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ALGEBRAICALLY EQUIVALENT CLONES

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Two functional clones F and G on a set A are said to be algebraically equivalent if sets of solutions for F- and G-equations coincide on A. It is proved that pairwise algebraically nonequivalent existentially additive clones on finite sets A are finite in number. We come up with results on the structure of algebraic equivalence classes, including an equationally additive clone, in the lattices of all clones on finite sets.

The notion of an algebraic set of a universal algebra $\mathfrak{A} = \langle A; \sigma \rangle$ ultimately depends not on its signature σ but on the clone of its term functions. Every functional clone F on a set A is the clone of term functions in an algebra $\mathfrak{A}_F = \langle A; F \rangle$ the functions in the signature of which are all functions in F. In [1], therefore, we proposed to treat algebraic sets for any functional clones on A and algebraic equivalence relations on clones as calques of geometric equivalence relations on universal algebras with identical universes. For any functional clone F on a set A, a subset $B \subseteq A^n$ is said to be *n-dimensional F-algebraic* if some system of function pairs $\{f_i^1(x_1, \ldots, x_n), f_i^2(x_1, \ldots, x_n) \mid i \in I\}$ in F satisfies the equality

$$B = \{\bar{b} \in A^n \mid f_i^1(\bar{b}) = f_i^2(\bar{b}), i \in I\}.$$

Denote by $\mathrm{Alg}_n F$ a collection (which is a complete lattice with respect to a set-theoretic relation $\subseteq$) of all n-dimensional F-algebraic subsets of A, and by $\mathrm{Alg}\, F$ a sequence $\langle \mathrm{Alg}_1 F, \ldots, \mathrm{Alg}_n F, \ldots \rangle$ which is defined below as *algebraic geometry of a clone F*. Two clones F_1 and F_2 on a set A are said to be *algebraically equivalent* (written $F_1 \sim_{\mathrm{alg}} F_2$) if $\mathrm{Alg}\, F_1 = \mathrm{Alg}\, F_2$. (In the language of algebraic geometry of universal algebras, this corresponds to geometric equivalence of universal algebras $\mathfrak{A}_{F_1} = \langle A; F_1 \rangle$ and $\mathfrak{A}_{F_2} = \langle A; F_2 \rangle$, corrected for conjugation of sequences $\mathrm{Alg}\, \mathfrak{A}_{F_1}$ and $\mathrm{Alg}\, \mathfrak{A}_{F_2}$ by some bijection of the set A onto A.)

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In [1], it was noted that on any set A consisting of at least three elements, there exist mutually different algebraically equivalent clones, and the question was posed whether such clones exist on a two-element set. In fact, a similar statement holds also for a two-element set A. To prove this statement, we need some concepts of universal algebra.

First we recall the concepts of a positive-conditional term for an algebra $\mathfrak{A} = \langle A; \sigma \rangle$ and of a corresponding positive-conditional term function on A created in [2]. A *positive-conditional term for* $\mathfrak{A} = \langle A; \sigma \rangle$ is a formula of the form

$$\bigwedge_{i=1}^{k} \Phi_i(x_1, \dots, x_n) \to y = t_i(x_1, \dots, x_n), \qquad (*)$$

where $\Phi_i(x_1, \dots, x_n)$ is some finite conjunction of term equations of a signature σ, and $t_i(x_1, \dots, x_n)$ are terms of the same signature if, moreover, the following formulas are true on $\mathfrak{A}$:

$$\forall x_1, \dots, x_n \bigvee_{i=1}^{k} \Phi_i(x_1, \dots, x_n)$$

$$\forall x_1, \dots, x_n (\Phi_i(x_1, \dots, x_n) \wedge \Phi_j(x_1, \dots, x_n) \to t_i(x_1, \dots, x_n) = t_j(x_1, \dots, x_n))$$

for any $1 \leq i, j \leq k$. A function $y = f(x_1, \dots, x_n)$ on a set A is called a *positive-conditional term function for* $\mathfrak{A}$ if it is defined on $\mathfrak{A}$ by some positive-conditional term; i.e., the equality $f(a_1, \dots, a_n) = t_i(a_1, \dots, a_n)$ holds for any $a_1, \dots, a_n \in A$ with $\mathfrak{A} \models \Phi_i(a_1, \dots, a_n)$.

For any algebra $\mathfrak{A} = \langle A; \sigma \rangle$, by $PCT(\mathfrak{A})$ we denote the collection (clone) of all positive-conditional term functions on A for $\mathfrak{A}$. Correspondingly, for a functional clone F on A, by $PCT(F)$ we denote the clone $PCT(\mathfrak{A}_F)$. An operator $PCT : F \to PCT(F)$ is a closure operator on a lattice L_A of all functional clones on a set A. In other words, for any clones F_1 and F_2 on A, the following hold:

(1) $F_1 \subseteq PCT(F_1)$,

(2) $PCT(PCT(F_1)) = PCT(F_1)$,

(3) if $F_1 \subseteq F_2$, then $PCT(F_1) \subseteq PCT(F_2)$.

Recall the definition of a *discriminator function* $d(x, y, z)$ on a set A: for $a, b, c \in A$,

$$d(a, b, c) = \begin{cases} c & \text{if } a = b, \\ a & \text{otherwise.} \end{cases}$$

A *normal transformation* on A is a function $n(x, y, z, u)$ defined by $n(x, y, z, u) = d(d(x, y, z), d(x, y, u), u)$. In this case, for any $a, b, c, d \in A$,

$$n(a, b, c, d) = \begin{cases} c & \text{if } a = b, \\ d & \text{otherwise.} \end{cases}$$

In view of the above, the equation $n(x, y, z, u) = z$ is equivalent to a disjunction of equations $x = y$ and $z = u$.

We call F a *discriminator* clone if the discriminator function d is contained in F. In [3], the algebra $\mathfrak{A} = \langle A; \sigma \rangle$ is defined as an *equational domain* if the collections $\mathrm{Alg}_n \mathfrak{A}$ (for any $n \in \omega$) are closed under unions of finite collections of their elements. There, also, numerous examples of equational domains are found among groups, rings, Lie algebras, and other classical algebras. A clone F on a set A is *equationally additive* if the algebra $\mathfrak{A}_F = \langle A; F \rangle$ is an equational domain. In view of what has been remarked in relation to the equation $n(x, y, z, u) = z$, for any normal transformation n on A, every discriminator clone F on a finite set A (finiteness is required for F-algebraic sets to be defined by a finite system of F-equations) will be equationally additive.

LEMMA 1. For every equationally additive clone F, $F \sim_{\mathrm{alg}} PCT(F)$.

Proof. Since $F \subseteq PCT(F)$,

$$\mathrm{Alg}_n F \subseteq \mathrm{Alg}_n PCT(F)$$

for any $n \in \omega$. The definition of positive-conditional terms implies that any $PCT(F)$-equation is equivalent to a finite disjunction of some systems of F-equations. Thus, for an equationally additive clone F, we have $\mathrm{Alg}_n PCT(F) \subseteq \mathrm{Alg}_n F$; i.e., the clones F and $PCT(F)$ are algebraically equivalent.

Denote by F_d a functional clone on a set $\{0, 1\}$ generated by the discriminator function d on $\{0, 1\}$. The clone F_d is equationally additive, and by virtue of Lemma 1, $F_d \sim_{\mathrm{alg}} PCT(F_d)$. In order to prove that the relation $\sim_{\mathrm{alg}}$ is nontrivial on clones on $\{0, 1\}$, it suffices to observe that $F_d \neq PCT(F_d)$.

Denote by $h(x, y, z)$ a function on $\{0, 1\}$ defined by the following positive-conditional term for $\mathfrak{A} = \langle \{0, 1\}; d \rangle$:

$$h(x, y, z) = \begin{cases} x = y \to x, \\ x = z \to x, \\ y = z \to y. \end{cases}$$

We show that $h \notin F_d$. First, note that $d(x, x, y) = y$, $d(x, y, x) = x$, and $d(y, x, x) = y$. By induction on the structural complexity of a term $t(x, y, z)$ in the signature consisting of the discriminator function $d(x, y, z)$, we prove that there exists an identification of variables x, y, z (x and y, or x and z, or y and z) such that the value of $t(x, y, z)$ will coincide with a third unidentifiable variable. On the other hand, by virtue of the definition of a function $h(x, y, z)$, its value will always coincide with the value of an identifiable variable. Hence $h \notin F_d$ and $F_d \neq PCT(F_d)$, whereas $PCT(F_d) \sim_{\mathrm{alg}} F_d$ by Lemma 1. Thus $\sim_{\mathrm{alg}}$ is not a trivial relation on the collection of all clones on a two-element set. In view of the above-mentioned nontriviality of $\sim_{\mathrm{alg}}$ on a collection of clones on any set consisting of at least three elements, we obtain

Assertion 1. On the collection of all clones on any nonsingleton set, the algebraic equivalence relation is nontrivial.

THEOREM 1 [4]. For any equational domains $\mathfrak{A}_0 = \langle A; \sigma_0 \rangle$ and $\mathfrak{A}_1 = \langle A; \sigma_1 \rangle$ with common universe A, the following conditions are equivalent:

(1) $\mathrm{Alg}\,\mathfrak{A}_0 = \mathrm{Alg}\,\mathfrak{A}_1$ and $\mathrm{Sub}\,\mathfrak{A}_0 = \mathrm{Sub}\,\mathfrak{A}_1$;

(2) $PCT(\mathfrak{A}_0) = PCT(\mathfrak{A}_1)$.

Here $\mathrm{Sub}\,\mathfrak{A}$ is the subalgebra lattice of $\mathfrak{A}$.

Recall that an *inner homomorphism* of an algebra $\mathfrak{A}$ is a homomorphism of some subalgebra of $\mathfrak{A}$ onto some subbialgebra of $\mathfrak{A}$.

A characterization of positive-conditional term functions on finite algebras is given in the following:

THEOREM 2 [2]. For every finite algebra $\mathfrak{A} = \langle A; \sigma \rangle$ and for an arbitrary function $f(x_1, \dots, x_n)$ on a set A, the following conditions are equivalent:

(a) f is a positive-conditional term function for $\mathfrak{A}$;

(b) subalgebras of $\mathfrak{A}$ are closed with respect to f and f commutes with all inner homomorphisms of $\mathfrak{A}$.

Note that subalgebras of $\mathfrak{A}$ can be identified with idempotents of a semigroup $Ihm\,\mathfrak{A}$ of inner homomorphisms of $\mathfrak{A}$.

Theorem 2 gives rise to

COROLLARY. For finite algebras $\mathfrak{A}_0 = \langle A; \sigma_0 \rangle$ and $\mathfrak{A}_1 = \langle A; \sigma_1 \rangle$ with common universe A, the conditions $PCT(\mathfrak{A}_1) = PCT(\mathfrak{A}_0)$ and $Ihm\,\mathfrak{A}_1 = Ihm\,\mathfrak{A}_0$ are equivalent.

Among the initial natural questions associated with the relation $\sim_{\mathrm{alg}}$ on the collection $\mathfrak{F}_A$ of all clones on the set A are questions on the cardinality of a factor set $|\mathfrak{F}_A/\sim_{\mathrm{alg}}|$ and on the structure of classes $F/\sim_{\mathrm{alg}}$ for $F \in \mathfrak{F}_A$ as subsets of the lattice L_A of all clones on A.

We start by proving the following:

THEOREM 3. For any finite set A, pairwise algebraically nonequivalent equationally additive clones on A are finite in number.

Proof. Let A be some finite n-element set, F_A be the collection of all equationally additive functional clones on A, and $F_1, F_2 \in F_A$. A relation $\sim$ on F_A is defined as follows: $F_1 \sim F_2$ iff $\mathrm{Alg}\,\mathfrak{A}_{F_1} = \mathrm{Alg}\,\mathfrak{A}_{F_2}$ and $\mathrm{Sub}\,\mathfrak{A}_{F_1} = \mathrm{Sub}\,\mathfrak{A}_{F_2}$. By virtue of Theorem 1, for any $F_1, F_2 \in F_A$, $F_1 \sim F_2$ implies $PCT(\mathfrak{A}_{F_1}) = PCT(\mathfrak{A}_{F_2})$, which in turn entails $Ihm\,\mathfrak{A}_{F_1} = Ihm\,\mathfrak{A}_{F_2}$ (in view of the corollary). Thus, since the number of semigroups of maps of subsets of A onto similar subsets (the number of potential semigroups of inner homomorphisms of algebras with universe A) is finite, the number of $\sim$-equivalence classes of the form $F/\sim_{\mathrm{alg}}$ for $F \in F_A$ is finite.

Algebraic equivalence on clones is rougher than $\sim$-equivalence; therefore, pairwise algebraically nonequivalent equationally additive clones on a finite set are also finite in number. Theorem 3 is proved.

Now we consider the question on the structure of classes of algebraically equivalent clones on A as subsets of the lattice L_A of all clones on A (w.r.t. $\subseteq$). Obviously, the collection $F/\sim_{\mathrm{alg}}$ is a convex set in L_A for any clone F on A.

In [1], on the collection $\mathfrak{F}_A$ of all clones on the set A, a metric d is naturally introduced by setting, for $F_1, F_2 \in F_A$,

$$d(F_1, F_2) = \begin{cases} \frac{1}{\min\{n \in \omega' \mid F_1^{(n)} \neq (F_2)^{(n)}\}} & \text{if } F_1 \neq F_2, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\omega' = \omega \setminus \{0\}$ and $F^{(n)}$ is the collection of all n-ary functions in F. In [1], also, it was noted that the operations $\wedge$ and $\vee$ of L_A are continuous on the space $\langle \mathfrak{F}_A; d \rangle$.

Again it is obvious that $F/\sim_{\text{alg}}$ are closed sets of the metric space $\langle \mathfrak{F}_A; d \rangle$.

The convexity of collections $F/\sim_{\text{alg}}$ in the lattice L_A allows us to dub the conjecture that $F/\sim_{\text{alg}}$ is an interval in L_A. However, the examples below show that this is not the case. Moreover, the collection $F/\sim_{\text{alg}}$ may not even be directed (neither upward nor downward) in L_A.

Example 1. Let be A an arbitrary set consisting of at least four elements and $B_1, B_2 \subseteq A$, with $|B_1|, |B_2| \geq 2$ and $B_1 \cap B_2 = \varnothing$. Let F_i be clones of all functions on A which either have values in the set B_i or are selectors on A. Then for any $n \in \omega$ we have $\text{Alg}_n F_i = P(A^n)$ (where $P(C)$ is the collection of all subsets of C). Therefore, $F_1 \sim_{\text{alg}} F_2$. At the same time, $F_1 \wedge F_2$ consists only of selector functions, and hence $\text{Alg}_n(F_1 \wedge F_2) \neq P(A^n)$; i.e., $F_1 \wedge F_2 \nsim_{\text{alg}} F_1 \sim_{\text{alg}} F_2$ and the class $F_i/\sim_{\text{alg}}$ is not a down-directed subset of the lattice L_A.

Example 2. Let A be as in Example 1 and $\{B_1, B_2, B_3\}$ be a partition of A into nonempty subsets, with $|B_3| \geq 2$. Let $b_i \in B_i$ for $i = 1, 3$. Unary functions g_1 and g_2 on A are defined as follows:

$$g_1(x) = \begin{cases} b_1 & \text{if } x \in B_1, \\ b_3 & \text{if } x \in B_2 \cup B_3, \end{cases} \qquad g_2(x) = \begin{cases} b_1 & \text{if } x \in B_1 \cup B_2, \\ b_3 & \text{if } x \in B_3. \end{cases}$$

Let F_i be functional clones on A generated by functions g_i. Since $g_i^2 = g_i$, the function $h(x_1, \ldots, x_n)$ is contained in F_i iff it is a selector function, or $h(x_1, \ldots, x_n) = g_i(x_j)$ for some $j \leq n$.

It is easy to see that $\text{Alg}_n F_1 = \text{Alg}_n F_2$ for any $n \in \omega$. Thus $F_1 \sim_{\text{alg}} F_2$. Moreover, $\text{Alg}_1 F_i = \{\varnothing, A, \{b_1, b_3\}\}$ (solution for $g_i(x) = x$).

At the same time, if F includes clones F_1 and F_2, then the set $B_1 \cup B_3$ (solution for $g_1(x) = g_2(x)$) is contained in $\text{Alg}_1 F$, and $F \nsim_{\text{alg}} F_1 \sim_{\text{alg}} F_2$. This means that the classes $F/\sim_{\text{alg}}$ are not generally up-directed subsets of the lattice L_A.

For the case where A is finite, however, for equationally additive clones F on a set A we can have some description of classes $F/\sim_{\text{alg}}$ as subsets of L_A.

In Lemma 1, it was mentioned that for the case of an equationally additive clone F, the clones F and $PCT(F)$ are algebraically equivalent. We show that such is not the case in general.

Example 3. Let $|A| = 9$ and $\{B_1, B_2, B_3\}$ be a partition of A such that $|B_1| = 2$, $|B_2| = 3$, and $|B_3| = 5$. Suppose that the function $g(x)$ is a 2-cycle on B_1, a 3-cycle on B_2, and a 5-cycle on B_3. Assume that F is a clone on A generated by the function g. Then $\text{Alg}_1 F = \{\varnothing, A, B_1, B_2, B_3\}$.

Let a $PCT(F)$-function on A be defined by the positive-conditional term

$$h(x) = \begin{cases} g^2(x) = x \to x, \\ g^3(x) = x \to x, \\ g^5(x) = x \to x^2. \end{cases}$$

Therefore, $B_1 \cup B_2$ is a solution for a $PCT(F)$-equation $h(x) = x$, and hence $\mathrm{Alg}_1 PCT(F) \neq \mathrm{Alg}_1 F$; i.e., $PCT(F) \nsim_{\mathrm{alg}} F$.

For any closure operator $g(x)$ on the lattice L, a convex subset B of L is called an *upper semi-interval generated by the operator* g if $g(b) = b$ and $B = \{c \in L : g(x) = b\}$ for some $b \in B$. Obviously, $b = \sup B$ in this case.

THEOREM 4. For any finite set A and for an arbitrary equationally additive clone F on A, the class $F/\sim_{\mathrm{alg}}$ is a union of finitely many upper semi-intervals generated by the closure operator PCT on the lattice L_A.

Proof. As noted, for every equationally additive clone F in $\mathfrak{F}_A$, the class $F/\sim_{\mathrm{alg}}$, along with any clone F_1 contained in it, includes the entire interval $[F_1, PCT(F_1)]$ of L_A (since $\mathrm{Alg}\, PCT(F_1) = \mathrm{Alg}\, F_1$) and, hence, the entire upper semi-interval generated by the operator PCT with the greatest element $PCT(F_1)$. It remains to note (as we did in the proof of Theorem 3) that the number of such upper semi-intervals is finite, or, which is the same, the number of PCT-closed clones on A (i.e., F in $\mathfrak{F}_A$ such that $PCT(F) = F$) is finite.

By Theorem 1, the number of different upper semi-intervals relative to the operator PCT that are contained in the class $F/\sim_{\mathrm{alg}}$ does not exceed $2^{2^{|A|}}$ for an equationally additive clone F on A.

In conclusion, once again we point out the still open question as to the cardinality of sets $\mathfrak{F}_A/\sim_{\mathrm{alg}}$, for finite A included.

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