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MAXIMALITY OF THE COUNTABLE SPECTRUM IN SMALL QUITE o-MINIMAL THEORIES

B. Sh. Kulpeshov*

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We give a criterion for the countable spectrum to be maximal in small binary quite o-minimal theories of finite convexity rank.

The present paper deals with the notion of *weak o-minimality*, which was initially deeply investigated in [1]. A subset A of a linearly ordered structure M is said to be *convex* if $c \in M$ whenever $a < c < b$ for any $a, b \in A$ and any $c \in A$. A *weakly o-minimal structure* is a linearly ordered structure $M = \langle M, =, <, \ldots \rangle$ such that every definable (with parameters) subset of M is the union of finitely many convex sets in M. Real closed fields with a proper convex valuation ring furnish an important example of weakly o-minimal structures.

In the definitions below, M is a weakly o-minimal structure, $A, B \subseteq M$, M is $|A|^+$-saturated, and $p, q \in S_1(A)$ are nonalgebraic.

Definition 1 [2]. We say that a type p is not *weakly orthogonal* to a type q ($p \not\perp^w q$) if there is an A-definable formula $H(x, y)$ and there are $\alpha \in p(M)$ and $\beta_1, \beta_2 \in q(M)$ such that $\beta_1 \in H(M, \alpha)$ and $\beta_2 \notin H(M, \alpha)$.

Definition 2 [3]. We say that a type p is not *quite orthogonal* to a type q ($p \not\perp^q q$) if there exists an A-definable bijection $f : p(M) \to q(M)$. We also say that a weakly o-minimal theory is *quite o-minimal* if the notions of weak orthogonality and quite orthogonality coincide for 1-types.

Quite o-minimal theories are a subclass of the class of weakly o-minimal theories which inherits many properties of o-minimal theories. The Vaught problem for quite o-minimal theories was solved in [4]: it was proved that every countable quite o-minimal theory either is countably categorical, or

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International Information Technologies University, Manas str. 34/1, Alma-Ata, 050040 Kazakhstan. Institute of Mathematics and Mathematical Modeling, Ministry of Education and Science RK, ul. Pushkina 125, Alma-Ata, 050010 Kazakhstan. Kazakh-British Technical University, ul. Tole bi 59, Alma-Ata, 050000 Kazakhstan; b.kulpeshov@iitu.kz. Translated from *Algebra i Logika*, Vol. 58, No. 2, pp. 200-209, March-April, 2019. Original article submitted November 27, 2017; accepted July 9, 2019.

is an Ehrenfeucht theory, or has the maximum number of countable models. This result generalizes a theorem of L. Mayer [5], which is a solution to the Vaught problem for o-minimal theories. Here we give a criterion for the number of countable models to be maximal in small binary quite o-minimal theories of finite convexity rank (Thm. 11).

Definition 3 [6]. Let M be a weakly o-minimal structure, $A \subseteq M$, M be $|A|^+$-saturated, and $p \in S_1(A)$ be nonalgebraic.

(1) An A-definable formula $F(x, y)$ is *p-stable* if there are $\alpha, \gamma_1, \gamma_2 \in p(M)$ such that $F(M, \alpha) \setminus \{\alpha\} \neq \varnothing$ and $\gamma_1 < F(M, \alpha) < \gamma_2$.

(2) A p-stable formula $F(x, y)$ is *convex to the right* (*left*) if there exists $\alpha \in p(M)$ such that $F(M, \alpha)$ is convex, α is a left (right) endpoint of the set $F(M, \alpha)$, and $\alpha \in F(M, \alpha)$.

If $F_1(x, y)$ and $F_2(x, y)$ are p-stable convex to the right (left) formulas, then we say that $F_2(x, y)$ is *bigger than* $F_1(x, y)$ if there exists $\alpha \in p(M)$ for which $F_1(M, \alpha) \subset F_2(M, \alpha)$.

Definition 4 [7]. We say that a p-stable convex to the right (left) formula $F(x, y)$ is *equivalence generating* if, for any $\alpha, \beta \in p(M)$ such that $M \models F(\beta, \alpha)$, the following holds:

$$M \models \forall x[x \geq \beta \to [F(x, \alpha) \leftrightarrow F(x, \beta)]]$$
$$(M \models \forall x[x \leq \beta \to [F(x, \alpha) \leftrightarrow F(x, \beta)]]).$$

LEMMA 5 [7]. Let M be a weakly o-minimal structure, $A \subseteq M$, $p \in S_1(A)$ be nonalgebraic, and M be $|A|^+$-saturated. Suppose that $F(x, y)$ is a p-stable convex to the right (left) formula, which is equivalence generating. Then:

(1) $G(x, y) := F(y, x)$ is a p-stable convex to the left (right) formula, which is also equivalence generating;

(2) $E(x, y) := F(x, y) \vee F(y, x)$ is an equivalence relation partitioning $p(M)$ into infinitely many infinite convex classes.

Definition 6 [8]. Let T be a weakly o-minimal theory, M a sufficiently saturated model of T, and $\phi(x)$ an arbitrary M-definable formula with one free variable. The *convexity rank of a formula* $\phi(x)$ ($RC(\phi(x))$) is defined as follows:

(1) $RC(\phi(x)) \geq 1$ if $\phi(M)$ is infinite;

(2) $RC(\phi(x)) \geq \alpha + 1$ if there exist a parametrically definable equivalence relation $E(x, y)$ and infinitely many elements b_i, $i \in \omega$, such that:

for any $i, j \in \omega$, $M \models \neg E(b_i, b_j)$ whenever $i \neq j$;

for every $i \in \omega$, $RC(E(x, b_i)) \geq \alpha$ and $E(M, b_i)$ is a convex subset of $\phi(M)$;

(3) $RC(\phi(x)) \geq \delta$ if $RC(\phi(x)) \geq \alpha$ for all $\alpha < \delta$ (δ is a limit ordinal).

If $RC(\phi(x)) = \alpha$ for some α, then we say that $RC(\phi(x))$ *is defined*; otherwise (i.e., $RC(\phi(x)) \geq \alpha$ for all α), we put $RC(\phi(x)) = \infty$.

For a 1-type p, we define the *convexity rank*

$$RC(p) := \inf\{RC(\phi(x)) \mid \phi(x) \in p\}.$$

Definition 7 [1]. Let $\mathcal{M}$ be a weakly o-minimal structure, $D \subseteq M$ be an infinite set, and $f : D \to K$ be a function. We say that f is *locally increasing* (*locally decreasing*, *locally constant*) on D if for any $x \in D$ there exists an infinite interval $J \subseteq D$ which contains x and is such that f is strictly increasing (strictly decreasing, constant) on J.

We also say that a function f is *locally monotone* on a set $D \subseteq M$ if f is either locally increasing or locally decreasing on D.

Let f be an A-definable function on $D \subseteq M$ and E an A-definable equivalence relation on D. We say that f is *strictly increasing* (*decreasing*) on D/E if $f(a) < f(b)$ ($f(a) > f(b)$) for any $a, b \in D$ with $a < b$ and $\neg E(a, b)$.

PROPOSITION 8 [9]. Let $\mathcal{M}$ be a weakly o-minimal structure, $A \subseteq M$, and $p \in S_1(A)$ be a nonalgebraic type. Then every A-definable function whose domain contains a set $p(\mathcal{M})$ is locally monotone or locally constant on $p(\mathcal{M})$.

Below we need the concept of a (p, q)-splitting formula introduced in [10]. Let $A \subseteq M$, $p, q \in S_1(A)$ be nonalgebraic types, and $p \not\perp^w q$. An A-definable formula $\phi(x, y)$ is called a (p, q)-*splitting formula* if there exists an element $a \in p(M)$ for which $\phi(a, M) \subset q(M)$, $\phi(a, M)$ is convex, and $\phi(a, M)^- = q(M)^-$, where $\phi(a, M)^- := \{b \in M \mid b < \phi(a, M)\}$. Let $\phi_1(x, y)$ and $\phi_2(x, y)$ be (p, q)-splitting formulas; then we say that $\phi_1(x, y)$ is *smaller than* $\phi_2(x, y)$ if there exists an element $a \in p(M)$ such that $\phi_1(a, M) \subset \phi_2(a, M)$.

Obviously, if $p, q \in S_1(A)$ are nonalgebraic types and $p \not\perp^w q$, then there exists a (p, q)-splitting formula, and the set of all (p, q)-splitting formulas is linearly ordered. It is also clear that for any (p, q)-splitting formula $\phi(x, y)$, the function $f(x) := \sup \phi(x, M)$ is not constant on $p(M)$.

LEMMA 9. Let T be a binary quite o-minimal theory, $p, q \in S_1(\varnothing)$ be nonalgebraic, and $RC(p) = n$. Suppose that every p-stable convex to the right (left) formula is equivalence generating. The relation $p \not\perp^w q$ holds if and only if there exists a unique $\varnothing$-definable bijection $f : p(M) \to q(M)$, and there are precisely $2n$ (p, q)-splitting formulas.

Proof. Since $RC(p) = n$, there exist $\varnothing$-definable equivalence relations $E_1(x, y), \ldots, E_{n-1}(x, y)$ partitioning $p(M)$ into infinitely many infinite convex classes, so that for every $1 \le i \le n-2$ the equivalence E_i partitions each E_{i+1}-class into infinitely many E_i-subclasses, and $E_1(a, M) \subset \ldots \subset E_{n-1}(a, M)$ for any $a \in p(M)$.

Suppose $p \not\perp^w q$. Then, in view of quite o-minimality, there exists a $\varnothing$-definable bijection $f : p(M) \to q(M)$. Consider the following formulas:

$$\phi^0_-(x, y) := y < f(x),$$
$$\phi^0_+(x, y) := y \le f(x),$$
$$\phi^i_-(x, y) := \forall t[E_i(x, t) \to y < f(t)],\ 1 \le i \le n-1,$$
$$\phi^i_+(x, y) := \exists t[E_i(x, t) \land y < f(t)],\ 1 \le i \le n-1.$$

Obviously, these are (p,q)-splitting formulas, and

$$\phi_-^{n-1}(a,M) \subset \ldots \subset \phi_-^1(a,M) \subset \phi_-^0(a,M) \\ \subset \phi_+^0(a,M) \subset \phi_+^1(a,M) \subset \ldots \subset \phi_+^{n-1}(a,M).$$

We claim that there are no other (p,q)-splitting formulas. In particular, there exist no other $\varnothing$-definable functions mapping $p(M)$ into $q(M)$. Assume to the contrary that there exists a (p,q)-splitting formula $\Phi(x,y)$ distinct from these $2n$ (p,q)-separating formulas. The following cases are possible:

$$\phi_-^{i+1}(a,M) \subset \Phi(a,M) \subset \phi_-^i(a,M) \text{ for some } 0 \le i \le n-2,$$
$$\phi_+^{i}(a,M) \subset \Phi(a,M) \subset \phi_+^{i+1}(a,M) \text{ for some } 0 \le i \le n-2,$$
$$\Phi(a,M) \subset \phi_-^{n-1}(a,M) \text{ or } \phi_+^{n-1}(a,M) \subset \Phi(a,M).$$

There is no loss of generality in assuming that $\phi_-^{i+1}(a,M) \subset \Phi(a,M) \subset \phi_-^i(a,M)$ for some $0 \le i \le n-2$ (the other cases can be treated analogously). Since f is $\varnothing$-definable, f is locally monotone on $p(M)$, and f should be strictly increasing or strictly decreasing on each $E_{i+1}(a,M)/E_i$ for any $a \in p(M)$. For definiteness, suppose that f is strictly increasing. Consider the formula

$$G^\Phi(z,a) := z \le a \wedge \forall y[\neg\phi_-^{i+1}(a,y) \wedge \phi_+^i(a,y) \wedge y < f(z) \to \Phi(a,y)].$$

It is not hard to see that $G^\Phi(z,x)$ is a p-stable convex to the left formula, and $G^\Phi(z,x)$ is smaller than $G_{i+1}(z,x)$ and is bigger than $G_i(z,x)$, where $G_{i+1}(z,x) := E_{i+1}(z,x) \wedge z \le x$ and $G_i(z,x) := E_i(z,x) \wedge z \le x$ are also p-stable convex to the left formulas. By the hypotheses of the lemma, $G^\Phi(z,x)$ should be equivalence generating, and by virtue of Lemma 5, we obtain $RC(p) \ge n+1$, a contradiction. Thus other (p,q)-splitting formulas are missing. □

Definition 10 [11-13]. Let $p_1(x_1), \ldots, p_n(x_n) \in S_1(T)$. A type $q(x_1,\ldots,x_n) \in S(T)$ is called a $(p_1,\ldots,p_n)$-*type* if $q(x_1,\ldots,x_n) \supseteq \bigcup_{i=1}^{n} p_i(x_i)$. The set of all $(p_1,\ldots,p_n)$-types of T is denoted by $S_{p_1,\ldots,p_n}(T)$. A countable theory T is said to be *almost ω-categorical* if for any types $p_1(x_1), \ldots,$ $p_n(x_n) \in S(T)$ there exist only finitely many types $q(x_1,\ldots,x_n) \in S_{p_1,\ldots,p_n}(T)$.

Recall some of the notions considered in [5, 14]. We say that $\Gamma \subseteq S_1(\varnothing)$ is *independent* if, for every set Γ' consisting of exactly one realization of each type in Γ, $c' \notin \operatorname{dcl}(\Gamma' \setminus \{c'\})$ holds with any $c' \in \Gamma'$. We say that $p \in S_1(\varnothing)$ *depends on* Γ (or p and Γ *are dependent*) if $\Gamma \cup \{p\}$ is not independent. The *dimension* of a set Γ (denoted $\dim(\Gamma)$) is the cardinality of a maximally independent subset of the set Γ.

THEOREM 11. Let T be a small binary quite o-minimal theory of finite convexity rank and Γ be the set of all nonisolated types from $S_1(\varnothing)$. The theory T has 2^ω countable models if and only if at least one of the following conditions holds:

(1) $\dim(\Gamma) = \omega$;

(2) there exist a nonalgebraic type $p \in S_1(\varnothing)$ and a p-stable convex to the right (left) formula $F(x, y)$ which is not equivalence generating.

Proof. If $\dim(\Gamma) = \omega$, then there exist countably many pairwise weakly orthogonal nonisolated 1-types ensuring the maximality of a countable spectrum. If (2) holds, then the conclusion follows from [4, Prop. 2.8].

Suppose now that T has 2^ω countable models and $\dim(\Gamma) < \omega$. Assume to the contrary that every p-stable convex to the right (left) formula $F(x, y)$ is equivalence generating for any nonalgebraic type $p \in S_1(\varnothing)$. We claim that in this case T is almost ω-categorical. By induction on $k \geq 2$, we show that for any family of nonalgebraic types $p_1, \ldots, p_k \in S_1(\varnothing)$ there exist only finitely many $(p_1, \ldots, p_k)$-types.

Step $k = 2$.

Case 1. Let $p_1 \perp^w p_2$. Then the set $p_1(x) \cup p_2(y)$ defines a complete 2-type over $\varnothing$.

Case 2. Let $p_1 \not\perp^w p_2$. In view of quite o-minimality, there exists a $\varnothing$-definable bijection $f_{1,2} : p_1(M) \to p_2(M)$, whence $RC(p_1) = RC(p_2)$ (their convexity rank is denoted n_p). Taking into account Lemma 9, we see that no other $\varnothing$-definable functions from $p_1(M)$ are in $p_2(M)$, and there exist precisely $2n$ (p, q)-splitting formulas. Possible extensions of the set $p_1(x) \cup p_2(y)$ are formed by joining to it the following $2n_p + 1$ formulas:

$$
\begin{aligned}
& f_{1,2}(x) = y, \\
& f_{1,2}(x) < y \wedge E_1^{p_2}(f_{1,2}(x), y), \\
& f_{1,2}(x) < y \wedge E_{i+1}^{p_2}(f_{1,2}(x), y) \wedge \neg E_i^{p_2}(f_{1,2}(x), y),\ 1 \leq i \leq n_p - 2, \\
& f_{1,2}(x) < y \wedge \neg E_{n_p-1}^{p_2}(f_{1,2}(x), y) \text{ (and similarly with } f_{1,2}(x) > y).
\end{aligned}
$$

Thus there exist exactly $2n_p + 1$ (p_1, p_2)-types.

Step $n + 1$. Take arbitrary nonalgebraic types $p_1, \ldots, p_n, p_{n+1} \in S_1(\varnothing)$.

Case 1. Let $p_{n+1} \perp^w p_i$ for every $1 \leq i \leq n$. In this case the number of $(p_1, \ldots, p_n, p_{n+1})$-types coincides with the number of $(p_1, \ldots, p_n)$-types.

Case 2. Let $p_{n+1} \not\perp^w p_i$ for every $1 \leq i \leq n$. Then $RC(p_1) = \ldots = RC(p_{n+1})$ (their convexity rank is denoted n_p) and there exists a unique $\varnothing$-definable bijection $f_{n,n+1} : p_n(M) \to p_{n+1}(M)$. Possible extensions of the set $p_1(x_1) \cup \ldots \cup p_n(x_n) \cup p_{n+1}(x_{n+1})$ are formed by joining to it the following $2n_p + 1$ formulas:

$$
\begin{aligned}
& f_{n,n+1}(x_n) = x_{n+1}, \\
& f_{n,n+1}(x_n) < x_{n+1} \wedge E_1^{p_{n+1}}(f_{n,n+1}(x_n), x_{n+1}), \\
& f_{n,n+1}(x_n) < x_{n+1} \wedge E_{i+1}^{p_{n+1}}(f_{n,n+1}(x_n), x_{n+1}) \wedge \neg E_i^{p_{n+1}}(f_{n,n+1}(x_n), x_{n+1}), \\
& \text{where } 1 \leq i \leq n_p - 2, \\
& f_{n,n+1}(x_n) < x_{n+1} \wedge \neg E_{n_p-1}^{p_{n+1}}(f_{n,n+1}(x_n), x_{n+1}) \\
& \text{(and similarly with } f_{n,n+1}(x_n) > x_{n+1}).
\end{aligned}
$$

By the inductive assumption, the number of $(p_1, \dots, p_n)$-types is finite (denote it by $S_{p_1,\dots,p_n}$). Then the number of $(p_1, \dots, p_{n+1})$-types is equal to the product of $S_{p_1,\dots,p_n}$ and $2n_p + 1$.

Case 3. Let $p_{n+1} \not\perp^w p_i$ and $p_{n+1} \perp^w p_j$ for some $1 \leq i, j \leq n$, $i \neq j$. Then (if necessary) there exist a renumbering of types p_i and an element k with the condition $1 \leq k < n$ such that $p_{n+1} \perp^w p_j$, for all $1 \leq j \leq k$, and $p_{n+1} \not\perp^w p_l$ for all $k+1 \leq l \leq n$. By the inductive assumption, both the number of $(p_1, \dots, p_k, p_{n+1})$-types and the number of $(p_{k+1}, \dots, p_n, p_{n+1})$-types are finite, in which case the number of $(p_1, \dots, p_k, p_{n+1})$-types coincides with the number of $(p_1, \dots, p_k)$-types. Denote these numbers by $S_{p_1,\dots,p_k}$ and $S_{p_{k+1},\dots,p_n,p_{n+1}}$, respectively. Then the number of $(p_1, \dots, p_{n+1})$-types is equal to the product of $S_{p_1,\dots,p_k}$ and $S_{p_{k+1},\dots,p_n,p_{n+1}}$.

Thus the theory T is almost ω-categorical, and by virtue of [14, Cor. 3.10], it will be an Ehrenfeucht theory, which is a contradiction with T having 2^ω countable models. □

Note that the condition of being finite for the convexity rank is essential in the following:

Example 12. Let $M = \langle \mathbb{Q}, < E_i^2 \rangle_{i \in \omega}$ be a dense linear order structure on the set $\mathbb{Q}$ of rational numbers, and let it be enriched with equivalence relations E_i, $i \in \omega$, where each relation E_{i+1}, $i \geq 2$, consists of infinitely many open convex E_i-classes, which are densely ordered.

We can prove that $\mathrm{Th}(M)$ is a small binary quite o-minimal theory of infinite convexity rank having 2^ω countable models, $p(x) := \{x = x\} \in S_1(\varnothing)$ is a unique nonalgebraic type, and every p-stable convex to the right (left) formula is equivalence generating.

Note also that there exists a small quite o-minimal theory of finite convexity rank that is not binary.

Example 13. Let $\mathcal{M} = \langle M; <, P_1^1, P_2^1, P_3^1, f^2 \rangle$ be a linearly ordered structure whose universe M is a disjoint union of interpretations of unary predicates P_1, P_2, and P_3, with $P_1(\mathcal{M}) < P_2(\mathcal{M}) < P_3(\mathcal{M})$. We identify each interpretation of P_i $(1 \leq i \leq 3)$ with the set $\mathbb{Q}$ of rational numbers ordered in the usual way. A symbol f is interpreted by a partial binary function with $\mathrm{Dom}(f) = P_1(\mathcal{M}) \times P_2(\mathcal{M})$ and $\mathrm{Range}(f) = P_3(\mathcal{M})$ and is defined by the equality $f(a, b) = a + b$ for all $(a, b) \in \mathbb{Q} \times \mathbb{Q}$.

Obviously, $\mathrm{Th}(M)$ has convexity rank 1.

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