

for some $b \in \bigcup_{i\in In} In(i)$. Naturally, we may assume that $b = 0$, i.e., $(\mathcal{F}, 0) \nVdash_V \varepsilon(x_1)$. Let d be the maximum temporal degree of formulas $\varepsilon(x_i)$ for all i. Suppose also that

$$w_s := \min\{n \mid n \in In, n > d\} + 1;$$

i.e., w_s exceeds by 1 the least number in the set In of indices strictly larger than d. Now we modify the valuation V, assuming that variables of all formulas $\varepsilon(x_i)$ are true at all $a \in \bigcup_{i\in In, i>w_s} [i, i(\text{next})]$, and in $[0, w_s]$, they have the same values as before for V. Denote the new valuation by V_0. Note that

$$\forall x_i \in \text{Var}(\phi)\ \forall a \in In(0) := [0, 0(\text{next})],\ [(\mathcal{F}, x) \Vdash_V \varepsilon(x_i) \Leftrightarrow (\mathcal{F}, a) \Vdash_{V_0} \varepsilon(x_i)];$$

i.e., this modification of V does not change truth values of all formulas $\varepsilon(x_i)$ at all states in $[0, 0(\text{next})]$. This is verified by standard induction on the length of formulas and their temporal degree.

The presence of possible lacunas in accessibility relations R_j (we allow that $R_j \subset (\le)$ in some time interval) does not violate inductive steps, since the operations U_j are bounded by time intervals. Furthermore, variables of all formulas $\varepsilon(x_i)$ for V_0 in $\bigcup_{i\in In, i\ge w_s} [i, i(\text{next})]$ have the same values. Therefore, every formula $\varepsilon(x_i)$ will have the same truth value for all $a \in \bigcup_{i\in In, i>w_s} [i, i(\text{next})]$ under V_0.

By the hypothesis of the lemma, $\varepsilon(\phi) \in L(m, \max)$, and so

$$\forall c \in |\mathcal{F}|\ \exists j\ (\mathcal{F}, c) \Vdash_{V_0} \varepsilon(\varphi_j).$$

Denote such unique φ_j by $\theta(c)$.

On the frame $\mathcal{F}$ we introduce another valuation now for the variables of the rule itself, setting $V_1(x_i) := V_0(\varepsilon(x_i))$. Then

$$\forall x_i \in \text{Var}(\phi)\ \forall a \in |\mathcal{F}|\ [(\mathcal{F}, x) \Vdash_{V_0} \varepsilon(x_i) \ \Leftrightarrow\ (\mathcal{F}, x) \Vdash_{V_1} x_i].$$

For such V_1, all conclusions of Lemma 11 are satisfied. □

We extend the lemma. As above, the rule $\mathbf{r_{nf}} = \bigvee_{1\le j\le l} \varphi_j/x_1 = \phi/x_1$ is not admissible and the map $x_i \to \varepsilon(x_i)$ is a substitution of formulas $\varepsilon(x_i)$ for the variables x_i in $\mathbf{r_{nf}}$. Moreover, after extending this substitution to the formulas, we have $\varepsilon(\phi) \in L(m, \max)$ and $\varepsilon(x_1) \notin L(m, \max)$, while φ_j are disjuncts in the premise of the rule. Let $t(w_s)$ be the number of intervals of the form $In(i)$, i.e., transitivity intervals, within $[0, w_s]$. Below we use the notation given in the proof of Lemma 11.

LEMMA 12. Let $\mathbf{r_{nf}}$ be an inadmissible rule, and let a frame $\mathcal{F}_1$ with valuation V_1 and a state $w_s \in \mathcal{F}_1$ be as in Lemma 11. Suppose also that $[0, a]$ is some initial segment of an arbitrary frame for a logic $L(m, \max)$, where the number of transitivity intervals within $[0, a]$ equals $t(w_s) + 3$ and V_s is a valuation for variables in all formulas $\varepsilon(x_i)$ in $[0, a]$, with all the variables being true under V_s. Then for any $w \in [0, w_s] \subset |\mathcal{F}_1|$ there exist a frame $\mathcal{F}_2(w)$ and a valuation V_2 of variables in $\mathbf{r_{nf}}$ such that:

(a) $|\mathcal{F}_2(w)| = [0, a] \cup \{b \mid b \in |\mathcal{F}_1|, b \ge w\}$, where $a = w$ and accessibility relations R_j at concatenation are any that are admissible with respect to the maximum length m of the transitivity interval;

(b) V_2 coincides with V_1 in $\{b \mid b \ge w\}$ and all formulas φ_j at states in $\{b \mid b \ge w\}$ with respect to V_1 and V_2 have the same truth values;

(c) for any x in the first transitivity interval in $[0, a]$, it is true that $(\mathcal{F}_2, x) \Vdash_{V_2} \varphi_{j_0}$, where φ_{j_0} is as in Lemma 11(ii);

(d) for any $x \in [0, a]$, there is φ_j such that $(\mathcal{F}_2, x) \Vdash_{V_2} \varphi_j$.

Proof. Let a model on $\mathcal{F}_1$ with V_1 and $w_s \in \mathcal{F}_1$ be as in Lemma 11. Take $\mathcal{F}_2(w)$, $|\mathcal{F}_2(w)| = [0, a] \cup \{b \mid b \ge w\}$, such as in item (a). For $\{b \mid b \ge w\}$, we repeat verbatim the proof of Lemma 11, and assume that such b satisfy all the facts mentioned. For $x \in [0, a]$, we repeat the fragment associated with use of the temporal degree of formulas in Lemma 11.

More precisely, we define the valuation V_0 for variables of all formulas $\varepsilon(x_i)$ as follows: at $x \in [0, a-1]$, the values are assumed to be true (at all states greater than w_s as in Lemma 11), and in $\{b \mid b \ge w\}$, they are the same as in the proof of Lemma 11. Then the truth value of any formula $\varepsilon(x_i)$ for V_0 will be the same at all $a \in \bigcup_{i\in In, i\ge w_s} In(i)$ in $\mathcal{F}_1$ under V_0 and at all states in the first transitivity interval in $[0, a]$ under V_0, in which case it will be equal for all of those states.

As in Lemma 11, we prove these facts by induction on the temporal degree of formulas. Introducing on $\mathcal{F}_2(w)$ (as we did on $\mathcal{F}$) another valuation for variables of the rule by setting $V_2(x_i) := V_0(\varepsilon(x_i))$, we obtain

$$\forall x_i \forall u \in |\mathcal{F}_2| \; [(\mathcal{F}_2, u) \Vdash_{V_0} \varepsilon(x_i) \Leftrightarrow (\mathcal{F}_2, u) \Vdash_{V_2} x_i].$$

Then statements (a)-(d) hold. □

LEMMA 13. Suppose that the hypotheses of Lemmas 11 and 12 are valid. We may assume that intervals of all states below w_s in the resulting frames $\mathcal{F}_1$ and $\mathcal{F}_2(w)$ have a finite size with an upper boundary effectively computable in the size of the rule.

Proof. We apply the drop-point technique. We start with frame $\mathcal{F}_1$. Moving from the least transitivity interval upward, we consider sequentially all transitivity intervals (as models under V_1) and formulas $\theta(j)$ true at states j of these intervals (such $\theta(j)$ exist by Lemma 11(iii)).

We find two transitivity intervals—the lowest one and its least successor with coincident sequences of formulas

$$[\theta(j), \dots, \theta(j+m)] \text{ and } [\theta(i), \dots, \theta(i+m)].$$

Thereafter, we remove all transitivity intervals between $[i, i+m]$ and $[j, j+m]$, and replace $[j, j+m]$ by $[i, i+m_i]$ redenoting all accessibility relations and the valuation of variables, respectively. This transformation preserves truth of formulas of the form $\theta(j)$. Taking into account the properties of model $\mathcal{F}_1$ in Lemma 11, we note that the model obtained via such a transformation will have the same properties.

If we apply the above transformation to all transitivity intervals below w_s and move to w_s, we see that in a finite number of steps (computable in the size of a rule), it will be completed, and the interval of all states below w_s will be finite with size computable in the size of the rule.

For models $\mathcal{F}_2(w)$ with V_2, we apply the same transformation first to the whole $\{b \mid b \geq w\}$ moving from w to $w_s + 3 \times m + 1$ (as indicated above), and then to $[0, a]$ in $\mathcal{F}_2(w)$ moving from 0 to $a = w$, using the properties specified in Lemma 12 and preserving the entry point in $\mathcal{F}_1$. $\Box$

Now we apply Lemmas 11, 12 (and Lemma 13 for computing upper bounds for initial segments of verifiable frames) to obtain a sufficient admissibility condition.

THEOREM 14. Let a rule $\mathbf{r_{nf}} = \bigvee_{1 \leq j \leq l} \varphi_j / x_1$ in normal reduced form be given. Suppose also that there exist a model $\mathcal{F}_1$ with valuation V_1 and a model $\mathcal{F}_2(w)$ with valuation V_2 for all variables in $\mathbf{r_{nf}}$ satisfying the conditions of Lemma 12. Then the rule $\mathbf{r_{nf}}$ is not admissible in $L(m, \max)$.

Proof. In view of Lemma 13, we may assume that intervals below w_s (stabilization points) in all the models mentioned are finite and have a size computable in the size of $\mathbf{r_{nf}} = \bigvee_{1 \leq j \leq l} \varphi_j / x_1$. First we describe the structure of a model on $\mathcal{F}_1$ with V_1 using the formulas given below. Thus, let $\mathcal{F}_1 := \left\langle \bigcup_{i \in In} In(i), R_1, \dots, R_k, \mathrm{Next} \right\rangle$ and let V_1 be a valuation defined on $\mathcal{F}_1$: for all $i \in In$, we assume that $Id(i) = [i, i(\mathrm{next})]$ are transitivity intervals and $\bigcup_{i \in In} In(i) = [0, w_s] \cup \bigcup_{i \in In, i \geq w_s} In(i)$.

For every $t \in |\mathcal{F}_1| \setminus \{u \mid u \geq w_s + 3m + 1\}$, we introduce a unique variable p_t. For $x, y \in |\mathcal{F}_1| \setminus \{u \mid u \geq w_s + 3m + 1\}$ and $x < y$, as above, $\mathrm{dist}\,(x, y)$ denotes the distance between x and y, i.e., $y - x$. For any fixed p_t, we define the following formulas:

$$A(p_t) := p_t \wedge \left[\bigwedge_{x > t, x \in [t+1, w_s + 3 \times m]} N^{\mathrm{dist}\,(t,x)} \left(p_x \wedge \bigwedge_{p_l \neq p_t} \neg p_l \right) \right].$$

Let $S(t) := \{x \mid x \in [t, w_s + 3 \times m]\}$ and let

$$B(p_t) := p_t \to \bigwedge_{x \in S(t), x \geq t, tR_j x, j \in [1,k]} [\Diamond_j p_x] \wedge \bigwedge_{x, \in S(t), (\mathcal{F}, t), \neg(tR_j x), j \in [1,k]} [\neg \Diamond_j p_x],$$

$$C(p_t) := A(p_t) \wedge \bigwedge_{x,y \in S(t), x \neq y} [N^{\mathrm{dist}\,(t,x)} p_x \to \neg p_y],$$

$$D(p_t) := A(p_t) \wedge \bigwedge_{x \geq t, x \in S(t)} N^{\mathrm{dist}\,(t,x)} A(p_x) \wedge B(p_x) \wedge C(p_x).$$

Suppose that $\mathcal{F} := \left\langle \bigcup_{i \in In} In(i), R_1, \ldots, R_k, \mathrm{Next} \right\rangle$ is a model with valuation V for the above variables p_t.

LEMMA 15. Let $a_t \in |\mathcal{F}|$, $(\mathcal{F}, a_t) \Vdash_V D(p_t)$, and V_p be a valuation on $\mathcal{F}$ of variables in $\mathbf{r_{nf}} = \bigvee_{1 \leq j \leq l} \varphi_j / x_1$ defined by the equality

$$V_p(x_i) := V\left(\bigvee \{D(p_t) \mid t \in |\mathcal{F}_1| \setminus \{u \mid u \geq w_s + 3 \times m + 1, (\mathcal{F}_1, t) \Vdash_{V_1} x_i\}\}\right).$$

Then $(\mathcal{F}, x) \Vdash_{V_p} \theta(c)$, where $\theta(c)$ is as in Lemma 11(iii), for all states $x \geq a_t$ and for a state c from $[t, w_s]$ in $\mathcal{F}_1$, where $\mathrm{dist}\,(a_t, x) = \mathrm{dist}\,(t, c)$.

Proof. We use the structure of formulas $A(p_t)$, $B(p_t)$, $C(p_t)$, and $D(p_t)$. We need to verify that the models in $[a_t, a_t + \mathrm{dist}\,(t, w_s)]$ inside $\mathcal{F}$ with respect to V_p and in $[t, t + \mathrm{dist}\,(t, w_s)]$ inside $\mathcal{F}_1$ with respect to V_1 are isomorphic. Indeed, since

$$(\mathcal{F}, a_t) \Vdash_V D(p_t),$$

we have

$$\forall a \in |\mathcal{F}|\ \forall t, t_1 \in [0, w_s + 3 \times m]\ (\mathrm{dist}\,(a_t, a) = \mathrm{dist}\,(t, t_1) \Rightarrow (\mathcal{F}, a) \Vdash_V p_{t_1}).$$

Therefore, the intervals $[t, t + \mathrm{dist}\,(t, w_s + 3 \times m)]$ and $[a_t, a_t + \mathrm{dist}\,(t, w_s + 3 \times m)]$ are isomorphic as frames. The valuations V_1 and V_p on the respective frames coincide by virtue of the following definition:

$$V_p(x_i) := V\left(\bigvee \{D(p_t) \mid t \in |\mathcal{F}_1| \setminus \{u \mid u \geq w_s + 3 \times m + 1, (\mathcal{F}_1, t) \Vdash_{V_1} x_i)\}\}\right).$$

Hence the models on these frames are also isomorphic.

Recall that $a_t \in |\mathcal{F}|$ and $(\mathcal{F}, a_t) \Vdash_V D(p_t)$. In Lemma 11, it was proved that $(\mathcal{F}_1, t) \Vdash_{V_1} \theta(t)$ for every element t in $\mathcal{F}_1$. Therefore, for any x, $x \geq a_t$, and for $b \in [t, w_s]$ in model $\mathcal{F}_1$, where $\mathrm{dist}\,(a_t, x) = \mathrm{dist}\,(t, b)$, we have $(\mathcal{F}, x) \Vdash_{V_p} \theta(b)$. □

Suppose that $\mathcal{F} := \left\langle \bigcup_{i \in In} In(i), R_1, \ldots, R_k, \mathrm{Next}, V \right\rangle$ is a model with valuation V for variables p_t. (Recall that a unique variable p_t has been introduced for every $t \in |\mathcal{F}_1| \setminus \{u \mid u \geq w_s + 3m + 1\}$.) Let m_w be the maximum possible number of a in all the obtained models $\mathcal{F}_2(w)$ which were introduced in Lemma 12 (after applying Lemma 13). Assume that there exists $b \in \mathcal{F}$ such that

$$b \in |\mathcal{F}| : (\mathcal{F}, b) \Vdash_V \neg \bigvee_{p_t} D(p_t) \wedge \left[\bigvee_{s \leq m_w, p_t} \left[N^s D(p_t) \wedge \neg \bigvee_{s_1 < s-1, t_1} N^{s_1} D(p_{t_1})\right]\right].$$

Let c be a least element greater than b, where

$$(\mathcal{F}, c) \Vdash_V D(p_t)$$

for some p_t.

Elsewhere above, we have shown that the model $[t, t + \operatorname{dist}(t, w_s)]$ with valuation V_1 in model $\mathcal{F}_1$ is isomorphic to the model $[c, c + \operatorname{dist}(t, w_s)]$ with valuation $V_p(x_i) := \{u \mid (\mathcal{F}, u) \Vdash_{V_p} (x_i)\}$ within $\langle \mathcal{F}, V_p \rangle$. In view of Lemma 11, we obtain

$$\forall d \in [c, c + \operatorname{dist}(t, w_s)]\ (\mathcal{F}, d) \Vdash_{V_p} \theta(q)$$

for some q.

For $d \in [0, c-1]$ inside $\mathcal{F}$, again we apply Lemma 12. In view of that lemma, we note, for any upper interval $[a_1, c-1]$ in the segment $[b, c-1]$ such that the number of elements inside $[a_1, c-1]$ does not exceed the number of elements inside $[0, a]$ specified in Lemma 12, there exists an extension of the valuation V_2 of variables x_i in $[a, \infty)$ to the interval $[a_1, c-1]$, and there exists an extension of the valuation V_1 of x_i in $[a, \infty)$ to $[0, c-1]$ such that the following facts hold. Some formula $\theta(u)$ is true on any element in $[0, c-1]$ under V_2. A special formula φ_{j_0} is true in the first transitive part in $[0, c-1]$ under V_2 if the length of $[a_1, c-1]$ is greater than the length of $[0(\text{next}), a]$. In other words, $(\mathcal{F}, x) \Vdash_{V_2} \varphi_{j_0}$, where φ_{j_0} is as in the formulation of Lemma 11, for any state x in such an interval.

It remains to observe that such a valuation V_2 for variables x_i is definable (defined by formulas) in the variables p_t and can be specified on $\{d \mid d \geq c, d \in [c, c + \operatorname{dist}(t, w_s + 3 \times m)]\}$ as above after introducing p_t (i.e., $V_2(x_i) := V(\bigvee\{D(p_t) \mid t \in |\mathcal{F}_1| \setminus \{u \mid u \geq w_s + 3 \times m + 1\}, (\mathcal{F}_1, t) \Vdash_{V_1} x_i)\}\})$. On $[b, c-1]$, V_2 is defined by formulas in just one variable p_c; we need to take into account the structure of the frame $[b, c-1]$, its finiteness, and computable bounded size. Thus the following holds:

LEMMA 16. Under the above definable valuation V_2,

$$\forall d \in [0, c]\ (\mathcal{F}, d) \Vdash_{V_2} \theta(q)$$

for some q.

The valuation of variables x_i can be summed up as follows:

$$S(x_i) := V_p(x_i) \cup V_2(x_i) \cup V_3(x_i),$$

where V_3 is a definable valuation for x_i given by the formula

$$\begin{aligned} G(x_i) := &\Big[\neg \Big[\bigvee \{D(p_t) \mid t \in |\mathcal{F}_1| \setminus \{u \mid u \geq w_s + 3 \times m + 1\}\Big]\Big] \\ &\wedge \neg \left[\neg \bigvee_{p_t} D(p_t) \wedge \left[\bigvee_{s \leq m_w, p_t} [N^s D(p_t) \wedge \neg \bigvee_{s_1 < s-1, t_1} N^{s_1} D(p_{t_1}]\right]\right] \wedge Sg_i. \end{aligned}$$

$Sg_i = \top$ if x_i occurs positively in θ_{j_0}, and $Sg_i = \bot$ otherwise.

LEMMA 17. The valuation $S(x_i)$ is definable, and the following statements hold:

(a) there exist a frame $\mathcal{F}_0$ and a valuation V_0 of variables in formulas defining $S(x_1)$ on $\mathcal{F}_0$ such that $(\mathcal{F}_0, 0) \nVdash_{V_0} x_1$;

(b) given any valuation V_4 of variables in formulas defining valuations $S(x_i)$ on any frame $\mathcal{F}$, the premise of the rule $\bigvee_{1\le j\le l} \varphi_j/x_1$ is valid on $\mathcal{F}$ under V_4.

Proof. That S is definable was shown above.

(a) Let $\mathcal{F} = \mathcal{F}_1$ be a model such as in Lemma 11. Recall that x_1 is false on 0 in $\mathcal{F}_1$ under V_1 (see Lemma 11), while $V_1(x_1)$ and $S(x_1)$ on $[0, w_s]$ in $\mathcal{F}_1$ coincide.

(b) Take any frame $\mathcal{F}$, an arbitrary valuation V_4 for all variables p_t introduced earlier, and any element x in $\mathcal{F}$.

(Step-1) First assume that for some $D(p_t)$,

$$x \in \mathcal{F},\ (\mathcal{F}, x) \Vdash_{V_4} D(p_t).$$

By Lemma 15, a formula $\theta(u)$ is true for x under V_p. Then the valuation S coincides with V_p in the interval $[x, x + \mathrm{dist}\,(t, w_s + 3 \times m)]$, and $\theta(u)$ will be true for x under S.

(Step-2) Consider the second case where

$$x \in |\mathcal{F}| :\ (\mathcal{F}, x) \Vdash_V \neg \bigvee_{\forall p_t} D(p_t) \wedge \left[\bigvee_{s\le m_w, p_t} (N^s p_t \wedge \neg N^{s-1} p_t) \right].$$

Let c be a least element greater than x with $(\mathcal{F}, c) \Vdash_V D(p_t)$. By Lemma 16, some formula $\theta(u)$ is true on u under V_2 for any $d \in [0, c]$, and then V_2 in the interval $[x, c-1]$ is again consistent with S (does not extend it in this interval); i.e., $\forall x \in [0, c-1]\ (\mathcal{F}, c) \Vdash_S \theta(u)$ for some $\theta(u)$.

(Step-3) Assume that (S1) and (S2) are invalid. Then the given element x is at a distance greater than m_w from those for which formulas $D(p_t)$ hold, and the truth of variables x_i at x is then defined only by the valuation V_3; hence $(\mathcal{F}, x) \Vdash_S \varphi_{j_0}$ (see Lemma 12(d)). □

Using Lemmas 11-13 and Theorem 14, we derive the following:

THEOREM 18. $L(m, \max)$ has a decidable admissibility problem. There exists an algorithm for verifying admissibility of inference rules in $L(m, \max)$.

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