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INTEGRAL CAYLEY GRAPHS

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Let G be a group and $S \subseteq G$ a subset such that $S = S^{-1}$, where $S^{-1} = \{s^{-1} \mid s \in S\}$. Then a Cayley graph $\mathrm{Cay}(G,S)$ *is an undirected graph Γ with vertex set $V(\Gamma) = G$ and edge set $E(\Gamma) = \{(g, gs) \mid g \in G, s \in S\}$. For a normal subset S of a finite group G such that $s \in S \Rightarrow s^k \in S$ for every $k \in \mathbb{Z}$ which is coprime to the order of s, we prove that all eigenvalues of the adjacency matrix of* $\mathrm{Cay}(G,S)$ *are integers. Using this fact, we give affirmative answers to Questions* 19.50(a) *and* 19.50(b) *in the Kourovka Notebook.*

INTRODUCTION

We assume that all groups and graphs under consideration are finite. The symbols G and Γ denote some group and some graph, respectively. Furthermore, $V = V(\Gamma)$ and $E = E(\Gamma)$ are the

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set of vertices and the set of edges of Γ. The *adjacency matrix* of a graph Γ with vertex set V and edge set E is a matrix $(a_{ij}) \in \mathrm{M}_{|V|}(\mathbb{C})$, whose rows and columns are indexed by elements of the set V, and

$$a_{ij} = \begin{cases} 1, & (i,j) \in E, \\ 0, & (i,j) \notin E \end{cases} \quad \text{for all} \quad i,j \in V.$$

A graph Γ is said to be *integral* if its *spectrum*, i.e., the spectrum of its adjacency matrix, consists of integers.

Suppose $S \subseteq G$. A *Cayley graph* $\mathrm{Cay}\,(G,S)$ *of a group G associated with a set S* is a graph Γ with vertex set $V(\Gamma) = G$ and edge set $E(\Gamma) = \{(g, gs) \mid g \in G, s \in S\}$. Below we assume that the set S with which the graph $\mathrm{Cay}\,(G,S)$ is associated satisfies the following conditions:

S does not contain 1, which is equivalent to having no loops in $\mathrm{Cay}\,(G,S)$;

S is *symmetric*, i.e., $S = S^{-1}$, where $S^{-1} = \{s^{-1} \mid s \in S\}$.

The latter condition allows us to think of the graph $\mathrm{Cay}\,(G,S)$ as being undirected.

We consider only undirected graphs with no loops. The adjacency matrix of such a graph is symmetric, and therefore its spectrum consists of real numbers.

We need another two definitions to state our main results.

A subset $S \subseteq G$ is said to be *normal* if $S = S^G = \{s^g \mid g \in G\}$ (here $s^g = g^{-1}sg$ is an element conjugate to s with respect to $g \in G$).

It is known that for an element $s \in G$ of order m, the set of generating elements of a cyclic group $\langle s \rangle$ coincides with the set

$$\{s^k \mid k \in \mathbb{Z}, (k,m) = 1\} = \{s^k \mid 0 \leq k \leq m-1, \, (k,m) = 1\},$$

and its cardinality equals $\phi(m)$, where ϕ is Euler's totient function which, for a natural number m with a canonical prime factorization $m = p_1^{\alpha_1} \dots p_t^{\alpha_t}$, takes the value

$$\phi(m) = p_1^{\alpha_1 - 1}(p_1 - 1) \dots p_t^{\alpha_t - 1}(p_t - 1).$$

A subset $S \subseteq G$ is said to be *Eulerian* if the set

$$\{x \in G \mid \langle x \rangle = \langle s \rangle\} = \{s^k \mid 0 \leq k \leq |s| - 1, (k, |s|) = 1\}$$

is contained in S for any $s \in S$. Every Eulerian set is symmetric since $\langle s \rangle = \langle s^{-1} \rangle$ for any $s \in G$. If a symmetric set $S \subseteq G$ is such that $\phi(|s|) \leq 2$, or, which is equivalent, $|s| \in \{2,3,4,6\}$ for every $s \in S$, then it is Eulerian.

We look into the following two questions.

Question 1 [1, Quest. 19.50(a)]. Is it true that if $S \subseteq G$ is a normal set of elements of order 2, then the graph $\mathrm{Cay}\,(G,S)$ is integral?

Question 2 [1, Quest. 19.50(b)]. Is it true that if A_n is the alternating group of degree n and $S = \{(12i)^{\pm 1} \mid i = 3, \dots, n\}$, then the graph $\mathrm{Cay}\,(A_n, S)$ is integral?

In the present paper, both of the questions will be answered in the affirmative.

Answers to these questions will follow from the following theorem.

THEOREM. If S is an Eulerian normal subset of a group G, then the graph $\mathrm{Cay}\,(G, S)$ is integral.

The proof of this theorem uses character-theoretic methods and does not depend on any other results on integral graphs.

A particular case of the theorem is [2, Thm. 1] with the additional assumption that G is nilpotent.

An affirmative answer to Question 1 is given by

COROLLARY 1. If orders of elements of a normal symmetric subset S in a group G belong to the set $\{2, 3, 4, 6\}$, then the graph $\mathrm{Cay}\,(G, S)$ is integral.

Note that Question 1 was independently answered by A. Abdollahi who also used character theory and results of [3, 4] (see [1, commentary to Quest. 19.50(a)]).

A particular case of a normal set of elements of order 2 (which is automatically symmetric) is the set of all transpositions in the symmetric group. Therefore, we give an independent proof of the following result obtained in [2, Thm. 2] by using the property of being integral for the so-called *star graph* $\mathrm{Cay}\,(G, S)$, where $G = S_n$ is the symmetric group of degree n, and $S = \{(1i) \mid 1 < i \leq n\}$ [5, Thm. 1; 6, Cor. 2.1].

COROLLARY 2. The graph $\mathrm{Cay}\,(G, S)$, where $G = S_n$ and $S = \{(ij) \mid 1 \leq i < j \leq n\}$, is integral.

The proof that a star graph is integral which does not depend on [5, Thm. 1; 6, Cor. 2.1] follows from Corollary 2 and

COROLLARY 3. Assume that R is an Eulerian normal subset in a group G, and H is a subgroup of G. Set $S = R \setminus (R \cap H)$. Then the graph $\mathrm{Cay}\,(G, S)$ is integral.

COROLLARY 4. The graph $\mathrm{Cay}\,(G, S)$, where $G = S_n$ and $S = \{(1i) \mid 1 < i \leq n\}$, is integral.

Similar considerations allow us to answer Question 2 in the affirmative. Namely, the following holds:

COROLLARY 5. The graph $\mathrm{Cay}\,(G, S)$, where $G = A_n$ is the alternating group of degree n and $S = R \cup R^{-1}$ for $R = \{(12i) \mid i = 3, \dots, n\}$, is integral.

We were informed by M. Muzychuk that he had independently answered Question 2 by using analogous methods. Moreover, the subsequent detailed use of results in the representation theory of the symmetric groups allowed him to identify an explicit form of eigenvalues of a corresponding graph.

A combination of Corollaries 1 and 3 might be useful for obtaining new integral Cayley graphs. Note that not every symmetric set S of elements of a group G for which the graph $\mathrm{Cay}\,(G, S)$ is integral can be found with the help of the method given in Corollary 3. For example, if a group

G has a normal Eulerian subset R and a family $H_1, \dots, H_n$ of subgroups satisfying the condition $[H_i, H_j] = 1$ for $i \neq j$, then $\mathrm{Cay}\,(G, S)$ will be integral for a set $S = R \setminus \cup(R \cap H_i)$. The set $S = \{(12i)^{\pm 1} \mid i = 3, \dots, n\}$ in Corollary 5 whose elements are conjugate in the alternating group A_n for $n > 3$ also cannot be obtained from the class R of all cycles of length 3 by eliminating those elements that belong to some subgroup and even to a family of elementwise commuting subgroups. This follows from the fact that the difference $R \setminus S$ contains all cycles of the form $(34i)$ generating the group A_n.

1. NOTATION

We will use standard facts, notions, and designations from the representation and character theory:

$\mathbb{C}G$ is the complex group algebra of a group G [7];

$\mathrm{Irr}\,(G)$ is the set of common irreducible characters of a group G [7];

V_χ is a $\mathbb{C}G$-module corresponding to a (not necessarily irreducible) character χ of a group G;

a_V for a $\mathbb{C}G$-module V and an element $a \in \mathbb{C}G$ is a linear transformation of a space V given by the rule $v \mapsto va$ [7];

a_χ for an element $a \in \mathbb{C}G$ is a linear transformation a_{V_χ} of a space V_χ;

$[\varphi]_B$ for a linear transformation φ of a vector space V with basis B is a matrix of the transformation φ relative to B;

$[\varphi]$ for a linear transformation φ of a vector space V is a matrix of the transformation φ relative to some basis of the space V;

ω_χ for $\chi \in \mathrm{Irr}\,(G)$ is a homomorphism of complex algebras $Z(\mathbb{C}G) \to \mathbb{C}$ such that $[a_\chi] = \omega_\chi(a)I$ for every $a \in Z(\mathbb{C}G)$ [7, p. 35]; here I is a $\chi(1) \times \chi(1)$ identity matrix;

$\overline{S}$ for a set S of elements of a group G is the element $\sum\limits_{s \in S} s \in \mathbb{C}G$.

By the *spectrum of an element* $a \in \mathbb{C}G$ *on a* $\mathbb{C}G$*-module* V we mean the spectrum of a linear transformation a_V.

We identify every element $a \in \mathbb{C}G$ with right multiplication of elements of a regular module by a. In particular, unless specified otherwise, the *spectrum of an element* $a \in \mathbb{C}G$ is the spectrum of a on a regular module.

2. PRELIMINARY RESULTS

We remind that a complex number is called an *algebraic integer* if it is a root of a monic polynomial with integer coefficients. It is known that algebraic integers form a subring of $\mathbb{C}$ [7, Cor. 3.5], and a number $\alpha \in \mathbb{Q}$ is an algebraic integer iff $\alpha \in \mathbb{Z}$ [7, Lemma 3.2].

LEMMA 1. Suppose that $\mathfrak{X} : G \to \mathrm{GL}_n(\mathbb{C})$ is a representation of a group G with a character χ and $g \in G$ is an element of order m. Then:

(1) the matrix $\mathfrak{X}(g)$ is similar to a diagonal matrix $\mathrm{diag}\,(\zeta_1, \dots, \zeta_n)$ such that $\zeta_i^m = 1$ for all $i = 1, \dots, n$;

(2) $\chi(g) = \zeta_1 + \dots + \zeta_n$ is an algebraic integer.

Proof. Statement (1) was proved in [7, Lemma 2.15]. Statement (2) follows from (1) and the fact that algebraic integers form a subring of $\mathbb{C}$. □

LEMMA 2. Let $g \in G$ be an element whose order divides $m \in \mathbb{Z}$, χ a character of the group G, and ζ a complex primitive mth root of unity. Then:

(1) $\chi(g) \in \mathbb{Q}(\zeta)$;

(2) $\mathbb{Q}(\zeta)/\mathbb{Q}$ is a normal extension;

(3) for every automorphism σ of the field $\mathbb{Q}(\zeta)$, there exists $k \in \mathbb{Z}$ such that $(k, m) = 1$ and $\zeta^\sigma = \zeta^k$;

(4) $\chi(g)^\sigma = \chi(g^k)$ for such σ and k.

Proof. Statement (1) follows from Lemma 1(1) and the fact that ζ is a primitive mth root of unity. Statement (2) follows from the fact that $\mathbb{Q}(\zeta)$ is the splitting field of the polynomial $x^m - 1$. Statement (3) is true since an automorphism of the field $\mathbb{Q}(\zeta)$ should map ζ to some primitive mth root of unity, i.e., to a number of the form ζ^k, where $(k, m) = 1$. If $\chi(g) = \zeta_1 + \dots + \zeta_n$ with ζ_i chosen as in Lemma 1(1), then they are mth roots of unity, and every ζ_i is a power of ζ. Therefore, $\zeta_i^\sigma = \zeta_i^k$. Now it follows from Lemma 1 that

$$\chi(g)^\sigma = \zeta_1^k + \dots + \zeta_n^k = \chi(g^k).$$

Statement (4) is proved. □

LEMMA 3. Let $\chi \in \mathrm{Irr}\,(G)$. Then, for any $x \in G$, the number $\omega_\chi(\overline{x^G})$ is an algebraic integer, and

$$\omega_\chi(\overline{x^G}) = \frac{\chi(x)|x^G|}{\chi(1)}.$$

Proof. See [7, p. 36]. □

LEMMA 4. Suppose that $S \subseteq G$ is a symmetric set, $H = \langle S \rangle$, and $|G : H| = n$. Let $\Gamma = \mathrm{Cay}\,(G, S)$ and $\Delta = \mathrm{Cay}\,(H, S)$. Denote by f_G and f_H the respective characteristic polynomials of the adjacency matrices of the graphs Γ and Δ. The following statements hold:

(1) the graph Γ is connected iff $G = H$;

(2) if $H \neq G$, then every connected component of the graph Γ is isomorphic to Δ, and the number of components is equal to n;

(3) $f_G = f_H^n$; in particular, the spectra of the graphs Γ and Δ coincide.

Proof. Statement (2) was proved in [2, Lemma 1]. Statement (2) implies both statement (3) and the property of Γ being disconnected for $G \neq H$ in statement (1). To complete the proof of (1), it suffices to note that every element $g \in G$ can be written as a product of elements of the set S if the graph Γ is connected. The connectivity of Γ implies that there exists a path

$$(x_0, x_1), (x_1, x_2), \dots, (x_{m-1}, x_m) \in E(\Gamma), \text{ where } x_0 = 1 \text{ and } x_m = g.$$

By the definition of a Cayley graph, for any $i = 1, \dots, m$ there is an element $s_i \in S$ such that $x_i = x_{i-1}s_i$. Therefore,

$$g = x_m = x_0 s_1 s_2 \dots s_m = s_1 s_2 \dots s_m. \ \square$$

3. SPECTRA OF ELEMENTS OF THE GROUP ALGEBRA

PROPOSITION 1. Suppose that G is a finite group and $S \subseteq G$ is a subset such that $S = S^{-1}$. Then the following statements hold:

(1) The adjacency matrix of the graph $\mathrm{Cay}\,(G, S)$ coincides with a matrix $[\overline{S}]_G$; the spectrum of $\mathrm{Cay}\,(G, S)$ coincides with the spectrum of an element $\overline{S}$ on a regular module.

(2) For every element $a \in \mathbb{C}G$, the matrix $[a]_G$ is similar to a block diagonal matrix with blocks $[a_\chi]$ on the diagonal, where $\chi \in \mathrm{Irr}\,(G)$, and every block $[a_\chi]$ appears $\chi(1)$ times.

(3) For every element $a \in \mathbb{C}G$, the spectrum of the matrix $[a]_G$ is the union of the spectra of the matrices $[a_\chi]$ taken over all $\chi \in \mathrm{Irr}\,(G)$.

(4) For every element $a \in \mathbb{C}G$, the following statements are equivalent:

(i) the spectrum of a is integral on a regular $\mathbb{C}G$-module;

(ii) the spectrum of a is integral on every irreducible $\mathbb{C}G$-module;

(iii) the spectrum of a is integral on every $\mathbb{C}G$-module.

(5) If λ is an eigenvalue of $[a]_G$, then the multiplicity of λ is

$$\sum_{\chi \in \mathrm{Irr}\,(G)} \chi(1) \mathrm{m}_\chi(\lambda),$$

where $\mathrm{m}_\chi(\lambda)$ is a (possibly zero) multiplicity of λ as an eigenvalue of the transformation a_χ.

(6) For a normal set S of G, the element $\overline{S}$ lies in the center of the algebra $\mathbb{C}G$.

(7) For every element $a \in Z(\mathbb{C}G)$, the spectrum of the matrix $[a]_G$ is equal to

$$\{\omega_\chi(a) \mid \chi \in \mathrm{Irr}\,(G)\}.$$

(8) If $x_1, \dots, x_t \in G$ are pairwise nonconjugate elements and

$$S = \bigcup_{i=1}^{t} K_i, \quad \text{where} \quad K_i = x_i^G,$$

then the spectrum of the matrix $[\overline{S}]_G$ is equal to

$$\left\{ \sum_{i=1}^{t} \omega_\chi(\overline{K}_i) \;\middle|\; \chi \in \mathrm{Irr}\,(G) \right\} = \left\{ \sum_{i=1}^{t} \frac{\chi(x_i)|K_i|}{\chi(1)} \;\middle|\; \chi \in \mathrm{Irr}\,(G) \right\}.$$

Proof. Statement (1) is verified by direct calculations. Statement (2) follows from Maschke's theorem [7, Thm. 1.9] and from a well-known decomposition of a regular module of a semisimple algebra into a direct sum of irreducible modules [7, Cor. 1.17]. Statements (3)-(5) follow from

(2). Statement (6) is a consequence of [7, Thm. 2.4]. It is known that $[a_\chi]$ is a scalar matrix for every $\chi \in \mathrm{Irr}\,(G)$ and equals $\omega_\chi(a)I$ (see [7, p. 35]). Therefore, (7) follows from (3). Since the transformation

$$\omega_\chi : Z(\mathbb{C}G) \to \mathbb{C}G$$

is a homomorphism of algebras [7, p. 35], statements (3), (6), (7) and Lemma 3 imply (8). □

An exact expression for the spectrum of $\mathrm{Cay}\,(G, S)$ in the case where $S \subseteq G$ is normal, which follows directly from statements (1) and (8) of Proposition 1, is also pointed out in [4, Thm. 9]: the spectrum consists of numbers

$$\lambda_\chi = \frac{1}{\chi(1)} \sum_{x \in S} \chi(x)$$

over all $\chi \in \mathrm{Irr}\,(G)$. There is an inaccuracy in the formulation of [4, Thm. 9], where it is stated that the multiplicity of λ_χ is equal to $\chi(1)^2$. It may so happen that $\lambda_\chi = \lambda_\psi$ for distinct $\chi, \psi \in \mathrm{Irr}\,(G)$. As follows from statement (5) of Proposition 1, the multiplicity of an eigenvalue λ of $\mathrm{Cay}\,(G, S)$ is equal to $\sum\limits_{\substack{\chi \in \mathrm{Irr}\,(G) \\ \lambda_\chi = \lambda}} \chi(1)^2$.

4. PROOF OF THEOREM 1

Let S be a normal Eulerian subset in G and

$$a = \overline{S} \in \mathbb{C}G.$$

In view of statements (1), (3), (6), and (7) of Proposition 1, it suffices to show that $\omega_\chi(a) \in \mathbb{Z}$ for every $\chi \in \mathrm{Irr}\,(G)$. Suppose that $x_1, \ldots, x_t$ are representatives of all conjugacy classes whose union is S. As in statement (8) of Proposition 1,

$$\omega_\chi(a) = \sum_{i=1}^{t} \frac{\chi(x_i)|x_i^G|}{\chi(1)}. \qquad (*)$$

Lemma 2 implies that $\omega_\chi(a) \in \mathbb{Q}(\zeta)$, where ζ is a primitive mth root of unity, and m is a least common multiple of the numbers $|x_i|$. Take an arbitrary automorphism σ of the field $\mathbb{Q}(\zeta)$. By virtue of Lemma 2, there is a number k such that $(m, k) = 1$ and $\chi(x_i)^\sigma = \chi(x_i^k)$ for all $i = 1, \ldots, t$. Since S is Eulerian, $x_i^k \in S$ for every i. The set S is normal, so the element x_i^k is conjugate to some x_j. We also have

$$\chi(x_i)^\sigma = \chi(x_i^k) = \chi(x_j).$$

The mapping $x \mapsto x^k$ is a bijection between the classes x_i^G and x_j^G, hence

$$|x_i^G| = |x_j^G|.$$

Thus σ permutes terms in the right part of equation $(*)$. Consequently, $\omega_\chi(a)^\sigma = \omega_\chi(a)$ for every automorphism σ of the field $\mathbb{Q}(\zeta)$, and $\omega_\chi(a) \in \mathbb{Q}$. The number $\omega_\chi(a)$ is a sum of algebraic integers by Lemma 3, so it is itself an algebraic integer and, hence, just an integer. □

5. PROOFS OF THE COROLLARIES

The **proof** of Corollaries 1 and 2. Since the symmetric set S for which $\{|s| \mid s \in S\} \subseteq \{2,3,4,6\}$ is Eulerian, the conclusion of Corollary 1 follows directly from the theorem. Corollary 2 is a particular case of Corollary 1. □

The **proof** of Corollary 3. In the group algebra $\mathbb{C}G$, we consider the following elements:

$$a = \overline{S}, \quad b = \overline{R}, \quad \text{and} \quad c = \overline{H \cap R}.$$

It is clear that $a = b - c$. By virtue of Proposition 1(6), $b \in Z(\mathbb{C}G)$. Hence the elements a, b, and c commute pairwise. Their matrices are symmetric; therefore, the regular module V of the group algebra has a basis of common eigenvectors for a, b, and c. It follows that every eigenvalue α of a is the difference $\beta - \gamma$ of some eigenvalues β and γ of b and c, respectively. The spectrum of the element b on V is integral by the theorem. The set $H \cap R$ is a normal Eulerian subset in H. In view of the theorem, the element c has an integral spectrum on a regular module of the group algebra $\mathbb{C}H$. Any $\mathbb{C}G$-module, in particular, V, can be treated as a $\mathbb{C}H$-module with compatible action, so the element c has an integral spectrum on V by Prop. 1(4). Hence the spectrum of a on V is also integral, and the graph Cay (G,S) is integral by Prop. 1(1). □

The **proof** of Corollary 4. Suppose that $G = S_n$, $R = \{(ij) \mid 1 \leq i < j \leq n\}$, and $H \cong S_{n-1}$ is the set of all permutations of $G = S_n$ that fix point 1. It is easy to see that

$$S = \{(1i) \mid 1 < i \leq n\} = R \setminus (R \cap H).$$

As in Corollary 2, R is a normal Eulerian subset in G. Corollary 3 implies that the graph Cay (G,S) is integral. □

The **proof** of Corollary 5. By virtue of Lemma 4, it suffices to show that the graph Cay (G^*,S), where $G^* = S_n$, will be integral. In the group algebra $\mathbb{C}G^*$, consider the following elements:

$$a = \overline{S} = \sum_{i=3}^{n} \big((12i) + (21i)\big), \qquad b = \sum_{1 \leq i < j \leq n} (ij),$$

$$c = \sum_{3 \leq i < j \leq n} (ij), \qquad d = (12).$$

By Proposition 1, we need only show that the spectrum of a on a regular $\mathbb{C}G^*$-module is integral. The elements c and d commute. The element b, which is the sum of all transpositions, is central in the algebra $\mathbb{C}G^*$; therefore, b, c, and d commute pairwise. It is straightforward to verify that

$$a = (12)\left(\sum_{i=3}^{n}(1i) + \sum_{i=3}^{n}(2i)\right) = d(b - c - d).$$

The regular $\mathbb{C}G^*$-module has a basis of common eigenvectors of the elements b, c, and d, which is also a basis of eigenvectors of the element a. In this case if, for such a vector v and for numbers $\beta, \gamma, \delta \in \mathbb{C}$, we have

$$vb = \beta v, \quad vc = \gamma v, \quad \text{and} \quad vd = \delta v,$$

then it is obvious that

$$va = \delta(\beta - \gamma - \delta)v.$$

To complete the proof, it suffices to show that the spectra of the elements b, c, and d are integral.

The spectrum of b is integral by Corollary 1 and Proposition 1(1), since $b = \overline{T}$, where T is the set of all transpositions in G^*. The element d has order 2, so its spectrum lies in the set of roots of the polynomial $x^2 - 1$ and consists of numbers ± 1. Finally, suppose that H is a subgroup of G^* which is isomorphic to S_{n-2} and consists of all permutations that fix points 1 and 2. Then $H \cap T$ is a normal Eulerian subset in H, $c = \overline{H \cap T}$, and as in the proof of Corollary 3, we conclude that the spectrum of c is integral. Hence the element a also has an integral spectrum. □

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