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UNIVERSAL EQUIVALENCE OF LINEAR GROUPS OVER LOCAL COMMUTATIVE RINGS WITH 1/2

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It is proved that the universal equivalence of general or special linear groups of orders greater than 2 over local commutative rings with 1/2 is equivalent to the coincidence of orders of groups and universal equivalence of respective rings.

INTRODUCTION

This paper is a continuation of [1]. We will look into universal equivalence of linear groups over local rings. Our objective is to prove an analog of Mal'tsev's theorem [2] stating that general and special linear groups over commutative local rings with 1/2 are universally equivalent iff the order of groups is at least 3.

A. I. Mal'tsev in [2] proved the theorem in which necessary and sufficient conditions are specified under which linear groups over fields are elementarily equivalent. Namely, the groups $\mathbf{G}_n(K)$ and $\mathbf{G}_m(L)$ ($\mathbf{G} = \mathrm{GL}, \mathrm{SL}, \mathrm{PGL}, \mathrm{PSL}$; K and L are fields characteristic 0) are elementarily equivalent iff $m = n$ and the fields K and L are elementarily equivalent. Logical properties of linear groups may be studied not only in the frames of an elementary theory, but also in the frames of a restricted—universal—theory, when formulas admit only one kind of quantifiers. For the case of linear groups over fields, a universal equivalence criterion, similar to the elementary equivalence criterion, holds; this result was expounded in [1]. A natural generalization of this case—universal equivalence of linear groups over local rings—is the subject of the present investigation.

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1. PRELIMINARIES

We assume that all rings are associative, commutative and contain an identity element.

Definition 1. A commutative associative ring with 1 is local if it contains only one maximal ideal.

The above definition implies that if R is a local ring and I is its maximal ideal, then I coincides with the set of all noninvertible elements of the ring R. Note also that if the sum of two elements is invertible in such a ring, then at least one of the two elements is invertible. For more information concerning local commutative rings, see [1].

We briefly recall basic notions associated with universal equivalence. For a more detailed presentation of definitions and auxiliary statements, see [3].

Definition 2. A formula φ of a signature Σ is *universal* (*existential*) if its prenex normal form is the following:

$$Q_1x_1 \dots Q_nx_n\psi(x_1, \dots, x_n),$$

where $Q_1 = \dots = Q_n = \forall$ ($Q_1 = \dots = Q_n = \exists$), and ψ is quantifier-free.

Definition 3. Two algebraic systems $\mathfrak{A}$ and $\mathfrak{B}$ of a signature Σ are said to be *universally equivalent* (*existentially equivalent*) if, for every universal (existential) sentence φ in the signature Σ, the following condition holds:

$$\mathfrak{A} \models \varphi \Longleftrightarrow \mathfrak{B} \models \varphi.$$

A set of universal (existential) sentences $\{\varphi \mid \mathfrak{A} \models \varphi\}$ of the signature Σ is called a *universal* (*existential*) *theory* of the system $\mathfrak{A}$ and is denoted by $\mathrm{Th}_\forall(\mathfrak{A})$ ($\mathrm{Th}_\exists(\mathfrak{A})$). Thus $\mathfrak{A} \equiv_\forall \mathfrak{B} \Leftrightarrow \mathrm{Th}_\forall(\mathfrak{A}) = \mathrm{Th}_\forall(\mathfrak{B}) \Leftrightarrow \mathrm{Th}_\exists(\mathfrak{A}) = \mathrm{Th}_\exists(\mathfrak{B}) \Leftrightarrow \mathfrak{A} \equiv_\exists \mathfrak{B}$. The last two relations hold in view of quantifier dependence.

We will use the following *criterion for universal equivalence*: two algebraic systems of the same finite signature are universally equivalent iff every finite submodel of one system has an isomorphic submodel in the other system and vice versa.

In the present paper, we prove the following:

THEOREM. Let R_1 and R_2 be local commutative rings with $1/2$. Groups $\mathbf{G}_n(R_1)$ and $\mathbf{G}_m(R_2)$ ($\mathbf{G} = \mathrm{GL}, \mathrm{SL}$, $m, n \geqslant 3$) are universally equivalent if and only if $n = m$ and the rings R_1 and R_2 are universally equivalent.

Proving the simpler implication that universal equivalence of rings implies universal equivalence of linear groups is similar to the proof for the field case given in [1]. Note that the statement is also true for $n = m < 3$.

Now we turn to a more difficult implication: universal equivalence of linear groups implies universal equivalence of respective rings and coincidence of sizes.

Note that if linear groups are universally equivalent, then the rings R_1 and R_2 are either both finite or both infinite. For finite systems, being universally equivalent coincides with being isomorphic. Hence the statement of the theorem follows immediately from [4], in which it was proved that for natural $n, m \geqslant 3$, the following conditions are equivalent:

(1) $n = m$ and $R_1 \cong R_2$;

(2) $\mathrm{GL}_n(R_1) \cong \mathrm{GL}_m(R_2)$,

(3) $\mathrm{SL}_n(R_1) \cong \mathrm{SL}_m(R_2)$,

where R_1 and R_2 are local commutative rings with 1/2.

Therefore, below we will assume that R_1 and R_2 are infinite.

2. PROOF OF THE THEOREM

The statement on coincidence of orders is a consequence of the fact that in a group $\mathbf{G}_n(R)$ ($\mathbf{G} = \mathrm{GL}, \mathrm{SL}$) over a local commutative ring R with 1/2, all pairwise commuting involutions are diagonalizable in a common basis; for local rings, this fact was proved in [5].

Below we assume that $n > 2$.

Denote by $\mathcal{MI}$ a fixed maximal set of pairwise commuting involutions. In a common basis, all involutions of $\mathcal{MI}$ are diagonal matrices with ± 1 on the diagonals. Fix one of such bases. Note that $\mathcal{MI}$ is partitioned into conjugacy classes, of which each consists of just those matrices that have an equal number of -1's on the diagonals in the basis chosen. We introduce the notation for some subsets of $\mathcal{MI}$.

First consider $\mathrm{SL}_n(R)$ with odd n and $\mathrm{GL}_n(R)$. In the set $\mathcal{MI}$, the smallest conjugacy classes, except for a one-element one (if any), consist of n elements. These are conjugacy classes composed of matrices with -1 occurring on the diagonal once or $n-1$ times. The classes are distinguished by an existential formula. Note that a formula calculating the number of elements in two conjugacy classes distinguishes each of the two classes, but does not say which. Denote by $\mathcal{I}_1$ one (no matter which) of these classes.

For $\mathrm{SL}_n(R)$ with even $n \geqslant 4$, we denote by $\mathcal{I}_2$ the following subset of $\mathcal{MI}$: matrices with -1 occurring exactly twice on the diagonal. The subset $\mathcal{I}_2$ is distinguished by an existential formula, as in [1, Lemma 26]. We give an appropriate argument. Subsets composed of matrices having $n-2$ or two -1 on the diagonal are conjugacy classes each of which consists of C_n^2 elements. The other conjugacy classes, except one that consists of a single element $-E$, have more elements. If $n = 4$, then the conjugacy class composed of C_n^2 elements is unique, and it is exactly the subset that we are interested in.

Let $n = 6$. By multiplying all elements of a conjugacy class of matrices with two -1's on the diagonal, we obtain $-E$, since this class contains $C_5^1 = 5$ matrices with -1 at a fixed place. If we multiply all elements of a conjugacy class containing matrices with four -1's, then we obtain an identity matrix, since among matrices of that class there are exactly C_5^3 matrices with -1 at a fixed place. Thus, for $n = 6$, among matrices of the system $\mathcal{MI}$ we can distinguish those on the diagonal of which -1 occurs exactly twice.

Let $n > 8$. Among the matrices with two -1's on the diagonal, there always exist three matrices the product of the first two of which equals the third. However, if we multiply any two matrices on the diagonal of which -1 occurs $n-2$ times we obtain a matrix of another conjugacy class.

The fact that the set $\mathcal{MI}$ of matrices is distinguished by an existential formula implies that the form of diagonal matrices is preserved under isomorphism of a submodel containing all matrices of $\mathcal{MI}$.

LEMMA 1. Let R_1 and R_2 be infinite local commutative rings with $1/2$ and let $\mathbf{G}_n(R_1) \equiv_\forall \mathbf{G}_n(R_2)$ ($\mathbf{G} = \mathrm{GL}, \mathrm{SL}$). Suppose also that M_1 is an arbitrary finite submodel of $\mathbf{G}_n(R_1)$ containing $\mathcal{MI}$, and M_2 is a finite submodel of $\mathbf{G}_n(R_2)$ isomorphic to M_1 (its existence follows from the universal equivalence criterion).

Then, for any isomorphism

$$\Phi\colon M_1 \to M_2,$$

the matrix $\sigma_{12} = E - E_{11} - E_{22} - E_{12} + E_{21}$ is mapped to a matrix of the form

$$\begin{pmatrix} 0 & -1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \pm 1 & & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & \pm 1 \end{pmatrix}$$

in a basis in which the form of matrices in $\mathcal{MI}$ will not be changed.

Proof. There are two cases to consider:

Case 1. Let $\mathbf{G} = \mathrm{GL}$ or $\mathbf{G} = \mathrm{SL}$ for odd n. We make use of the fact that σ_{12} commutes with all but two matrices in the set $\mathcal{I}_1$. Denote the two matrices by I_1 and I_2, assuming that the jth place on the diagonal in I_j is occupied by an element that differs from all other diagonal elements. Below, in dealing with this case, we will use the notation $I_{ij} = I_i I_j$, $i \neq j$.

Case 2. Let $\mathbf{G} = \mathrm{SL}$ and n be even. First, among matrices of the system $\mathcal{I}_2$, we choose one, say, I_{12}, assuming that the first and second places on the diagonal are occupied by (-1). Then we consider matrices in $\mathcal{I}_2$ whose product with I_{12} yields a matrix not in $\mathcal{I}_2$. All of these matrices, as well as I_{12}, commute with σ_{12}. In any case we will have an image of the following form:

$$\Phi(\sigma_{12}) = \begin{pmatrix} d_{11} & d_{12} & 0 & \dots & 0 \\ d_{21} & d_{22} & 0 & \dots & 0 \\ 0 & 0 & d_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & d_{nn} \end{pmatrix}.$$

Next we use the relations $\sigma_{12}^2 = I_{12}$. For images, we obtain

$$\begin{pmatrix} d_{11}^2 + d_{12}d_{21} & d_{11}d_{12} + d_{12}d_{22} & 0 & \dots & 0 \\ d_{21}d_{11} + d_{22}d_{21} & d_{21}d_{12} + d_{22}^2 & 0 & \dots & 0 \\ 0 & 0 & d_{33}^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & d_{nn}^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

which, in view of the fact that a local ring contains $1/2$, immediately implies that $d_{ii} = \pm 1$ for $2 < i \leqslant n$.

Now consider the first case. We make use of yet another relation $\sigma_{12} I_1 \sigma_{12} = I_1$. Using this, for images we derive the following (writing out only the corner block):

$$\begin{aligned} \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} &= \begin{pmatrix} -d_{11}^2 + d_{12}d_{21} & -d_{11}d_{12} + d_{12}d_{22} \\ -d_{21}d_{11} + d_{22}d_{21} & -d_{21}d_{12} + d_{22}^2 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Adding and subtracting the respective elements of the matrices in the second and third relations, we conclude that $2d_{12}d_{21} = -2$; i.e., $d_{12}d_{21} = -1$, $2d_{22}d_{21} = 0$, and $2d_{11}d_{12} = 0$, which, in view of d_{12} and d_{21} being invertible, implies that $d_{11} = d_{22} = 0$.

Consider the second case. Here we use the relation $\sigma_{12} I_{13} \sigma_{12}^{-1} = I_{23}$. (In system $\mathcal{J}_2$, we fixed a matrix which under multiplication by I_{12} yields a matrix in $\mathcal{J}_2$ and denoted it by I_{13}; then $I_{23} = I_{12} I_{13}$.) For images, we have the following (again we write out only the corner block):

$$\begin{aligned} &\frac{1}{d_{11}d_{22} - d_{12}d_{21}} \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d_{22} & -d_{12} \\ -d_{21} & d_{11} \end{pmatrix} \\ &= \frac{1}{d_{11}d_{22} - d_{12}d_{21}} \begin{pmatrix} -d_{11}^2 - d_{12}d_{21} & d_{11}d_{12} + d_{12}d_{22} \\ -d_{11}d_{21} - d_{22}d_{21} & d_{12}d_{21} + d_{22}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

Notice first that $d_{11}d_{22} - d_{12}d_{21} = \pm 1$ (which follows from the second relation). If $d_{11}d_{22} - d_{12}d_{21} = 1$, then similarly we derive $2d_{12}d_{21} = -2$ and $d_{22} = d_{11} = 0$. If $d_{11}d_{22} - d_{12}d_{21} = -1$, then we obtain a contradiction. In fact, consequences of the second and third relations hold, i.e., $d_{11}^2 = d_{22}^2 = 1$ and $d_{11}d_{12} = d_{22}d_{21} = 0$, while the invertibility of d_{11} and d_{22} yields $d_{12} = d_{21} = 0$. Then $\Phi(\sigma_{12})$ commutes with all matrices in $\mathcal{MJ}$, which is not true in our case.

Thus

$$\Phi(\sigma_{12}) = \begin{pmatrix} 0 & -\alpha & 0 & \dots & 0 \\ \frac{1}{\alpha} & 0 & 0 & \dots & 0 \\ 0 & 0 & \pm 1 & & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & \pm 1 \end{pmatrix},$$

where α is an invertible element of the ring. It remains to make a change of basis generated by the matrix $\operatorname{diag}[1/\alpha, 1, \dots, 1]$, commuting with all elements of $\mathcal{MI}$. $\square$

Remark. In exactly the same way, we can prove that a matrix σ_{23} with a block $-E_{12} + E_{21}$ at the intersection of the second and third rows with the second and third columns preserves its form. Arguing as in Lemma 1, first we see that

$$\Phi(\sigma_{23}) = \begin{pmatrix} \pm 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & -\beta & 0 & \dots & 0 \\ 0 & \frac{1}{\beta} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \pm 1 & & \\ \vdots & \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & & \pm 1 \end{pmatrix},$$

and then make a change of basis generated by the matrix $\operatorname{diag}[1, 1, \beta, 1, \dots, 1]$, which commutes with all elements of $\mathcal{MI}$ and with σ_{12}.

LEMMA 2. Let R_1 and R_2 be infinite local commutative rings with $1/2$ and let $\mathbf{G}_n(R_1) \equiv_\forall \mathbf{G}_n(R_2)$ ($\mathbf{G} = \mathrm{GL}, \mathrm{SL}$). Suppose also that M_1 is an arbitrary finite submodel of $\mathbf{G}_n(R_1)$ containing a finite set of matrices such as in the previous lemma and the matrix $\operatorname{diag}[2, 1, 1/2, 1, \dots, 1]$, and M_2 is a finite submodel of $\mathbf{G}_n(R_2)$ isomorphic to M_1. Then, for any isomorphism

$$\Phi\colon M_1 \to M_2,$$

the matrix $E + E_{12}$ is mapped into a matrix of the same form in a basis in which the matrices σ_{12} and σ_{23} in $\mathcal{MI}$ are form preserving.

Proof. As in the proof of Lemma 1,

$$\Phi(E + \alpha E_{12}) = \begin{pmatrix} t_{11} & t_{12} & 0 & \dots & 0 \\ t_{21} & t_{22} & 0 & \dots & 0 \\ 0 & 0 & t_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & t_{nn} \end{pmatrix}.$$

The matrix $\operatorname{diag}[2, 1, 1/2, 1, \dots, 1]$ commutes with all matrices in $\mathcal{MI}$, and so $\Phi(\operatorname{diag}[2, 1, 1/2, 1, \dots, 1]) = \operatorname{diag}[\alpha_1, \dots, \alpha_n]$, where all α_i are invertible. From the relation $\operatorname{diag}[2, 1, 1/2, 1, \dots, 1](E + \alpha E_{12})(\operatorname{diag}[2, 1, 1/2, 1, \dots, 1])^{-1} = (E + \alpha E_{12})^2$, for images we conclude that $t_{ii}^2 = t_{ii}$ with all $3 \leqslant i \leqslant n$. All t_{ii} with $3 \leqslant i \leqslant n$ are invertible and, consequently, are equal to 1.

Now we can confine ourselves to just corner blocks of dimension 2 or (wherever necessary) 3. Use will be made of the following three relations:

$$((E + E_{12})I_{23})^2 = E,$$

$$(\sigma_{12}(E + E_{12}))^3 = I_{12},$$

$$E + E_{12} \text{ commutes with } \sigma_{23}(E + E_{12})\sigma_{23}^{-1}.$$

(The second relation implies that all diagonal elements of $\Phi(\sigma_{12})$ except the first and second ones equal 1.)

We will write these relations for images with due regard for the fact that some matrices are form preserving, as proved above. Since σ_{12} and σ_{23} are conjugate, the first diagonal element of $\Phi(\sigma_{23})$ equals 1. Indeed,

$$\left(\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)^2 = \begin{pmatrix} t_{11}^2 - t_{12}t_{21} & t_{12}(-t_{11} + t_{22}) \\ t_{21}(t_{11} - t_{22}) & t_{22}^2 - t_{12}t_{21} \end{pmatrix} = E,$$

$$\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}\right)^3 = \begin{pmatrix} -t_{21} & -t_{22} \\ t_{11} & t_{12} \end{pmatrix}^3$$

$$= \begin{pmatrix} -t_{21}^3 + 2t_{11}t_{21}t_{22} - t_{11}t_{12}t_{22} & -t_{21}^2 t_{22} + t_{11}t_{22}^2 + t_{12}t_{21}t_{22} - t_{12}^2 t_{22} \\ * & t_{11}t_{21}t_{22} - 2t_{11}t_{12}t_{22} + t_{12}^3 \end{pmatrix} = -E,$$

$$\begin{pmatrix} t_{11} & t_{12} & 0 \\ t_{21} & t_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} t_{11} & t_{12} & 0 \\ t_{21} & t_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} t_{11} & t_{12} & 0 \\ t_{21} & t_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} t_{11} & t_{12} & 0 \\ t_{21} & t_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Or, which is the same,

$$\begin{pmatrix} t_{11}^2 & t_{12} & t_{11}t_{12} \\ t_{11}t_{21} & t_{22} & t_{12}t_{21} \\ t_{21} & 0 & t_{22} \end{pmatrix} = \begin{pmatrix} t_{11}^2 & t_{11}t_{12} & t_{12} \\ t_{21} & t_{22} & 0 \\ t_{11}t_{21} & t_{12}t_{21} & t_{22} \end{pmatrix}.$$

The last relation implies $t_{12}t_{21} = 0$, while the first one entails $t_{11}^2 = t_{22}^2 = 1$. Computing the determinant in the second relation, we obtain $1 = (t_{11}t_{22} - t_{12}t_{21})^3 = (t_{11}t_{22})^3 = t_{11}^2 t_{22}^2 t_{11}t_{22} = t_{11}t_{22}$. The same relation yields $t_{21}^2 t_{22} - t_{12}t_{21}t_{22} + t_{12}^2 t_{22} = t_{11}t_{22}^2$. In view of $t_{12}t_{21} = 0$ and the invertibility of t_{22}, we see that $t_{12}^2 + t_{21}^2 = t_{11}t_{22}$. Since the ring is local, one of the elements t_{12} and t_{21} is invertible, and the other equals zero.

If $t_{21} = 0$, then $t_{11}t_{12}t_{22} = 1$ (which follows from the second relation), and hence $t_{12} = 1$. Furthermore, $t_{12} = t_{11}t_{12}$ (which follows from the third) yields $t_{11} = 1$ and $t_{22} = 1$, as required.

If $t_{12} = 0$, then again the second relation implies $t_{11}t_{21}t_{22} = -1$. Hence $t_{21} = -1$ and $t_{11} = 1 = t_{22}$. It remains to apply a contragradient automorphism which assigns $(A^T)^{-1}$ to each element of A such that all elements of $\mathcal{MJ}$, as well as σ_{12} and σ_{23}, are left fixed. □

LEMMA 3. Let R_1 and R_2 be infinite local commutative rings with 1/2 and let $\mathbf{G}_n(R_1) \equiv_\forall \mathbf{G}_n(R_2)$ ($\mathbf{G} = \mathrm{GL}, \mathrm{SL}$). Suppose also that M_1 is an arbitrary finite submodel of $\mathbf{G}_n(R_1)$

containing a finite set of matrices such as in the previous lemmas and the matrix diag $[2,1,1/2,1,\dots,1]$, and M_2 is a finite submodel of $\mathbf{G}_n(R_2)$ isomorphic to M_1. Then, for any isomorphism

$$\Phi\colon M_1 \to M_2,$$

every finite set of matrices of the form $E+\alpha_i E_{12}$, $i=1,\dots,k$, is mapped to a set of matrices of the form $E+\beta_i E_{12}$, $i=1,\dots,k$, in a basis in which the form of the matrices from the previous lemmas will not be changed.

Proof. As in Lemma 2, it is easy to show that

$$\Phi(E+\alpha_i E_{12}) = \begin{pmatrix} s^i_{11} & s^i_{12} & 0 \\ s^i_{21} & s^i_{22} & 0 \\ 0 & 0 & E \end{pmatrix}.$$

Since the matrices $E+E_{12}$ and $E+\alpha_i E_{12}$ commute, we obtain $s^i_{21}=0$ and $s^i_{11}=s^i_{22}$.

It remains to show that the first two places on the diagonal are occupied by 1's. We write the relations with a diagonal matrix for a corner block, taking into account that $\Phi(\mathrm{diag}\,[2,1,1/2,1,\dots,1]) = \mathrm{diag}\,[d_1,\dots,d_n]$:

$$\begin{pmatrix} s^i_{11} & \frac{d_1}{d_2}s^i_{12} \\ 0 & s^i_{22} \end{pmatrix} = \begin{pmatrix} (s^i_{11})^2 & 2s^i_{12} \\ 0 & (s^i_{22})^2 \end{pmatrix}.$$

Now the invertibility of s^i_{11} and s^i_{22} implies that both of these elements are equal to 1. □

We make to remarks on a connection between multiplication of elements in a local ring R and operations over elements of $\mathbf{G}_n(R)$ ($\mathbf{G}=\mathrm{GL},\mathrm{SL}$). First,

$$[E+\alpha E_{12}, E+\beta E_{23}] = E+\alpha\beta E_{13}.$$

Second,

$$E+\alpha E_{23} = \sigma_{12}\sigma_{23}^{-1}(E+\alpha E_{12})^{-1}\sigma_{23}\sigma_{12}^{-1},$$
$$E+\alpha E_{13} = [E+E_{12}, E+\alpha E_{23}].$$

Now the theorem follows from the previous lemmas, as in [1]. More precisely, let $S_1\subset R_1$ be a finite submodel of the ring R_1. Our goal is to find a submodel S_2 isomorphic to S_1 in R_2. Let $M_1\subset \mathbf{G}_n(R_1)$ ($\mathbf{G}=\mathrm{GL},\mathrm{SL}$) be a finite submodel containing all matrices of the form $E+\alpha E_{12}$ for all $\alpha\in S_1$ and finitely many auxiliary matrices from the previous lemmas. The groups are universally equivalent; therefore, for M_1 there exists an isomorphic submodel $M_2\subset\mathbf{G}_n(R_2)$ ($\mathbf{G}=\mathrm{GL},\mathrm{SL}$) such that any isomorphism $\Phi\colon M_1\to M_2$ preserves the form of the matrices. Therefore, the images of $E+\alpha E_{12}$, $E+\alpha E_{13}$, and $E+\alpha E_{23}$ will be $E+\beta_1 E_{12}$, $E+\beta_2 E_{13}$, and $E+\beta_3 E_{23}$, respectively, in some common basis, with $\beta_1=\beta_2=\beta_3\in R_2$. Then S_2 is the set of all elements of R_2 that correspond to image triplets of $E+\alpha E_{12}$, $E+\alpha E_{13}$, and $E+\alpha E_{23}$ for all elements $\alpha\in S_1$.

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