DOI 10.1007/s10469-017-9419-8
*Algebra and Logic*, Vol. 55, No. 6, January, 2017
(Russian Original Vol. 55, No. 6, November-December, 2016)

# SPECTRUM OF THE FIELD OF COMPUTABLE REAL NUMBERS

**M. V. Korovina[1*] and O. V. Kudinov[2*]**

UDC 510.65

Keywords: *field of computable real numbers, Turing degree.*

*Necessary and sufficient conditions for a Turing degree to be an element of the spectrum of the classical field of computable real numbers are established.*

Within the framework of computable analysis, we investigate properties of computable elements employing numbering theory [1] and computable model theory [2, 3]—namely, the spectrum of the ordered field of computable real numbers.

The results in [4-6] show that many topological spaces, for instance, real numbers and complete computable metric spaces, have no computable numberings of computable elements. This gives rise to the natural problem of describing all Turing degrees $\mathbf{a}$ such that the structure of computable elements of a topological space admits an $\mathbf{a}$-computable presentation, i.e., a presentation in which basic relations and functions will be $\mathbf{a}$-computable. In other words, the question on characterization of the spectrum of this structure arises naturally.

We look into this question for the ordered field of computable real numbers $\mathbb{R}_c = \langle \mathbb{R}_c, +, \cdot, 0, 1, \leq \rangle$. That $\mathbf{0} \notin \mathrm{Spec}\,(\mathbb{R}_c)$ was well known for a long time [4, 5], yet a full characterization of $\mathrm{Spec}\,(\mathbb{R}_c)$ remained an open problem. We will prove that $\mathbf{a} \in \mathrm{Spec}\,(\mathbb{R}_c)$ iff $\mathbf{a}' \geq \mathbf{0}''$. This class of degrees was explored in [7], where it was shown that it is the spectrum of some computable structures.

*Supported by RFBR (project No. 14-01-00376) and by DFG/RFBR joint grant CAVER BE 1267/14-1 and 14-01-91334.

[1]Institute of Informatics Systems, Siberian Branch, Russian Academy of Sciences, pr. Akad. Lavrent'eva 6, Novosibirsk, 630090 Russia; rita.korovina@gmail.com. [2]Sobolev Institute of Mathematics, pr. Akad. Koptyuga 4, Novosibirsk, 630090 Russia. Novosibirsk State University, ul. Pirogova 2, Novosibirsk, 630090 Russia; kud@math.nsc.ru. Translated from *Algebra i Logika*, Vol. 55, No. 6, pp. 738-759, November-December, 2016. Original article submitted March 13, 2015; revised May 14, 2016.

We proceed to definitions, notation, and auxiliary results. The reader is assumed to be familiar with the basics of numbering theory [1], recursion theory [8], and computable analysis [9].

**Definition 1.** A real number $x \in \mathbb{R}$ is said to be *computable* if one of the following equivalent conditions holds:

(1) there exists a computable sequence of rational numbers $\{q_n\}_{n\in\omega}$ such that $|q_{i+1} - q_i| < \frac{1}{2^i}$ for all $i \in \omega$ and $\lim\limits_{i\to\infty} q_i = x$;

(2) the left and right Dedekind cuts, i.e., $x^- = \{q \in \mathbb{Q} \mid q < x\}$ and $x^+ = \{q \in \mathbb{Q} \mid q > x\}$, are computably enumerable.

Put $\mathbb{R}_c = \langle \mathbb{R}_c, +, \cdot, 0, 1, \leq \rangle$. That $\mathbb{R}_c$ is a field was first noted in [10]. Moreover, $\mathbb{R}_c$ is an Archimedean real-closed field.

**Definition 2.** Let $\mathbf{a}$ be a Turing degree. We say that a structure $\mathcal{A} = \langle A, \sigma \rangle$ admits an **a**-*computable presentation* if there is a numbering $\nu : \omega \to A$ such that relations and operations in the signature, including equality, are **a**-computable (computable relative to some oracle in **a**) with respect to the numbering $\nu$. A pair $(\mathcal{A}, \nu)$ is called an **a**-*computable structure*, and we refer to the numbering $\nu$ as its **a**-*computable presentation* (**a**-*constructivization*).

**Definition 3.** The *spectrum* of a countable structure $\mathcal{A}$, denoted $\operatorname{Spec}(\mathcal{A})$, is the collection of all Turing degrees **a** such that $\mathcal{A}$ admits some **a**-computable presentation.

If $F$ and $G$ are functions from $\omega$ to $\omega$, then we say that $F$ *dominates* $G$ if $F(n) > G(n)$ for all sufficiently large $n$. A function from $\omega$ to $\omega$ is called a *dominant* if it dominates every computable function from $\omega$ to $\omega$. The following characterization of Turing degrees of a dominant is well known.

**PROPOSITION 1** [11]. For any Turing degree $\mathbf{a}$, $\mathbf{a}' \geq \mathbf{0}''$ if and only if there exists a dominant of degree less than or equal to $\mathbf{a}$.

As usual, we denote the family of all total computable functions from $\omega$ to $\omega$ by $S_{\mathrm{tot}}$, and its index set by Tot [12]. Let $\varphi_n$ denote the $n$th partial computable function in the Kleene numbering and $\varphi_n^s(x)$ be the result of the computation of $\varphi_n(x)$ in no more than $s$ steps. We say that $S_{\mathrm{tot}}$ *admits* an **a**-*computable numbering* if there exists a numbering $\nu : \omega \to S_{\mathrm{tot}}$ which is **a**-computable as a function of two variables.

By analogy with Definition 3, we define the spectrum of the family $S_{\mathrm{tot}}$, i.e., $\operatorname{Spec}(S_{\mathrm{tot}})$ is the collection of all Turing degrees **a** such that $S_{\mathrm{tot}}$ admits some **a**-computable numbering.

**PROPOSITION 2.** If $\mathbf{a} \in \operatorname{Spec}(S_{\mathrm{tot}})$, then $\mathbf{a}' \geq \mathbf{0}''$.

**Proof.** Let $\nu_{\mathrm{tot}} : \omega \to S_{\mathrm{tot}}$ be an appropriate numbering. It is not hard to see that the **a**-computable function

$$F(n) = \max_{i \leq n} \nu(i)(n) + 1$$

is a dominant. From Proposition 1, it follows that $\mathbf{a}' \geq \mathbf{0}''$. □

**PROPOSITION 3.** If $\mathbf{a} \in \operatorname{Spec}(\mathbb{R}_c)$, then $\mathbf{a} \in \operatorname{Spec}(S_{\mathrm{tot}})$.

**Proof.** We define a function $\langle\cdot\rangle : \mathbb{R} \to (0,1]$ as

$$\langle x\rangle = \begin{cases} \{x\}, & x \notin \mathbb{Z}, \\ 1, & x \in \mathbb{Z}, \end{cases}$$

where $\{x\}$ is the fractional part of $x$.

Define a numbering $\hat{\nu}$ of the set

$$\hat{S} = \{f \in S_{\mathrm{tot}} \mid \mathrm{im}(f) \subseteq \{0,1\} \text{ and } \exists^\infty i f(i) = 1\}$$

by the following rule:

$$\hat{\nu}(n)(i) = c_i, \text{ where } \langle(\nu(n))\rangle = \sum_{k=1}^{i} \frac{c_k}{2^k} + y \text{ and } 0 < y \leq \frac{1}{2^i}.$$

Put

$$\lambda(n)(0) = 0;$$
$$\lambda(n)(s+1) = \mu\, i\, [i > \lambda(n)(s) \wedge \hat{\nu}(n)(i) = 1].$$

Let $\nu_{\mathrm{tot}}(n)(i) = \lambda(n)(i+1) - \lambda(n)(i) - 1$. By construction, the function $\nu_{\mathrm{tot}}$ is the required **a**-computable numbering of $S_{\mathrm{tot}}$. $\square$

**THEOREM 1.** If $\mathbf{a} \in \mathrm{Spec}\,(\mathbb{R}_c)$, then $\mathbf{a}' \geq \mathbf{0}''$.

The **proof** follows from Prop. 3. $\square$

Below, for $A \subseteq \omega$, we use the notation $A^{[n]} = \{k \mid c(n,k) \in A\}$ and write $A(x) = 1$ if $\chi_A(x) = 1$. A refinement of Jockusch's theorem [13] is

**THEOREM 2.** The following assertions are equivalent:

(1) $\mathbf{a}' \geq \mathbf{0}''$;

(2) $\mathbf{a} \in \mathrm{Spec}\,(S_{\mathrm{tot}})$;

(3) there exist an **a**-computable numbering $\nu_{\mathrm{tot}} : \omega \to S_{\mathrm{tot}}$ and a computable surjection $h : \omega \to \omega$ such that:

(3a) $\forall \varphi_n \in S_{\mathrm{tot}})\, \exists t (h(t) = n \wedge \nu_{\mathrm{tot}}(t) = \varphi_n)$;

(3b) $\varphi_n \notin S_{\mathrm{tot}} \to \forall t (h(t) = n \to \exists N (\forall s \geq N)\, \nu_{\mathrm{tot}}(t)(s) = 0)$;

(3c) $(\varphi_n \in S_{\mathrm{tot}} \wedge h(t) = n \wedge \nu_{\mathrm{tot}}(t) \neq \varphi_n) \to \exists N (\forall s \geq N)\, \nu_{\mathrm{tot}}(t)(s) = 0$.

**Proof.** (1) $\to$ (3) Assume $\mathbf{a}' \geq \mathbf{0}''$. By Proposition 1, there exists an **a**-computable dominant $F : \omega \to \omega$. Put

$$A^{[n]}(k) = 1 \leftrightarrow (\exists s \leq F(k))(\forall l \leq k)\, \varphi_n^s(l)\downarrow \wedge\, \varphi_n^s(l) \leq F(k).$$

The set $A$ has the following properties:

if $n \in \mathrm{Tot}$ then $A^{[n]} =^* \omega$, i.e., $\omega \setminus A^{[n]}$ is finite;

if $n \notin \mathrm{Tot}$ then $A^{[n]} =^* \varnothing$, i.e., $A^{[n]}$ is finite;

$A$ is **a**-computable.

Indeed, if $\varphi_n$ is total then functions $h_n(x) = \max\limits_{l \le x} \varphi_n(l)$ and $s_n(x) = \min\{s \mid \forall(l \le x)\varphi_n^s(l)\downarrow\}$ are also total and computable. Therefore, $F$ dominates the functions, i.e., $\exists N(\forall k \ge N)F(k) \ge h_n(k) \wedge F(k) \ge s_n(k)$. By definition, $A^{[n]}(k) = 1$ for all sufficiently large $k$, i.e., the first property holds. Assume $\varphi_n$ is not total and $N$ is the maximal number such that $[0, \dots, N] \subseteq \mathrm{dom}(\varphi_n)$. Then $A^{[n]}(k) = 0$ for all $k > N$, i.e., the second property holds. The third property follows from the definitions.

Now we construct $\nu_{\mathrm{tot}}(t)$ for $t = c(n, k)$. First we define the relation

$$P(t, m, z) \leftrightharpoons (z = 0 \wedge (\exists l \ge k) A^{[n]}(l) = 0) \vee (z = \varphi_n(m) \\ \wedge \exists s(\forall i \le m)\, \varphi_n^s(i)\downarrow \wedge \forall k(k < l \le m \to A^{[n]}(l) = 1)).$$

We have $P \in \Sigma_1^0[\mathbf{a}]$ and $\mathrm{dom}(P) = \{(t, m) \mid \exists z P(t, m, z)\} = \omega^2$. Therefore, by uniformization, there exists a total **a**-computable function $g : \omega^2 \to \omega$ such that:

$g_t(m) \in \{0, \varphi_n(m)\}$;

if $g_t(m) \ne \varphi_n(m)$ then $(\exists l \ge k)\, A^{[n]}(l) = 0$;

if $g_t(m) = \varphi_n(m)$ then $(\exists s)(\forall i \le m)\, \varphi_n^s(i)\downarrow$;

if $(\exists l \ge k)\, A^{[n]}(l) = 0$ then $(\exists N)(\forall m \ge N)\, g_t(m) = 0$;

if $\varphi_n$ is total and $(\forall l \ge k)\, A^{[n]}(l) = 1$, then $g_t = \varphi_n$, where $g_t(m) = g(t, m)$.

Put $\nu_{\mathrm{tot}}(t) = g_t$ and $h(t) = l(t)$. These, in view of the above properties, are the required functions. Indeed, if $\varphi \in S_{\mathrm{tot}}$ then, by the construction of $A$, there exists $k$ such that $(\forall l \ge k)\, A^{[n]}(l) = 1$. Put $t = c(n, k)$; then we have $\nu_{\mathrm{tot}}(t) = \varphi_n$ and $h(t) = n$. Verification of (3a) is completed. Properties (3b) and (3c) are straightforward.

(3) $\to$ (2) Follows from the definition of Spec.

(2) $\to$ (1) Follows from Prop. 2. $\square$

Before describing basic constructions, we state a proposition, which is interesting in its own right.

**PROPOSITION 4.** Suppose $\mathfrak{L} = \langle L, \le, c_{i|i \in \omega} \rangle$ is a linear order with distinguished constants and $\nu : \omega \to L$ is a numbering such that:

(1) $C = \{c_i^{\mathfrak{L}} \mid i \in \omega\}$ is dense in $L$, with $c_i^{\mathfrak{L}} \ne c_j^{\mathfrak{L}}$ for $i \ne j$;

(2) there exists a computable function $f : \omega \to \omega$ such that $\nu(f(i)) = c_i^{\mathfrak{L}}$;

(3) $\nu^{-1}(<) \in \Sigma_1^0$;

(4) $\nu^{-1}(C) \in \Delta_2^0$.

Then there exists a numbering $\mu : \omega \to L$ such that $(\mathfrak{L}, \mu)$ is a computable structure.

**Proof.** Let $E = \nu^{-1}(C)$. By the limit lemma [12], there exists a strongly computable sequence of finite sets $E_0, E_1, E_2, \dots$ such that $E = \lim\limits_{s \to \infty} E_s$. We construct a required $\mu$ employing the finite injury method.

At every stage $s \in \omega$, we construct a finite function $\mu^s : \{0, \dots, 2s - 1\} \to \omega$ and finite sets $C_s \subset \mathrm{im}(f)$ and $B_s \subset \omega$ such that:

$\nu(\mu^s(2i)) = \nu(f(i))$ for $i < s$;

$\nu \circ \mu^s$ restricted to $\{i \mid i \leq 2s-1$ and $i$ is odd$\}$ is injective;

$B_s = f^{-1}(C_s)$.

Stage 0. Let $\mu^0 = \perp$, $C_0 = \varnothing$, and $B_0 = \varnothing$.

Before embarking on the next stages, we note that for every $n \leq s+1$ there exists $t \geq s+1$ such that one of the following properties holds:

(a) $n \in E_t$;

(b) at a stage no higher than $t$, we computed $\nu(n) \neq \nu(i)$ for all $\nu(i)|_{i \in C_s}$.

For every $n \leq s+1$, we denote the least $t$ satisfying (a) or (b) by $t(n,s)$. If, for $t(n,s)$, property (b) holds then $n$ is called a *b-number* If, for $t(n,s)$, property (a) holds and property (b) does not, then $n$ is called an *a-number*. We say that an $a$-number $n$ is *active at stage* $s+1$ if there exists $m_n$ for which $n = \mu^s(m_n)$ and $n \notin C_s$. We say that a number $n \leq s+1$ *requires attention at stage* $s+1$ if one of the following cases holds:

(c) $n$ is an $a$-number which is active at stage $s+1$;

(d) $n$ is a $b$-number such that:

$n \notin \operatorname{im}(\mu^s)$;

at stage $\leq s+1$, we computed that $\nu(n) \neq \nu(l)$ for all $l < n$.

Stage $s+1$. We search for a least number $n \leq s+1$ that requires attention at stage $s+1$. If there is no such $n$ then we proceed to step $A$. If there is such $n$ and $n$ is an $a$-number then we proceed to step $B$. If there is such $n$ and $n$ is a $b$-number then we proceed to step $C$.

Step $A$. Using the density of $C$ in $L$, we search for a least $i_0$ such that $f(i_0) \notin C_s$ and $f(i_0) \neq f(s)$. Put

$$\mu^{s+1}(2i+1) = \begin{cases} \mu^s(2i+1), & i < s, \\ f(i_0), & i = s, \end{cases}$$

$$C_{s+1} = C_s \cup \{f(i_0)\} \cup \{f(s)\}, \; B_{s+1} = B_s \cup \{i_0\} \cup \{s\}.$$

Proceed to the next stage.

To describe the other parts of stage $s+1$, we associate $\mu^s$ with a structure $\mathfrak{L}_s = \langle L_s, \sigma_s \rangle$, where $L_s = \nu(\operatorname{im}(\mu^s))$ and $\sigma_s = (<, \{c_i\}_{i \in B_s}, \{d_i\}_{i \leq s})$. Its partial positive atomic diagram $D_s$ is defined as follows:

—the interpretation of constants: $c_i^{L_s} = \nu(f(i))$ and $d_i^{L_s} = \nu(\mu^s(2i+1))$;

—$D_s$ contains all the results obtained under the replacement in the formula $x < y$ of variables $x, y$ by constants from $\sigma_s$ whose truth in $\mathfrak{L}_s$ was computed at stage $\leq s+1$.

Step $B$. Since $n$ is active at stage $s+1$, using the density of $C$ in $L$, we search for a least $j$ such that $j \neq s$ and $\langle L, \sigma_s \rangle \models D_s$, where interpretations of constants are as follows:

$$c_i^L = \nu(f(i)) \text{ and } d_i^L = \begin{cases} \nu(\mu^s(2i+1)), & i \neq m_n, \\ \nu(f(j)), & i = m_n. \end{cases}$$

Then we search for a least $i_0$ such that $i_0 \neq j$, $i_0 \neq s$, and $f(i_0) \notin C_s$. Put

$$\mu^{s+1}(2i+1) = \begin{cases} \mu^s(2i+1), & i \neq m_n,\ i < s, \\ f(j), & i = m_n, \\ f(i_0), & i = s, \end{cases}$$

$$C_{s+1} = C_s \cup \{f(j)\} \cup \{f(i_0)\} \cup \{f(s)\},\ B_{s+1} = B_s \cup \{j\} \cup \{i_0\} \cup \{s\}.$$

Proceed to the next stage.

Step $C$. Let

$$F_s = \{m \mid m \in \mathrm{dom}(\mu^s),\ f(m) \notin C_s,\ \mu^s(m) > n,\ m \text{ is odd}\}.$$

Using the density of $C$ in $L$, we construct a finite injection $j : F_s \to \omega$ such that $j(m) = j_m$, $f(j_m) \notin C_s$, $\nu(f(j_m)) \neq \nu(n)$, for all $m \in F_s$, and $\langle L, \sigma_s \rangle \models D_s$, where interpretations of constants are as follows:

$$c_i^L = \nu(f(i)) \text{ and } d_i^L = \begin{cases} \nu(\mu^s(2i+1)), & 2i+1 \notin F_s, \\ \nu(f(j_m)), & 2i+1 \in F_s. \end{cases}$$

Define

$$\mu^{s+1}(2i+1) = \begin{cases} \mu^s(2i+1), & 2i+1 \notin F_s,\ i < s, \\ f(j(2i+1)), & 2i+1 \in F_s, \\ n, & i = s, \end{cases}$$

$$C_{s+1} = C_s \cup \{f(j_m) \mid m \in F_s\} \cup \{f(s)\},\ B_{s+1} = B_s \cup j(F_s) \cup \{s\}.$$

Proceed to the next stage. The construction is completed.

It is clear that $\mu^s$ meets the inductive requirements. Let

$$\mu(m) = \nu(\lim_{s\to\infty} \mu^s(m)).$$

The following lemmas establish properties of $\mu$.

**LEMMA 1.** (a) Along the construction process, the value $\mu^s(m)$ can be changed only finitely many times for all $m$.

(b) For all $m$, if there exist $s$ and $j$ such that $\mu^s(m) = f(j)$ and $f(j) \in C_s$, then $\mu^{s'}(m) = f(j)$ for all $s' \geq s$.

(c) If, for some $m$, we have $\mu^s(m) = n$ and $\mu^{s+1}(m) \neq n$, then $n \notin \mathrm{im}(\mu^{s+1})$.

(d) For all $n$, there exists $s_0$ such that $(\forall s \geq s_0)(n \in \mathrm{im}(\mu^{s_0}) \leftrightarrow n \in \mathrm{im}(\mu^s))$.

**Proof.** Claims (a)-(c) are straightforward from the construction.

(d) Assume $n \in \mathrm{im}(\mu^s) \setminus \mathrm{im}(\mu^{s+1})$. This means that at every stage $s+1$, either there exists $l < n$ such that $l \notin \mathrm{im}(\mu^s)$ or there exists an $a$-number $k < n$ which is active at stage $s+1$. By inductive reasoning, it is easy to see that after some stage $s_0$ both of the cases will not hold. $\square$

**LEMMA 2.** The function $\mu(m) = \nu(\lim\limits_{s\to\infty} \mu^s(m))$ is well defined and it is a numbering of $L$.

**Proof.** The well-definedness follows from Lemma 1. We show that $\mathrm{im}(\mu) = L$. Assume $n$ is a least number such that $\nu(n) \notin \mathrm{im}(\mu)$. In particular, $n \notin \mathrm{im}(\lim\limits_{s\to\infty} \mu_s)$, $\nu(n) \notin C$, and $n \notin E$. In view of Lemma 1 and the description of step $C$, it follows by assumption that after every stage $s_1$ such that $n \notin \mathrm{im}(\mu^{s_1})$ there exists a stage $s_2 > s_1$ such that $n \in \mathrm{im}(\mu^{s_2})$. Therefore, there exists $s_0$ such that, for all $s \geq s_0$, $n \in \mathrm{im}(\mu^s) \wedge n \notin E_s$, all $a$-numbers $k < n$ are nonactive at stage $s+1$, and all $b$-numbers $z < n$ are already in $\mathrm{im}(\mu^s)$. For some $m$, the value $\mu^s(m) = n$ will not be changed at stages $s \geq s_0$, i.e., $n = \lim\limits_{s\to\infty} \mu^s(m)$, which contradicts the assumption. □

**LEMMA 3.** $(\mathfrak{L}, \mu)$ is a computable structure.

**Proof.** Since $(\mathfrak{L}, \sigma_{s_0}) \models D_{s_0}$ implies $(\mathfrak{L}, \sigma_{s_0+1}) \models D_{s_0}$ under constant interpretation changes according to the construction, it is clear that the truth value of $\nu(\mu^s(m)) < \nu(\mu^s(l))$ is preserved at all stages $s \geq s_0$. Therefore, $\mu^{-1}(<)$ is a computably enumerable relation. Since $\nu(\mu^s(2i+1)) \neq \nu(\mu^s(2j+1))$ for $i \neq j$, it is sufficient to construct an algorithm which, given $i$ and $j$, decides whether $\nu(\mu(2i+1)) = c_j$. Using the construction, we can effectively find $s > i$ such that $f(j) \in C_s$. Thus $\mu(2i+1) = c_j$ iff $\mu^{s+1}(2i+1) = f(j)$. Therefore, the equality is computable with respect to $\mu$. The lemma is completed proving the proposition. □

It is worth observing that Proposition 4 can be relativized to any oracle $a$. We show that for $(\mathbb{R}_c \cap [-1,1], \leq, c_{r|r\in \mathrm{Dyadic}\cap[-1,1]})$ and $\mathbf{a}' \geq \mathbf{0}''$, there exists a numbering $\nu$ satisfying the conditions of the relativized version of Prop. 4 (here Dyadic denotes all dyadic numbers). Using the function

$$H(n) = \begin{cases} 0, & n = 0, \\ 1 & n \text{ is even}, \\ -1, & n \text{ is odd}, \end{cases}$$

and the numbering $\nu_{\mathrm{tot}}$ in Theorem 2, we define

$$\nu(n) = \sum_{k\in\omega} b_k 2^{-(k+1)}, \text{ where } b_k = H(\nu_{\mathrm{tot}}(n)(k)).$$

It is easy to see that $D = \{c_i^{\mathbb{R}_c} \in [-1,1]\}_{i\in\omega}$ is dense in $\mathbb{R}_c \cap [-1,1]$ and $\nu^{-1}(<) \in \Sigma_1^0[\mathbf{a}]$. We show that $\nu^{-1}(D) \in \Delta_2^0[\mathbf{a}]$. Let $x = \nu(m)$ be associated with $\nu_{\mathrm{tot}}(m)$. Using $h$ such as in Theorem 2, we write the condition that $m \in \nu^{-1}(D)$ in an equivalent form for $n = h(m)$:

$$(\varphi_n \in S_{\mathrm{tot}} \wedge ((\exists N)(\forall s \geq N)\, \varphi_n(s) \text{ is even} \vee (\exists N)(\forall s \geq N)\, \varphi_n(s) \text{ is odd}$$
$$\vee (\exists N)(\forall s \geq N)\, \varphi_n(s) = 0 \vee \nu_{\mathrm{tot}}(m) \neq \varphi_n)) \vee \varphi_n \notin S_{\mathrm{tot}}.$$

The properties of $\nu_{\mathrm{tot}}$ (see Thm. 2) and the $\Pi_2^0$-completeness of Tot imply that $\nu^{-1}(D)$ is $\mathbf{a}'$-computable, i.e., $\nu^{-1}(D) \in \Delta_2^0[\mathbf{a}]$. This entails the following:

**PROPOSITION 5.** If $\mathbf{a}' \geq \mathbf{0}''$ then there exists a numbering $\mu : \omega \to \mathbb{R}_c$ such that $(\langle \mathbb{R}_c, \leq, c_{r|r\in \mathrm{Diad}} \rangle, \mu)$ is an $\mathbf{a}$-computable structure.

The **proof** follows from Proposition 4 and the argument above. □

Proposition 5 highlights fundamental ideas and steps in the proof of Theorem 4, which is the main result of the present paper. In order to establish Theorem 4, we prove the following statements. The real closure of an ordered field $(F, \leq)$ will be denoted by $[F]_{rcl}$. First note that the field $[\mathbb{Q}]_{rcl}$ admits a computable presentation [2] and can be embedded in any real-closed field [14].

Let $([\mathbb{Q}]_{rcl}, \nu_0)$ be a computable structure. In the lemmas and proposition that follow, we assume that $\mathcal{F} = \langle F, \sigma_c \rangle$ is an Archimedean countable real-closed field with signature $\sigma_c = \{\leq, +, \cdot, 0, 1, c_{i|i\in\omega}\}$. Suppose that $\nu : \omega \to F$ is a numbering and an intermediate subfield $\mathbb{Q} \leq C \leq [\mathbb{Q}]_{rcl}$ admits a computable presentation $\nu_c$ for which $c_i^{\mathcal{F}} = \nu_c(i)$. Since the field $[\mathbb{Q}]_{rcl}$ is rigid and computably categorical, we have $\nu_c \leq \nu_0$; i.e., there exists a computable function $f : \omega \to \omega$ such that $\nu_c = \nu_0 \circ f$. Put

$$AD_\nu = \{\bar{n} \in \omega^{<\omega} \mid \nu(\bar{n}) \text{ is a tuple of algebraically dependent elements}\}.$$

**LEMMA 4.** (1) Let $\{p_i(x) > 0 \mid i \leq k\}$ be a finite set of formulas where $p_i \in F[x]$ and $F[x]$ is the ring of polynomials in one variable. The following statements hold:

(1.1) if there exists $a \in F$ such that

$$F \models \bigwedge_{i=0}^{k} p_i(a) > 0,$$

then there exists an Archimedean ordering of $F(x)$ such that

$$F(x) \models \bigwedge_{i=0}^{k} p_i(x) > 0;$$

(1.2) if there exists $a \in F$ such that

$$F \models \bigwedge_{i=0}^{k} p_i(a) > 0,$$

then there exists $b \in \mathbb{Q}$ such that

$$F \models \bigwedge_{i=0}^{k} p_i(b) > 0.$$

(2) Let $\{p_i(\bar{x}) > 0 \mid i \leq k\}$ be a finite set of formulas where $p_i \in F[\bar{x}]$ and $F[\bar{x}]$ is the ring of polynomials in several variables $\bar{x}$. The following statements are equivalent:

(2.1) there exists a tuple $\bar{a} \in F^{<\omega}$ such that

$$F \models \bigwedge_{i=0}^{k} p_i(\bar{a}) > 0;$$

(2.2) there exist a tuple $\bar{b} \in \mathbb{Q}^{<\omega}$ such that

$$F \models \bigwedge_{i=0}^{k} p_i(\bar{b}) > 0.$$

**Proof.** The statements follow from quantifier elimination in strictly ordered fields [15] and topological properties of Archimedean fields. □

**PROPOSITION 6.** Suppose $(\mathcal{F}, \nu)$ and $AD_\nu$ satisfy the following properties:

(1) $AD_\nu \in \Delta_2^0$;

(2) $(\mathcal{F}, \nu)$ is a numerated (effective) algebra, i.e., there are computable functions $f : \omega \to \omega$, $g_+ : \omega \times \omega \to \omega$, and $g_* : \omega \times \omega \to \omega$ such that $\nu(f(i)) = c_i^{\mathcal{F}}$, $\nu(n) + \nu(m) = \nu(g_+(n, m))$, and $\nu(n) \cdot \nu(m) = \nu(g_*(n, m))$;

(3) the sets $\{(m, r) \in \omega \times \mathbb{Q} \mid \mathcal{F} \models \nu(m) > r\}$ and $\{(m, r) \in \omega \times \mathbb{Q} \mid \mathcal{F} \models \nu(m) < r\}$ are computably enumerable.

Then there exists a numbering $\eta : \omega \to F$ such that $(\mathcal{F}, \eta)$ is a computable structure.

**Proof.** In order to simplify the proof, we assume that $C = \mathbb{Q}$. First, employing the finite injury method, we construct a numbering $\mu$ of an appropriate $X \subseteq \omega$ for which $F$ is algebraic over $\mathbb{Q}(\nu(X))$.

The existence of a required $\eta$ will follow from Ershov's theorem [2] formulated below as Theorem 3. By the limit lemma (see [12]), there is a strongly computable sequence of finite sets $E_0, E_1, E_2, \ldots$ such that $AD_\nu = \lim_{s\to\infty} E_s$.

Let $\lambda : \omega \to V$ be a standard numbering of $V = \mathbb{Q}[\bar{x}] \setminus \{0\}$, where $\mathbb{Q}[\bar{x}] = \mathbb{Q}[x_i \mid i \in \omega]$. For simplicity, the elements of $V$ are called *rational polynomials*. At every stage $s \in \omega$, we construct a finite function $\mu^s : \{0, \ldots, 2s - 1\} \to \omega$, finite sets $C_s \subset \mathrm{im}(f)$, $B_s \subset \omega$, and $B_s = f(C_s)$, a language $\sigma_s$, a finite structure with partial positive diagram $(F_s, D_s)$, and a value $t(s) \in \omega$.

Stage 0. Let $\mu^0 = \perp$, $C_0 = \varnothing$, $B_0 = \varnothing$, $F_s = \{0, 1\}$, $D_s = \varnothing$, and $\sigma_0 = \varnothing$.

For every $n \le s + 1$, under the assumption that a tuple $\bar{m}$ contains all elements of $\mathrm{im}(\mu^s)$ which are no higher than $n$ and are not contained in $C_s$, there exists $t \ge s + 1$ such that one of the following properties holds:

(a) $\bar{m} \smile n \in E_t$;

(b) at a stage no higher than $t$, for polynomials $p(\bar{x})$ such that $\lambda^{-1}(p) \le s$ we verified that $p(\nu(\bar{m} \smile n)) \ne 0$,

where the symbol $\smile$ denotes the concatenation of a tuple and an element.

For every $n \le s + 1$, we denote the least $t$ satisfying (a) or (b) by $t(n, s)$ and put $t(s) \rightleftharpoons \max_{n\in\mathrm{im}(\mu^s)} (t(n, s))$. If, for $t(n, s)$, property (b) holds, then $n$ is called a *b-number*. If, for $t(n, s)$, property (a) holds and property (b) does not, then $n$ is called an *a-number*. We say that an $a$-number $n$ is *active at stage* $s + 1$ if there exists $m_n$ such that $n = \mu^s(m_n)$ and $n \notin C_s$. We say that a number $n \le s + 1$ *requires attention at stage* $s + 1$ if one of the following cases holds:

(c) $n$ is an $a$-number which is active at stage $s + 1$;

(d) $n$ is a $b$-number such that:

—$n \notin \operatorname{im}(\mu^s)$,

—for a tuple $\bar{m}$ of elements of $\operatorname{im}(\mu^s)$ which are $\le n$ and $\notin C_s$, $\bar{m} \smile n \notin E_t$.

We construct $\mu_s$ with the following inductive properties:

—$\nu(\mu^s(2i)) = \nu(f(i))$ holds for $i < s$;

—$\nu \circ \mu^s$ restricted to $\{i \mid i \le 2s-1 \text{ and } i \text{ is odd}\}$ is injective;

—for rational polynomials $p(\bar{x})$ with $\lambda^{-1}(p) \le s$ we verified that $p(\nu(\bar{w}_s)) \neq 0$ in no more than $t(s)$ stages for all tuples $\bar{w}_s$ such that $\operatorname{set}(\bar{w}_s) = w_s$, where $w_s = \operatorname{im}(\mu^s) \setminus C_s$.

We associate $\mu^s$ with a structure $\mathfrak{F}_s = \langle F_s, \sigma_s \rangle$, where

$$\begin{aligned}
F_s &= \nu(\operatorname{im}(\mu^s)),\\
\Gamma^s &= \{\Gamma_p \mid \text{where } p \text{ is a rational polynomial such that } \lambda^{-1}(p) \le s,\\
&\qquad \text{in no more than } t(s) \text{ stages we verified that } p(\nu(\bar{w})) > 0\},\\
\sigma_s &= (<, \{c_i\}_{i \in B_s},\ \{d_i\}_{i \le s},\ \Gamma^s),
\end{aligned}$$

and with its positive diagram $D_s$ defined as follows:

—the interpretation of constants: $c_i^{F_s} = \nu(f(i))$ and $d_i^{F_s} = \nu(\mu^s(2i+1))$;

—the interpretation of predicates: $\mathfrak{F}_s \models \Gamma_p(\nu(\bar{w}))$ if in no more than $t(s)$ stages we verified that $p(\nu(\bar{w})) > 0$;

—$D_s$ contains all the results obtained under the replacement in the formulas $x_1 < x_2$ and $\Gamma_p(x_1, \ldots, x_l)$ of variables $x_1, \ldots, x_l$ by constants from $\sigma_s$ whose truth in $\mathfrak{F}_s$ was computed in no more than $t(s)$ stages.

It is easy to see that $D_s$ is completely defined by the language $\sigma_s$. At later stages, we define $\sigma_s$ via $\mu^s$ and construct $\mu^{s+1}$ using $\sigma_s$.

Stage $s+1$. We search for a least number $n \le s+1$ that requires attention at step $s+1$. If there is no such $n$, then we proceed to step $A$. If there is such $n$ and $n$ is an $a$-number, then we proceed to step $B$. If there is such $n$ and $n$ is a $b$-number, then we proceed to step $C$.

Step $A$. Using the density of $C$ in $F$, we search for a least $i_0 > s$ such that $f(i_0) \notin C_s$ and the inductive properties defined above hold for $\mu^{s+1}$.

Put

$$\mu^{s+1}(2i+1) = \begin{cases} \mu^s(2i+1), & i < s,\\ f(i_0), & i = s, \end{cases}$$

$$C_{s+1} = C_s \cup \{f(i_0)\} \cup \{f(s)\},\ B_{s+1} = B_s \cup \{i_0\} \cup \{s\}.$$

Proceed to the next stage.

Step $B$. Since $n$ is active at stage $s+1$, using the density of $C$ in $L$, being Archimedean for $\mathfrak{F}$, and Lemma 4, we search for a least $j$ such that $j \neq s$ and $\langle F, \sigma_s \rangle \models D_s$, where interpretations of constants are as follows:

$$c_i^F = \nu(f(i)) \text{ and } d_i^F = \begin{cases} \nu(\mu^s(2i+1)), & i \neq m_n, \\ \nu(f(j)), & i = m_n. \end{cases}$$

Next we search for a least $i_0$ such that $i_0 \neq j$, $i_0 \neq s$, $f(i_0) \notin C_s$, and the inductive properties defined above hold for $\mu^{s+1}$. Put

$$\mu^{s+1}(2i+1) = \begin{cases} \mu^s(2i+1), & i \neq m_n,\ i < s, \\ f(j), & i = m_n, \\ f(i_0), & i = s, \end{cases}$$

$$C_{s+1} = C_s \cup \{f(j)\} \cup \{f(i_0)\} \cup \{f(s)\},\ B_{s+1} = B_s \cup \{j\} \cup \{i_0\} \cup \{s\}.$$

Proceed to the next stage.

Step $C$. Let

$$F_s = \{m \mid m \in \mathrm{dom}(\mu^s),\ f(m) \notin C_s,\ \mu^s(m) > n,\ m \text{ is odd}\}.$$

Using the density of $C$ in $F$, being Archimedean for $\mathfrak{F}$, and Lemma 4, we construct a finite injection $j : F_s \to \omega$ such that $j(m) = j_m$, $f(j_m) \notin C_s$, $\nu(f(j_m)) \neq \nu(n)$ for all $m \in F_s$, and $\langle F, \sigma_s \rangle \models D_s$, where interpretations of constants are as follows:

$$c_i^F = \nu(f(i)) \text{ and } d_i^F = \begin{cases} \nu(\mu^s(2i+1)), & 2i+1 \notin F_s, \\ \nu(f(j_m)), & 2i+1 \in F_s. \end{cases}$$

Put

$$\mu^{s+1}(2i+1) = \begin{cases} \mu^s(2i+1), & 2i+1 \notin F_s,\ i < s, \\ f(j(2i+1)), & 2i+1 \in F_s, \\ n, & i = s, \end{cases}$$

$$C_{s+1} = C_s \cup \{f(j_m) \mid m \in F_s\} \cup \{f(s)\},\ B_{s+1} = B_s \cup j(F_s) \cup \{s\}.$$

Proceed to the next stage. The construction is completed.

It is clear that $\mu^s$ meets the inductive requirements. Let $\mu_0(m) = \lim\limits_{s\to\infty} \mu^s(m)$.

The following lemmas establish properties of $\mu$.

**LEMMA 5.** (a) Along the construction process, the value $\mu^s(m)$ can be changed only finitely many times for all $m$.

(b) For all $m$, if there exist $s$ and $j$ such that $\mu^s(m) = f(j)$ and $f(j) \in C_s$, then $\mu^{s'}(m) = f(j)$ for all $s' \geq s$.

(c) If, for some $m$, we have $\mu^s(m) = n$ and $\mu^{s+1}(m) \neq n$, then $n \notin \mathrm{im}(\mu^{s+1})$.

(d) For all $n$, there exists $s_0$ such that $(\forall s \geq s_0)(n \in \mathrm{im}(\mu^{s_0}) \leftrightarrow n \in \mathrm{im}(\mu^s))$.

**Proof.** Claims (a)-(c) are straightforward from the construction.

(d) Assume $n \in \mathrm{im}(\mu^s) \setminus \mathrm{im}(\mu^{s+1})$. This means that at every stage $s+1$, either there exists $l < n$ such that $l \notin \mathrm{im}(\mu^s)$ or there exists an $a$-number $k < n$ which is active at stage $s+1$. By inductive reasoning, it is easy to see that after some stage $s_0$ both of the cases will not hold. □

**LEMMA 6.** (1) The function $\mu(m) = \nu(\lim\limits_{s\to\infty} \mu^s(m))$ is well defined. Put $X = \mu_0(\omega) \setminus \mathrm{im}(f)$. Then $\nu(X)$ is an algebraically independent set in $F$.

(2) $F$ is algebraic over $\tilde{F} = \mathbb{Q}(\nu(X))$.

**Proof.** (1) Follows from Lemma 5 and inductive properties.

(2) Assume the contrary. Let $n$ be a least number for which $\nu(n)$ is not algebraic over $\tilde{F}$. In particular, $\nu(n) \notin C$ and $n \notin X$.

There exists $s_0$ such that for all $s \geq s_0$ and for all $m \leq n$ such that $m \notin C_{s_0}$, the following hold:

—$m \in X \leftrightarrow m \in \mathrm{im}(\mu^{s_0})$;

—$m \in \mathrm{im}(\mu^{s_0}) \leftrightarrow m \in \mathrm{im}(\mu^s)$;

—$n \notin \mathrm{im}(\mu^s)$ and $m \notin C_s$;

—$\bar{m} \smile n \notin E_s$, where $\bar{m}$ contains all elements of $\mathrm{im}(\mu^s)$ which are $\leq n$ and $\notin C_s$;

—for every $l < n$ and for an $n$-tuple $\bar{k}$ of different numbers less than $n$, if $\nu(l)$ is algebraically dependent on $\nu(\bar{k})$, then $\bar{m} \smile n \in E_s$ for all $s \geq s_0$.

At stage $s_0$, therefore, $n$ is the least number that requires attention. Applying step $C$ of the construction, we obtain $n \in \mathrm{im}(\mu^{s_0+1})$, i.e., $n \in X$, a contradiction. □

**LEMMA 7.** The positive diagram $D = \bigcup\limits_{s\in\omega} D_s$ is computably enumerable under the following interpretations:

(1) $d_i^F = \nu(\mu_0(2i+1))$;

(2) $c_i^F = \nu(f(i))$.

Moreover, $F \models p(d_{i_1}, \ldots, d_{i_k}) > 0 \leftrightarrow \Gamma_p(d_{i_1}, \ldots, d_{i_k}) \in D$ for any rational polynomial $p(\bar{x}) \not\equiv 0$.

The **proof** is straightforward. □

**LEMMA 8.** There exists an algorithm to check the validity of

$$p(\nu(\mu_0(i_1)), \ldots, \nu(\mu_0(i_k))) = 0$$

for rational polynomials $p$.

**Proof.** We construct an algorithm by recursion on $k$. Without loss of generality, we will assume that every $i_l$ is odd.

If stage $s$ is sufficiently large, i.e., for all $l \leq k$ and all $s_1 \geq s$, $\mu_0(i_l) = \mu^{s_1}(i_l)$ holds, and $\lambda^{-1}(p) \leq s$, then

either in no more than $t(s)$ stages we verified that

$$p(\nu(\mu^s(i_1)), \ldots, \nu(\mu^s(i_1))) \neq 0,$$

or there exists $l \leq k$ such that $\mu^s(i_k) \in C_s$.

Using minimization on $s$, we check if one of the following cases holds:

(1) $\Gamma_p \in \sigma_s$ and $\Gamma_p(d_{j_1}, \dots, d_{j_1}) \in D_s$;

(2) $\Gamma_{-p} \in \sigma_s$ and $\Gamma_{-p}(d_{j_1}, \dots, d_{j_1}) \in D_s$;

(3) there exists $l \leq k$ such that $\mu^s(i_l) \in C_s$, where $i_l = 2j_l + 1$ for $1 \leq l \leq k$.

In cases (1) and (2), the algorithm stops working. In case (3), we replace the $l$th variable in $p$ by $\mu^s(i_l)$ and proceed by recursion. □

**LEMMA 9.** Let $\tilde{F} = \mathbb{Q}(\nu(X)) = \mathbb{Q}(\mu_0(\omega))$. The field $(\tilde{F}, <)$ admits a computable presentation in the language of strictly ordered fields.

**Proof.** The construction of a natural numbering of $\mathbb{Q}(\mu_0(\omega))$ is standard and can be found, for instance, in [16]. That such a presentation is computable follows from Lemma 8. □

The kernel theorem entails the following:

**THEOREM 3** [2]. If an ordered field $L_0$ admits a computable presentation $\lambda_0$, then a computable presentation $\lambda_1$ of its real closure $L_1 = [L_0]_{rcl}$ can be constructed effectively so that $\lambda_0(n) = \lambda_1(g(n))$ for some computable function $g : \omega \to \omega$.

Theorem 3 completes the proof of Prop. 6. □

It is worth observing that we can relativize Proposition 6 to any degree $\mathbf{a}$.

Now we show that for a strictly ordered field $(\mathbb{R}_c, \sigma_c)$ and for $\mathbf{a}' \geq \mathbf{0}''$, there exists a numbering $\nu$ satisfying the conditions of the relativized version of Prop. 6.

First, using $H_z : \mathbb{Z} \to \{0, 1, -1\}$ defined as

$$H_z(n) = \begin{cases} 0, & n = 0, \\ 1, & n > 0, \\ -1, & n < 0, \end{cases}$$

and $\nu_{\text{tot}}$ such as in Theorem 2, we construct $\tilde{\nu}_{\text{tot}} : \omega \to \tilde{S}_{\text{tot}}$, where $\tilde{S}_{\text{tot}}$ is the set of all computable functions $f : \omega \to \mathbb{Z}$.

It is well known how to add and multiply real numbers in presentations in which numbers from $\{-1, 0, 1\}$ are involved (so-called sign-digit representations). Therefore, we can assume that there exist computable partial operators $G_+$ and $G_*$ such that, for all computable functions $f, g : \omega \to \mathbb{Z}$,

$$\left(l + \sum_{i=0}^{\infty} \frac{H_z(f(i))}{2^{i+1}}\right) + \left(m + \sum_{i=0}^{\infty} \frac{H_z(g(i))}{2^{i+1}}\right) = \sum_{i=0}^{\infty} \frac{G_+(f, g, l, m)}{2^i},$$
$$\left(l + \sum_{i=0}^{\infty} \frac{H_z(f(i))}{2^{i+1}}\right) \cdot \left(m + \sum_{i=0}^{\infty} \frac{H_z(g(i))}{2^{i+1}}\right) = \sum_{i=0}^{\infty} \frac{G_*(f, g, l, m)}{2^i},$$

where $G_+(f, g, l, m)(i) \in \{0, 1, -1\}$, $G_*(f, g, l, m)(i) \in \{0, 1, -1\}$ for $i > 0$, and $l, m \in \mathbb{Z}$.

It follows from [9] that there exists a computable sequence $\{f_i\}_{i\in\omega}$ of computable functions $f_i : \omega \to \mathbb{Z}$ such that

$$c_i = f_i(0) + \sum_{k=1}^{\infty} \frac{H_z(f_i(k))}{2^k}.$$

We define an operator $T^-(f)(i) = f(i+1)$. The numbering $\tilde{\nu}_{\text{tot}}$ can be constructed as follows:
—if $n = c(0,m)$ then $\tilde{\nu}_{\text{tot}}(n) = \nu_{\text{tot}}(m)$;
—if $n = c(1,m)$ then $\tilde{\nu}_{\text{tot}}(n) = -\nu_{\text{tot}}(m)$;
—if $n = c(2,m)$ then $\tilde{\nu}_{\text{tot}}(n) = f_m$;
—if $n = c(k,m)$, $k$ is even, $k > 2$, and $m = c(a,b)$, then

$$\tilde{\nu}_{\text{tot}}(n) = G_+(T^-(\tilde{\nu}_{\text{tot}}(a)), T^-(\tilde{\nu}_{\text{tot}}(b)), \tilde{\nu}_{\text{tot}}(a)(0), \tilde{\nu}_{\text{tot}}(b)(0));$$

—if $n = c(k,m)$, $k$ is odd, $k > 1$, and $m = c(a,b)$, then

$$\tilde{\nu}_{\text{tot}}(n) = G_*(T^-(\tilde{\nu}_{\text{tot}}(a)), T^-(\tilde{\nu}_{\text{tot}}(b)), \tilde{\nu}_{\text{tot}}(a)(0), \tilde{\nu}_{\text{tot}}(b)(0)).$$

Now we define $\nu : \omega \to \mathbb{R}_c$ as follows:

$$\nu(n) = \tilde{\nu}_{\text{tot}}(n)(0) + \sum_{i=1}^{\infty} \frac{H_z(\tilde{\nu}_{\text{tot}}(n)(i))}{2^i}.$$

It is not hard to see that

$$\nu(n) = \varphi_{h(m)}(0) + \sum_{i=1}^{\infty} \frac{H_z(\varphi_{h(m)}(i))}{2^i} \text{ for } n = c(0,m).$$

We show that the numbering $\nu$ satisfies the conditions of the relativized version of Prop. 6.

Consider condition (2) in Proposition 6. It is not hard to see that $g_+(n_1,n_2) = c(4,m)$, where $m = c(n_1,n_2)$. Indeed,

$$\begin{aligned}\nu(n_1) + \nu(n_2) &= \tilde{\nu}_{\text{tot}}(n_1)(0) + \sum_{i=1}^{\infty} \frac{H_z(\tilde{\nu}_{\text{tot}}(n_1)(i))}{2^i} + \tilde{\nu}_{\text{tot}}(n_2)(0) \\ &\quad + \sum_{i=1}^{\infty} \frac{H_z(\tilde{\nu}_{\text{tot}}(n_2)(i))}{2^i} \\ &= \sum_{i=0}^{\infty} \frac{G_+(T^-(\tilde{\nu}_{\text{tot}}(n_1)), T^-(\tilde{\nu}_{\text{tot}}(n_2)), \tilde{\nu}_{\text{tot}}(n_1)(0), \tilde{\nu}_{\text{tot}}(n_2)(0))}{2^i}.\end{aligned}$$

In a similar way, we can define $g_*$ and $f$. Therefore, $g_+$, $g_*$, and $f$ are computable. Condition (3) in Proposition 6 is straightforward. Condition (1), i.e., $AD_\nu \in \Delta^0_2$, follows from the following lemmas.

**LEMMA 10.** The set of $\Delta^0_2[\mathbf{a}]$-computable functions is closed under superposition.

**LEMMA 11.** (1) The set $A = \{m \mid \nu(c(0,m)) \in \text{Dyadic}\}$ belongs to $\Delta^0_2[\mathbf{a}]$.

(2) There exists a partial $\Delta_2^0[\mathbf{a}]$-computable function on $A$ that computes $\nu(c(0,m))$ as an element of Dyadic.

**LEMMA 12.** If $m \notin A$, then $\tilde{\nu}_{\rm tot}(c(0,m)) = \nu_{tot}(m) = \varphi_{h(m)}$, where $h$ is defined in Theorem 2.

**LEMMA 13.** Every element $\nu(n)$ can be effectively represented as

$$t(\nu(n_1), \dots, \nu(n_k)),$$

where $t(x_1, \dots, x_k)$ are terms in the language $(*^2, +^2, -^1)$ and $n_i = c(0, m_i)$.

**LEMMA 14.** Let $J \subseteq \{1, \dots, s\}$, $n_i$ be of the form $c(0, m_i)$ for $1 \le i \le s$, and $r_j \in$ Dyadic for $j \in J$. Suppose also that $J = \{j \le s \mid \nu(n_j) \in \text{Dyadic}\}$, $t_1, \dots, t_k$ are terms in the language $(*, +, -)$ having the form

$$\widetilde{t_i} = [t_i(\bar{x})] \begin{pmatrix} x_i \text{ if } i \notin J, & x_j \text{ if } j \in J \\ \nu(n_i), & r_j \end{pmatrix}.$$

Then the condition of being algebraically dependent for $\widetilde{t}_1, \dots, \widetilde{t}_k$ is a $\Sigma_2^0$-relation with arguments $\overline{n}$ and $\overline{r}$ and a parameter $J$.

Note that $\Sigma_2^0 \subset \Delta_3^0 \subseteq \Delta_2^0[a]$ for $\mathbf{a}' \ge \mathbf{0}''$. Therefore, the properties of $\nu_{\rm tot}$ (see Thm. 2) and $\nu$ and the previous lemmas imply that $AD_\nu \in \Delta_2^0[a]$, where $\mathbf{a}' \ge \mathbf{0}''$.

**COROLLARY 1.** If $\mathbf{a}' \ge \mathbf{0}''$, then there exists a numbering $\mu : \omega \to \mathbb{R}_c$ such that $(\langle \mathbb{R}_c, \le, +, \cdot, 0, 1 \rangle, \mu)$ is an $\mathbf{a}$-computable structure.

The **proof** follows from Proposition 6 and the argument above. □

**THEOREM 4.** The relation $\mathbf{a}' \ge \mathbf{0}''$ holds if and only if $\mathbf{a} \in \operatorname{Spec}(\mathbb{R}_c)$.

**Proof.** The characterization follows from Theorem 1 and Corollary 1. □

## REFERENCES


1. Yu. L. Ershov, *Numeration Theory* [in Russian], Nauka, Moscow (1977).
2. S. S. Goncharov and Yu. L. Ershov, *Constructive Models, Sib. School Alg. Log.* [in Russian], Nauch. Kniga, Novosibirsk (1999).
3. C. J. Ash and J. F. Knight, *Computable Structures and the Hyperarithmetical Hierarchy, Stud. Log. Found. Math.*, **144**, Elsevier, Amsterdam (2000).
4. G. S. Tsejtin, "Mean value theorems in constructive analysis," *Am. Math. Soc., Translat., II. Ser.*, **98**, 11-40 (1971).
5. P. Martin-Löf, *Notes on Constructive Mathematics*, Almqvist and Wiksell, Stockholm (1971).
6. V. Brattka, "Computable versions of Baire's category theorem," in *Mathematical Foundations of Computer Science*, J. Sgall et al. (Eds.), *Lect. Notes Comput. Sci.*, **2136**, Springer, Berlin, (2001), pp. 224-235.

7. A. Frolov, I. Kalimullin, V. Harizanov, O. Kudinov, and R. Miller, "Spectra of high$_n$ and non-low$_n$ degrees," *J. Log. Comput.*, **22**, No. 4, 755-777 (2012).

8. H. Rogers, *Theory of Recursive Functions and Effective Computability*, McGraw-Hill, New York (1967).

9. K. Weihrauch, *Computable Analysis. An Introduction, Texts Theoret. Comput. Sci. EATCS Ser.*, Springer-Verlag, Berlin (2000).

10. H. G. Rice, "Recursive real numbers," *Proc. Am. Math. Soc.*, **5**, No. 5, 784-791 (1954).

11. D. A. Martin, "Classes of recursively enumerated sets and degrees of unsolvability," *Z. Math. Log. Grund. Math.*, **12**, No. 4, 295-310 (1966).

12. J. R. Shoenfield, *Degrees of Unsolvability, North-Holland Math. Stud.*, **2**, Elsevier, New York (1971).

13. C. G. Jockusch, Jr., "Degrees in which the recursive sets are uniformly recursive," *Can. J. Math.*, **24**, 1092-1099 (1972).

14. A. Robinson, *Introduction to Model Theory and to the Metamathematics of Algebra*, North Holland, Amsterdam (1963).

15. J. Bochnak, M. Coste, and M.-F. Roy, *Real Algebraic Geometry*, Springer-Verlag, Berlin (1998).

16. A. Fröhlich and J. Shepherdson, "Effective procedures in field theory," *Philos. Trans. Roy. Soc. London, Ser. A*, **248**, 407-432 (1956).