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PERIODIC GROUPS SATURATED WITH FINITE SIMPLE GROUPS OF TYPES U_3 AND L_3

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Suppose that $\mathfrak{M}$ is a set whose elements are simple three-dimensional unitary groups $U_3(q)$ and linear groups $L_3(q)$ over finite fields. We prove that a periodic group saturated with groups of $\mathfrak{M}$ is locally finite and isomorphic to $U_3(Q)$ or $L_3(Q)$ for some locally finite field Q.

A group G is said to be *saturated* with groups of a set $\mathfrak{R}$ of groups if every finite subgroup of G is contained in a subgroup of G isomorphic to a group of $\mathfrak{R}$.

In [1], it was proved that a periodic group saturated with groups of a finite set $\mathfrak{F}$ of groups isomorphic to finite simple groups $U_3(q)$ or $L_3(q)$ is isomorphic to an element of $\mathfrak{F}$. In [2], it was shown that a periodic group saturated with simple groups of $\mathfrak{T} = \{U_3(q), L_3(q) \mid q \text{ is even}\}$ is isomorphic to a unitary or linear group of degree 3 over some locally finite field of characteristic 2. Our goal is to generalize these results.

THEOREM. Suppose that a periodic group G is saturated with groups of the set

$$\mathfrak{M} = \{U_3(q), L_3(q) \mid q \text{ is a power of a prime, } q \geqslant 3\}.$$

Then G is isomorphic to $U_3(Q)$ or $L_3(Q)$ for some locally finite field Q.

1. PRELIMINARY RESULTS

Let $GF(q)$ be a finite field of order q, $SL_3(q) = SL_3^+(q)$ be a group of matrices of degree 3 with determinants equal to 1, and $SU_3(q) = SL_3^-(q)$ be a group of unitary matrices of degree 3 over a

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field $GF(q^2)$, i.e., a subgroup of $SL_3(q^2)$ consisting of matrices m such that $m\overline{m}^T$ is the identity matrix, where T denotes transposition, and $\overline{m}$ is obtained from m by replacing every element m_{ij} with m_{ij}^q.

Denote by φ the natural homomorphism of $SL_3(q^2)$ onto $PSL_3(q^2)$ (with kernel consisting of scalar matrices), and we will use the same notation for the restriction of φ to $SL_3(q)$ and $SU_3(q)$.

Thus

$$SL_3(q)^\varphi = PSL_3(q) = L_3(q) = L_3^+(q),$$
$$SU_3(q)^\varphi = PSU_3(q) = U_3(q) = L_3^-(q).$$

Now let q be odd.

Put

$$i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^\varphi, \qquad j = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^\varphi,$$
$$b = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}^\varphi, \qquad w = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^\varphi.$$

Obviously, $i, j, b, w \in L_3^\varepsilon(q)$, where $\varepsilon \in \{+, -\}$. Define

$$A = \langle i, j\rangle, \quad B = \langle w, j\rangle, \quad V = \langle b, w\rangle.$$

PROPOSITION 1. Suppose that $L = L_3^\varepsilon(q)$, where q is odd, and i, j, b, w and A, B, V are elements and subgroups of L defined as above. Then:

(1) A and B are four-groups, i.e., elementary Abelian subgroups of order 4, AB is the dihedral group of order 8, b is of order 3, and V is isomorphic to the symmetric group of degree 3.

(2) $D = C_L(A)$ is the direct product of a cyclic group of order $q - \varepsilon 1$ and a cyclic group of order $(q - \varepsilon 1)/(3, q - \varepsilon 1)$, and

$$N_L(A) = N_L(D) = D \leftthreetimes V.$$

(3) If an element of $N_L(A)$ induces by conjugation an automorphism of A of order 3, then its order equals 3.

(4) All involutions of L are conjugate in L, every four-group of L is conjugate to A, L has an element of order 8, and every Abelian section of a Sylow 2-subgroup of L is generated by three elements.

(5) There exists $v \in L$ for which $j^v = j$ and $i^v = w$.

(6) If $(q, \varepsilon) \notin \{(3, +), (5, -)\}$, then for every four-subgroup $C \neq A$ of $N_L(A)$ it is true that $L = \langle N_L(A), C_L(C)\rangle$. If $(q, \varepsilon) = (3, +)$, then $L = \langle N_L(A), N_L(C)\rangle$. If $(q, \varepsilon) = (5, -)$, then $\langle N_L(A), N_L(C)\rangle \simeq A_7$.

Proof. Items (1)-(3) can be verified by direct calculations (see, e.g., [1]). Item (4) was proved in [3].

(5) By (4), there exists $v_1 \in L$ such that $A^{v_1} = B$. According to (2), V^{v_1} induces via conjugation in B the full automorphism group of B, acting doubly transitively on the set of involutions in B. This yields the desired result.

(6) Obviously, C contains an involution t not belonging to $C_L(A)$, and

$$|C_{N_L(A)}(t)| = |C_{C_L(A)}(t)\langle t\rangle| = 2|C_{C_L(A)}(w)|.$$

It is straightforward to verify that $|C_{C_L(A)}(w)| = (q - \varepsilon 1)/(3, q - \varepsilon 1)$. Therefore, $|C_{N_L(A)}(C)| \leqslant 2(q - \varepsilon 1)/(3, q - \varepsilon 1)$. Since A and C are conjugate in L, we have $|C_L(C)| = (q - \varepsilon 1)^2/(3, q - \varepsilon 1)$, which implies that $C_L(C) \not\leqslant N_L(A)$ except for the case $q = 3$, $\varepsilon = +$. $N_L(A)$ is not maximal in L only if $L \simeq U_3(5)$ [4, pp. 378, 379]. Therefore, it remains to consider the cases where $(q, \varepsilon) \in \{(3, +), (5, -)\}$, for which (6) is readily verifiable using [5].

For a group G and a set $\mathfrak{L}$ of groups, we denote by $\mathfrak{L}(1)$ the set of subgroups of G that are isomorphic to elements of $\mathfrak{L}$.

PROPOSITION 2. Let $\mathfrak{A}$ and $\mathfrak{B}$ be nonempty sets of finite groups of even orders and G be a periodic group saturated with groups of $\mathfrak{A} \cup \mathfrak{B}$. Suppose that the following conditions hold:

(1) if $A \in \mathfrak{A}$, $B \in \mathfrak{B}$, and S_A and S_B are Sylow 2-subgroups of A and B, respectively, then S_A is not isomorphic to any subgroup of S_B, and S_B is not isomorphic to any subgroup of S_A;

(2) $\mathfrak{A}(1) \neq \varnothing \neq \mathfrak{B}(1)$.

Then for every natural number t there exist $A_t, B_t \leq G$ such that $A_t \in \mathfrak{A}(1)$, $B_t \in \mathfrak{B}(1)$, and $|A_t \cap B_t|$ is divisible by 2^t.

The **proof** is by induction on t. Let $A \in \mathfrak{A}(1)$, $B \in \mathfrak{B}(1)$, a be an involution in A, and b be an involution in B. Then $\langle a, b\rangle$ is a finite group contained in some subgroup C of $(\mathfrak{A} \cup \mathfrak{B})(1)$. If $C \in \mathfrak{A}(1)$, then we set $A_1 = C$ and $B_1 = B$; if, however, $C \in \mathfrak{B}(1)$, then we set $A_1 = A$ and $B_1 = C$. In any case $|A_1 \cap B_1|$ is divisible by 2, and the conclusion of the proposition is true for $t = 1$.

Suppose that we have already found $A_m \in \mathfrak{A}(1)$ and $B_m \in \mathfrak{B}(1)$ such that $n = |A_m \cap B_m|$ is divisible by 2^{t-1}. If n is divisible by 2^t, then the conclusion of the proposition is true for $A_t = A_m$ and $B_t = B_m$. Let n not be divisible by 2^t and S be a Sylow 2-subgroup of $A_m \cap B_m$. By hypothesis, S is a Sylow 2-subgroup neither in A_m nor in B_m; hence $N_{A_m}(S)$ has an element x such that Sx is an involution in $N(S)/S$, and $N_{B_m}(S)$ has an element y such that Sy is an involution in $N(S)/S$. The subgroup $\langle S, x, y\rangle$ is finite and, therefore, lies in $C \in \mathfrak{A}(1) \cup \mathfrak{B}(1)$. If $C \in \mathfrak{A}(1)$, then we set $A_t = C$ and $B_t = B_m$; if $C \in \mathfrak{B}(1)$, then we set $A_t = A_m$ and $B_t = C$. In any case $|A_t \cap B_t|$ is divisible by 2^t. The proposition is proved.

PROPOSITION 3 (V. D. Mazurov). Let H be a proper normal subgroup of a group G. If $x^3 = 1$ for every element of $G \setminus H$, then H is nilpotent.

Proof. Suppose $x \in G \setminus H$. Then $(hx^{-1})^3 = 1$ for every $h \in H$. Since

$$(hx^{-1})^3 = hh^x h^{x^2} x^{-3} = hh^x h^{x^2},$$

x induces in H a splitting automorphism of order 3. By Lemma 6 in [6], which was proved by Mazurov, the conclusion of the proposition is true.

2. PROOF OF THE THEOREM

Suppose that the theorem is false. Set $\mathfrak{A} = \{L_3(q), U_3(q) \mid q \text{ is odd}\}$ and $\mathfrak{B} = \{L_3(2^m), U_3(2^m) \mid m \geqslant 2\}$.

LEMMA 1. G is saturated with groups of $\mathfrak{A}$, i.e., $\mathfrak{B}(1) = \varnothing$.

Proof. Assume the contrary. By virtue of [1], $\mathfrak{A}(1) \neq \varnothing$. We show that the conditions of Proposition 2 are satisfied. Indeed, by Proposition 1(4), Sylow subgroups of groups of the set $\mathfrak{A}$ have elements of order 8, and the periods of Sylow 2-subgroups of $\mathfrak{B}$ equal 4. On the other hand, groups of $\mathfrak{B}$ contain elementary Abelian sections of order 16, while Sylow 2-subgroups of groups of $\mathfrak{A}$ lack such sections.

By Proposition 2, $\mathfrak{M}(1)$ contains subgroups A and B, where $A \in \mathfrak{A}(1)$ and $B \in \mathfrak{B}(1)$, such that $|A \cap B|$ is divisible by 2^{12}, which is impossible. Indeed, on the one hand, a Sylow 2-subgroup of $A \cap B$ (being a subgroup of B) contains an elementary Abelian section of order 2^4, and on the other hand, the rank of every elementary Abelian 2-section of A is at most three. The lemma is proved.

LEMMA 2. Let $\mathfrak{A}_0$ be a set of groups isomorphic to groups of $\mathfrak{A}(1)$. Then $\mathfrak{A}_0$ is infinite.

Proof. If $\mathfrak{A}_0$ is finite, then, by [1], G is a finite group of the set $\mathfrak{M}$, which is a contradiction with the assumption. The lemma is proved.

LEMMA 3. All involutions in G are conjugate. All four-groups in G are conjugate.

Proof. If a and b are involutions in G, then $\langle a, b\rangle$ is a finite subgroup in $R \in \mathfrak{M}(1)$ and a and b are conjugate in R by Prop. 1(4). If K_1 and K_2 are four-groups in G, then by the above there exists $g \in G$ such that $K_1 \cap K_2^g \neq 1$. Hence $\langle K_1, K_2^g\rangle$ is a finite subgroup, and again the statement follows from Prop. 1(4). The lemma is proved.

By Lemma 2, $\mathfrak{A}(1)$ contains a subgroup L_0 isomorphic to $U_3(q)$, where $q > 5$ and is odd, or to $L_3(q)$, where $q > 3$ and is odd. We identify L_0 with L defined in Proposition 1 and borrow the notation from that proposition. Let $N = N_G(A)$, $C_A = C_G(A)$, and $C_B = C_G(B)$.

LEMMA 4. $N = C_A \cdot V$ and C_A is an Abelian group of rank 2. In particular, N is countable and locally finite.

Proof. Let $d \in C_A$. Then $db \in N_G(A)$ and $\langle A, db\rangle$ lies in a finite subgroup L_1 of $\mathfrak{M}(1)$. Since db induces by conjugation in A an automorphism of order 3, by Proposition 1(3) applied to L_1 instead of L, we have $(db)^3 = 1$. According to Proposition 3, C_A is nilpotent and is therefore locally finite. If now $d_1, d_2 \in C_A$, then $K = \langle A, d_1, d_2\rangle$ lies in $C_R(A)$ for some subgroup $R \in \mathfrak{M}(1)$. By

Proposition 1(2), K is Abelian and its rank is at most two. Thus C_A is an Abelian group of rank at most two; in particular, it is countable. Since N is a finite extension of C_A, N is locally finite and countable. The lemma is proved.

In view of Lemma 4, N is countable, i.e., $N = \{n_1, n_2, n_3, \dots\}$. Consider now two sequences of subgroups

$$N_0, N_1, N_2, \dots; \quad L_0, L_1, \dots$$

constructed according to the following rules: $L_0 = L$ and $N_0 = N \cap L_0 = N_{L_0}(A)$. If $N_0 = N$, then the process ends. Otherwise, we suppose that L_1 is a subgroup of $\mathfrak{A}(1)$ that contains N_0 and the first numbered element n_i, which is not in N_0. Let $N_1 = N \cap L_1 = N_{L_1}(A)$. If $N_1 = N$, then the process ends. Otherwise, we choose in $\mathfrak{A}(1)$ a subgroup L_2 containing N_1 and the first numbered element in N, which is not in N_1. Set $N_2 = N \cap L_2 = N_{L_2}(A)$. If we continue this process eventually we arrive at a sequence $L_0, L_1, L_2, \dots$ of subgroups of $\mathfrak{A}(1)$ such that the union of a sequence

$$N_0 < N_1 < N_2 < \dots,$$

where $N_l = N \cap L_l$, coincides with N.

LEMMA 5. $L_{l-1} \leq L_l$ for every $l = 1, 2, \dots$.

Proof. By Proposition 1, $V, A, B \leq N_0 \leq N_0 \cap L_1$. Let $v \in L_0$, $j^v = j$, $i^v = w$, and $v_1 \in L_1$, $j^{v_1} = j$, $i^{v_1} = w$. Then $c = v_1 v^{-1} \in C$, i.e., $v_1 = cv$. Since C_A is Abelian, $L_1 \geq C_{L_1}(A)^{v_1} = C_{L_1}(A)^{cv} = C_{L_1}(A)^v = C_{N_1}(A)^v \geq C_N(A)^v = C_{L_0}(A)^v = C_{L_0}(A^v) = C_{L_0}(B)$. Thus $C_{L_0}(B) \leq L_1$, and by Proposition 1,

$$L_0 = \langle N_0, C_{L_0}(B) \rangle \leq \langle N_1, L_1 \rangle = L_1.$$

If we have already stated that $L_{l-1} \geq L_0$ and $N_{l-1} \neq N_l$, then the same argument shows that $L_{l-1} = \langle N_{l-1}, C_{L_{l-1}}(B) \rangle \leq \langle N_l, L_l \rangle = L_l$. The lemma is proved.

The union X of an ascending chain L_0, L_1, $\dots$ is a locally finite group, which, by [7], is a group of Lie type over some locally finite field Q. Clearly, $X \simeq U_3(Q)$ or $L_3(Q)$. Furthermore, $N \leq X$.

LEMMA 6. If T is a dihedral subgroup of order 8 in X, then $N_G(T) \leq X$.

Proof. We may assume that $T \geq A$, so $T \leq N$, and hence $T \leq N_l \leq L_l$ for some l. If C is the second four-subgroup of T, then L_l has an element v mapping A into C. If now $x \in N(T)$, then either $A^x = A$ and $x \in N(A) \leq N \leq X$, or $A^x = C$ and $x = nv$, where $n \in N(A)$. In any case $x \in X$. The lemma is proved.

LEMMA 7. $X = G$.

Proof. Suppose the contrary. If every involution of G belongs to X, then $X \trianglelefteq G$ and $G = XN_G(A) \leq X$. Therefore, there exist an involution $g \in G \setminus X$ and a finite subgroup containing $\langle j, g \rangle$ and not lying in X. Thus there exists a subgroup $M \in \mathfrak{A}(1)$ not lying in X and containing j. We show that M can be chosen so that $M \cap X$ will contain a four-subgroup. Otherwise, $i \notin M$ and $C_M(j) \setminus X$ contains an involution $m \notin X$. Now we can replace M with a subgroup $M_1 \in \mathfrak{A}(1)$, containing $\langle i, j, m \rangle$. Without loss of generality, we may assume that M contains A. By Proposition 1,

$N_M(A)$ has a four-subgroup C, which is distinct from A, and $A^x = C$ for some $x \in M$. On the other hand, $C \leq N \leq X$, and so there exists $y \in X$ such that $A^y = C$; i.e., $x = ny$, where $n \in N$. Hence $x \in X$.

Thus $S = \langle N_M(A), N_M(C) \rangle \leq X$. Since $S \neq M$, by Proposition 1(6), $M \simeq U_3(5)$ and $S \simeq A_7$ is a maximal subgroup in M. Now a Sylow 2-subgroup T of S is a dihedral group of order 8. By Lemma 6, its normalizer $R = N_M(T)$ in M lies in X but not in S. Therefore, $M = \langle R, S \rangle \leq X$, a contradiction. The lemma is completed, which proves the theorem.

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