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MAXIMAL AND SUBMAXIMAL $\mathfrak{X}$-SUBGROUPS

W. Guo[1*] **and D. O. Revin**[2**]

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Let $\mathfrak{X}$ be a class of finite groups closed under taking subgroups, homomorphic images, and extensions. Following H. Wielandt, we call a subgroup H of a finite group G a submaximal $\mathfrak{X}$-subgroup if there exists an isomorphic embedding $\phi : G \hookrightarrow G^*$ of G into some finite group G^* under which G^ϕ is subnormal in G^* and $H^\phi = K \cap G^\phi$ for some maximal $\mathfrak{X}$-subgroup K of G^*. In the case where $\mathfrak{X}$ coincides with the class of all π-groups for some set π of prime numbers, submaximal $\mathfrak{X}$-subgroups are called submaximal π-subgroups. In his talk at the well-known conference on finite groups in Santa Cruz in 1979*, Wielandt emphasized the importance of studying submaximal π-subgroups, listed (without proof) certain of their properties, and formulated a number of open questions regarding these subgroups. Here we prove properties of maximal and submaximal $\mathfrak{X}$- and π-subgroups and discuss some open questions both Wielandt's and new ones. One of such questions due to Wielandt reads as follows: Is it always the case that all submaximal $\mathfrak{X}$-subgroups are conjugate in a finite group G in which all maximal $\mathfrak{X}$-subgroups are conjugate?*

1. HALL, MAXIMAL AND SUBMAXIMAL π-SUBGROUPS: DEFINITION, PROPERTIES, HISTORY

The present paper is based on the ideas and concepts introduced in different years by H. Wielandt—primarily in his plenary report at the well-known conference on finite groups held in

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[1]University of Science and Technology of China, Hefei, 230026 P.R. China; wguo@ustc.edu.cn. [2]Sobolev Institute of Mathematics, pr. Akad. Koptyuga 4, Novosibirsk, 630090 Russia. Novosibirsk State University, ul. Pirogova 1, Novosibirsk, 630090 Russia. University of Science and Technology of China, Hefei, 230026 P.R. China; revin@math.nsc.ru. Translated from *Algebra i Logika*, Vol. 57, No. 1, pp. 14-42, January-February, 2018. Original article submitted April 25, 2017; revised December 6, 2017.

Santa Cruz in 1979, which preceded the announcement of the completion of the classification of finite simple groups [1], and in his lectures [2, 3]. The important and useful concept of a submaximal $\mathfrak{X}$-subgroup holds a central place in the works mentioned. Many statements given there were left unproved. Certain of the statements can now be revised relying on progress in studying so-called $\mathscr{D}_\pi$-groups. Moreover, Wielandt formulated a number of open questions, whose possible solutions or approaches to which, as well as some new questions, will be discussed in what follows.

Whenever we use the term a "group" we mean a finite group. Throughout, we denote by p a prime number, and by π a fixed set of prime numbers. Furthermore, π' is the set of all prime numbers not in π, $\pi(n)$ is the set of all prime divisors of a natural number n, and $\pi(G)$ is the set $\pi(|G|)$ for a group G.

Recall that a group G with the condition $\pi(G) \subseteq \pi$ is a called a π-*group*. For the case $\pi = \{p\}$ it is common practice to call a π-group a p-group and denote the set π' by p' omitting braces.

A well-known theorem of Sylow says that for an arbitrary prime p and for any finite group G, the following conditions hold:

(E_p) G contains a *Sylow p-subgroup*, a p-subgroup of index not divisible by p;

(C_p) every two Sylow p-subgroups are conjugate in G;

(D_p) every p-subgroup is contained in some Sylow p-subgroup.

In particular, it turns out that Sylow p-subgroups and maximal p-subgroups are one and the same thing, and studying p-subgroups for a given group reduces to examining subgroups of a fixed Sylow p-subgroup.

We will use the following notation introduced by Ph. Hall [4] in 1956. For a given set π of prime numbers, a finite group G is said to *possess the $\mathscr{D}_\pi$-property*, and we write $G \in \mathscr{D}_\pi$, if:

(E_π) G contains a *Hall π-subgroup*, a π-subgroup of index not divisible by any number in π;

(C_π) every two Hall π-subgroups are conjugate in G;

(D_π) every π-subgroup is contained in some Hall π-subgroup.

Thus, in Hall's notation, Sylow's theorem can be formulated thus: $G \in \mathscr{D}_p$ for an arbitrary prime p and for any finite group G. This explains the fact that along with Hall's terminology, frequent use is made of Wielandt's—when instead of $G \in \mathscr{D}_\pi$ we say that the *Sylow π-theorem* holds for a group G. Note that for a group $G \in \mathscr{D}_\pi$, the concept of a maximal π-subgroup coincides with the concept of a Hall π-subgroup. Moreover, Wielandt observed that in view of the Sylow theorem, the $\mathscr{D}_\pi$-property can be defined not appealing to the concept of a Hall π-subgroup, and namely:

$G \in \mathscr{D}_\pi$ *if and only if every two maximal π-subgroups are conjugate in G.*

In 1928 Hall [5] showed that the Sylow π-theorem holds for every finite *solvable* group G and for any set π of prime numbers.

Note that for each of the following cases there is an example of a pair (G, π), where G is an unsolvable group and π is some set of prime numbers, such that:

G does not satisfy (E_π), i.e., G does not contain Hall π-subgroups;

G satisfies (E_π) but does not satisfy (C_π), i.e., G contains more than one conjugacy class of Hall π-subgroups;

G satisfies (E_π) and (C_π) but does not satisfy (D_π), i.e., G contains exactly one conjugacy class of Hall π-subgroups, but not every π-subgroup is contained in a Hall π-subgroup;

G satisfies (E_π), (C_π), and (D_π), but G is unsolvable.

Thus in the case of unsolvable groups, the Hall theorem does not hold in general. Moreover, Hall [6] in 1937 and S. A. Chunikhin [7] in 1938 independently showed that the converse of Hall's theorem also holds. More exactly, for any unsolvable group G, there always exists a set π such that G contains no Hall π-subgroups.

Hall π-subgroups of unsolvable groups were intensively investigated by different authors. In particular, a fundamental contribution to their study was made by Wielandt [1-3, 8-14]; see also B. Hartly's essay [15]. In recent years certain progress in this direction has been achieved (see the review paper [16]). We discuss it in some more detail.

A set of Hall π-subgroups of G is denoted by $\mathrm{Hall}_\pi(G)$.

The existence and properties of Hall π-subgroups are closely connected to the normal structure of a group, as could be seen from just the theorems of Hall and Chunikhin. The following proposition is well known and easy to prove.

PROPOSITION 1 [4, Lemma 1]. If N is a normal subgroup of a finite group G and $H \in \mathrm{Hall}_\pi(G)$, then $H \cap N \in \mathrm{Hall}_\pi(N)$ and $HN/N \in \mathrm{Hall}_\pi(G/N)$.

As a consequence, if a group possesses a Hall π-subgroup, then any composition factor of the group also possesses a Hall π-subgroup. Although the converse statement is untrue, a necessary and sufficient condition for the existence of Hall π-subgroups in a finite group G is the existence of such subgroups in G-induced automorphism groups $\mathrm{Aut}_G(G_{i-1}/G_i)$ of each of the factors G_{i-1}/G_i of the fixed composition series

$$G = G_0 > G_1 > \cdots > G_n = 1 \tag{1}$$

of G which refines some principal series of G (see [16, Cor. 4.6; 17, Cor. 5; 18, Chap. 2, Thm. 6.11]). Recall that the *G-induced automorphism group* $\mathrm{Aut}_G(G_{i-1}/G_i)$ of a factor G_{i-1}/G_i in series (1) is the image of a homomorphism $N_G(G_{i-1}) \cap N_G(G_i) \to \mathrm{Aut}\,(G_{i-1}/G_i)$ given by the rule $x \mapsto \phi_x$, where $\phi_x : G_{i-1}g \mapsto G_{i-1}x^{-1}gx$ for all $x \in N_G(G_{i-1}) \cap N_G(G_i)$ and $g \in G_i$.

It is also worth mentioning the following results. In simple groups, Hall π-subgroups were described, and in each case the action of an automorphism group on the set of conjugacy classes of Hall π-subgroups was explored. Moreover, this description made it possible to reveal a number of useful latent properties of Hall subgroups in simple groups (pronormality [19], theorem on the number of conjugacy classes [20], and so on), from which in turn important properties of Hall π-subgroups in arbitrary groups were derived (for details, cf. [16]). In particular, Proposition 2 (see below) was proved which plays here an important role and gives a positive solution to the problem of closedness under taking extensions of normal subgroups and homomorphic images for the class of groups with the $\mathscr{D}_\pi$-property. This problem was posed by Wielandt [8] at the XIII International Congress of Mathematicians held in Edinburgh (see also [12, 21-23; 24, Quest. 3.62]). It goes back to a known theorem of Hall [4, Thm. D5], which says that an extension of a $\mathscr{D}_\pi$-group with nilpotent Hall π-subgroup by a $\mathscr{D}_\pi$-group with solvable Hall π-subgroup possesses the $\mathscr{D}_\pi$-property.

PROPOSITION 2 [16, Thm. 6.6; 18, Chap. 2, Thm. 6.12; 25, Thm. 7.7]. Let G be a finite group and N its normal subgroup. The inclusion $G \in \mathscr{D}_\pi$ holds if and only if $N \in \mathscr{D}_\pi$ and $G/N \in \mathscr{D}_\pi$.

Thus a group possesses the $\mathscr{D}_\pi$-property iff each composition factor of the group has this property. Furthermore, a description of all simple groups with the $\mathscr{D}_\pi$-property was obtained in [26]. This made it possible to solve some other known problems. In [27], for instance, Wielandt's conjecture [11] was proved which holds that if $\pi = \sigma \cup \tau$ for disjoint subsets σ and τ, then a group G having a Hall π-subgroup $H = H_\sigma \times H_\tau$, where $H_\sigma \in \mathrm{Hall}_\sigma(H)$ and $H_\tau \in \mathrm{Hall}_\tau(H)$, possesses the $\mathscr{D}_\pi$-property iff $G \in \mathscr{D}_\sigma \cap \mathscr{D}_\tau$. The result mentioned generalizes Wielandt's results in [9, 11], in particular his known theorem saying that $G \in \mathscr{D}_\pi$ if G contains a nilpotent Hall π-subgroup.

The description obtained shows also that for an arbitrary non-Abelian simple group, not only the $\mathscr{D}_\pi$-property, but even the existence in it of Hall π-subgroups for a given π will most probably be an exception. At the same time, in an arbitrary unsolvable finite group, we can obviously guarantee the presence of maximal π-subgroups. Following [1], we denote the set of maximal π-subgroups of G by $\mathrm{m}_\pi(G)$. It is clear that

$$\mathrm{Hall}_\pi(G) \subseteq \mathrm{m}_\pi(G) \tag{2}$$

for any group G. It is precisely the concept of a maximal π-subgroup that can be considered as a more relevant (compared to Hall π-subgroups) generalization of the concept of a Sylow p-subgroup, which, in addition, is meaningful for infinite groups as well.

However, the problem is that compared to Hall π-subgroups, studying maximal π-subgroups in finite groups reduces to simple groups in a worser way. In particular, nothing similar to Proposition 1 can be asserted for maximal π-subgroups. Thus, Wielandt observed that in the factor group G/N, the image of a maximal π-subgroup of G is, generally speaking, not only a maximal π-subgroup, but it may coincide with any π-subgroup in G/N, as shown in

PROPOSITION 3 [1, 4.2 (without proof)]. Let a proper subset π of the set of all prime numbers contain at least two elements. Then the following statements hold:

(i) not all finite groups possess the $\mathscr{D}_\pi$-property;

(ii) let X and Y be finite groups, with $X \notin \mathscr{D}_\pi$; then any π-subgroup in Y is the image of some maximal π-subgroup of the regular wreath product $X \wr Y$ under the natural epimorphism $X \wr Y \to Y$.

The fact that a basis for the wreath product $X \wr Y$ does not possess the $\mathscr{D}_\pi$-property plays an important part in Prop. 3. For the sake of comparison, we prove

PROPOSITION 4. Let G be a finite group and $N \in \mathscr{D}_\pi$ its normal subgroup. Then the following statements hold:

(i) the rule $H \mapsto H \cap N$ specifies a surjection $\mathrm{m}_\pi(G) \to \mathrm{m}_\pi(N) = \mathrm{Hall}_\pi(N)$;

(ii) the rule $H \mapsto HN/N$ specifies a surjection $\mathrm{m}_\pi(G) \to \mathrm{m}_\pi(G/N)$ and induces a bijection between the conjugacy classes of maximal π-subgroups in G and G/N.

Note that the given assertion uses the classification of finite simple groups, for it relies on Prop. 2.

In the general case the intersection of a maximal π-subgroup H and a normal subgroup N must not necessarily be a maximal π-subgroup in N. For instance, it is easy to show that a Sylow 2-subgroup of $G = PGL_2(7)$ is a maximal $\{2, 3\}$-subgroup, but its intersection with $N = PSL_2(7)$, a Sylow 2-subgroup of N, is contained in two nonconjugate $\{2, 3\}$-subgroups isomorphic to S_4.

The situation with intersection of a maximal π-subgroup and a normal subgroup, yet, does not look as hopeless as it does in the case with homomorphic images. A fundamental result in this direction is the Wielandt–Hartley theorem asserting that the intersection of a maximal π-subgroup H of G and a normal subgroup N may be trivial only in the case where N is a π'-group. This theorem was formulated by Wielandt in [2, 8.1; 1, 4.5] (without proof) and in [3, 13.2] (with proof). It was independently obtained by Hartley [28, Lemmas 2, 3]. All known proofs of the theorem make use of O. Schreier's conjecture on the solvability of the outer automorphism group of any finite simple group, which has been confirmed modulo the classification of finite simple groups. Following Wielandt, we say that a π-subgroup H of G is π-*selfnormalizable* if $N_G(H)/H$ is a π'-group. Here we cite the Wielandt–Hartley theorem omitting those additional requirements in the hypothesis whose validity follows from Schreier's conjecture. The theorem is given in the formulation in which it has been found in sources with proof with which the authors are familiar: Wielandt [3, 13.2], Hartly [28, Lemmas 2, 3], M. Suzuki [23, Chap. V, (3.20)] where statement (ii) is proved, and L. A. Shemetkov [29, Thm. 7] where statement (i) of Proposition 5 is proved. A somewhat stronger version of Proposition 5 is Proposition 8 below.

PROPOSITION 5 (Wielandt–Hartley theorem). Let G be a finite group and N its subnormal subgroup. Then, for any $H \in \mathrm{m}_\pi(G)$, the following statements hold:

(i) if N is not a π'-group, then $H \cap N \neq 1$;

(ii) if N is normal in G, then the intersection $H \cap N$ is a π-selfnormalizable subgroup in N.

The Wielandt–Hartley theorem, together with the remarks made above, allowed Wielandt to introduce a new object—submaximal π-subgroups and, wider, submaximal $\mathfrak{X}$-subgroups, which are the main subject for study in the present paper.

Following Wielandt, we say that a class $\mathfrak{X}$ of finite groups closed under taking subgroups, homomorphic images, and extensions is *complete.*[1] Complete classes may be exemplified by the following: $\mathfrak{S}$ the class of all solvable groups, $\mathfrak{G}_\pi$ the class of all π-groups, and $\mathfrak{S}_\pi$ the class of all solvable π-groups. A class of all groups with composition factors isomorphic to groups in the set $\{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5, A_5\}$ is also complete, while a class of groups with factors $\{\mathbb{Z}_2, \mathbb{Z}_3, A_5\}$ is not. Groups in $\mathfrak{X}$ are called $\mathfrak{X}$-*groups.* A set of maximal $\mathfrak{X}$-subgroups of a finite group G is denoted by $\mathrm{m}_\mathfrak{X}(G)$.

[1] Wielandt [3] used the term 'vollständig,' which can also be translated as 'perfect.'

Definition 1. Let $\mathfrak{X}$ be a complete class of finite groups. A subgroup H of a finite group G is called a *submaximal $\mathfrak{X}$-subgroup* (denoted $H \in \mathrm{sm}_{\mathfrak{X}}(G)$) if there exists an isomorphic embedding

$$\phi : G \hookrightarrow G^*$$

of the group G into some finite group G^* such that the subgroup G^ϕ is subnormal in G^* and $H^\phi = X \cap G^\phi$ for some $X \in \mathrm{m}_{\mathfrak{X}}(G^*)$.

In the case where $\mathfrak{X}$ is the class of all π-groups, submaximal $\mathfrak{X}$-subgroups are referred to as *submaximal π-subgroups*, and for a group G, we put $\mathrm{sm}_\pi(G) = \mathrm{sm}_{\mathfrak{X}}(G)$.

Evidently, the term a 'submaximal $\mathfrak{X}$-subgroup' was initially introduced in Wielandt's lectures [3], where fundamental significance of such subgroups was revealed. Also these groups hold a central place in Wielandt's talk [1] at the well-known conference on finite groups held in Santa Cruz (U.S.A.), which preceded the announcement of the completion of the classification of finite simple groups. Wielandt in his talk (and then Hartly in [15]) expressed a viewpoint that it is exactly the study of maximal and submaximal $\mathfrak{X}$-subgroups that is the right direction toward extending the theorems of Sylow and Hall to arbitrary finite groups and sets of prime numbers within the frames of the Hölder program.

It follows immediately from the definition that the following statements hold:

$$\mathrm{m}_{\mathfrak{X}}(G) \subseteq \mathrm{sm}_{\mathfrak{X}}(G) \text{ for any group } G; \tag{3}$$

$$\text{if } N \trianglelefteq\trianglelefteq G, \text{ then } H \cap N \in \mathrm{sm}_{\mathfrak{X}}(N) \text{ for any subgroup } H \in \mathrm{sm}_{\mathfrak{X}}(G). \tag{4}$$

Thus submaximal π-subgroups, as distinct from merely maximal ones, behave well (i.e., similar to Hall π-subgroups) in relation to normal subgroups. An equivalent formulation of the Wielandt–Hartley theorem maintains that π-submaximal subgroups may be trivial only in π'-groups. Clearly, if G is a π-group then $\mathrm{sm}_\pi(G) = G$.

The above example with $\{2,3\}$-subgroups of $G = PGL_2(7)$ and $N = PSL_2(7)$ shows that a Sylow 2-subgroup of N is a submaximal but not maximal $\{2,3\}$-subgroup in N. This example shows also that a submaximal π-subgroup (unlike a Hall or maximal π-subgroup), generally speaking, does not remain such in relation to its overgroups.

In terms of submaximal π-subgroups we can give a characterization of groups with the $\mathscr{D}_\pi$-property which is fully analogous to the above-mentioned characterization of Wielandt, as shown in

PROPOSITION 6. For a finite group G, the following statements are equivalent:

(i) $G \in \mathscr{D}_\pi$;

(ii) all subgroups of $\mathrm{m}_\pi(G)$ are conjugate;

(iii) all subgroups of $\mathrm{sm}_\pi(G)$ are conjugate.

In particular, if G is a $\mathscr{D}_\pi$-group, then $\mathrm{sm}_\pi(G) = \mathrm{m}_\pi(G) = \mathrm{Hall}_\pi(G)$.

Here, as noted, statements (i) and (ii) are equivalent (according to Wielandt's remark) in view of the Sylow theorem. The implication (iii) $\Rightarrow$ (ii) follows from inclusion (3). We prove the implication (ii) $\Rightarrow$ (iii) from Prop. 2. Notice that this implication gives a partial (for the case where $\mathfrak{X}$ is the class of all π-groups) solution to the following open problem posed by Wielandt in 1964 [3, p. 642, open question to 15.4] for an arbitrary complete class $\mathfrak{X}$.

Problem 1. Will all submaximal $\mathfrak{X}$-subgroups be conjugate in a group in which all maximal $\mathfrak{X}$-subgroups are conjugate?

Wielandt motivates this problem by the observation that if N is a normal subgroup of G, then the conjugacy in N of all elements of the set $\mathrm{sm}_{\mathfrak{X}}(N)$ (implying also the conjugacy of $\mathrm{m}_{\mathfrak{X}}(N)$, as can be seen from (3)) entails the existence of a bijection between the conjugacy classes of maximal $\mathfrak{X}$-subgroups of G and G/N (cf. [3, 15.4])—similar to statement (ii) in Prop. 4. Discussion of possible approaches to solving Problem 1 is the main content of [30] which continues the present paper. We will see that a positive solution of Problem 1 for the class of all π-groups is equivalent to stating that the $\mathscr{D}_\pi$-property is inherited under extensions—the famous conjecture with which Wielandt came up at the XIII International Mathematical Congress [3. p. 271]. Great efforts were made to prove it which became possible only by using the classification of finite simple groups; see Prop. 2 above. Therefore, in the general setting, the problem (in case of its positive solution) is probably difficult. Here we focus on just the assertion that reduces the study of Problem 1 to simple groups.

THEOREM 1. Let $\mathfrak{X}$ be a complete class of groups. The following statements are equivalent:

(i) submaximal $\mathfrak{X}$-subgroups are conjugate in any finite group in which all maximal $\mathfrak{X}$-subgroups are conjugate;

(ii) submaximal $\mathfrak{X}$-subgroups are conjugate in any finite simple group in which all maximal $\mathfrak{X}$-subgroups are conjugate.

In this connection, the natural question arises as to how submaximal $\mathfrak{X}$-subgroups look in simple groups. Some idea is given by

PROPOSITION 7 [1, 5.3 (without proof)]. Let $\mathfrak{X}$ be a complete class of finite groups. Then, for a subgroup H of a non-Abelian simple group S, the following statements are equivalent:

(i) $H \in \mathrm{sm}_{\mathfrak{X}}(S)$;

(ii) there exists an almost simple group G with socle S such that $G/S \in \mathfrak{X}$ and $H = S \cap X$ for some $X \in \mathrm{m}_{\mathfrak{X}}(G)$;

(iii) $H = S \cap X$ for some $X \in \mathrm{m}_{\mathfrak{X}}(\mathrm{Aut}\,(S))$, where S is identified with $\mathrm{Inn}\,(S)$.

Recall that a group G is said to be *almost simple* if its socle S is a non-Abelian simple group. In other words, G is isomorphic to a subgroup in $\mathrm{Aut}\,(S)$ that contains $\mathrm{Inn}\,(S)$ for some non-Abelian simple group S.

Relying on Proposition 7, we can obtain the following version of the Wielandt–Hartley theorem.

PROPOSITION 8. Let $\mathfrak{X}$ be a complete class of finite groups and

$$\pi = \bigcup_{X \in \mathfrak{X}} \pi(X).$$

Then, for any group G, for its subnormal subgroup A, and for a maximal $\mathfrak{X}$-subgroup H, the following statements hold:

(i) either $H \cap A \neq 1$ or A is a π'-group;

(ii) if A is normal in G or is minimal subnormal, then $H \cap A$ is a π-selfnormalizable subgroup of A.

In the light of Problem 1, Theorem 1, and Proposition 7, of interest is

Problem 2. Describe submaximal $\mathfrak{X}$-subgroups in any non-Abelian simple group G for every complete class $\mathfrak{X}$. In other words, obtain a description of maximal $\mathfrak{X}$-subgroups in automorphism groups of finite simple groups.

In the remaining part of this section, we discuss in more detail Problem 2 as applied to the case where $\mathfrak{X}$ coincides with the class of all finite π-groups.

Problem 3. Obtain a description of submaximal π-subgroups in every non-Abelian simple group G for every π.

This problem is also important beyond the context of Problem 1.

In the proposition below, we show that submaximal π-subgroups in a finite group G can be studied modulo any radical $G_{\mathfrak{F}}$ corresponding to a Fitting class $\mathfrak{F} \subseteq \mathscr{D}_\pi$ (e.g., modulo the solvable radical, the nilpotent radical, the π-separable radical, the $\mathscr{D}_\pi$-radical, and so on). Recall that a class $\mathfrak{F}$ of finite groups is called a *Fitting class* if $N \in \mathfrak{F}$ for any (sub)normal subgroup N of $G \in \mathfrak{F}$, and $\langle N_1, \ldots, N_s \rangle \in \mathfrak{F}$ if $N_1, \ldots, N_s$ are subnormal subgroups of some group G such that $N_1, \ldots, N_s \in \mathfrak{F}$. For a Fitting class $\mathfrak{F}$, in any finite group G there exists the $\mathfrak{F}$-*radical* $G_{\mathfrak{F}}$, the greatest normal subgroup belonging to $\mathfrak{F}$. It follows from Proposition 2 that the class $\mathscr{D}_\pi$ is a Fitting class.

PROPOSITION 9. Let $\mathfrak{F}$ be some Fitting class such that $\mathfrak{F} \subseteq \mathscr{D}_\pi$, and let G be a finite group and N its normal subgroup. Then the following statements hold:

(i) if $N \in \mathscr{D}_\pi$, then the rule $H \mapsto H \cap N$ specifies a surjection $\mathrm{sm}_\pi(G) \to \mathrm{sm}_\pi(N) = \mathrm{m}_\pi(N) = \mathrm{Hall}_\pi(N)$;

(ii) if $N = G_{\mathfrak{F}}$, then the rule $H \mapsto HN/N$ specifies a map $\mathrm{sm}_\pi(G) \to \mathrm{sm}_\pi(G/N)$.

Propositions 4 and 9 imply that the study of maximal and submaximal π-subgroups of an arbitrary finite group G can be reduced to a situation where G contains no nontrivial normal $\mathscr{D}_\pi$-subgroups (in particular, its solvable radical, π- and π'-radicals are trivial). Property (4) also shows that an important role should be played by submaximal π-subgroups of lower composition factors of the group that arise from its socle—the subgroup generated by minimal (sub)normal subgroups. Since the socle itself and all minimal normal subgroups are direct products of simple groups, submaximal π-subgroups of such groups are constructed naturally from submaximal π-subgroups of simple components, as shown in

PROPOSITION 10. Let $G = G_1 \times \cdots \times G_n$. Then, for any complete class $\mathfrak{X}$,

$$\mathrm{m}_{\mathfrak{X}}(G) = \{\langle H_1, \ldots, H_n \rangle \mid H_i \in \mathrm{m}_{\mathfrak{X}}(G_i),\ i = 1, \ldots, n\}.$$

If, in addition, $G_1, \ldots, G_n$ are simple groups, then

$$\mathrm{sm}_{\mathfrak{X}}(G) = \{\langle H_1, \ldots, H_n \rangle \mid H_i \in \mathrm{sm}_{\mathfrak{X}}(G_i),\ i = 1, \ldots, n\}.$$

Therefore, it is important to be able to find submaximal $\mathfrak{X}$- and π-subgroups in simple groups.

If we solve Problem 3 we would be brought closer to the possibility for finding maximal π-subgroups in arbitrary groups. Indeed, suppose that submaximal π-subgroups in all simple groups have been revealed. And we want to find all maximal π-subgroups in some group G. Since submaximal as well as maximal π-subgroups in simple groups are known, we may assume that G is not simple. Proposition 9 allows us to deal not with G but with its factor group with respect to the solvable radical. We can therefore suppose that G does not contain normal solvable subgroups other than the identity. Similarly, we may assume that G does not contain nontrivial normal π- and π'-subgroups.

Let A be a normal subgroup of G, which is a direct product of simple groups (e.g., a minimal normal one or socle). Submaximal π-subgroups of A are known (Prop. 10) and an arbitrary maximal π-subgroup H of G is contained in $N_G(H \cap A)$; so $H \cap A \in \mathrm{sm}_\pi(A)$ is a "known" subgroup. In addition, H is a maximal π-subgroup of $N_G(H \cap A)$, a group of order strictly less than $|G|$. (Recall that A is not a π'-group, and by the Wielandt–Hartley theorem $N_G(H \cap A) \trianglerighteq H \cap A \neq 1$, while G does not contain normal π-subgroups.) Moreover, $N_A(H \cap A) \in \mathscr{D}_\pi$, and in view of Proposition 9 there exists a bijection between the conjugacy classes of maximal π-subgroups in $N_G(H \cap A)$, and $N_G(H \cap A)/N_A(H \cap A) \cong AN_G(H \cap A)/A$. Clearly, the group G/A acts naturally on the set of conjugacy classes of submaximal π-subgroups of A, and $AN_G(H \cap A)/A$ is the stabilizer of a class $\{(H \cap A)^x \mid x \in A\}$ containing the image of the subgroup H. If we manage to find stabilizers in G/A of the A-conjugacy classes into which the set $\mathrm{sm}_\pi(A)$ is divided, and if in these stabilizers (groups of order strictly less than the order of G), we succeed in determining maximal π-subgroups, then we will be able to find in them the images of maximal π-subgroups of G. Therefore, along with studying Problem 3, it is important to examine the action of a group on conjugacy classes of submaximal π-subgroups in minimal normal subgroups. Probably, the case of almost simple groups should play the key role here.

It should be observed that the reasoning above relies on Proposition 9 and, therefore, uses the fact that the class $\mathscr{D}_\pi$ is closed under extensions. As noted, an analog of such closedness for an arbitrary complete class $\mathfrak{X}$ would be a positive solution to Problem 1. At the same time, it is easy to show that Problem 1 is solved positively for groups possessing a normal series each factor of which either is an $\mathfrak{X}$-group or is a π'-group for $\pi = \pi(\mathfrak{X})$ (an analog of so-called π-separable groups). Above, the closedness of the class $\mathscr{D}_\pi$ under extensions is required in just this case. Therefore, our reasoning could well be done in a more general situation. This would show that for finding maximal $\mathfrak{X}$-subgroups of an arbitrary group, it is important to solve Problem 2.

Proposition 7 indicates that Problem 3 is equivalent to the problem of describing maximal π-subgroups in almost simple groups, or to the problem of characterizing maximal π-subgroups in complete automorphism groups of simple groups. However, to obtain a full description of maximal π-subgroups is a challenging task even for simple groups. For instance, if M is merely a maximal subgroup of a simple group S with the condition $\pi(M) \neq \pi(S)$ (which is often so for a randomly taken pair (S, M); cf. [31, Prop. 1]), then $M \in \mathrm{m}_\pi(S) \subseteq \mathrm{sm}_\pi(S)$ for $\pi = \pi(M)$. Therefore, the solution to Problem 3 in essence includes the problem of describing all maximal subgroups in finite simple groups. Even for simple linear groups, this involves, in particular, a knowledge of all absolutely irreducible representations over finite fields for all almost simple and quasisimple groups [32]; but an exact description of such representations, in specialists opinion [33, 2.3, p. 11], will (in the best case) be the problem in the quite distant future.

For this reason, it is meaningful to consider Problem 3 imposing some natural restrictions. There exists a description of maximal subgroups of odd index in almost simple groups [34-40]. Therefore, we can expect that the following particular case of Problem 3 will be solved in the foreseeable future.

Problem 4. Let $2 \in \pi$. Describe submaximal π-subgroups containing a Sylow 2-subgroup of G for each non-Abelian simple group G.

For alternating groups, the given problem was in essence solved by C. Cooper [41]. More precisely, Cooper obtained a description of maximal π-subgroups of symmetric groups containing a Sylow p-subgroup, where $p = \min \pi$. Notice that these subgroups coincide with Sylow's if $p > 2$.

Another particular case where it seems possible to solve Problem 3 is

Problem 5. Let $2 \notin \pi$. Describe submaximal π-subgroups for each non-Abelian simple group G.

Recall that a subgroup H of a group G is said to be *pronormal* if H and H^g are conjugate in $\langle H, H^g \rangle$ for any element $g \in G$. In [19], it was proved that Hall π-subgroups in simple groups are pronormal. Therefore, of interest is the following:

Problem 6. Will submaximal π-subgroups in simple finite groups be pronormal?

Wielandt drew the attention to the importance of studying the given problem (see, e.g., Problem 8 below). In connection with the results in [42], this question is especially interesting for submaximal π-subgroups containing a Sylow 2-subgroup of a simple group. Techniques for investigating pronormality for subgroups of odd index in simple groups were developed in [42-44].

In his report Wielandt [1] formulated ten well-known open questions of which some relate to maximal and submaximal π-subgroups. Among these is question (i)—one of the equivalent formulations of the famous Kegel–Wielandt conjecture—which reads as follows: Will the converse of the Wielandt–Hartley theorem[2] be true? More precisely, will a subgroup A of a finite group G be subnormal if the intersection of A and any maximal π-subgroup is π-selfnormalizable in A for each set π? P. Kleidman [45] gave an affirmative answer to this question in the general case by using the classification of finite simple groups.

Question (h) in [1] is associated with attempts to carry over the theory of submaximal $\mathfrak{X}$-subgroups to the maximally possible complete class $\mathfrak{X} \subseteq \mathscr{D}_\pi$ (the class $\mathscr{D}_\pi$ itself is not complete since, generally speaking, it is not closed under taking subgroups; see, e.g., [46, Lemma 7]). The question reads as follows: In which known simple groups does the "strong Sylow π-theorem" hold? More specifically, for which two π-subgroups A and B does there exist an element $t \in \langle A, B \rangle$ such

[2] We mean a strong formulation of this theorem; see remark at the end of the introductory part of the paper.

that $\langle A, B^t \rangle$ is a π-group? An equivalent formulation is given in [24, 17.43a; 16, Probl. 7.6; 47, Probl. 1]. We cite it.

Problem 7. In which simple groups does every subgroup possess the $\mathscr{D}_\pi$-property?

Since such groups should possess the $\mathscr{D}_\pi$-property themselves, we have to be concerned with some groups from the list of simple $\mathscr{D}_\pi$-groups in [26]. The solution to Problem 7 is known for alternating [48, Thm. 3] and sporadic [46, Thm. 2] groups; but it presents difficulties for simple groups of Lie type.

Finally, we mention question (g) in [1], whose solution was obtained by the authors in [49] by using results of the present paper.

Problem 8. Describe submaximal π-subgroups of minimal unsolvable groups. Study their conjugacy in the automorpism group, pronormality, intravariance, and so on.

Recall that Wielandt refers to a subgroup H of a group G as *intrainvariant* if its conjugacy class in G is invariant under the natural action of the group $\mathrm{Aut}\,(G)$ on the conjugacy classes of subgroups.

The motivation for this question is clear: of special interest is the critical case where all proper subgroups of a group possess the $\mathscr{D}_\pi$-property, whereas the group itself lacks it.

As is known, G is a minimal unsolvable group (i.e., an unsolvable groups all of whose proper subgroups are solvable) iff its factor group with respect to a Frattini subgroup, $G/\Phi(G)$, is a simple minimal unsolvable group isomorphic (as shown by J. Thompson [50]) to a group S in the following list:

$PSL_3(3)$;

$PSL_2(2^p)$, where p is a prime;

$PSL_2(3^p)$, where p is an odd prime;

$PSL_2(p)$, where p is a prime such that $p > 3$ and $p^2 + 1 \equiv 0 \pmod 5$;

$Sz(2^p)$, where p is an odd prime.

In view of Proposition 9, it suffices to solve Problem 8 for groups with trivial Frattini subgroup (which, for minimal unsolvable subgroups, coincides with the nilpotent radical), i.e., for the groups S in Thompson's list.

The problem of completely describing minimal non-$\mathscr{D}_\pi$-groups is, obviously, essentially more complex than Problem 7.

Finally, note that in Wielandt's report [1, 5.4(a)] and in Hartley's essay [15, Thm. 6] devoted to the contribution made by Wielandt to studying the π-structure of finite groups, statements (ii) in Propositions 5 and 8 are formulated in a stronger form than in the present paper. Wielandt says that in *any* finite group, submaximal $\mathfrak{X}$-subgroups are π-selfnormalizable for any complete class $\mathfrak{X}$ and for a set $\pi = \pi(\mathfrak{X})$. Hartly reformulated this statement for the case where $\mathfrak{X}$ coincides with the class of all π-groups: namely,

if a subgroup N is subnormal in a group G and $H \in \mathrm{m}_\pi(G)$, then the intersection $H \cap N$ is a π-selfnormalizable subgroup in N.

This strengthens statement (ii) of Prop. 5 in an obvious way. We have not managed to find a proof for such strengthening; yet S. V. Skresanov has recently announced such proof. The importance of this strengthening to further applications can be readily understood if we address, for example, Prop. 10. Comparing in this proposition the characterization of maximal and submaximal $\mathfrak{X}$-subgroups in direct products, it is natural to ask to what extent the condition for simplicity of direct factors is essential in the case of submaximal $\mathfrak{X}$-subgroups. Lemma 4 shows that modulo [1, 5.4(a); 15, Thm. 6], this condition may be omitted.

2. PRELIMINARY RESULTS

LEMMA 1. Let $H_1, \ldots, H_n$ be subnormal subgroups of a finite group G. Then $K = \langle H_1, \ldots, H_n \rangle$ is a subnormal subgroup of G, and each composition factor of K is isomorphic to a composition factor of one of the groups $H_1, \ldots, H_n$.

Proof. See [51, items 7, 9]. □

LEMMA 2. Let $\mathfrak{X}$ be a complete class, $\phi : G \to G_0$ be a group homomorphism, and $K \in \mathfrak{X}$ for a subgroup $K \leq G^\phi$. Then $K = H^\phi$ for an $\mathfrak{X}$-subgroup $H \leq G$. In particular, $\mathrm{m}_\mathfrak{X}(G^\phi) \subseteq \mathrm{m}_\mathfrak{X}(G)^\phi$.

Proof. Let $N = \ker \phi$. Consider a set

$$\Omega = \{L \leq G \mid L^\phi = K\}$$

(the set is nonempty since the preimage of the subgroup K belongs to Ω) and choose in it a subgroup H of least order. Then $H \cap N \leq \Phi(H)$. Indeed, suppose that there exists a maximal subgroup M of H which does not contain $H \cap N$. By the maximality of M, we have $H = M(H \cap N)$, and hence $M^\phi = H^\phi = K$. Therefore, $M \in \Omega$ and $|M| < |H|$, a contradiction with the choice of H. Thus $H \cap N$ is contained in the intersection of all maximal subgroups, which coincides with the Frattini subgroup $\Phi(H)$.

Let $\pi = \pi(K) = \pi(H^\phi) = \pi(H/H \cap N)$. By the properties of the Frattini subgroup, $H \cap N$ is a nilpotent π-group. Since $K \in \mathfrak{X}$ and the class $\mathfrak{X}$ is closed under taking subgroups, $\mathfrak{X}$ contains groups of all prime orders $p \in \pi$, and since $\mathfrak{X}$ is closed under extensions, it contains all solvable π-groups. In particular, $H \cap N \in \mathfrak{X}$. Then $H \in \mathfrak{X}$ as an extension of the $\mathfrak{X}$-group $H \cap N$ by the $\mathfrak{X}$-group $H/(H \cap N) \cong H^\phi$.

If now $K \in \mathrm{m}_\mathfrak{X}(G^\phi)$, $H \leq G$ is an $\mathfrak{X}$-subgroup, and $H^\phi = K$, then $K = H^\phi = M^\phi$ for any subgroup $M \in \mathrm{m}_\mathfrak{X}(G)$ containing H. Therefore, $K = M^\phi \in \mathrm{m}_\mathfrak{X}(G)^\phi$. Hence $\mathrm{m}_\mathfrak{X}(G^\phi) \subseteq \mathrm{m}_\mathfrak{X}(G)^\phi$. □

LEMMA 3. Let $\mathfrak{X}$ be a complete class of finite groups, $\pi = \pi(\mathfrak{X})$, G be a group, $A \trianglelefteq G$, and $H \in \mathrm{m}_\mathfrak{X}(G)$. Then:

(i) either $H \cap A \neq 1$ or A is a π'-group;

(ii) the subgroup $H \cap A$ is π-selfnormalizable in A.

Proof. See [3, 13.2]. □

This lemma is a particular case of Proposition 8 whenever the subgroup A is not merely subnormal but is normal in G. Proposition 8 is deduced from Lemma 3 and Prop. 7.

3. PROVING SOME PROPOSITIONS

We will prove only those propositions whose proofs we have not managed to find in other sources.

Proof of Proposition 3. (i) Let π be distinct from the set of all prime numbers and $|\pi| > 1$. Choose a number n so that π would contain at least two numbers not exceeding n, and one number in π' would not exceed n. Then $S_n \notin \mathscr{D}_\pi$ (see, e.g., main result in [26]).

(ii) A basis for the regular wreath product $G = X \wr Y$ is the direct product

$$B = \prod_{y \in Y} X_y$$

of $|Y|$ isomorphic copies of the group X, and a map $x \mapsto x_y$ specifies an isomorphism $X \cong X_y$. Let $H, K \in \mathrm{m}_\pi(X)$ be two maximal π-subgroups of X that are nonconjugate in X. Given a π-subgroup V of Y, we consider the V-invariant subgroup

$$U = \langle H_v, K_w \mid v \in V, w \in Y \setminus V \rangle \cong \prod_{v \in V} H_v \times \prod_{w \in Y \setminus V} K_w$$

of B. It is easy to see that $U \in \mathrm{m}_\pi(B)$. Now $UV \in \mathrm{m}_\pi(G)$. In fact, if $UV \leq M$, where M is a π-subgroup, then $M \cap B = U$ and the group M permutes projections of the group U on direct factors X_y. The projections coincide with H_v for $v \in V$ or with K_w for $w \in Y \setminus V$. If we assume that there exists an element $g \in M \setminus UV$, then

$$g = \left(\prod_{y \in Y} x^{(y)} \right) w,$$

where $x^{(y)} \in X_y$ and $w \in Y \setminus V$. On the one hand,

$$H_1^g = H_1^{x^{(1)} w} = H_w^{w^{-1} x^{(1)} w} = H_w^t,$$

where $t = w^{-1} x^{(1)} w \in X_w$, and on the other hand, it is true that $H_1^g = K_w$ since M permutes projections of U on direct factors, $w \notin V$, and $H_1^g \leq X_w$, a contradiction with H_w and K_w being nonconjugate in $X_w \cong X$. $\square$

Proof of Proposition 4. Let $N \trianglelefteq G$ and $N \in \mathscr{D}_\pi$.

(i) Suppose $H \in \mathrm{m}_\pi(G)$. We show that $H \cap N \in \mathrm{Hall}_\pi(N)$. Since H is a π-group, by Proposition 2 we have $HN \in \mathscr{D}_\pi$, and $H \in \mathrm{m}_\pi(HN) = \mathrm{Hall}_\pi(HN)$. Therefore, $H \cap N \in \mathrm{Hall}_\pi(N)$ in view of Prop. 1. Now let $K \in \mathrm{Hall}_\pi(N)$. Since $N \in \mathscr{D}_\pi$, while K and $H \cap N$ are two Hall π-subgroups in N, it follows that $K = H^x \cap N$ for some element $x \in N$. In addition, $H^x \in \mathrm{m}_\pi(G)$. Hence the rule $H \mapsto H \cap N$ specifies a surjection $\mathrm{m}_\pi(G) \to \mathrm{Hall}_\pi(N) = \mathrm{m}_\pi(N)$.

(ii) Again let $H \in \mathrm{m}_\pi(G)$. We show that $HN/N \in \mathrm{m}_\pi(G/N)$. Assume that $HN/N \leq K/N$ and K/N is a π-group. By Proposition 2, it is true that $K \in \mathscr{D}_\pi$, and since H is a maximal π-subgroup in K, we have $H \in \mathrm{Hall}_\pi(K)$. Then $HN/N \in \mathrm{Hall}_\pi(K/N) = \{K/N\}$ by Proposition 1, whence $HN/N = K/N$. Thus $HN/N \in \mathrm{m}_\pi(G/N)$. Now let $L/N \in \mathrm{m}_\pi(G/N)$. Again $L \in \mathscr{D}_\pi$ and $L/N = UN/N$ for $U \in \mathrm{Hall}_\pi(L)$. We show that $U \in \mathrm{m}_\pi(G)$. If $U \leq V$ and V is a π-subgroup in G, then $L \leq VN$ and $L = VN$, since $L/N \in \mathrm{m}_\pi(G/N)$. By virtue of $U \in \mathrm{Hall}_\pi(L)$ and $U \leq V$, we have $U = V$. Therefore, $U \in \mathrm{m}_\pi(G)$. Hence the rule $H \mapsto HN/N$ specifies a surjection $\mathrm{m}_\pi(G) \to \mathrm{m}_\pi(G/N)$. This also implies that the corresponding map between the sets of conjugacy classes of maximal π-subgroups in the groups G and G/N is surjective. We prove that it is injective. Let H and H_1 be maximal π-subgroups in G such that $H_1N = H^xN$ for some $x \in G$. By Proposition 2, $H_1N = H^xN \in \mathscr{D}_\pi$ and $H_1, H^x \in \mathrm{m}_\pi(H_1N)$. Therefore, H_1 and H are conjugate. □

Proof of Proposition 6. As mentioned in Section 1, it suffices to show that if $G \in \mathscr{D}_\pi$ then all submaximal subgroups of G are conjugate. To do this, in turn, we need only state that $\mathrm{sm}_\pi(G) \subseteq \mathrm{Hall}_\pi(G)$. Let $H \in \mathrm{sm}_\pi(G)$. We may assume that $G \trianglelefteq\trianglelefteq X$ and $H = K \cap G$ for some $K \in \mathrm{m}_\pi(X)$. Let $Y = \langle G^X \rangle$ be the normal closure of G in X. By Lemma 1, every composition factor of the group Y is isomorphic to a composition factor of the group G, and $Y \in \mathscr{D}_\pi$ in view of Prop. 2. By the same proposition, we obtain $KY \in \mathscr{D}_\pi$, and hence $K \in \mathrm{m}_\pi(HY) = \mathrm{Hall}_\pi(HY)$. If we apply Proposition 1 and the fact that $G \trianglelefteq\trianglelefteq Y$ we conclude that $H = K \cap G \in \mathrm{Hall}_\pi(G)$. □

Proof of Proposition 7.

(iii)⇒(ii) It suffices to take $G = SX$, where X is as in statement (3).

(ii)⇒(i) Follows from the definition.

(i)⇒(iii) Among all groups X such that S is subnormally embedded into X and $H = S \cap K$ for some $K \in \mathrm{m}_{\mathfrak{X}}(X)$, we choose X of least order. Let $A = \langle S^X \rangle$. Then the simplicity of S implies that A is a minimal normal subgroup of X. By the choice of X, we obtain $X = KA$.

Our present goal is to state that K normalizes S and, as a consequence, $A = \langle S^X \rangle = \langle S^K \rangle = S$. Clearly, $H = K \cap S \leq N_K(S)$, so $H \leq S \cap N_K(S) \leq S \cap K = H$ and $H = S \cap N_K(S)$. We show that $N = N_K(S)$ is a maximal $\mathfrak{X}$-subgroup in $X_0 = SN$, and hence $K = N_K(S)$ and $X = X_0 \leq N_X(S)$. Suppose that $N \leq U \leq X_0$, with $U \in \mathfrak{X}$. Since $SN \leq SU \leq X_0 = SN$, we obtain $SU = SN$. We also show that $U \cap S = N \cap S$. Let $U_0 = U \cap S$ and $g_1, \dots, g_m$ be a right transversal of K in N. Subgroups $S_i = S^{g_i}$, $i = 1, \dots, m$, are pairwise distinct and

$$A = \langle S_1, \dots, S_m \rangle \cong S_1 \times \cdots \times S_m.$$

Put $V = \langle U_0^{g_1}, \dots, U_0^{g_m} \rangle$. The subgroup K normalizes V. Indeed, let $x \in K$. Since K acts by right shifts on the set $\{Ng_1, \dots, Ng_m\}$, there exist a permutation σ on the set $\{1, \dots, m\}$ and elements $t_1, \dots, t_m \in N$ such that

$$g_i x = t_i g_{i\sigma}.$$

Then

$$\begin{aligned} V^x &= \langle U_0^{g_1 x}, \dots, U_0^{g_m x} \rangle = \langle U_0^{t_1 g_{1\sigma}}, \dots, U_0^{t_m g_{m\sigma}} \rangle \\ &= \langle U_0^{g_{1\sigma}}, \dots, U_0^{g_{m\sigma}} \rangle = \langle U_0^{g_1}, \dots, U_0^{g_m} \rangle = V, \end{aligned}$$

and so K normalizes V. In view of K being maximal and $\mathfrak{X}$ being complete, we obtain $V \le K$. Now

$$U \cap S = U_0 \le V \cap S \le K \cap S = H = N \cap S \le U \cap S.$$

Consequently, $U \cap S = N \cap S$. With $US = NS$ in mind, we derive

$$U/(U \cap S) \cong US/S = NS/S \cong N/(N \cap S).$$

Comparing the orders of N and U, we conclude that $U = N$.

Now $C_K(S) \trianglelefteq X$, and it is an $\mathfrak{X}$-subgroup. Let $^{-} : X \to X/C_K(S)$ be a natural homomorphism. Note that the restriction of this homomorphism to S is an embedding of S into the group $\overline{X}$, which in turn is an almost simple group with socle $\overline{S} \cong S$. It is easy to see that $\overline{K} \in \mathrm{m}_{\mathfrak{X}}(\overline{X})$ and $\overline{H} = \overline{K} \cap \overline{S}$. By the choice of X, $C_K(S) = 1$ and X is an almost simple group.

Thus we may assume that $X \le \mathrm{Aut}\,(S)$. Let a subgroup $M \in \mathrm{m}_{\mathfrak{X}}(\mathrm{Aut}\,(S))$ be such that $K \le M$. Then $M \cap S \le S \le X$ and K normalizes $M \cap S$. Consequently, $M \cap S \le K$ and $M \cap S = K \cap S = H$. □

Proof of Proposition 8. Let $A \trianglelefteq\trianglelefteq G$, $H \in \mathrm{m}_{\mathfrak{X}}(G)$ for some complete class $\mathfrak{X}$, and $\pi = \pi(\mathfrak{X})$. We need to show that if $H \cap A = 1$ then A is a π'-group, and that if $A \trianglelefteq G$ or A is a minimal subnormal subgroup in A then $N_A(H \cap A)/(H \cap A)$ is a π'-group.

For the case where $A \trianglelefteq G$, the required result follows from Lemma 3.

Suppose that A is a minimal subnormal subgroup of G. Then A is simple. Assume that A is Abelian. Then $|A| = p$ for some prime p. If $p \in \pi'$, then the desired result is obvious. If, however, $p \in \pi$, then A is an $\mathfrak{X}$-group; it is contained in the greatest normal $\mathfrak{X}$-subgroup and, consequently, in H. Then $H \cap A = A$ and the desired result is again obvious. Therefore, we assume that A is non-Abelian. Then $H \cap A \in \mathrm{sm}_{\mathfrak{X}}(A)$ and $H \cap A = K \cap A$ for some $K \in \mathrm{m}_{\mathfrak{X}}(X)$, where $X = \mathrm{Aut}\,(A)$ in view of Prop. 7. Since $A \trianglelefteq X$, the result follows from the case where A is a normal subgroup.

Finally, let A be an arbitrary subnormal subgroup of G and A not be a π'-group. We show that $H \cap A \ne 1$.

Suppose that $N = O_{\pi'}(G)$ is the greatest normal π'-subgroup in G and that $^{-} : G \to G/N$ is a natural epimorphism. Then $\overline{H} \in \mathrm{m}_{\mathfrak{X}}(\overline{G})$. In fact, it follows from Lemma 2 that if $\overline{H} \le \overline{K}$ for some $\overline{K} \in \mathrm{m}_{\mathfrak{X}}(\overline{G})$ then we may assume that $K \in \mathrm{m}_{\mathfrak{X}}(G)$ and $H \le KN$. Now $(|H|, |N|) = (|K|, |N|) = 1$, and the Schur–Zassenhaus theorem implies that $H \le K^x$ for some $x \in N$, whence $H = K^x$ and $\overline{H} = \overline{K}$. Furthermore, $\overline{A} \ne 1$, $\overline{A}$ does not contain subnormal π'-subgroups, and in view of the above $\overline{H} \cap \overline{B} \ne 1$ for some minimal subnormal subgroup $\overline{B}$ of G such that $\overline{B} \le \overline{A}$. Thus $\overline{H \cap A} = \overline{H} \cap \overline{A} \ne 1$ and $H \cap A \ne 1$. □

Proof of Proposition 9. Let $\mathfrak{F}$ be a Fitting class contained in the class $\mathscr{D}_\pi$, and let G be a finite group and $N \in \mathscr{D}_\pi$ its normal subgroup.

(i) The required result is obvious in virtue of the fact that if $H \in \mathrm{sm}_\pi(G)$ then $H \cap N \in \mathrm{sm}_\pi(N)$ by the property of submaximal π-subgroups, and $\mathrm{sm}_\pi(N) = \mathrm{m}_\pi(N) = \mathrm{Hall}_\pi(N)$ by Prop. 6. The conjugacy in N of the subgroups of $\mathrm{sm}_\pi(N) = \mathrm{m}_\pi(N) = \mathrm{Hall}_\pi(N)$ implies that $H \mapsto H \cap N$ is surjective being the map $\mathrm{sm}_\pi(G) \to \mathrm{sm}_\pi(N)$.

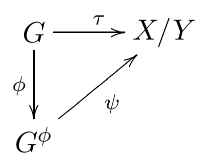
(ii) Let $N = G_{\mathfrak{F}}$. Denote by ϕ the natural homomorphism $G \to G/N$.

We need to show that the rule $H \mapsto H^\phi$ specifies a map $\mathrm{sm}_\pi(G) \to \mathrm{sm}_\pi(G^\phi)$; i.e, if $H \in \mathrm{sm}_\pi(G)$ then $H^\phi \in \mathrm{sm}_\pi(G^\phi)$. We may assume that there exists a group X for which $G \trianglelefteq\trianglelefteq X$ and $H = G \cap K$ for some subgroup $K \in \mathrm{m}_\pi(X)$. Let $Y = \langle N^X \rangle$ be the normal closure of N in X. Since $N \trianglelefteq G \trianglelefteq\trianglelefteq X$ and $\mathfrak{F}$ is a Fitting class, it is true that $Y \in \mathfrak{F}$. In addition, $G \cap Y \trianglelefteq\trianglelefteq Y$, and for $\mathfrak{F}$ is a Fitting class, we obtain $G \cap Y \in \mathfrak{F}$. Furthermore, $G \cap Y \trianglelefteq G$ entails

$$N \leq G \cap Y \leq G_{\mathfrak{X}} = N,$$

and hence $G \cap Y = N$.

Consider the restriction τ to G of the natural homomorphism $X \to X/Y$. The kernel of τ coincides with N, and by the homomorphism theorem there exists an injective homomorphism $\psi : G^\phi = G/N \to X$ such that the following diagram commutes:



Notice that

$$G^{\phi\psi} = G^\tau = GY/Y \trianglelefteq\trianglelefteq X/Y.$$

Furthermore, in view of Proposition 4, it is true that

$$H^{\phi\psi} = H^\tau = HY/Y = (G \cap K)Y/Y = (GY/Y) \cap (KY/Y) = G^{\phi\psi} \cap (KY/Y),$$

where $KY/Y \in \mathrm{m}_\pi(X/Y)$, and $H^\phi \in \mathrm{sm}_\pi(G^\phi)$ by definition. □

Proof of Proposition 10. The required result follows from Propositions 7 and 8 and from Lemma 4 below. We use the following notation: for a group G, let[3]

$$\mathrm{ng}_\pi(G) = \{H \leq G \mid N_G(H)/H \text{ is a } \pi'\text{-group}\}$$

denote the set of all π-selfnormalizable subgroups.

LEMMA 4. Let $\mathfrak{X}$ be a complete class of finite groups, $\pi = \pi(\mathfrak{X})$, and $G = G_1 \times \cdots \times G_n$. Then the following statements hold:

(i) $\mathrm{m}_{\mathfrak{X}}(G) = \{\langle H_1, \dots, H_n \rangle \mid H_i \in \mathrm{m}_{\mathfrak{X}}(G_i),\ i = 1, \dots, n\}$;

(ii) if $\mathrm{sm}_{\mathfrak{X}}(G_i) \subseteq \mathrm{ng}_\pi(G_i)$ for all $i = 1, \dots, n$, then $\mathrm{sm}_{\mathfrak{X}}(G) = \{\langle H_1, \dots, H_n \rangle \mid H_i \in \mathrm{sm}_{\mathfrak{X}}(G_i),\ i = 1, \dots, n\}$.

Proof. (i) Is readily proved using the definition of a maximal $\mathfrak{X}$-subgroup.

[3]The notation derives from the German term 'π-normalisatorgleich,' i.e, π-selfnormalizable, used by Wielandt [2].

(ii) First let $H \in \mathrm{sm}_{\mathfrak{X}}(G)$. In view of (4), $H \cap G_i \in \mathrm{sm}_{\mathfrak{X}}(G_i)$ for all $i = 1, \ldots, n$. We show that $H = \langle H \cap G_1, \ldots, H \cap G_n \rangle$. On the one hand,

$$\langle H \cap G_1, \ldots, H \cap G_n \rangle \leq H.$$

Now let $\rho_i : G \to G_i$ be a coordinate projection mapping and $H_i = H^{\rho_i}$. Since $H \cap G_i \trianglelefteq H$ and ρ_i acts identically on G_i, we have

$$H \cap G_i = (H \cap G_i)^{\rho_i} \trianglelefteq H^{\rho_i} = H_i.$$

Therefore, $H_i \leq N_{G_i}(H \cap G_i)$. Since $H \cap G_i \in \mathrm{sm}_{\mathfrak{X}}(G_i)$, the condition $\mathrm{sm}_{\mathfrak{X}}(G_i) \subseteq \mathrm{ng}_{\pi}(G_i)$ implies that $N_{G_i}(H \cap G_i)/(H \cap G_i)$ is a π'-group, while H_i is an $\mathfrak{X}$-group and, hence, a π-group. Therefore, $H_i = H \cap G_i$. This entails

$$\langle H \cap G_1, \ldots, H \cap G_n \rangle \leq H \leq \langle H_1, \ldots, H_n \rangle = \langle H \cap G_1, \ldots, H \cap G_n \rangle$$

and $H = \langle H \cap G_1, \ldots, H \cap G_n \rangle$, as desired. Thus

$$\mathrm{sm}_{\mathfrak{X}}(G) \subseteq \{\langle H_1, \ldots, H_n \rangle \mid H_i \in \mathrm{sm}_{\mathfrak{X}}(G_i),\ i = 1, \ldots, n\}.$$

Conversely, let $H_i \in \mathrm{sm}_{\mathfrak{X}}(G_i)$, $i = 1, \ldots, n$. We may assume that for every i there exists a group X_i in which G_i is subnormal, and $H_i = K_i \cap G_i$ for a suitable $K_i \in \mathrm{m}_{\mathfrak{X}}(X_i)$. In virtue of the first statement of the proposition, for a group $X = X_1 \times \cdots \times X_n$ we have

$$K = \langle K_1, \ldots, K_n \rangle = K_1 \times \cdots \times K_n \in \mathrm{m}_{\mathfrak{X}}(X).$$

In addition, $H_i = K_i \cap G_i {\trianglelefteq\trianglelefteq} K_i \trianglelefteq K$, whence $H_i {\trianglelefteq\trianglelefteq} K \cap G$ and $H_i = H_i^{\rho_i} {\trianglelefteq\trianglelefteq} (K \cap G)^{\rho_i}$. Now the condition $\mathrm{sm}_{\mathfrak{X}}(G_i) \subseteq \mathrm{ng}_{\pi}(G_i)$ implies that $N_{(K \cap G)^{\rho_i}}(H_i) = H_i$, which is possible only if $H_i = (K \cap G)^{\rho_i}$. Thus

$$\langle H_i \mid i = 1, \ldots, n \rangle \leq K \cap G \leq \langle (K \cap G)^{\rho_i} \mid i = 1, \ldots, n \rangle = \langle H_i \mid i = 1, \ldots, n \rangle.$$

In view of $K \in \mathrm{m}_{\mathfrak{X}}(X)$ and $G {\trianglelefteq\trianglelefteq} X$, we obtain $\langle H_i \mid i = 1, \ldots, n \rangle = K \cap G \in \mathrm{sm}_{\mathfrak{X}}(G)$. □

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