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A SUFFICIENT CONDITION FOR NONPRESENTABILITY OF STRUCTURES IN HEREDITARILY FINITE SUPERSTRUCTURES

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We introduce a class of existentially Steinitz structures containing, in particular, the fields of real and complex numbers. A general result is proved which implies that if $\mathfrak{M}$ is an existentially Steinitz structure then the following structures cannot be embedded in any structure Σ-presentable with trivial equivalence over $\mathbb{HF}(\mathfrak{M})$: the Boolean algebra of all subsets of ω, its factor modulo the ideal consisting of finite sets, the group of all permutations on ω, its factor modulo the subgroup of all finitary permutations, the semigroup of all mappings from ω to ω, the lattice of all open sets of real numbers, the lattice of all closed sets of real numbers, the group of all permutations of $\mathbb{R}$ Σ-definable with parameters over $\mathbb{HF}(\mathbb{R})$, and the semigroup of such mappings from $\mathbb{R}$ to $\mathbb{R}$.

Questions on the existence of computable presentations for particular algebraic structures have always occupied an important place in computable structure theory. The same questions arise in studying its various generalizations. For example, a generalization of the notion of a computable structure is the notion of a Σ-definable structure over an admissible set introduced by Yu. L. Ershov [1]. This notion arises naturally if, in the definition of a computable structure, the concept of computability is replaced with its analog, Σ-definability (see [2, 3]). Therefore, the issues mentioned are a natural extension of the range of problems related to constructive models. A most interesting class of admissible sets for which it is quite reasonable to pose such questions is the class of hereditarily finite superstructures over structures.

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First of all, the author was interested in Σ-presentability of some general mathematical structures over $\mathbb{HF}(\mathbb{R})$, the hereditarily finite superstructure over the ordered field $\mathbb{R}$ of real numbers. Motivation for the study of such questions can be found, for example, in [4, 5]. Previously, the author together with M. V. Korovina obtained some results on Σ-presentability of countable structures over $\mathbb{HF}(\mathbb{R})$. It is known that any countable structure is Σ-presentable over $\mathbb{HF}(\mathbb{R})$ with a single parameter from R (see, e.g., [6]); in the same paper, we gave a characterization of countable structures that possess such presentations without parameters. Currently, the countable case is thought of as being understood, to a greater of lesser extent; therefore, questions on Σ-presentability of uncountable structures over $\mathbb{HF}(\mathbb{R})$ become most interesting. For example, a series of matrix groups, polynomial rings over $\mathbb{R}$, etc., have such presentations. It is also worth mentioning some results on presentations of the natural ordering on the reals [7, 8].

In this paper we deal with Σ-presentability of natural general uncountable mathematical structures, such as the Boolean algebra $\mathcal{P}(\omega)$ of all subsets of a countable set, its factor $\mathcal{P}(\omega)^*$ modulo the ideal of all finite sets, the lattices of all open and closed sets of the reals, the group $\mathtt{Sym}(\omega)$ of all permutations on ω, its quotient $\mathtt{Sym}(\omega)^*$ modulo the subgroup of all finitary permutations, the semigroup ω^ω under composition of all functions from ω to ω, the group of all permutations on $\mathbb{R}$ Σ-definable with parameters over $\mathbb{HF}(\mathbb{R})$, and the semigroup of all mappings from $\mathbb{R}$ to $\mathbb{R}$ Σ-definable with parameters over $\mathbb{HF}(\mathbb{R})$.

The analysis of formerly obtained proofs for $\mathbb{HF}(\mathbb{R})$ has shown that these results can be formulated and proven in a more general form—namely, for hereditarily finite superstructures over so-called existentially Steinitz structures. The ordered field $\mathbb{R}$ and the field $\mathbb{C}$ of complex numbers are particular cases of such structures. Moreover, some general and easily verifiable sufficient conditions for non-Σ-presentability of structures and even for the absence of embeddings in such structures have been revealed. Eventually the original proofs have been simplified and more general results have been obtained. More exactly, in the paper we use a unified method to prove that there are no Σ-presentations over $\mathbb{HF}(\mathfrak{M})$ with trivial equivalence for all structures from the list above (and even for all their extensions), provided that the structure $\mathfrak{M}$ has a property, which here we call the $\exists$-Steinitz or existential Steinitz property. Inasmuch as $\mathbb{R}$ and $\mathbb{C}$ are existentially Steinitz, all these results also hold for $\mathbb{R}$ and $\mathbb{C}$. For some of the structures above, the absence of presentations of a particular kind (so-called one-dimensional Σ-presentations over $\mathbb{HF}(\mathbb{R})$) was proven in [5]; for ω^ω, the absence of Σ-presentations with trivial equivalence over $\mathbb{HF}(\mathbb{R})$ was proven by a more complicated method in [9].

Now we pass to relevant definitions, notation, and auxiliary results. The reader is assumed to be familiar with foundations of admissible set theory [2, 3], primarily with basic results concerning hereditarily finite superstructures.

If $\varphi(x_1,\dots,x_k,\bar{z})$ is a formula, $\mathfrak{A}$ is a structure, and $\bar{p}\in\mathfrak{A}$, then the set $\{\langle a_1,\dots,a_k\rangle \mid \mathfrak{A}\models \varphi(a_1,\dots,a_k,\bar{p})\}$ is denoted by $\varphi^{\mathfrak{A}}[x_1,\dots,x_k,\bar{p}]$. We write $A\uplus B=C$ to signify the fact that two conditions $A\cup B=C$ and $A\cap B=\varnothing$ are satisfied simultaneously. As usual, $\mathtt{sp}(a)$ denotes the *support of a* [3], which is the set of all urelements taking part in the building of a. The cardinality of a set A will be denoted by $\|A\|$.

Recall the basic definition. Although it is given for predicate signatures only, every time we speak about operations we will mean their graphs.

Definition 1 [1]. We say that the structure

$$\mathfrak{A} = \langle A; P_0^{m_0}, \ldots, P_{k-1}^{m_{k-1}} \rangle$$

is Σ-*definable over* $\mathbb{HF}(\mathfrak{M})$ if there exist $\bar{p} \in \mathbb{HF}(\mathfrak{M})$ and Σ-formulas $V(x, \bar{z})$, $E^+(x, y, \bar{z})$, $E^-(x, y, \bar{z})$, $P_0^+(x_1, \ldots, x_{m_0}, \bar{z})$, $P_0^-(x_1, \ldots, x_{m_0}, \bar{z})$, $\ldots$, $P_{k-1}^+(x_1, \ldots, x_{m_{k-1}}, \bar{z})$, and $P_{k-1}^-(x_1, \ldots, x_{m_{k-1}}, \bar{z})$ such that:

(1) $(E^+)^{\mathbb{HF}(\mathfrak{M})}[x, y, \bar{p}] \uplus (E^-)^{\mathbb{HF}(\mathfrak{M})}[x, y, \bar{p}] = \left(V^{\mathbb{HF}(\mathfrak{M})}[x, \bar{p}]\right)^2$;

(2) for all $i < k$,

$$\left(P_i^+\right)^{\mathbb{HF}(\mathfrak{M})}[\bar{x}, \bar{p}] \uplus \left(P_i^-\right)^{\mathbb{HF}(\mathfrak{M})}[\bar{x}, \bar{p}] = \left(V^{\mathbb{HF}(\mathfrak{M})}[x, \bar{p}]\right)^{m_i};$$

(3) the set $E = (E^+)^{\mathbb{HF}(\mathfrak{M})}[x, y, \bar{p}]$ is a congruence on the structure

$$\mathfrak{B} = \left\langle V^{\mathbb{HF}(\mathfrak{M})}[\bar{x}, \bar{p}]; \left(P_0^+\right)^{\mathbb{HF}(\mathfrak{M})}[\bar{x}, \bar{p}], \ldots, \left(P_{k-1}^+\right)^{\mathbb{HF}(\mathfrak{M})}[\bar{x}, \bar{p}] \right\rangle,$$

and $\mathfrak{B}/\mathrm{E} \cong \mathfrak{A}$.

We need some refinement of this definition. Namely, we will distinguish between structures of kind $\mathfrak{B}/\mathrm{E}$ (referred to as Σ-*definable* as before) and their isomorphic copies (referred to as Σ-*presentable*). The relationship between these two notions is approximately the same as between computable structures and structures having computable isomorphic copies.

Presentations in which the equivalence E in Definition 1 is trivial are said to be *simple*. In this case, in the definition of Σ-presentability we can assume that $\mathfrak{B}/_E$ coincides with $\mathfrak{B}$. As usual, in what follows we will suppose that operations on a structure are just relations corresponding to their graphs.

Let $\mathfrak{M}$ be a structure and $A \subseteq \mathfrak{M}$. An element $a \in \mathfrak{M}$ is said to be $\exists$-*algebraic over* A if there exist an $\exists$-formula $\varphi(x, \bar{y})$ and parameters $\bar{b} \in A$ such that the set $\varphi^{\mathfrak{M}}[x, \bar{b}]$ is finite and contains a. The set of all elements of $\mathfrak{M}$ that are $\exists$-algebraic over A is denoted by $\mathbf{C}_{\exists}^{\mathfrak{M}}(A)$. When working with algebraic dependency, it will be convenient to use the notions and formulations, which (as far as the author knows) were first employed by Metakides and Nerode in [10]. Under this approach, of crucial importance are so-called Steinitz closure systems. The present paper also contains references to some earlier approaches and formulations.

PROPOSITION 1. For any $S, U \subseteq \mathfrak{M}$, the following conditions are satisfied:

(1) $S \subseteq \mathbf{C}_{\exists}^{\mathfrak{M}}(S)$;

(2) $S \subseteq U \to \mathbf{C}_{\exists}^{\mathfrak{M}}(S) \subseteq \mathbf{C}_{\exists}^{\mathfrak{M}}(U)$;

(3) if $a \in \mathbf{C}_{\exists}^{\mathfrak{M}}(S)$, then there is a finite $S_0 \subseteq S$ such that $a \in \mathbf{C}_{\exists}^{\mathfrak{M}}(S_0)$;

(4) $\mathbf{C}_\exists^{\mathfrak{M}}(\mathbf{C}_\exists^{\mathfrak{M}}(S)) = \mathbf{C}_\exists^{\mathfrak{M}}(S)$.

Proof. The first three properties are obvious. We prove the fourth one. Assume that $a \in \mathbf{C}_\exists^{\mathfrak{M}}(\mathbf{C}_\exists^{\mathfrak{M}}(S))$. Then there exists an $\exists$-formula with parameters $\varphi(x, b_0, \ldots, b_{n-1})$ defining a finite set containing a, and $b_0, \ldots, b_{n-1} \in \mathbf{C}_\exists^{\mathfrak{M}}(S)$. Fix some parameters $\bar{s} \in S$ and $\exists$-formulas $\varphi_i(y_i, \bar{s})$, $i < n$, so that each set $\varphi_i^{\mathfrak{M}}[y_i, \bar{s}]$ is finite and contains b_i. For $i < n$, denote by $c_i^0, \ldots, c_i^{k_i-1}$ all pairwise distinct elements of the set $\varphi_i^{\mathfrak{M}}[y_i, \bar{s}]$. Define

$$I = \left\{ \langle m_0, \ldots, m_{n-1} \rangle \in k_0 \times \cdots \times k_{n-1} \middle| \|\varphi^{\mathfrak{M}}[x, c_0^{m_0}, \ldots, c_{n-1}^{m_{n-1}}]\| < \omega \right\},$$
$$I' = (k_0 \times \cdots \times k_{n-1}) \setminus I.$$

Let

$$p = \left\{ \left\| \varphi^{\mathfrak{M}} \left[x, c_0^{m_0}, \ldots, c_{n-1}^{m_{n-1}} \right] \right\| \,\middle|\, \langle m_0, \ldots, m_{n-1} \rangle \in I \right\}.$$

It is not hard to see that the following $\exists$-formula with parameters $\bar{s}$ defines a finite set containing a (here we use $\exists^{>p} z$ as an abbreviation for "there exist at least $p+1$ elements z such that ..."):

$$\exists\, x_0^0 \ldots x_0^{k_0-1} \ldots\ x_{n-1}^0 \ldots x_{n-1}^{k_{n-1}-1} \left[\bigwedge_{i<n} \bigwedge_{v<w<k_i} x_i^v \neq x_i^w \wedge \bigwedge_{i<n, j<k_i} \varphi_i(x_i^j, \bar{s}) \right.$$
$$\wedge \bigvee_{\langle m_0, \ldots, m_{n-1} \rangle \in I} \varphi\left(x, x_0^{m_0}, \ldots, x_{n-1}^{m_{n-1}}\right)$$
$$\left. \wedge \bigwedge_{\langle m_0, \ldots, m_{n-1} \rangle \in I'} \exists^{>p} z \varphi\left(z, x_0^{m_0}, \ldots, x_{n-1}^{m_{n-1}}\right) \right].$$

Consequently $a \in \mathbf{C}_\exists^{\mathfrak{M}}(S)$, i.e., $\mathbf{C}_\exists^{\mathfrak{M}}(\mathbf{C}_\exists^{\mathfrak{M}}(S)) \subseteq \mathbf{C}_\exists^{\mathfrak{M}}(S)$. The converse inclusion $\mathbf{C}_\exists^{\mathfrak{M}}(S) \subseteq \mathbf{C}_\exists^{\mathfrak{M}}(\mathbf{C}_\exists^{\mathfrak{M}}(S))$ follows from (1). The proposition is complete.

The property below is called the *exchange property*:

If $a \in \mathbf{C}_\exists^{\mathfrak{M}}(A \cup \{b\}) \setminus \mathbf{C}_\exists^{\mathfrak{M}}(A)$ then $b \in \mathbf{C}_\exists^{\mathfrak{M}}(A \cup \{a\})$.*

Recall that a pair consisting of a set with an operator on it is called a *Steinitz closure system* [10] if this operator possesses the properties given in Proposition 1 and the above-mentioned exchange property.

Definition 2. A structure $\mathfrak{M}$ is said to be $\exists$-*Steinitz* if the ordered pair $\langle \mathfrak{M}; \mathbf{C}_\exists^{\mathfrak{M}} \rangle$ is a Steinitz closure system.

The following statement, which derives from general properties of Steinitz closure systems, is mentioned in [10] and can be checked readily.

PROPOSITION 2. Let $\mathfrak{M}$ be an $\exists$-Steinitz structure and $B \subseteq \mathfrak{M}$. Then this structure remains $\exists$-Steinitz if we enrich the signature of $\mathfrak{M}$ with constants for all elements of B.

*Sometimes an equivalent formulation is used in which the conclusion is replaced with $b \in \mathbf{C}_\exists^{\mathfrak{M}}(A \cup \{a\}) \setminus \mathbf{C}_\exists^{\mathfrak{M}}(A)$.

Recall that a set A is said to be *independent with respect to an operator* $\mathbf{C}_{\exists}^{\mathfrak{M}}$ or simply independent (whenever it is clear which operator is meant) if $a \notin \mathbf{C}_{\exists}^{\mathfrak{M}}(A \setminus \{a\})$ for any $a \in A$.

Here we will exploit the following properties of $\exists$-Steinitz structures, which derive from the general definition of Steinitz closure systems.

THEOREM 1. Let $\mathfrak{M}$ be an $\exists$-Steinitz structure.

(1) If a (finite or infinite) sequence $a_0, a_1, \ldots$ satisfies the condition $a_i \notin \mathbf{C}_{\exists}^{\mathfrak{M}}(\{a_0, \ldots, a_{i-1}\})$ for all i, then the set $\{a_0, a_1, \ldots\}$ is independent.

(2) Let A and B be independent sets and $A \subseteq \mathbf{C}_{\exists}^{\mathfrak{M}}(B)$. Then $\|A\| \leqslant \|B\|$.

(3) Any subset B of A that is independent with respect to $\mathbf{C}_{\exists}^{\mathfrak{M}}$ is contained in a maximal independent subset B' of A. In this case $\mathbf{C}_{\exists}^{\mathfrak{M}}(B') \supseteq A$. All such sets have the same cardinality.

Proof. The first statement follows easily from the basic properties of the operator $\mathbf{C}_{\exists}^{\mathfrak{M}}$. The proof of the second statement can be found, for instance, in [11, Lemma 6.1.9]. The third statement follows from the first two and the basic properties of $\mathbf{C}_{\exists}^{\mathfrak{M}}$.

The next theorem gives us some examples of existentially Steinitz structures.

THEOREM 2. (1) Any model of a strongly minimal model complete theory is $\exists$-Steinitz.

(2) Any model complete field is an $\exists$-Steinitz structure.

(3) Any model complete ordered field is an $\exists$-Steinitz structure.

(4) Any real-closed field (in particular, the ordered field $\mathbb{R}$ of real numbers) and any algebraically closed field (in particular, the field $\mathbb{C}$ of complex numbers) are $\exists$-Steinitz structures.

Proof. (1) Follows immediately from [11, Lemma 6.1.4]; (2) and (3) are quite obvious; (4) is an immediate consequence of the previous two. The theorem is complete.

A set of *s-terms* is the smallest set that contains $\varnothing$ and all variables and is such that for any its elements $t_1, \ldots, t_k$, the expression $\{t_1, \ldots, t_k\}$ also belongs to this set. If $a_1, \ldots, a_n$ are urelements and $\tau(x_1, \ldots, x_n)$ is an s-term, then the value $\tau(a_1, \ldots, a_n)$ in a hereditarily finite superstructure is defined in an obvious way. It is also clear that each element of a hereditarily finite superstructure is a value for an appropriate s-term on some urelements.

The basic result, which will imply all other statements of the paper, is the following:

THEOREM 3. Assume that $\mathfrak{M}$ is an $\exists$-Steinitz structure of a finite signature. Let $\mathfrak{A}$ be an arbitrary structure of a finite signature such that there exist a family $(F_i)_{i<\omega}$ of unary operations definable by terms with parameters and a family $(A_i)_{i<\omega}$ of subsets of $\mathfrak{A}$ satisfying the following conditions:

(1) all sets $F_i\,[A_i]$ are uncountable;

(2) for any sequence $(a_i)_{i<\omega} \in \prod\limits_{i<\omega} A_i$, there exists an element $b \in \mathfrak{A}$ such that $F_i(b) = F_i(a_i)$ for all $i < \omega$.

Then $\mathfrak{A}$ cannot be embedded in any structure possessing a simple Σ-presentation with parameters over $\mathbb{HF}(\mathfrak{M})$.

Proof. Suppose that the structures $\mathfrak{A}$ and $\mathfrak{M}$ satisfy the conditions of the theorem, but nevertheless there exist parameters $\bar{p} \in \mathfrak{M}$, a structure $\mathfrak{B}$ Σ-definable with parameters $\bar{p}$ and trivial equivalence over $\mathbb{HF}(\mathfrak{M})$, and an isomorphism γ from $\mathfrak{A}$ to $\mathfrak{B}$. For each operation F_i, fix its termal presentation with parameters, $F_i(x) = t_i(x, c_0^i, \dots, c_{n_i-1}^i)$. Put

$$S = \{\bar{p}\} \cup \bigcup_{i<\omega,\ j<n_i} \mathtt{sp}\left(\gamma c_j^i\right).$$

Note that the set S is at most countable. Expand the structure $\mathfrak{M}$ by adding to its signature new constants for elements of S. Denote the so obtained structure by $\mathfrak{M}^*$. In view of Proposition 2, $\mathfrak{M}^*$ will also be an $\exists$-Steinitz structure.

Define elements $a_i \in \mathfrak{A}$ by induction on $i < \omega$. Assume that all a_j, $j < i$, are already defined. Take a_i to be any element of A_i such that $\mathtt{sp}(\gamma F_i(a_i))$ contains at least one element of the set $\mathfrak{M}^* \setminus \mathbf{C}_{\exists}^{\mathfrak{M}^*}\left(\bigcup_{j<i} \mathtt{sp}\left(\gamma F_j(a_j)\right)\right)$. This is possible in virtue of the following:

(1) $\|S\| \leqslant \omega$ implies that the set $\mathbf{C}_{\exists}^{\mathfrak{M}^*}\left(\bigcup_{j<i} \mathtt{sp}\left(\gamma F_j(a_j)\right)\right)$ is at most countable;

(2) γ is a bijection, and by assumption, the set of elements of kind $\gamma F_i(x)$, $x \in A_i$, is uncountable.

Therefore, at most countably many elements of A_i cannot be taken as the desired a_i. We choose a_i from among the remaining uncountably many elements of A_i. The description of the construction of a_i is complete.

For this sequence $(a_i)_{i<\omega}$, we take an element b such as in the hypothesis of the theorem. In fact, in the rest of the proof, we will find out that the cardinality of the set $\mathtt{sp}(\gamma b)$ cannot be finite, which is a contradiction.

First we construct some sequence of elements of $\mathfrak{M}^*$. Assume that we have already defined all elements $r_0, \dots, r_{i-1}$ so that:

(i) each member of the sequence $r_0, \dots, r_{i-1}$ does not belong to the value of $\mathbf{C}_{\exists}^{\mathfrak{M}^*}$ on the set of all its predecessors in the sequence;

(ii) for all $j < i$, $r_j \in \mathtt{sp}(\gamma F_j(a_j))$.

Let r_i be an arbitrary element in $\mathtt{sp}(\gamma F_i(a_i))$ which does not belong to the set

$$\mathbf{C}_{\exists}^{\mathfrak{M}^*}\left(\bigcup_{j<i} \mathtt{sp}\left(\gamma F_j(a_j)\right)\right).$$

Such an element exists by the choice of a_i. It is easy to verify that all the conditions above will be satisfied if we replace i with $i+1$, and the construction can be continued to infinity.

The sequence $(r_i)_{i<\omega}$ that we have constructed is infinite and consists of pairwise distinct elements. By Theorem 1, the set of its elements is independent with respect to $\mathbf{C}_{\exists}^{\mathfrak{M}^*}$.

Furthermore, by the choice of b, for each $i < \omega$ we have $r_i \in \mathtt{sp}(\gamma F_i(a_i)) = \mathtt{sp}(\gamma F_i(b))$. We show that $r_i \in \mathbf{C}_{\exists}^{\mathfrak{M}^*}(\mathtt{sp}(\gamma b))$. For the operation F_i, we have fixed a term $t_i(x, \bar{c})$ built from signature operations of the structure $\mathfrak{A}$ with parameters $\bar{c} \in \mathfrak{A}$ so that $F_i(x) = t_i(x, \bar{c})$ for all $x \in \mathfrak{A}$. We have

$$\gamma F_i(b) = \gamma t_i(b, \bar{c}) = t_i(\gamma b, \gamma \bar{c}).$$

Since γb and all elements of $\gamma\bar{c}$ are values of s-terms from $\mathtt{sp}(\gamma b) \cup \mathtt{sp}(\gamma\bar{c})$, and $\mathtt{sp}(\gamma\bar{c}) \subseteq S$, the property $x \in \mathtt{sp}(\gamma F_i(b))$, which is equivalent to $x \in \mathtt{sp}(t_i(\gamma b, \gamma\bar{c}))$, can be expressed by a Σ-formula with parameters in $\mathtt{sp}(\gamma b) \cup S$. For the property mentioned is Σ-presentable, we can express it as an infinite disjunction of $\exists$-formulas [12]. Taking into account this fact and the finiteness of the set $\mathtt{sp}(\gamma F_i(b))$, we see that $\mathtt{sp}(\gamma F_i(b)) \subseteq \mathbf{C}_{\exists}^{\mathfrak{M}^*}(\mathtt{sp}(\gamma b))$. Since $r_i \in \mathtt{sp}(\gamma F_i(b))$ and i is an arbitrary number, we conclude that $\{r_i \mid i < \omega\}$ is an infinite independent subset of a finite set $\mathbf{C}_{\exists}^{\mathfrak{M}^*}(\mathtt{sp}(\gamma b))$. Let m be the cardinality of an arbitrary maximal independent subset of $\mathtt{sp}(\gamma b)$. Clearly, $m < \omega$. Since $\{r_i \mid i < \omega\} \subseteq \mathbf{C}_{\exists}^{\mathfrak{M}^*}(\mathtt{sp}(\gamma b))$, it follows from Theorem 1 that $\omega \leqslant m$, which is a contradiction. The theorem is complete.

COROLLARY 1. Let $\mathfrak{M}$ be an arbitrary $\exists$-Steinitz structure and $\mathfrak{A} = \prod\limits_{i<\omega} \mathfrak{A}_i$ be the Cartesian product of a countable family of uncountable Boolean algebras $\mathfrak{A}_i$, $i < \omega$. Then $\mathfrak{A}$ has no embedding in any structure having a simple Σ-presentation with parameters over $\mathbb{HF}(\mathfrak{M})$.

Proof. It suffices to apply Theorem 3 taking functions $x \cap c_i$ as $F_i(x)$, where c_i are units of respective Boolean algebras $\mathfrak{A}_i$, and taking the whole set $\mathfrak{A}$ as A_i, $i < \omega$.

COROLLARY 2. Let $\mathfrak{M}$ be an arbitrary $\exists$-Steinitz structure. Then the Boolean algebra $\mathcal{P}(\omega)$ of all subsets of ω has no embedding in any structure having a simple Σ-presentation with parameters over $\mathbb{HF}(\mathfrak{M})$.

Proof. Since the Boolean algebra $\mathcal{P}(\omega)$ is uncountable and is isomorphic to its ωth Cartesian power, the result follows from Corollary 1.

COROLLARY 3. Let $\mathfrak{M}$ be an arbitrary $\exists$-Steinitz structure. Then the factor $\mathcal{P}(\omega)^*$ of the Boolean algebra $\mathcal{P}(\omega)$ of all subsets of ω modulo the ideal of finite sets has no embedding in any structure having a simple Σ-presentation with parameters over $\mathbb{HF}(\mathfrak{M})$.

Proof. The statement follows from the fact that $\mathcal{P}(\omega)$ is isomorphically embedded in $\mathcal{P}(\omega)^*$, and Corollary 2.

Before we formulate further results, it is pertinent to note that the lattice of open sets has cardinality 2^{ω} since any open set is the union of an at most countable family of open balls with rational centers and rational radii.

COROLLARY 4. Let $\mathfrak{M}$ be an arbitrary $\exists$-Steinitz structure. Then:

(1) The lattice of all open subsets of $\mathbb{R}^m$, $m > 0$, has no embedding in any structure having a simple Σ-presentation with parameters over $\mathbb{HF}(\mathfrak{M})$.

(2) The lattice of all closed subsets of $\mathbb{R}^m$, $m > 0$, has no embedding in any structure having a simple Σ-presentation with parameters over $\mathbb{HF}(\mathfrak{M})$.

Proof. (1) Apply Theorem 3, taking $F_i(x)$ to be the function $x \cap C_i$, where $C_i = (i, i+1/2) \times \mathbb{R}^{m-1}$, and letting A_i be the class of all open sets.

(2) Follows from (1) and the property that the lattice of closed subsets is dually isomorphic to the lattice of open subsets. The corollary is complete.

Corollary 5(1) was previously proved in [5] for the one-dimensional case and the superstructure $\mathbb{HF}(\mathbb{R})$.

COROLLARY 5. Let $\mathfrak{M}$ be an arbitrary $\exists$-Steinitz structure. Then the following structures have no embeddings in any structure having a simple Σ-presentation with parameters over $\mathbb{HF}(\mathfrak{M})$:

(1) the group $\mathtt{Sym}\,(\omega)$ of all permutations on ω;

(2) the group $\mathtt{Sym}\,(\omega)^*$ (the factor group of $\mathtt{Sym}\,(\omega)$ modulo the subgroup of all finitary permutations).

Proof. (1) Since the Cartesian power $\mathtt{Sym}\,(\omega)^\omega$ is embeddable in $\mathtt{Sym}\,(\omega)$, it suffices to prove our corollary for the group $\mathtt{Sym}\,(\omega)^\omega$. We view the group $\mathtt{Sym}\,(\omega)^\omega$ as the set of all functions from ω to $\mathtt{Sym}\,(\omega)$, which in turn will sometimes be viewed as infinite sequences $\langle f(0), f(1), \ldots\rangle$. Define projections $\mathtt{pr}_i : \mathtt{Sym}\,(\omega)^\omega \to \mathtt{Sym}\,(\omega)$, $i < \omega$, by the rule

$$\mathtt{pr}_i\,(\langle f(0), f(1), \ldots\rangle) = f(i).$$

For an arbitrary $q \in \mathtt{Sym}\,(\omega)$, define elements $q_i \in \mathtt{Sym}\,(\omega)^\omega$ by setting

$$\mathtt{pr}_j(q_i) = \begin{cases} q & \text{if } i = j, \\ 1 & \text{otherwise.} \end{cases}$$

Define the *support of an element* $f \in \mathtt{Sym}\,(\omega)$ as follows:

$$\mathtt{Supp}\,(f) = \{x \in \omega \mid f(x) \neq x\}.$$

Define also an element $p \in \mathtt{Sym}\,(\omega)$ as

$$p = \prod_{i<\omega} (2i, 2i+1),$$

and define the set

$$A = \left\{f \in \mathtt{Sym}\,(\omega) \mid f^2 = 1 \wedge \mathtt{Supp}\,(f) \text{ consists of even numbers}\right\}.$$

Let

$$A_i = \left\{f \in \mathtt{Sym}\,(\omega)^\omega \mid f^2 = 1 \wedge \mathtt{pr}_i(f) \in A\right\},$$
$$F_i(x) = p_i x p_i x.$$

It remains to verify the conditions of Theorem 3. Since $p_i^2 = 1$, it follows that for any $x \in A_i$, the element $F_i(x) = p_i x p_i x$ is the product of x conjugated via p_i and the x itself, and $\mathtt{pr}_i(F_i(x))$ is the product of $\mathtt{pr}_i(x) \in A$ (whose support is a subset of the set of even numbers) and its conjugate pxp (whose support is a subset of the set of odd numbers); moreover, $\mathtt{pr}_i(F_j(x)) = 1$ for all $i \neq j$. Therefore, condition (1) in Theorem 3 is satisfied.

Consider a family $(a_i)_{i<\omega} \in \prod\limits_{i<\omega} A_i$. Define b as $b = \langle \mathtt{pr}_0(a_0), \mathtt{pr}_1(a_1) \ldots\rangle$. Note that $b \in \bigcap\limits_{i<\omega} A_i$. In view of the above remark on the structure of $F_i(b)$, we obtain $F_i(b) = F_i(a_i)$ for all $i < \omega$.

(2) Since the group $\mathtt{Sym}\,(\omega)$ is embeddable in the group $\mathtt{Sym}\,(\omega)^*$, condition (2) follows from (1). The proof is complete.

Remark. In our proof, we did not use the group operation of inversion $^{-1}$. Therefore, the group $\mathtt{Sym}\,(\omega)$ being considered in the signature consisting of the multiplication operation and the constant for the unit element cannot be embedded in a structure with a simple Σ-presentation over an $\exists$-Steinitz structure.

The following corollary was previously proved in [9] for the particular case $\mathfrak{M} = \mathbb{R}$ but using a more complicated method.

COROLLARY 6. Let $\mathfrak{M}$ be an arbitrary $\exists$-Steinitz structure. Then the semigroup ω^ω of all functions from ω to ω cannot be embedded in a structure having a simple Σ-presentation with parameters over $\mathbb{HF}(\mathfrak{M})$.

Proof. The statement follows from Corollary 5 and the fact that the group $\mathtt{Sym}\,(\omega)$ being considered without the inversion operation is embeddable in ω^ω. The corollary is complete.

COROLLARY 7. Let $\mathfrak{M}$ be an arbitrary $\exists$-Steinitz structure.

(1) The group of all permutations on $\mathbb{R}$ that are Σ-definable over $\mathbb{HF}(\mathbb{R})$ cannot be embedded in any structure having a simple Σ-presentation with parameters over $\mathbb{HF}(\mathfrak{M})$.

(2) The semigroup of all mappings from $\mathbb{R}$ to $\mathbb{R}$ that are Σ-definable over $\mathbb{HF}(\mathbb{R})$ cannot be embedded in any structure having a simple Σ-presentation with parameters over $\mathbb{HF}(\mathfrak{M})$.

Proof. It is not hard to see that for any permutation $f \in \mathtt{Sym}\,(\omega)$, the mapping

$$\widehat{f}(x) = \begin{cases} f(x) & \text{if } x \in \omega, \\ x & \text{otherwise} \end{cases}$$

is a permutation on $\mathbb{R}$ Σ-definable with appropriate parameters over $\mathbb{HF}(\mathbb{R})$, and the mapping $f \mapsto \widehat{f}$ is an isomorphic embedding.

(1) Follows from Corollary 5.

(2) Follows immediately from (1).

The corollary is complete.

Question. Will similar statements hold true for the Σ-presentability of structures over hereditarily finite superstructure without the extra requirement of triviality for the equivalence E?

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