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THE d-RANK OF A TOPOLOGICAL SPACE

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It is shown that for any ordinal α, there exists a T_0-space whose d-rank is equal to α.

1. PRELIMINARY INFORMATION

The concept of a d-space was introduced by O. Wyler [1]; in [2], such spaces were called *monotone convergence spaces*. The concept of a d-completion was introduced in [1]. Different ways of constructing d-completions were found in [1] and [3]. In the present paper, we point out yet another way for constructing a d-completion. We cite relevant information on d-spaces from [3].

Definition 1.1. A topological T_0-space $\mathbb{X}$ is called a *d-space* if, for any nonempty upward directed set $D \subseteq X$ under the specialization order $\leq_{\mathcal{T}(\mathbb{X})}$, there exists $\sup D$, and $\sup D \in \mathrm{cl}_{\mathbb{X}} D$.

Definition 1.2. A d-space $\mathbb{Y}$ is called a *d-completion* of a space $\mathbb{X}$ if there exists a homeomorphic embedding $\lambda\colon \mathbb{X} \to \mathbb{Y}$, and for any d-space $\mathbb{Z}$ and any continuous map $f\colon \mathbb{X} \to \mathbb{Z}$, there is a unique continuous map $g\colon \mathbb{Y} \to \mathbb{Z}$ such that $g\lambda = f$.

Obviously, every d-space is its d-completion. If a d-completion of a space $\mathbb{X}$ exists, then we denote it by $\mathbb{H}_d(\mathbb{X})$.

Let $\mathbb{X} = \langle X, \mathcal{T}\rangle$ be an arbitrary T_0-space. Denote by $\overline{D}(X)$ the family of all nonempty upward directed subspaces under the specialization order $\leq$ in $\mathbb{X}$. Consider an equivalence relation $\sim$ on $\overline{D}(X)$ defined as follows:

$$S_0 \sim S_1 \text{ if and only if } S_0 \cap U \neq \varnothing \text{ is equivalent to } S_1 \cap U \neq \varnothing \text{ for any } U \in \mathcal{T}.$$

Put

$$[S] = \{S' \in \overline{D}(X) \mid S \sim S'\},\ S \in \overline{D}(X),$$

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$$D(\mathbb{X}) = \{[S] \mid S \in \overline{D}(X)\},$$
$$U^* = \{[S] \mid S \cap U \neq \varnothing\},\ U \in \mathcal{T},$$
$$\mathcal{T}^* = \{U^* \mid U \in \mathcal{T}\}.$$

For arbitrary open sets $U_0, U_1 \in \mathcal{T}$, we have $U_0^* \cap U_1^* = (U_0 \cap U_1)^* \in \mathcal{T}^*$. It is easy to verify that $\bigcup\{U_i^* \mid i \in I\} = \left(\bigcup\{U_i \mid i \in I\}\right)^* \in \mathcal{T}^*$ for any family $\{U_i \in \mathcal{T} \mid i \in I\}$ of open sets. Since $\varnothing^* = \varnothing$ and $X^* = D(\mathbb{X})$, $\mathcal{T}^*$ is a topology. Moreover, the topology $\mathcal{T}^*$ is T_0-separable. Put $\mathbb{D}(\mathbb{X}) = \langle D(\mathbb{X}), \mathcal{T}^*\rangle$.

Define a map $\lambda\colon \mathbb{X} \to \mathbb{D}(\mathbb{X})$ using the rule

$$\lambda(x) = \big[\{x\}\big].$$

It is not hard to see that the map λ is a homeomorphic embedding.

A transfinite sequence of extensions is constructed thus:

$\mathbb{D}_0(\mathbb{X}) = \mathbb{X}$,

$\mathbb{D}_{\alpha+1}(\mathbb{X}) = \mathbb{D}(\mathbb{X}_\alpha)$,

$\mathbb{D}_\alpha(\mathbb{X}) = \varinjlim\langle \alpha, \mathbb{X}_\beta, e_{\beta\gamma}\rangle$ if α is a limit ordinal,

where $e_{\beta\gamma}$ is a natural embedding of $\mathbb{D}_\beta(\mathbb{X})$ in $\mathbb{D}_\gamma(\mathbb{X})$ for any $\beta \leqslant \gamma < \alpha$, whose continuity was established in [3]. Without loss of generality, we may identify $\mathbb{D}_\beta(\mathbb{X})$ with the corresponding subspace of $\mathbb{D}_\gamma(\mathbb{X})$ for all ordinals $\beta \leqslant \gamma$.

THEOREM 1.3 [3]. For every T_0-space $\mathbb{X}$, there exists its d-completion—namely, there exists an ordinal α such that $\mathbb{H}_d(\mathbb{X}) \cong \mathbb{D}_\alpha(\mathbb{X}) = \mathbb{D}_{\alpha+1}(\mathbb{X})$.

Definition 1.4. The *d-rank* of a topological T_0-space $\mathbb{X}$ is the least ordinal α such that $\mathbb{D}_\alpha(\mathbb{X}) = \mathbb{D}_{\alpha+1}(\mathbb{X})$.

A basic result of the present paper, Theorem 3.6, says that every ordinal is the d-rank of some T_0-space.

2. BASIC CONSTRUCTION AND ITS PROPERTIES

Consider the following construction. Let topological spaces $\mathbb{X}$ and $\mathbb{Y}_x$, $x \in X$, be given. Put

$$Z = \bigcup_{x \in X} Y_x \times \{x\},$$
$$\mathcal{T} = \{U \subseteq Z \mid\ U_x \in \mathcal{T}(\mathbb{Y}_x) \text{ for any } x \in X \text{ and } U_X \in \mathcal{T}(\mathbb{X})\},$$

where $U_x = \{y \in Y_x \mid (y, x) \in U\}$ for any $x \in X$ and $U_X = \{x \in X \mid U_x \neq \varnothing\}$.

LEMMA 2.1. Let $\mathbb{X}$ be a T_0-space and $\mathbb{Y}_x$ be an irreducible T_0-space for every $x \in X$. Then:

(i) $\mathcal{T}$ is a T_0-separable topology on Z;

(ii) the map $y \mapsto (y, x)$ determines a homeomorphic embedding of $\mathbb{Y}_x$ in $\mathbb{Z} = \langle Z, \mathcal{T}\rangle$ for any $x \in X$;

(iii) if the space $\mathbb{X}$ is irreducible, then the space $\mathbb{Z} = \langle Z, \mathcal{T} \rangle$ is also irreducible.

Proof. (i) It is straightforward to verify that $\mathcal{T}$ is closed under arbitrary unions and finite intersections; i.e., it is a topology on Z. Let $z_0 \neq z_1$ in Z. If $z_0, z_1 \in Y_x \times \{x\}$ for some $x \in X$, then, in view of $\mathcal{T}(\mathbb{Y}_x)$ being T_0-separable, there exists $U \in \mathcal{T}(\mathbb{Y}_x)$ such that $z_i \in U \times \{x\}$ and $z_{1-i} \notin U \times \{x\}$ for some $i < 2$. In this case $z_i \in V = U \times \{x\} \cup \bigcup\limits_{x' \neq x} Y_{x'} \times \{x'\} \in \mathcal{T}(\mathbb{Z})$ and $z_{1-i} \notin V$. If, however, $z_0 \in Y_{x_0} \times \{x_0\}$ and $z_1 \in Y_{x_1} \times \{x_1\}$ for some distinct $x_0, x_1 \in X$, then there exists $U \in \mathcal{T}(\mathbb{X})$ such that $x_i \in U$ and $x_{1-i} \notin U$ for some $i < 2$, since the topology $\mathcal{T}(\mathbb{X})$ is T_0-separable. In this case $z_i \in V = \bigcup\limits_{x \in U} Y_x \times \{x\} \in \mathcal{T}$ and $z_{1-i} \notin V$.

(ii) Is obvious.

(iii) Let sets $U, V \in \mathcal{T}$ be nonempty. Then the sets $U_X, V_X \in \mathcal{T}(\mathbb{X})$ are also nonempty. The irreducibility of $\mathbb{X}$ entails $U_X \cap V_X \neq \varnothing$. Moreover, $U_x \cap V_x \neq \varnothing$ for any $x \in U \cap V$ in view of $\mathbb{Y}_x$ being irreducible. Therefore, $(U \cap V)_X = U_X \cap V_X \neq \varnothing$. Consequently, $\mathbb{Z}$ is an irreducible space. □

The space $\mathbb{Z} = \langle Z, \mathcal{T} \rangle$ dealt with in Lemma 2.1 is denoted also by $\sum\limits_{\mathbb{X}} \mathbb{Y}_x$. The specialization order $\leq_{\mathbb{Z}}$ is described by the following:

LEMMA 2.2. Let $\mathbb{X}$ be a T_0-space, $\mathbb{Y}_x$ an irreducible T_0-space for any $x \in X$, and $\mathbb{Z} = \sum\limits_{\mathbb{X}} \mathbb{Y}_x$. For all $(y_0, x_0), (y_1, x_1) \in Z$, we have $(y_0, x_0) \leq_{\mathbb{Z}} (y_1, x_1)$ if and only if one of the following two alternatives holds:

(i) $x_0 = x_1$ and $y_0 \leq_{\mathbb{Y}_{x_0}} y_1$;

(ii) $x_0 <_{\mathbb{X}} x_1$ and $y_1 = \top_{x_1}$ is the greatest element in $\mathbb{Y}_{x_1}$.

Proof. Sufficiency. Suppose that condition (i) is satisfied and $(y_0, x_0) \in U \in \mathcal{T}(\mathbb{Z})$. Then $y_0 \in U_{x_0} \in \mathcal{T}(\mathbb{Y}_{x_0})$. Therefore, $y_1 \in U_{x_0}$, i.e., $(y_1, x_1) \in U$. Now let condition (ii) be satisfied and $(y_0, x_0) \in U \in \mathcal{T}(\mathbb{Z})$. In this event $x_0 \in U_X \in \mathcal{T}(\mathbb{X})$, and $x_1 \in U_X$. Consequently, there exists $y \in U_{x_1} \in \mathcal{T}(\mathbb{Y}_{x_1})$. The equality $y_1 = \top_{x_1}$ entails $y_1 \in U_{x_1}$, hence $(y_1, x_1) \in U$.

Necessity. Let $(y_0, x_0) \leq_{\mathbb{Z}} (y_1, x_1)$. There are two cases to consider:

Case 1. Let $x_0 = x_1$. We show that in this event $y_0 \leq_{\mathbb{Y}_{x_0}} y_1$. Indeed, let $y_0 \in V \in \mathcal{T}(\mathbb{Y}_{x_0})$. Then $(y_0, x_0) \in U = (V \times \{x_0\}) \cup \bigcup\limits_{x \neq x_0} Y_x \times \{x\} \in \mathcal{T}(\mathbb{Z})$. Consequently, $(y_1, x_1) \in U$, i.e., $y_1 \in U_{x_0} = V$.

Case 2. Let $x_0 \neq x_1$. First we show that y_1 is the greatest element in $\mathbb{Y}_{x_1}$. Indeed, let $y \in Y_{x_1}$ be an arbitrary element, and $y \in V \in \mathcal{T}(\mathbb{Y}_{x_1})$. Then $x_0 \neq x_1$, and so $(y_0, x_0) \in U = (V \times \{x_1\}) \cup \bigcup\limits_{x \neq x_1} Y_x \times \{x\} \in \mathcal{T}(\mathbb{Z})$. Hence $(y_1, x_1) \in U$, i.e., $y_1 \in U_{x_1} = V$, which proves that $y \leq_{\mathbb{Y}_{x_1}} y_1$ for any $y \in Y_{x_1}$. Now we show that $x_0 <_{\mathbb{X}} x_1$. Indeed, let $x_0 \in W \in \mathcal{T}(\mathbb{X})$. Then $(y_0, x_0) \in U = \bigcup\limits_{x \in W} Y_x \times \{x\} \in \mathcal{T}(\mathbb{Z})$, and so $(y_1, x_1) \in U$, i.e., $x_1 \in U_X = W$, as required. □

Put $\tilde{X} = \{x \in X \mid \mathbb{Y}_x \text{ has the greatest element } \top_x\}$. Obviously, the space $\tilde{\mathbb{X}}$ with the induced topology is a subspace of $\mathbb{X}$.

LEMMA 2.3. Let $\mathbb{X}$ be a T_0-space, $\mathbb{Y}_x$ an irreducible T_0-space for any $x \in X$, and $\mathbb{Z} = \sum\limits_{\mathbb{X}} \mathbb{Y}_x$. Then an arbitrary set $S' \in \overline{D}(\mathbb{Z})$ contains a cofinal subset $S \subseteq S'$ having one of the following forms:

(I) $S = \{(y,x) \mid y \in S_x\}$ for some fixed $x \in X$ and $S_x \in \overline{D}(\mathbb{Y}_x)$;

(II) $S = \{(\top_x, x) \mid x \in S_*\}$ for some $S_* \in \overline{D}(\tilde{\mathbb{X}})$.

Proof. For a nonempty upward directed set $S' \subseteq Z$ under the specialization order, one of the following cases holds:

Case 1. There exists $x \in X$ such that for any $s \in S'$ there exists $y \in Y_x$ with the condition that $s \le (y,x)$. Put $S = \{(y,x) \in Z \mid (y,x) \in S'\}$. According to Lemma 2.2 and our hypothesis, the set $S_x = \{y \in Y_x \mid (y,x) \in S\}$ is nonempty and upward directed under the specialization order in $\mathbb{Y}_x$; so S has form (I).

Case 2. Suppose that the hypothesis of Case 1 does not hold. This means that whatever element $(y,x) \in S'$ we choose, there exists an element $(y',x') \in S'$ for which $(y',x') \nleq (\overline{y},x)$ with any $\overline{y} \in Y_x$. Since the set S' is upward directed, there is an element $(\tilde{y},\tilde{x}) \in S'$ such that $(y,x) \le (\tilde{y},\tilde{x})$ and $(y',x') \le (\tilde{y},\tilde{x})$. The inequality $(y',x') \le (\tilde{y},\tilde{x})$ means that $\tilde{x} \ne x$. If we apply Lemma 2.2 to $(y,x) \le (\tilde{y},\tilde{x})$ we obtain $\tilde{y} = \top_{\tilde{x}}$ and $x < \tilde{x} \in \tilde{X}$. Thus, whatever element $(y,x) \in S'$ we choose, there exists an element $\tilde{x} \in \tilde{X}$ for which $x < \tilde{x}$ and $(y,x) \le (\top_{\tilde{x}},\tilde{x}) \in S'$. Put $S = \{(\top_x,x) \in S' \mid x \in \tilde{X}\}$ and $S_* = \{x \in \tilde{X} \mid (\top_x,x) \in S\}$. Since $S' \ne \varnothing$, what has been stated above implies that $S_* \ne \varnothing$ and the set S_* is upward directed under the specialization order. Thus S has form (II). □

An immediate consequence of Lemmas 2.2 and 2.3 is the following:

COROLLARY 2.4. Let $\mathbb{X}$ be a T_0 space, $\mathbb{Y}_x$ an irreducible T_0-space for any $x \in X$, and $\mathbb{Z} = \sum_{\mathbb{X}} \mathbb{Y}_x$. If the set $S \in \overline{D}(\mathbb{Z})$ simultaneously contains a cofinal subset of type (I) and a cofinal subset of type (II), then $[S_*] = x$ in $\mathbb{D}(\mathbb{X})$ and $[S_x] = \top_x$ in $\mathbb{D}(\mathbb{Y}_x)$ for some $x \in \tilde{X}$.

Proof. Suppose that for some $x_0 \in X$, $S_{x_0} \in \overline{D}(\mathbb{Y}_{x_0})$, and $S_* \in \overline{D}(\tilde{\mathbb{X}})$, the sets $S_0 = \{(y,x_0) \mid y \in S_{x_0}\}$ and $S_1 = \{(\top_x,x) \mid x \in S_*\}$ are cofinal in S. Hence, for any $y_0 \in S_{x_0}$, there is $x \in S_*$ such that $(y_0,x_0) \le (\top_x,x)$, and for any $x \in S_*$, there is $y_1 \in S_{x_0}$ such that $(\top_x,x) \le (y_1,x_0)$. Summing up the above, we have

$$(y_0,x_0) \le (\top_x,x) \le (y_1,x_0).$$

By virtue of Lemma 2.1, we obtain $[S_*] = x_0$ and $y_1 = \top_{x_0}$. □

For any irreducible topological space $\mathbb{Y}$, put

$$\mathbb{Y}^\top = \begin{cases} \mathbb{Y} & \text{if } \mathbb{Y} \text{ has a greatest element,} \\ \big\langle Y \cup \{\top\}, \mathcal{T}(\mathbb{Y})^\top \big\rangle & \text{otherwise,} \end{cases}$$

where $\mathcal{T}(\mathbb{Y})^\top = \{\varnothing\} \cup \big\{U \cup \{\top\} \mid \varnothing \ne U \in \mathcal{T}(\mathbb{Y})\big\}$. According to our definition, for any irreducible T_0-space $\mathbb{Y}$, $\mathbb{Y}^\top$ is a T_0-separable topological space and has a greatest element. Also put

$$S_* = \big\{x \in \tilde{X} \mid (\top_x,x) \in S\big\} \text{ for any set } S \subseteq Z,$$
$$S_x = \big\{y \in Y_x \mid (y,x) \in S\big\} \text{ for any } x \in X,$$

$$X' = X \cup D(\tilde{\mathbb{X}}) \subseteq D(\mathbb{X}).$$

Then $\mathbb{X}'$ with the induced topology is obviously a subspace of $\mathbb{D}(\mathbb{X})$. Let $x \in X$, and let sets U_0 and $U_1 \in \mathcal{T}(\mathbb{Y}_x)$ be such that $U_0^*, U_1^* \neq \varnothing$ in $\mathbb{D}(\mathbb{Y}_x)$. This means that $U_0, U_1 \neq \varnothing$ in $\mathbb{Y}_x$, i.e., $U_0 \cap U_1 \neq \varnothing$. Thus $U_0^* \cap U_1^* = (U_0 \cap U_1)^* \neq \varnothing$ in $\mathbb{D}(\mathbb{Y}_x)$. Therefore, the space $\mathbb{D}(\mathbb{Y}_x)$ is irreducible for any $x \in X$. Furthermore, for every $x' \in X'$ we put

$$\mathbb{Y}'_{x'} = \begin{cases} \mathbb{D}(\mathbb{Y}_x) & \text{if } x' = x \in \tilde{X}, \\ \mathbb{D}(\mathbb{Y}_x)^\top & \text{if } x' = x \in X \backslash \tilde{X}, \\ \mathbb{T} & \text{if } x' \in D(\tilde{\mathbb{X}}) \backslash X, \end{cases}$$

where $\mathbb{T} = \big\langle \{\top\}, \{\varnothing, \{\top\}\} \big\rangle$.

THEOREM 2.5. Let $\mathbb{X}$ be a T_0-space, $\mathbb{Y}_x$ an irreducible T_0-space for any $x \in X$, and $\mathbb{Z} = \sum\limits_{\mathbb{X}} \mathbb{Y}_x$. Then the spaces $\mathbb{D}(\mathbb{Z})$ and $\mathbb{Z}' = \sum\limits_{\mathbb{X}'} \mathbb{Y}'_{x'}$ are homeomorphic.

Proof. Define a map $f\colon D(\mathbb{Z}) \to Z'$ setting

$$f([S]) = \begin{cases} \big([S_x], x\big) & \text{if } S \text{ has type (I) for some } x \in X, \\ (\top, x) & \text{if } S \text{ has type (II) and } [S_*] = x \in X, \\ \big(\top, [S_*]\big) & \text{if } S \text{ has type (II) and } [S_*] \notin X. \end{cases}$$

In view of Lemmas 2.2, 2.3, Corollary 2.4, and the definition of a space $\mathbb{Z}'$, the map f is well defined and is one-to-one.

Claim 1. The map f is continuous.

Proof. Suppose $V \in \mathcal{T}(\mathbb{Z}')$. Then $V_{X'} \in \mathcal{T}(\mathbb{X}')$, so $U = V_{X'} \cap X \in \mathcal{T}(\mathbb{X})$. For an arbitrary $x \in U$, put $W_x = V_x \cap Y_x$ and $W = \bigcup\limits_{x \in U} W_x \times \{x\}$. Then $W_x \neq \varnothing$ for any $x \in U$, and hence $W_X = U$ and $W \in \mathcal{T}(\mathbb{Z})$, i.e., $W^* \in \mathcal{T}\big(\mathbb{D}(\mathbb{Z})\big)$. It suffices to state that $W^* = f^{-1}(V)$. Indeed, let $S \in \overline{D}(\mathbb{Z})$ be such that $[S] \in W^*$, i.e., $(y_0, x_0) \in S \cap W$ for some $x_0 \in U$ and $y_0 \in W_{x_0}$. According to the definition of a map f, there are two cases to consider:

Case 1. S has type (I) for some $x \in X$. This means that there exists $y \in S_x$ such that $(y_0, x_0) \leq_{\mathbb{Z}} (y, x)$; in particular, $(y, x) \in W$, i.e., $y \in W_x \subseteq V_x \in \mathcal{T}\big(\mathbb{D}(\mathbb{Y}_x)\big)$. Consequently, $y \leq_{\mathbb{D}(\mathbb{Y}_x)} [S_x]$ and $[S_x] \in V_x$. Thus $f([S]) = \big([S_x], x\big) \in V_x \times \{x\} \subseteq V$, i.e., $[S] \in f^{-1}(V)$.

Case 2. S has type (II). This means that there exists $x_1 \in S_*$ with the condition that $(y_0, x_0) \leq_{\mathbb{Z}} (\top_{x_1}, x_1)$; in particular, $x_0 \leq_{\mathbb{X}'} x_1 \leq_{\mathbb{X}'} [S_*]$. Since $x_0 \in U \subseteq V_{X'}$, we have $[S_*] \in V_{X'}$. Three options are possible:

Case 2.1. $[S_*] = x \in \tilde{X}$. The space $\mathbb{Y}_x$, and hence $\mathbb{Y}'_x = \mathbb{D}(\mathbb{Y}_x)$, has the greatest element $\top$. Thus $x \in U$, $\top \in V_x$, and $f([S]) = \big(\top, x\big) \in V_x \times \{x\} \subseteq V$, i.e., $[S] \in f^{-1}(V)$.

Case 2.2. $[S_*] = x \in X \backslash \tilde{X}$. This means that $x \in U$, $\mathbb{Y}'_x = \mathbb{D}(\mathbb{Y}_x)^\top$, $\top \in V_x$, and $f([S]) = \big(\top, x\big) \in V_x \times \{x\} \subseteq V$.

Case 2.3. $[S_*] \notin X$. Since $[S_*] \in V_{X'}$, it follows that $f([S]) = \big(\top, [S_*]\big) \in V$.

In any case we have $[S] \in f^{-1}(V)$. Conversely, let $S \in \overline{D}(\mathbb{Z})$ be such that $f([S]) \in V$. According to the definition of a map f, there are again two cases to consider:

Case 1. S has type (I) for some $x \in X$. This means that $f([S]) = \big([S_x], x\big) \in V$, i.e., $[S_x] \in V_x$. Therefore, there exists $y \in S_x$ such that $y \in S_x \cap V_x \cap Y_x = S_x \cap W_x$, i.e., $(y, x) \in S \cap W$. Consequently, $[S] \in W^*$.

Case 2. S has type (II). This means that $f([S]) = \big(\top, [S_*]\big) \in V$. Thus $[S_*] \in V_{X'}$. Therefore, there exists $x \in S_* \cap V_{X'} \cap \tilde{X} \subseteq S_* \cap U$. Consequently, $V_x \neq \varnothing$ and $W_x \neq \varnothing$. Then $\top_x \in W_x$, i.e., $(\top_x, x) \in S \cap (W_x \times \{x\}) \subseteq S \cap W$ and $[S] \in W^*$. □

Claim 2. The map f is open.

Proof. Let $W \in \mathcal{T}(\mathbb{Z})$. Then $W_X \in \mathcal{T}(\mathbb{X})$, so $W' = W_X^* \cap X' \in \mathcal{T}(\mathbb{X}')$. For an arbitrary $x' \in W'$, put

$$V_{x'} = \begin{cases} \{\top_{x'}\} & \text{if } x' \in W' \backslash X, \\ W_x^* \cup \{\top\} & \text{if } x' = x \in X \backslash \tilde{X}, \\ W_x^* & \text{if } x' = x \in \tilde{X}, \end{cases}$$

and $V = \bigcup\limits_{x' \in W'} V_{x'} \times \{x'\}$; then $V \in \mathcal{T}(\mathbb{Z}')$. It suffices to state that $f(W^*) = V$. By Claim 1, $f^{-1}(V) = W^*$, i.e., $f(W^*) = V$ since f is one-to-one. □

This completes the proof of the theorem. □

3. SPECIAL SPACES

Definition 3.1. Let α be an ordinal. A topological T_0-space $\mathbb{X}$ is said to be α-*special* if its d-rank is equal to α, the space $\mathbb{D}_\alpha(\mathbb{X})$ has a greatest element, while the space $\mathbb{D}_\beta(\mathbb{X})$ does not have a greatest element for any ordinal $\beta < \alpha$.

Remark 3.2. If $\mathbb{X}$ is an α-special space, then α will not be a limit ordinal by the definition of $\mathbb{D}_\alpha(\mathbb{X})$.

LEMMA 3.3. For any nonlimit ordinal α, every α-special space is irreducible.

Proof. Suppose that $\mathbb{X}$ is an α-special space, but there exist nonempty sets $U_0, U_1 \in \mathcal{T}(\mathbb{X})$ such that $U_0 \cap U_1 = \varnothing$. By induction on β, it is not hard to verify that for any ordinal β, there exist (nonempty) sets $U_0^\beta, U_1^\beta \in \mathcal{T}\big(\mathbb{D}_\beta(\mathbb{X})\big)$ such that $U_0^\beta \cap X = U_0$, $U_1^\beta \cap X = U_1$, and $U_0^\beta \cap U_1^\beta = \varnothing$. In particular, the space $\mathbb{D}_\alpha(\mathbb{X})$ is not irreducible. This is impossible since every space containing a greatest element is irreducible. □

For an arbitrary ordinal $\alpha > 0$, consider the topological T_0-space

$$\mathbb{O}_\alpha = \big\langle \alpha, \{\varnothing\} \cup \{\uparrow\beta \mid \beta < \alpha \text{ is not limit}\}\big\rangle.$$

For any nonlimit ordinals $\beta_0, \beta_1 < \alpha$, we have $\uparrow\beta_0 \cap \uparrow\beta_1 = \uparrow\beta \neq \varnothing$, where $\beta = \max\{\beta_0, \beta_1\}$. Thus $\mathbb{O}_\alpha$ is an irreducible T_0-space.

PROPOSITION 3.4. Let $\alpha > 0$ be an ordinal.

(i) If α is limit, then $\mathbb{H}_d(\mathbb{O}_\alpha) = \mathbb{O}_\alpha^\top = \mathbb{D}(\mathbb{O}_\alpha)$, i.e., the space $\mathbb{O}_\alpha$ is 1-special.

(ii) If α is not limit, then $\mathbb{H}_d(\mathbb{O}_\alpha) = \mathbb{O}_\alpha$, i.e., the d-rank of $\mathbb{O}_\alpha$ is equal to 0.

(iii) If $\mathbb{Y}$ is an $(\alpha+1)$-special space for some ordinal α, then $\mathbb{D}_\beta(\mathbb{Y}^\top) \cong \mathbb{D}_\beta(\mathbb{Y})^\top$ for any $\beta \leqslant \alpha$ and $\mathbb{D}_{\alpha+1}(\mathbb{Y}^\top) \cong \mathbb{D}_{\alpha+1}(\mathbb{Y}) = \mathbb{H}_d(\mathbb{Y})$.

(iv) If α is limit, γ is not a limit ordinal, and a T_0-space $\mathbb{Y}_\beta$ is γ-special for any $\beta < \alpha$, then the space $\mathbb{Z} = \sum\limits_{\mathbb{O}_\alpha} \mathbb{Y}_\beta$ is $(\gamma+1)$-special.

(v) If α is limit, and a T_0-space $\mathbb{Y}_\beta$ is $(\beta+1)$-special for any $\beta < \alpha$, then the space $\mathbb{Z} = \sum\limits_{\mathbb{O}_\alpha} \mathbb{Y}_\beta$ is $(\alpha+1)$-special and the d-rank of a space $\mathbb{Z}^\top$ is equal to α.

Proof. (i)-(iii) Are obvious.

(iv) First we show that the spaces $\mathbb{D}_\delta(\mathbb{Z})$ and $\sum\limits_{\mathbb{O}_\alpha} \mathbb{D}_\delta(\mathbb{Y}_\beta)$ are homeomorphic for any ordinal $\delta \leqslant \gamma$. We use induction on δ. For $\delta = 0$, the statement follows from the definition of a space $\mathbb{Z}$. Let δ be such that $\delta + 1 \leqslant \gamma$, and let $\mathbb{D}_\delta(\mathbb{Z})$ and $\sum\limits_{\mathbb{O}_\alpha} \mathbb{D}_\delta(\mathbb{Y}_\beta)$ be homeomorphic. In view of the inequality $\delta < \gamma$ and by the choice of $\mathbb{Y}_\beta$, $\beta < \alpha$, the space $\mathbb{D}_\delta(\mathbb{Y}_\beta)$ does not contain a greatest element for any $\beta < \alpha$. This means that $\tilde{\alpha} = \varnothing$, and so $(\mathbb{O}_\alpha)' = \mathbb{O}_\alpha$; i.e., according to Theorem 2.5,

$$\mathbb{D}_{\delta+1}(\mathbb{Z}) \cong \mathbb{D}\left(\sum_{\mathbb{O}_\alpha} \mathbb{D}_\delta(\mathbb{Y}_\beta)\right) \cong \sum_{\mathbb{O}_\alpha} \mathbb{D}\big(\mathbb{D}_\delta(\mathbb{Y}_\beta)\big) = \sum_{\mathbb{O}_\alpha} \mathbb{D}_{\delta+1}(\mathbb{Y}_\beta).$$

Suppose now that $\delta \leqslant \gamma$ is a limit ordinal and that the required statement holds for any $\delta' < \delta$. Then

$$\begin{aligned}\mathbb{D}_\delta(\mathbb{Z}) &= \varinjlim_{\delta'<\delta} \mathbb{D}_{\delta'}(\mathbb{Z}) \cong \varinjlim_{\delta'<\delta} \sum_{\mathbb{O}_\alpha} \mathbb{D}_{\delta'}(\mathbb{Y}_\beta) \\ &\cong \sum_{\mathbb{O}_\alpha} \varinjlim_{\delta'<\delta} \mathbb{D}_{\delta'}(\mathbb{Y}_\beta) = \sum_{\mathbb{O}_\alpha} \mathbb{D}_\delta(\mathbb{Y}_\beta).\end{aligned}$$

Thus $\mathbb{D}_\gamma(\mathbb{Z}) \cong \sum\limits_{\mathbb{O}_\alpha} \mathbb{D}_\gamma(\mathbb{Y}_\beta) \cong \sum\limits_{\mathbb{O}_\alpha} \mathbb{H}_d(\mathbb{Y}_\beta)$. Since $\mathbb{H}_d(\mathbb{Y}_\beta)$ contains a greatest element for any $\beta < \alpha$, we have $\tilde{\alpha} = \alpha$ and $(\mathbb{O}_\alpha)' \cong \mathbb{O}_\alpha^\top \cong \mathbb{O}_{\alpha+1}$. By Theorem 2.5, we obtain

$$\begin{aligned}\mathbb{D}_{\gamma+1}(\mathbb{Z}) &\cong \mathbb{D}\left(\sum_{\mathbb{O}_\alpha} \mathbb{H}_d(\mathbb{Y}_\beta)\right) \cong \sum_{\mathbb{O}_{\alpha+1}} \mathbb{D}(\mathbb{W}_\beta) \cong \sum_{\mathbb{O}_{\alpha+1}} \mathbb{W}_\beta \\ &\cong \left(\sum_{\mathbb{O}_\alpha} \mathbb{H}_d(\mathbb{Y}_\beta)\right)^\top \cong \mathbb{D}_\gamma(\mathbb{Z})^\top, \\ \mathbb{D}_{\gamma+2}(\mathbb{Z}) &= \mathbb{D}\left(\mathbb{D}_{\gamma+1}(\mathbb{Z})\right) \cong \mathbb{D}\left(\sum_{\mathbb{O}_{\alpha+1}} \mathbb{W}_\beta\right) \cong \sum_{\mathbb{O}_{\alpha+1}} \mathbb{D}(\mathbb{W}_\beta) \\ &= \sum_{\mathbb{O}_{\alpha+1}} \mathbb{W}_\beta = \mathbb{D}_{\gamma+1}(\mathbb{Z}),\end{aligned}$$

where $\mathbb{W}_\beta = \mathbb{H}_d(\mathbb{Y}_\beta)$ if $\beta < \alpha$, and $\mathbb{W}_\alpha = \mathbb{T}$. Furthermore, the space $\mathbb{D}_\delta(\mathbb{Z})$ does not contain a greatest element for any $\delta \leqslant \gamma$. This proves that the space $\mathbb{Z}$ is $(\gamma+1)$-special.

(v) Using induction on δ, we state that for any ordinal $\delta < \alpha$, the spaces $\mathbb{D}_\delta(\mathbb{Z})$ and $\sum_{\mathbb{O}_\alpha} \mathbb{W}^\delta_\beta$ are homeomorphic, where

$$\mathbb{W}^\delta_\beta = \begin{cases} \mathbb{D}_\delta(\mathbb{Y}_\beta) & \text{if } \delta \leqslant \beta < \alpha, \\ \mathbb{H}_d(\mathbb{Y}_\beta) & \text{if } \beta < \delta < \alpha. \end{cases}$$

For $\delta = 0$, the statement follows from the definition of a space $\mathbb{Z}$. Let δ be such that $\delta + 1 < \alpha$, and $\mathbb{D}_\delta(\mathbb{Z}) \cong \sum_{\mathbb{O}_\alpha} \mathbb{W}^\delta_\beta$. In view of the inequality $\delta < \alpha$ and by the choice of $\mathbb{Y}_\beta$, $\beta < \alpha$, the space $\mathbb{D}_\delta(\mathbb{Y}_\beta)$ does not contain a greatest element for any ordinal β such that $\delta \leqslant \beta < \alpha$. This means that $\tilde{\alpha} = \delta$ and $(\mathbb{O}_\alpha)' = \mathbb{O}_\alpha$; i.e., in view of Theorem 2.5 and the induction hypothesis, we have

$$\mathbb{D}_{\delta+1}(\mathbb{Z}) \cong \mathbb{D}\big(\mathbb{Z}_\delta\big) \cong \mathbb{D}\left(\sum_{\mathbb{O}_\alpha} \mathbb{W}^\delta_\beta\right) \cong \sum_{\mathbb{O}_\alpha} \mathbb{D}\big(\mathbb{W}^\delta_\beta\big) = \sum_{\mathbb{O}_\alpha} \mathbb{W}^{\delta+1}_\beta.$$

Suppose now that $\delta < \alpha$ is a limit ordinal and that the required statement holds for any ordinal $\delta' < \delta$. By Theorem 2.5 and the induction hypothesis, we have

$$\mathbb{D}_\delta(\mathbb{Z}) = \varinjlim_{\delta'<\delta} \mathbb{D}_{\delta'}(\mathbb{Z}) \cong \varinjlim_{\delta'<\delta} \sum_{\mathbb{O}_\alpha} \mathbb{W}^{\delta'}_\beta \cong \sum_{\mathbb{O}_\alpha} \varinjlim_{\delta'<\delta} \mathbb{W}^{\delta'}_\beta = \sum_{\mathbb{O}_\alpha} \mathbb{W}^\delta_\beta.$$

Thus

$$\mathbb{D}_\alpha(\mathbb{Z}) = \varinjlim_{\delta<\alpha} \mathbb{D}_\delta(\mathbb{Z}) \cong \varinjlim_{\delta<\alpha} \sum_{\mathbb{O}_\alpha} \mathbb{W}^\delta_\beta \cong \sum_{\mathbb{O}_\alpha} \varinjlim_{\delta<\alpha} \mathbb{W}^\delta_\beta = \sum_{\mathbb{O}_\alpha} \mathbb{H}_d(\mathbb{Y}_\beta).$$

Furthermore, the space $\mathbb{H}_d(\mathbb{Y}_\beta)$ contains a greatest element for any $\beta < \alpha$. Then $\tilde{\alpha} = \alpha$ and $(\mathbb{O}_\alpha)' \cong \mathbb{O}^\top_\alpha \cong \mathbb{O}_{\alpha+1}$. In view of Theorem 2.5, we obtain

$$\begin{aligned} \mathbb{D}_{\alpha+1}(\mathbb{Z}) &\cong \mathbb{D}\left(\mathbb{D}_\alpha(\mathbb{Z})\right) \cong \mathbb{D}\left(\sum_{\mathbb{O}_\alpha} \mathbb{H}_d(\mathbb{Y}_\beta)\right) \cong \sum_{\mathbb{O}_{\alpha+1}} \mathbb{D}(\mathbb{W}_\beta) \\ &\cong \sum_{\mathbb{O}_{\alpha+1}} \mathbb{W}_\beta \cong \mathbb{D}_\alpha(\mathbb{Z})^\top, \\ \mathbb{D}_{\alpha+2}(\mathbb{Z}) &\cong \mathbb{D}\left(\mathbb{D}_{\alpha+1}(\mathbb{Z})\right) \cong \mathbb{D}\left(\sum_{\mathbb{O}_{\alpha+1}} \mathbb{W}_\beta\right) \cong \sum_{\mathbb{O}_{\alpha+1}} \mathbb{D}(\mathbb{W}_\beta) \\ &\cong \sum_{\mathbb{O}_{\alpha+1}} \mathbb{W}_\beta \cong \mathbb{D}_{\alpha+1}(\mathbb{Z}), \end{aligned}$$

where $\mathbb{W}_\beta = \mathbb{H}_d(\mathbb{Y}_\beta)$ if $\beta < \alpha$, and $\mathbb{W}_\alpha = \mathbb{T}$. By Lemma 2.2, the space $\mathbb{D}_\beta(\mathbb{Z})$ does not contain a greatest element, and hence $\mathbb{D}_\beta(\mathbb{Z}) < \mathbb{D}_{\alpha+1}(\mathbb{Z})$ for any ordinal $\beta \leqslant \alpha$. Thus the space $\mathbb{Z}$ is $(\alpha+1)$-special.

Assertion 1. For any ordinal γ with the condition that $\gamma \leqslant \alpha$, the space $\mathbb{D}_\gamma(\mathbb{Z}^\top)$ is homeomorphic to a space $\sum\limits_{\mathbb{O}_{\alpha+1}} \mathbb{W}^\gamma_\beta$, where

$$\mathbb{W}^\gamma_\beta = \begin{cases} \mathbb{D}_\gamma(\mathbb{Y}_\beta) & \text{if } \gamma \leqslant \beta < \alpha, \\ \mathbb{H}_d(\mathbb{Y}_\beta) & \text{if } \beta < \gamma \leqslant \alpha, \\ \mathbb{T} & \text{if } \beta = \alpha. \end{cases}$$

The **proof** is by induction on γ. If $\gamma = 0$, then $\mathbb{Z}^\top \cong \sum\limits_{\mathbb{O}_{\alpha+1}} \mathbb{Y}_\beta$, where $\mathbb{Y}_\alpha = \mathbb{T}$. Suppose now that the required assertion holds for an ordinal $\gamma < \alpha$. In this case we have $(\alpha \tilde{+} 1) = \gamma \cup \{\alpha\}$ and $(\mathbb{O}_{\alpha+1})' = \mathbb{O}_{\alpha+1}$. By Theorem 2.5 and the induction hypothesis, we obtain

$$\mathbb{D}_{\gamma+1}(\mathbb{Z}^\top) \cong \mathbb{D}(\mathbb{D}_\gamma(\mathbb{Z}^\top)) \cong \mathbb{D}\left(\sum_{\mathbb{O}_{\alpha+1}} \mathbb{W}^\gamma_\beta\right) \cong \sum_{\mathbb{O}_{\alpha+1}} \mathbb{D}(\mathbb{W}^\gamma_\beta) = \sum_{\mathbb{O}_{\alpha+1}} \mathbb{W}^{\gamma+1}_\beta.$$

Assume that γ is a limit ordinal and that the required assertion holds for any ordinal $\delta < \gamma$. In view of Theorem 2.5, Proposition 3.4(iii), and the induction hypothesis, we have

$$\mathbb{D}_\gamma(\mathbb{Z}^\top) = \varinjlim_{\delta<\gamma} \mathbb{D}_\delta(\mathbb{Z}^\top) \cong \varinjlim_{\delta<\gamma} \sum_{\mathbb{O}_{\alpha+1}} \mathbb{W}^\delta_\beta \cong \sum_{\mathbb{O}_{\alpha+1}} \varinjlim_{\delta<\gamma} \mathbb{W}^\delta_\beta = \sum_{\mathbb{O}_{\alpha+1}} \mathbb{W}^\gamma_\beta. \ \square$$

By Assertion 1, it is true that $\mathbb{W}^\alpha_\beta = \mathbb{H}_d(\mathbb{Y}_\beta)$ for $\beta < \alpha$, and $\mathbb{W}^\alpha_\alpha = \mathbb{T}$. Moreover, if $\beta < \alpha$, then $\mathbb{W}^\beta_\beta = \mathbb{D}_\beta(\mathbb{Y}_\beta) < \mathbb{H}_d(\mathbb{Y}_\beta) = \mathbb{W}^\alpha_\beta$, and so $\mathbb{D}_\beta(\mathbb{Z}^\top) < \mathbb{D}_\alpha(\mathbb{Z}^\top)$ for any $\beta < \alpha$. Finally,

$$\mathbb{D}_{\alpha+1}(\mathbb{Z}^\top) \cong \mathbb{D}(\mathbb{D}_\alpha(\mathbb{Z}^\top)) \cong \mathbb{D}\left(\sum_{\mathbb{O}_{\alpha+1}} \mathbb{W}^\alpha_\beta\right) \cong \sum_{\mathbb{O}_{\alpha+1}} \mathbb{D}(\mathbb{W}^\alpha_\beta)$$
$$\cong \sum_{\mathbb{O}_{\alpha+1}} \mathbb{W}^\alpha_\beta \cong \mathbb{D}_\alpha(\mathbb{Z}^\top),$$

i.e., the d-rank of the space $\mathbb{Z}^\top$ is equal to α. $\square$

THEOREM 3.5. For any nonlimit ordinal α, there exists an α-special T_0-space.

The **proof** is by induction on α. For $\alpha \in \{0, 1\}$, the required statement follows from Prop. 3.4(i), (ii). Suppose that $\alpha = \gamma + 1$ and that the statement of the theorem is valid for any nonlimit ordinal $\beta \leqslant \gamma$. There are two cases to consider:

Case 1. Let γ be a limit ordinal. In view of the induction hypothesis, there exists a $(\beta+1)$-special space $\mathbb{Y}_\beta$ for any ordinal $\beta < \gamma$. By Proposition 3.4(v), the space $\sum\limits_{\mathbb{O}_\gamma} \mathbb{Y}_\beta$ is $(\gamma + 1)$-special.

Case 2. Let γ not be a limit ordinal. In view of the induction hypothesis, there exists a γ-special space $\mathbb{Y}$. By Proposition 3.4(v), the space $\sum\limits_{\mathbb{O}_\omega} \mathbb{Y}_n$, where $\mathbb{Y}_n = \mathbb{Y}$ for any $n < \omega$, is $(\gamma+1)$-special. $\square$

THEOREM 3.6. For any ordinal α, there exists an irreducible T_0-space whose d-rank is equal to α.

Proof. If α is a nonlimit ordinal, then the statement of the theorem follows from Theorem 3.5. If α is a limit ordinal, then the statement of the theorem follows from Theorem 3.5 and Prop. 3.4(v). $\square$

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