DOI 10.1007/s10469-016-9384-7
Algebra and Logic, Vol. 55, No. 2, May, 2016
(Russian Original Vol. 55, No. 2, March-April, 2016)

COMPACTNESS CONDITIONS IN UNIVERSAL ALGEBRAIC GEOMETRY

P. Modabberi[1] **and M. Shahryari**[2]

UDC 512.54.0+512.57

Keywords: *algebraic structures, equations, algebraic sets, radical ideal, coordinate algebra, Zariski topology, equationally Noetherian algebras, q_ω-compactness, u_ω-compactness, metacompact algebras, metacompact spaces, equationally Artinian algebras, prevarieties, varieties, free algebras, equational domains, Hilbert's basis theorem.*

Different types of compactness in the Zariski topology are explored: for instance, equational Noetherianity, equational Artinianity, q_ω-compactness, and u_ω-compactness. Moreover, general results on the Zariski topology over algebras and groups are proved.

INTRODUCTION

Universal algebraic geometry is a new direction of research in modern algebra. The subject of universal algebraic geometry is studying equations over an arbitrary algebraic structure A (as distinct from classical algebraic geometry, where A is a field). Many papers concerning algebraic geometry over groups have been published to date (see [1-5]). In a series of papers, O. Kharlampovich and A. Myasnikov developed algebraic geometry over free groups, which made it possible to find a positive solution to A. Tarski's well-known problem on elementary theories of free groups (see [6] and also [7] containing an independent solution to Tarski's problem). Moreover, in [8], it was proved that the elementary theory of free groups is decidable.

Algebraic geometry over other types of algebraic structures is presented, for instance, in [9] (Lie algebras) and in [10, 11] (additive monoid of natural numbers). A systematic presentation of the foundations of algebraic geometry was initiated by E. Daniyarova, A. Myasnikov, and V. Remeslennikov [12-15].

[1]Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran; p_modabberi@tabrizu.ac.ir. [2]Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran; mshahryari@tabrizu.ac.ir. Translated from *Algebra i Logika*, Vol. 55, No. 2, pp. 219-256, March-April, 2016. Original article submitted January 31, 2014; revised October 3, 2015.

In this paper, we deal with different compactness conditions in universal algebraic geometry. In other words, we are interested in how properties of a system of equations depend on properties of its subsystems. Compactness conditions may be exemplified by concepts such as equational Noetherianity, weak equational Noetherianity, q_ω-compactness, and u_ω-compactness.

First we cite basic notions and results from universal algebraic geometry; we give the definition of a Zariski topology, and also of basic types of compactness of systems of equations (equational Noetherianity, q_ω- and u_ω-compactness). Then we look into the concept of a metacompact algebra and show how it is related to the concept of a metacompact Zariski topology. Next we couch the definition of equationally Artinian algebras and point out several equivalent definitions of being equationally Artinian. Finally, we study compactness conditions for free algebras in arbitrary varieties.

1. BASIC DEFINITIONS

Following [12-15], we give basic definitions and concepts in universal algebraic geometry over algebraic structures in functional languages. (For a deeper acquaintance with universal algebra and model theory, we recommend [16-18].) Note also that many results of our paper hold not only for functional languages, but also for arbitrary languages that may contain predicate symbols.

1.1. Systems of equations and algebraic sets. Let $\mathcal{L}$ be a functional language and A an algebraic structure in the language $\mathcal{L}$ (an $\mathcal{L}$-algebra). Adding to the language new constant symbols $a \in A$ corresponding to all elements of the algebra A, we obtain an extended language $\mathcal{L}(A)$. An algebra B in the language $\mathcal{L}(A)$ is called an A-*algebra* if the map $a \mapsto a^B$ is an embedding of A in B (here, a^B denotes an interpretation of constant symbols a in the $\mathcal{L}(A)$-algebra B). Denote by $X = \{x_1, \ldots, x_n\}$ a finite set of variables, and by $T_{\mathcal{L}}(X)$ the term $\mathcal{L}$-algebra generated by X.

For simplicity, we give our definitions in the language $\mathcal{L}$, but these can be readily extended to the case of $\mathcal{L}(A)$.

An equation in the language $\mathcal{L}$ (an $\mathcal{L}$-equation) in variables $X = \{x_1, \ldots, x_n\}$ is a pair (p, q), where $p, q \in T_{\mathcal{L}}(X)$. In this paper, we identify the concepts of an atomic formula in $\mathcal{L}$, of an equation, and of a pair of $\mathcal{L}$-terms (p, q). Thus the set $At_{\mathcal{L}}(X)$ of atomic formulas in the language $\mathcal{L}$ and the product algebra $T_{\mathcal{L}}(X) \times T_{\mathcal{L}}(X)$ are assumed to be equal.

Any subset $S \subseteq At_{\mathcal{L}}(X)$ of atomic formulas is called a *system of equations in the language* $\mathcal{L}$ (an $\mathcal{L}$-*system*). A system S is said to be *consistent* over an algebra A if there is an element $(a_1, \ldots, a_n) \in A^n$ such that for all equations $(p \approx q) \in S$, the following equality holds:

$$p^A(a_1, \ldots, a_n) = q^A(a_1, \ldots, a_n),$$

where p^A and q^A denote interpretations of terms p and q in A. Otherwise, we say that S is *inconsistent* over A. An $\mathcal{L}$-system S is called an *ideal* if it coincides with some congruence on the algebra $T_{\mathcal{L}}(X)$. For an arbitrary system S of $\mathcal{L}$-equations, an ideal generated by S is the smallest congruence containing S, and we denote it by $[S]$.

For an arbitrary $\mathcal{L}$-algebra A, an element $(a_1, \dots, a_n) \in A^n$ is often written as $\overline{a}$. Let S be a system of $\mathcal{L}$-equations. Then the set

$$V_A(S) = \{\overline{a} \in A^n : \forall (p \approx q) \in S,\ p^A(\overline{a}) = q^A(\overline{a})\}$$

is said to be *algebraic over an $\mathcal{L}$-algebra A*. It is clear that for any nonempty family $\{S_i\}_{i \in I}$ of $\mathcal{L}$-systems,

$$V_A\left(\bigcup_{i \in I} S_i\right) = \bigcap_{i \in I} V_A(S_i).$$

We say that a subset in A^n is *closed* if it is an arbitrary intersection of finite unions of algebraic sets over an $\mathcal{L}$-algebra A. In view of this definition, we obtain a topology on A^n, which is called the *Zariski topology*. For an arbitrary subset $Y \subseteq A^n$, its closure with respect to the Zariski topology is denoted by $\overline{Y}$. Also we denote by Y^{ac} the smallest algebraic set containing Y. The equality $\overline{Y} = Y^{ac}$ is untrue in general; however, it holds for the following class of algebraic structures.

An $\mathcal{L}$-algebra A is called an *equational domain* if the union of two algebraic sets in A^n $(n \in \mathbb{M})$ is again an algebraic set for any n. Examples of equational domains and their properties were treated in [15].

In the language $\mathcal{L}(A)$, all definitions of universal algebraic geometry are given by analogy with the language $\mathcal{L}$. Below $\mathcal{L}(A)$-equations $p \approx q$ are also called *equations with coefficients in A*.

Note that the class of equational domains may change in passing from $\mathcal{L}$ to $\mathcal{L}(A)$. For example, as shown in [15], in the group language without constants $\mathcal{L} = (\cdot, ^{-1}, 1)$, there exist no nontrivial equational domains; however, any non-Abelian free group F is an equational domain in the language $\mathcal{L}(F)$.

1.2. Radicals and coordinate algebras. Let $\mathcal{L}$ be an arbitrary functional language and A an algebra in $\mathcal{L}$. For any set $Y \subseteq A^n$, we put

$$\mathrm{Rad}_A(Y) = \{(p, q) : \forall\, \overline{a} \in Y,\ p^A(\overline{a}) = q^A(\overline{a})\}.$$

It is easy to see that the set $\mathrm{Rad}_A(Y)$ is an ideal in the term algebra. Every ideal of this type is called an *A-radical ideal* (or a *radical ideal* for short). The radical of a system S of $\mathcal{L}$-equations is defined as $\mathrm{Rad}_A(V_A(S))$. In what follows, we omit the subscript in $\mathrm{Rad}_A(Y)$ if it is known which algebra A is involved.

Note that any ideal in the term algebra $T_{\mathcal{L}}(X)$ is in fact an A-radical ideal for some $\mathcal{L}$-algebra A. Indeed, if, for any ideal R in the term algebra $T_{\mathcal{L}}(X)$, we consider the quotient algebra $B(R) = T_{\mathcal{L}}(X)/R$, then we obtain $\mathrm{Rad}_{B(R)}(R) = R$.

It is easy to see that a set Y is algebraic iff $V_A(\mathrm{Rad}(Y)) = Y$. For an arbitrary set Y, we have $V_A(\mathrm{Rad}(Y)) = Y^{ac}$ (see [13]). The *coordinate algebra* of a set Y is the quotient algebra

$$\Gamma(Y) = \frac{T_{\mathcal{L}}(X)}{\mathrm{Rad}(Y)}.$$

An arbitrary element of $\Gamma(Y)$ is denoted by $[p]_Y$. We define a function $p^Y : Y \to A$ by setting

$$p^Y(\overline{a}) = p^A(a_1, \dots, a_n).$$

The function p^Y is a *term function* on Y. The set of all such functions will be denoted by $T(Y)$. It is easy to show that $\Gamma(Y) \cong T(Y)$.

Two $\mathfrak{L}$-systems S and S' are *equivalent* over an $\mathfrak{L}$-algebra A if $V_A(S) = V_A(S')$. Thus $\mathrm{Rad}_A(S)$ is the largest $\mathfrak{L}$-system which is equivalent to S. Note also that $[S] \subseteq \mathrm{Rad}_A(S)$.

One of the major problems in universal algebraic geometry over an $\mathfrak{L}$-algebra A is to describe all $\mathfrak{L}$-algebras isomorphic to coordinate algebras of algebraic sets over A. There are many necessary and sufficient conditions for an $\mathfrak{L}$-algebra to be a coordinate algebra of some algebraic set over an $\mathfrak{L}$-algebra A (see Sec. 1.4).

1.3. Equationally Noetherian algebras. The most important type of compactness is equational Noetherianity.

Definition 1. An $\mathfrak{L}$-algebra A is said to be *equationally Noetherian* if, for any $\mathfrak{L}$-system S of equations, there exists a finite subsystem $S_0 \subseteq S$ such that $V_A(S) = V_A(S_0)$.

An $\mathfrak{L}$-algebra A is *n-equationally Noetherian* if, for any $\mathfrak{L}$-system S with at most n variables, there exists a finite subsystem $S_0 \subseteq S$ such that $V_A(S) = V_A(S_0)$.

If an A-algebra is equationally Noetherian in the language $\mathfrak{L}(A)$, then we say that it is *A-equationally Noetherian*. Many examples of equationally Noetherian algebras can be found in [13]. Among these are Noetherian rings, linear groups over Noetherian rings, and free groups. In [13], it was proved that the following four assertions are equivalent:

(i) an $\mathfrak{L}$-algebra A is equationally Noetherian;

(ii) for any $\mathfrak{L}$-system S, there exists a finite subsystem S_0 in $[S]$ such that $V_A(S) = V_A(S_0)$;

(iii) for any n, the Zariski topology on A^n is Noetherian, i.e., A^n has no infinite strictly descending chain of closed subsets;

(iv) for any chain of epimorphisms of coordinate $\mathfrak{L}$-algebras

$$\Gamma(Y_1) \to \Gamma(Y_2) \to \Gamma(Y_3) \to \dots,$$

there is a natural number n such that all epimorphisms $\Gamma(Y_i) \to \Gamma(Y_{i+1})$ are isomorphisms for any $i \geq n$.

Over an equationally Noetherian $\mathfrak{L}$-algebra A, an arbitrary closed set in A^n decomposes into a finite union of irreducible algebraic subsets, and this decomposition is unique up to a permutation. (A set Y is *irreducible* if it is not contained in any finite union of closed sets each of which has a nonempty intersection with Y.)

THEOREM 1 [13]. Let A be an equationally Noetherian $\mathfrak{L}$-algebra. Then the following $\mathfrak{L}$-algebras are also equationally Noetherian:

(i) any subalgebra and any filtered power of A;

(ii) any coordinate algebra over A;

(iii) any fully residually A-algebra;

(iv) any algebra in a quasivariety generated by A;

(v) any algebra universally equivalent to A;

(vi) any limit algebra over A;

(vii) any finitely generated algebra defined by a complete atomic type in the universal theory for A.

The next definition is a generalization of the concept of equationally Noetherian algebras.

Definition 2. An $\mathcal{L}$-algebra A is *weakly equationally Noetherian* if, for any $\mathcal{L}$-system S, there exists a finite system S_0 of $\mathcal{L}$-equations that is equivalent to S over A. Note that here we do not require that $S_0 \subseteq S$.

Clearly, an $\mathcal{L}$-algebra A is weakly equationally Noetherian iff the radical ideal $\mathrm{Rad}_A(S)$ is finitely generated for any $\mathcal{L}$-system S; i.e., there exists a finite subset $S_0 \subseteq \mathrm{Rad}_A(S)$ such that $\mathrm{Rad}_A(S) = \mathrm{Rad}_A(S_0)$.

The logical meaning of the last equality can be conveyed as follows. Let $\mathrm{QId}(A)$ be the set of all quasi-identities true in A. Then an $\mathcal{L}$-algebra A is weakly equationally Noetherian iff for any $\mathcal{L}$-system S there exist finitely many equations $p_1 \approx q_1, \ldots, p_m \approx q_m$ such that

$$\mathrm{Rad}_A(S) = \left\{ (p,q) : \left(\forall x_1 \ldots \forall x_n \left(\bigwedge_{i=1}^{m} p_i \approx q_i \Rightarrow p \approx q \right) \right) \in \mathrm{QId}(A) \right\}.$$

1.4. Unification theorems. In [12-14], the so-called *unification theorems* are proved which contain seven equivalent approaches to the problem of describing coordinate $\mathcal{L}$-algebras of algebraic sets over an arbitrary $\mathcal{L}$-algebra A. Below we cite a full formulation of just one of these unification theorems (all notions with which the reader is unfamiliar can be found in the papers mentioned).

THEOREM 2. Let A and Γ be algebras in the language $\mathcal{L}$. Suppose A is equationally Noetherian and Γ is finitely generated. Then the following assertions are equivalent:

(i) an $\mathcal{L}$-algebra Γ is the coordinate $\mathcal{L}$-algebra of some irreducible algebraic set over A;

(ii) an $\mathcal{L}$-algebra Γ is a fully residually A-algebra, i.e., for any finite subset $C \subseteq \Gamma$, there exists a homomorphism $\alpha : \Gamma \to A$ such that the restriction of α to C is an injective map;

(iii) an $\mathcal{L}$-algebra Γ embeds in some ultrapower of an $\mathcal{L}$-algebra A;

(iv) an $\mathcal{L}$-algebra Γ belongs to the universal closure of an $\mathcal{L}$-algebra A, i.e., $\mathrm{Th}_\forall(A) \subseteq \mathrm{Th}_\forall(\Gamma)$;

(v) an $\mathcal{L}$-algebra Γ is a limit algebra over A;

(vi) an $\mathcal{L}$-algebra Γ is defined by a complete type of $\mathrm{Th}_\forall(A)$.

2. TYPES OF COMPACTNESS

We consider the most important types of compactness related to properties of the Zariski topology.

The notions of q_ω- and u_ω-compact algebras were introduced in [14] where, also, unification theorems for these classes were proved. Below we give a topological characterization of equational domains in the classes of q_ω- and u_ω-compact algebras. We also introduce yet another generalization of the notion of being equational Noetherian for algebras using the concept of *metacompact* topological spaces.

2.1. q_ω-Compactness and u_ω-compactness. Following [14], we give definitions of q_ω- and u_ω-compact algebras.

Definition 3. An $\mathcal{L}$-algebra A is said to be q_ω-*compact* if, for any system S of equations and any equation $p \approx q$ such that $V_A(S) \subseteq V_A(p \approx q)$, there exists a finite subsystem $S_0 \subseteq S$ for which $V_A(S_0) \subseteq V_A(p \approx q)$.

An $\mathcal{L}$-algebra A is u_ω-*compact* if, for an arbitrary system S of equations and for finitely many equations $\{p_1 \approx q_1, \ldots, p_m \approx q_m\}$ such that

$$V_A(S) \subseteq \bigcup_{i=1}^{m} V_A(p_i \approx q_i),$$

there exists a finite subsystem $S_0 \subseteq S$ for which

$$V_A(S_0) \subseteq \bigcup_{i=1}^{m} V_A(p_i \approx q_i).$$

It is easy to show that any equationally Noetherian $\mathcal{L}$-algebra is u_ω-compact and that every u_ω-compact $\mathcal{L}$-algebra is q_ω-compact. Similarly to an n-equationally Noetherian algebra, we can define q_ω^n- and u_ω^n-compact algebras.

As noted, the results of [12] give rise to a topological description of equationally Noetherian $\mathcal{L}$-algebras.

PROPOSITION 1. An $\mathcal{L}$-algebra A is equationally Noetherian if and only if all subsets of A^n are compact for every n.

Below we give a topological description of q_ω- and u_ω-compact $\mathcal{L}$-algebras.

2.2. q_ω-Compactness and Zariski topology. Consider an $\mathcal{L}$-algebra A and an $\mathcal{L}$-equation $p \approx q$. Denote the set $A^n \setminus V_A(p \approx q)$ by $C_A(p \approx q)$.

PROPOSITION 2. Let an $\mathcal{L}$-algebra A be an equational domain. Then A is q_ω-compact if and only if $C_A(p \approx q)$ is compact for all $\mathcal{L}$-equations $p \approx q$.

Proof. Assume that an $\mathcal{L}$-algebra A is q_ω-compact. Let $C_A(p \approx q) \subseteq \bigcup_{i\in I} C_i$, with $C_i \subseteq A^n$ open. Since A is an equational domain, $C_i = A^n \setminus V_A(S_i)$ for some system S_i. We have

$$A^n \setminus V_A(p \approx q) \subseteq \bigcup_{i\in I}(A^n \setminus V_A(S_i)) = A^n \setminus \bigcap_{i\in I} V_A(S_i),$$

hence

$$\bigcap_{i\in I} V_A(S_i) \subseteq V_A(p \approx q),$$

and so

$$V_A\left(\bigcup_{i\in I} S_i\right) \subseteq V_A(p \approx q).$$

The definition of q_ω-compactness implies that there exists a finite subsystem $S' \subseteq \bigcup_{i\in I} S_i$ with the property $V_A(S') \subseteq V_A(p \approx q)$. We have

$$S' \subseteq S_{i_1} \cup \ldots \cup S_{i_m}$$

for some $i_1, \ldots, i_m \in I$. Hence

$$\bigcap_{j=1}^{m} V_A(S_{i_j}) \subseteq V_A(S') \subseteq V_A(p \approx q),$$

and so $C_A(p \approx q) \subseteq \bigcup_{j=1}^{m} C_{i_j}$, which shows that $C_A(p \approx q)$ is compact.

Suppose now that any $C_A(p \approx q)$ is compact. Consider a system S and an equation $p \approx q$ for which $V_A(S) \subseteq V_A(p \approx q)$. Then

$$\begin{aligned} C_A(p \approx q) &\subseteq A^n \setminus V_A(S) \\ &= A^n \setminus \bigcap_{(p\approx q)\in S} V_A(p \approx q) \\ &= \bigcup_{(p\approx q)\in S} (A^n \setminus V_A(p \approx q)). \end{aligned}$$

Hence

$$C_A(p \approx q) \subseteq \bigcup_{i=1}^{m} (A^n \setminus V_A(p_i \approx q_i))$$

for some $p_1 \approx q_1, \ldots, p_m \approx q_m \in S$. Therefore,

$$V_A(p_1 \approx q_1, \ldots, p_m \approx q_m) \subseteq V_A(p \approx q),$$

i.e., A is q_ω-compact. $\square$

A similar result is true for u_ω-compact equational domains. We can show that an equational domain A is u_ω-compact iff any finite intersection of sets of the form $C_A(p \approx q)$ is compact.

2.3. Metacompact algebras. Recall that a topological space is *metacompact* if every open covering of the space has a subcovering such that each point of the space belongs to just finitely many elements of the covering. The notion of metacompactness for topological spaces gives rise to a similar notion for algebras.

Let S be a system of $\mathfrak{L}$-equations and A be an $\mathfrak{L}$-algebra. Denote by $V_A^*(S)$ the set of all points $(a_1, \ldots, a_n) \in A^n$ such that $p^A(a_1, \ldots, a_n) \neq q^A(a_1, \ldots, a_n)$ for all but finitely many equations $(p \approx q) \in S$.

Definition 4. An $\mathcal{L}$-algebra A is *metacompact* if, for any $\mathcal{L}$-system S inconsistent over A, there exists an inconsistent subsystem $S' \subseteq S$ with the property $V_A^*(S') = A^n$.

It is easy to show that every equationally Noetherian $\mathcal{L}$-algebra A is metacompact. Indeed, if an $\mathcal{L}$-system S is inconsistent over A, then there is a finite inconsistent system $S_0 \subseteq S$ with the property $V_A(S_0) = V_A(S)$. Since S_0 is finite, we have $V_A^*(S_0) = A^n$, and so the $\mathcal{L}$-algebra A is metacompact.

PROPOSITION 3. Let A be an equational domain in the language $\mathcal{L}$. Then the space A^n is metacompact for any natural n.

Proof. Consider a covering $A^n = \bigcup\limits_{\alpha \in I} C_\alpha$, indexed by a set of ordinals $I = \{\alpha : \alpha \leq \kappa\}$. Since A is an equational domain, every C_α has the form

$$C_\alpha = A^n \setminus V_A(S_\alpha)$$

for some $\mathcal{L}$-system S_α, and $\bigcap\limits_\alpha V_A(S_\alpha) = \varnothing$. Suppose $S = \bigcup\limits_\alpha S_\alpha$. Then $V_A(S) = \varnothing$, and so there exists an inconsistent subsystem S' such that $V_A^*(S') = A^n$. Let

$$S'_0 = S' \cap S_0,$$
$$S'_{\alpha^+} = (S' \cap S_{\alpha^+}) \setminus S'_\alpha.$$

For any limit ordinal, put

$$S'_\lambda = (S' \cap S_\lambda) \setminus \bigcup_{\alpha < \lambda} S'_\alpha.$$

Clearly, $S' = \bigcup\limits_\alpha S'_\alpha$, $S'_\alpha \subseteq S_\alpha$, and $S'_\alpha \cap S'_\beta = \varnothing$ for distinct α and β. Now, put

$$C'_\alpha = A^n \setminus V_A(S'_\alpha).$$

Then $C'_\alpha \subseteq C_\alpha$ and $\bigcup\limits_\alpha C'_\alpha = A^n$. We have thus obtained a subcovering C'_α, and it remains to show that each point of the space A^n belongs to just finitely many elements of the covering.

Let $\overline{a} \in A^n$. Then $\overline{a} \in V_A^*(S') = A^n$. Consequently, there are finitely many equations

$$p_1 \approx q_1, \ldots, p_m \approx q_m \in S'$$

such that $p_i^A(\overline{a}) \neq q_i^A(\overline{a})$ $(1 \leq i \leq m)$. For each number i, there exists a unique $\alpha_i \in I$ with the property $(p_i \approx q_i) \in S'_{\alpha_i}$. Therefore, $\overline{a}$ does not belong to the set

$$V_A(S'_{\alpha_1}) \cup \ldots \cup V_A(S'_{\alpha_m}).$$

For other $\alpha \in I$, we have $\overline{a} \in V_A(S'_\alpha)$ (since $S'_{\alpha_i} \cap S'_\alpha = \varnothing$). Hence the point $\overline{a}$ belongs only to $C'_{\alpha_1}, \ldots, C'_{\alpha_m}$, and so A^n is metacompact. $\square$

3. FREE ALGEBRAS IN VARIETIES

Consider universal algebraic geometry over a free algebra $F_{\mathbf{V}}$ in a variety $\mathbf{V}$. A decisive role here is played by the following simple observation: every solution for an equation over $F_{\mathbf{V}}$ corresponds to an identity in $\mathbf{V}$.

3.1. Free algebras in prevarieties. Let $\mathbf{V}$ be a prevariety of $\mathcal{L}$-algebras; i.e., $\mathbf{V}$ is closed under taking subalgebras and direct products. Denote by $\mathrm{Cong}(A)$ the family of all congruences of an $\mathcal{L}$-algebra A. Define an ideal of the term $\mathcal{L}$-algebra $T_{\mathcal{L}}(X)$ by setting

$$R_{\mathbf{V}}(X) = \bigcap \left\{ R \in \mathrm{Cong}(T_{\mathcal{L}}(X)) : \ \frac{T_{\mathcal{L}}(X)}{R} \in \mathbf{V} \right\}.$$

In other words, $R_{\mathbf{V}}(X)$ is the smallest congruence on the term algebra $T_{\mathcal{L}}(X)$, for which the quotient

$$F_{\mathbf{V}}(X) = \frac{T_{\mathcal{L}}(X)}{R_{\mathbf{V}}(X)}$$

belongs to $\mathbf{V}$. An $\mathcal{L}$-algebra $F_{\mathbf{V}}(X)$ is called the *free algebra* of the prevariety $\mathbf{V}$ generated by the set X. We may show that a free algebra in $\mathbf{V}$ can be defined in a different way: an $\mathcal{L}$-algebra $F_{\mathbf{V}}(X)$ is generated by a set $\overline{X} = \{x/R_{\mathbf{V}}(X) : x \in X\}$ and any map from $\overline{X}$ to an $\mathcal{L}$-algebra $A \in \mathbf{V}$ extends uniquely to a homomorphism of the $\mathcal{L}$-algebra $F_{\mathbf{V}}(X)$ (for more details about properties of free algebras over prevarieties, we refer the reader to [16]).

A logical characterization of the set $R_{\mathbf{V}}(X)$ is given in

LEMMA 1. Let $\mathbf{V}$ be a prevariety and X be a set of variables. Then

$$R_{\mathbf{V}}(X) = \{(p, q) : \ \mathbf{V} \vDash (\forall x_1 \ldots \forall x_n \, p \approx q)\}.$$

Proof. Note that $\mathbf{V} \vDash (\forall x_1 \ldots \forall x_n p \approx q)$ in the formulation of the lemma means that for every $\mathcal{L}$-algebra $A \in \mathbf{V}$ and for arbitrary $a_1, \ldots, a_n \in A$, the following equality holds:

$$p^A(a_1, \ldots, a_n) = q^A(a_1, \ldots, a_n).$$

We prove the statement of the lemma for a finite set X (by a slight modification, the proof is readily extended to the case where X is infinite).

Put

$$K = \{(p, q) : \ V \vDash (\forall x_1 \ldots \forall x_n p \approx q)\}.$$

Let R be an ideal in the term $\mathcal{L}$-algebra $T_{\mathcal{L}}(X)$ such that $A = T_{\mathcal{L}}(X)/R \in \mathbf{V}$, and let $(p, q) \in K$. We have

$$p^A(x_1/R, \ldots, x_n/R) = q^A(x_1/R, \ldots, x_n/R),$$

where x/R denotes the equivalence class containing x. The last equality is equivalent to $p/R = q/R$, and so $(p, q) \in R$. Thus $K \subseteq R_{\mathbf{V}}$.

Now we prove the inverse inclusion $K \supseteq R_{\mathbf{V}}$. Denote by F the quotient $T_{\mathcal{L}}(X)/K$. Clearly, an $\mathcal{L}$-algebra F is generated by a set $X^* = \{x/K : \ x \in X\}$.

Consider an arbitrary map $\alpha : X^* \to A$, where A is some $\mathcal{L}$-algebra in the variety $\mathbf{V}$. Define $\alpha_0 : X \to A$ as $\alpha_0(x) = \alpha(x/K)$. The properties of a term algebra imply that there exists a homomorphism $\alpha_0' : T_{\mathcal{L}}(X) \to A$ extending α_0. It is easy to see that for all terms $p \in T_{\mathcal{L}}(X)$, we have $\alpha_0'(p) = p^A(\alpha(x_1/K), \ldots, \alpha(x_n/K))$. The last equality shows that for $(p, q) \in K$, we have $\alpha_0'(p) = \alpha_0'(q)$, and hence $(p, q) \in \ker \alpha_0'$. Thus we obtain a homomorphism $\alpha' : F \to A$ with the property

$$\alpha'(t/K) = \alpha_0'(t).$$

Clearly, α' coincides with α over the set X^*. We prove that α' is a unique extension of α to the $\mathcal{L}$-algebra F. Let $h : F \to A$ be another homomorphism such that h coincides with α over X^*. Using induction on the complexity of a term $t = f(t_1, \ldots, t_m)$, we obtain

$$\begin{aligned} h(t/K) &= h\left(\frac{f(t_1, \ldots, t_m)}{K}\right) \\ &= f^A(h(t_1/K), \ldots, h(t_m/K)) \\ &= f^A(\alpha'(t_1/K), \ldots, \alpha'(t_m/K)) \\ &= \alpha'\left(\frac{f(t_1, \ldots, t_m)}{K}\right) \\ &= \alpha'(t/K). \end{aligned}$$

Thus the $\mathcal{L}$-algebra F is a free algebra in the prevariety $\mathbf{V}$, and so $F \in \mathbf{V}$. Consequently, $R_{\mathbf{V}} \subseteq K$. □

COROLLARY 1. Let $\mathbf{V}$ be a variety of $\mathcal{L}$-algebras, $F = F_{\mathbf{V}}(X)$, and $p \approx q$ be an $\mathcal{L}$-equation with variables $X = \{x_1, x_2, \ldots, x_n\}$. A point $(\bar{t}_1, \ldots, \bar{t}_n) \in F^n$ is a solution for $p \approx q$ if and only if

$$\forall x_1 \ldots \forall x_m \ p(t_1, \ldots, t_n) \approx q(t_1, \ldots, t_n)$$

is true in each $\mathcal{L}$-algebra of the prevariety $\mathbf{V}$.

Proof. Let a point $(\bar{t}_1, \ldots, \bar{t}_n) \in F^n$ satisfy an equation $p \approx q$. Then

$$p^F(\bar{t}_1, \ldots, \bar{t}_n) = q^F(\bar{t}_1, \ldots, \bar{t}_n),$$

and so

$$\frac{p(t_1, \ldots, t_n)}{R_{\mathbf{V}}(X)} = \frac{q(t_1, \ldots, t_n)}{R_{\mathbf{V}}(X)}.$$

Consequently, $(p(t_1, \ldots, t_n), q(t_1, \ldots, t_n)) \in R_{\mathbf{V}}(X)$, and by the above lemma,

$$\mathbf{V} \vDash \forall x_1 \ldots \forall x_m p(t_1, \ldots, t_n) \approx q(t_1, \ldots, t_n).$$

The converse statement can be proved similarly. □

COROLLARY 2. Let $\mathbf{V}$ be a variety, $X = \{x_1, \ldots, x_n\}$, and $F = F_{\mathbf{V}}(X)$. Then

$$\mathrm{Rad}_F(\overline{x}_1, \ldots, \overline{x}_n) = \mathrm{Id}_X(\mathbf{V}),$$

where $\mathrm{Id}_X(\mathbf{V})$ is the set of identities of $\mathbf{V}$ with X as the set of variables.

3.2. Finitely axiomatizable classes of $\mathcal{L}$-algebras. The property of being equationally Noetherian can be used to construct a list of axioms for some universal classes of algebras. In the theorem below, we give a list of axioms for the universal closure $\mathrm{Ucl}(A)$ generated by some $\mathcal{L}$-algebra A. Recall that the *universal closure* of an $\mathcal{L}$-algebra A is the class of all $\mathcal{L}$-algebras such that any universal formula ϕ true in A is true in these algebras.

THEOREM 3. Let A be an $\mathcal{L}$-algebra and $\mathbf{V} = \mathrm{Var}(A)$ be the variety generated by some $\mathcal{L}$-algebra A. Assume that a free $\mathcal{L}$-algebra $F_{\mathbf{V}}(X)$ of the variety $\mathbf{V}$ is equationally Noetherian for any finite set X. Suppose that a class of $\mathcal{L}$-algebras $\mathbf{W} \subseteq \mathrm{Ucl}(A)$ is axiomatized by a set of identities $\Sigma \subseteq At_{\mathcal{L}}(x_1, \ldots, x_n)$. Then there exists a finite subset of identities $\Sigma_0 \subseteq \Sigma$ which axiomatizes $\mathbf{W}$. In other words,

$$\mathbf{W} = \{B \in \mathrm{Ucl}(A) : \ B \vDash \Sigma_0\}.$$

Proof. Let $\Sigma = \{p_i \approx q_i : \ i \in I\}$. Then

$$\mathbf{W} = \left\{ B \in \mathrm{Ucl}(A) : \ B \vDash \bigwedge_{i \in I} \forall x_1 \ldots \forall x_n p_i \approx q_i \right\}.$$

Put $X = \{x_1, \ldots, x_n\}$ and $F = F_{\mathbf{V}}(X)$. We can consider Σ as a system of $\mathcal{L}$-equations over F, and since F is equationally Noetherian, there exists a finite subset

$$\Sigma_0 = \{p_i \approx q_i : \ i \in I_0\} \subseteq \Sigma$$

with the property $V_F(\Sigma) = V_F(\Sigma_0)$.

Now, by virtue of Corollary 1 in Sec. 3.1, for any $t_1, \ldots, t_n$, we have

$$\mathbf{V} \vDash \bigwedge_{i \in I_0} \forall x_1 \ldots \forall x_n p_i(t_1, \ldots, t_n) \approx q_i(t_1, \ldots, t_n)$$

iff

$$\mathbf{V} \vDash \bigwedge_{i \in I} \forall x_1 \ldots \forall x_n p_i(t_1, \ldots, t_n) \approx q_i(t_1, \ldots, t_n).$$

Since $\mathbf{V} = \mathrm{Var}(A)$, we obtain

$$A \vDash \left(\bigwedge_{i \in I_0} \forall x_1 \ldots \forall x_n p_i \approx q_i \Rightarrow \bigwedge_{i \in I} \forall x_1 \ldots \forall x_n p_i \approx q_i \right).$$

This shows that for any $j \in I$, the universal formula

$$\bigwedge_{i \in I_0} \forall x_1 \ldots \forall x_n p_i \approx q_i \Rightarrow \forall x_1 \ldots \forall x_n p_j \approx q_j$$

is true in A. Therefore,

$$\mathrm{Th}_\forall(A) + \left(\bigwedge_{i \in I_0} \forall x_1 \ldots \forall x_n p_i \approx q_i \right) \models \forall x_1 \ldots \forall x_n p_j \approx q_j,$$

and hence

$$\mathbf{W} = \left\{ B \in \mathrm{Ucl}(A) : \; B \models \bigwedge_{i \in I_0} \forall x_1 \ldots \forall x_n p_i \approx q_i \right\}.$$

Thus the set of formulas Σ_0 is a list of axioms for $\mathbf{W}$ within $\mathrm{Ucl}(A)$. $\square$

Example 1. Let $\mathbf{V}$ be the variety of all groups. It is known that $\mathbf{V}$ is generated by the free group F_2 of rank two; i.e., $\mathbf{V} = \mathrm{Var}(F_2)$. Denote by $\mathrm{Id}(F_2)$ the set of all true identities in F_2.

Let $n \geq 1003$ be an odd number and consider the following family of identities:

$$\Sigma = \{[x^{pn}, y^{pn}]^n \approx 1 : \; p \text{ is a prime number}\}.$$

Denote by $\mathbf{W}_1$ the variety of all groups axiomatized by Σ. According to [19], $\mathbf{W}_1$ is not finitely based, i.e., it is not axiomatized by any finite list of identities. Consider the following class of groups:

$$\mathbf{W} = \{B \in \mathrm{Ucl}(F_2) : \; B \models \Sigma\} = \mathbf{W}_1 \cap \mathrm{Ucl}(F_2).$$

For every finite set X, the free group $F(X)$ is equationally Noetherian, and in view of the above theorem, $\mathbf{W}$ can be axiomatized by a finite subset of Σ. This means that there are prime numbers $p_1, \ldots, p_m$ such that

$$\mathrm{Th}_\forall(F_2) + \left(\bigwedge_{i=1}^{m} \forall x \forall y [x^{p_i n}, y^{p_i n}]^n \approx 1 \right) \vdash \Sigma.$$

Thus the set Σ of identities independent over $\mathrm{Id}(F_2)$ turns out dependent in the universal theory $\mathrm{Th}_\forall(F_2)$.

Remark 1. We know from model theory (see, e.g., [16]) that any algebra in the universal closure $\mathrm{Ucl}(A)$ is isomorphic to a subalgebra of some ultrapower of an algebra A. If an $\mathcal{L}$-algebra A is finite, then

$$\mathrm{Ucl}(A) = \{B : \; B \leq A\}.$$

Thus the theorem proved above becomes a trivial statement for finite algebras.

4. EQUATIONALLY NOETHERIAN GROUPS

We look at properties of equationally Noetherian groups. Below, unless otherwise stated, we consider group equations with coefficients.

Recall that an A-*group* is a pair (G, λ), where G is a group and $\lambda : A \to G$ is an embedding. In other words, G is an A-group if it contains a distinguished copy of A.

From a model-theoretic standpoint, an A-group is any group in the language $\mathcal{L}(A)$, in which an interpretation of constant symbols of $\mathcal{L}(A)$ forms a subgroup isomorphic to A. In the category of A-groups, any congruence must not identify different elements a_1 and a_2 of the subgroup of A. Thus every congruence of A-groups can be identified with a normal subgroup K of G for which $A \cap K = 1$. Groups K with such properties will be called A-*ideals*. The results obtained above imply that an A-group G is equationally Noetherian iff there are no infinite strictly ascending chains of A-ideals over G (the *maximal property for A-ideals*).

For a finite set $X = \{x_1, \ldots, x_n\}$ of variables, a free A-group generated by X is isomorphic to the free product $A[X] = A * F[X]$, where $F[X]$ is a free group generated by X, and subsets $S \subseteq A[X]$ correspond to $\mathcal{L}(A)$-systems of equations.

THEOREM 4. Let $\mathbf{V}$ be a variety of A-groups. All elements of $\mathbf{V}$ are equationally Noetherian if and only if a free A-group $F_{\mathbf{V}}(X)$ in $\mathbf{V}$ is equationally Noetherian for all finite sets X.

Proof. Let $R_{\mathbf{V}}(X)$ be an A-ideal in $A[X]$ such that $A[X]/R_{\mathbf{V}}(X) \cong F_{\mathbf{V}}(X)$. Then an arbitrary element of the A-group $F_{\mathbf{V}}(X)$ is represented as $\overline{w} = wR_{\mathbf{V}}(X)$. For an A-group $H \in \mathbf{V}$ and for $h_1, \ldots, h_n \in H$, we define a homomorphism $\varphi : F_{\mathbf{V}}(X) \to H$ by setting

$$\varphi(\overline{w}) = w(h_1, \ldots, h_n).$$

The homomorphism φ is well defined. Indeed, for $\overline{w}_1 = \overline{w}_2$, we obtain $w_1^{-1}w_2 \in R_{\mathbf{V}}(X)$, i.e., the identity $w_1^{-1}w_2 \approx 1$ is true in $\mathbf{V}$, and so

$$w_1(h_1, \ldots, h_n) = w_2(h_1, \ldots, h_n).$$

For $S \subseteq F_{\mathbf{V}}(X)$, we put

$$V_H(S) = \{(h_1, \ldots, h_n) \in H^n : \ \forall \overline{w} \in S \ \ w(h_1, \ldots, h_n) = 1\}.$$

(Formally, S is not a system of $\mathcal{L}(A)$-equations since the elements $\overline{w}$ belong to the A-group $F_{\mathbf{V}}(X)$ and not to $A[X]$; nevertheless, it is easy to show that the set $V_H(S)$ will be algebraic over H.)

Now we pass directly to the proof of the theorem. The equational Noetherianity of an A-group $F_{\mathbf{V}}(X)$ yields the maximal property for A-ideals. Let $H \in \mathbf{V}$, $S \subseteq A[X]$, and R be the normal closure of S in $A[X]$. Clearly, $V_H(S) = V_H(R)$, and every element of R is represented as $\prod\limits_{i=1}^{N} u_i w_i^{\pm 1} u_i^{-1}$, where $w_i \in S$ and $u_i \in A[X]$. Below we prove that there is a finite subset $R_0 \subseteq R$ such that $V_H(R) = V_H(R_0)$.

Assume first that the required set

$$R_0 = \{v_1, \ldots, v_k\}$$

is already constructed. Then, for any i, we obtain

$$v_i = \prod_{j=1}^{N_i} u_{ij} w_{ij}^{\pm 1} u_{ij}^{-1}$$

for some $u_{ij} \in A[X]$ and some $w_{ij} \in S$. Put

$$S_0 = \{w_{ij} : \ 1 \leq i \leq k, \ 1 \leq j \leq N_i\}.$$

Then $S_0 \subseteq S$ and $V_H(S) = V_H(S_0)$, which proves the equational Noetherianity of an A-group H.

Now we come back to the proof of the existence of a set R_0. Let

$$\overline{S} = \{\overline{w} \in F_{\mathbf{V}}(X) : w \in S\}$$

and $\overline{R}$ be the normal closure of $\overline{S}$ in the A-group $F_{\mathbf{V}}(X)$. We have

$$\overline{R} = \{\overline{w} : \ w \in R\}.$$

On the other hand,

$$V_H(S) = V_H(R) = V_H(\overline{R}) = V_H(\overline{S}),$$

and we have exactly two cases to consider.

Case 1. Let $A \cap \overline{R} \neq 1$. Then there exists an element $1 \neq a \in A \cap \overline{R}$ such that $\overline{a} \neq 1$, and so $V_H(\overline{R}) = \varnothing$. Hence we can put $R_0 = \{a\} \subseteq R$ in this case.

Case 2. Let $A \cap \overline{R} = 1$. Then $\overline{R}$ is an A-ideal of $F_{\mathbf{V}}(X)$, and so it is finitely generated as an A-ideal. Consequently, there exists a finite subset $\overline{R}_0 \subseteq \overline{R}$ generating $\overline{R}$. We have $V_H(\overline{R}) = V_H(\overline{R}_0)$. Suppose R_0 is a set of preimages of elements of $\overline{R}_0$. Then $V_H(\overline{R}_0) = V_H(R_0)$, and so $V_H(R) = V_H(R_0)$, which proves the equational Noetherianity of an A-group H.

Now assume that every element of the variety $\mathbf{V}$ is equationally Noetherian. We must prove that an A-group $F_{\mathbf{V}}(X)$ is equationally Noetherian for any finite set X. Let K be an arbitrary A-ideal in $F_{\mathbf{V}}(X)$ and $H = F_{\mathbf{V}}(X)/K$. Then H is an A-group belonging to $\mathbf{V}$. Let $\overline{w} \in K$. Then

$$\begin{aligned} w(\overline{x}_1 K, \ldots, \overline{x}_n K) &= w(\overline{x}_1, \ldots, \overline{x}_n)K \\ &= \overline{w}K \\ &= K. \end{aligned}$$

Hence the point $(\overline{x}_1 K, \ldots, \overline{x}_n K) \in H^n$ belongs to $V_H(\overline{w} \approx 1)$. If, however, $\overline{w} \in F_{\mathbf{V}}(X)$ and $(\overline{x}_1 K, \ldots, \overline{x}_n K)$ is a solution for $\overline{w} \approx 1$, then $\overline{w} \in K$. Thus

$$\overline{w} \in K \Leftrightarrow (\overline{x}_1 K, \ldots, \overline{x}_n K) \in V_H(\overline{w} \approx 1). \tag{$*$}$$

Now assume that $K_1 \subsetneq K_2 \subsetneq K_3 \subsetneq \ldots$ is a strictly ascending chain of A-ideals in $F_{\mathbf{V}}(X)$. For every i, we choose $\overline{w}_i \in K_{i+1} \setminus K_i$ and let L_i be the normal closure of a set $K_i \cup \{\overline{w}_i\}$. We have $K_i \subsetneq L_i \subseteq K_{i+1}$. Let $H_i = F_{\mathbf{V}}(X)/K_i$ and $H = \prod_{i=1}^{\infty} H_i$. By assumption, H is equationally Noetherian. For every i, $H_i \leq H$, and so

$$(\overline{x}_1 K_i, \ldots, \overline{x}_n K_i) \in V_{H_i}(K_i) \subseteq V_H(K_i).$$

On the other hand, $(\overline{x}_1K_i, \ldots, \overline{x}_nK_i)$ does not belong to $V_H(L_i)$ since $(\overline{x}_1K_i, \ldots, \overline{x}_nK_i) \in V_{H_i}(\overline{w}_i \approx 1)$ via $(*)$ implies $\overline{w}_i \in K_i$. Hence

$$V_H(K_1) \supsetneqq V_H(L_1) \supseteq V_H(K_2) \supsetneqq V_H(L_2) \supseteq \ldots,$$

and so H^n contains the following strictly descending chain of algebraic sets:

$$V_H(K_1) \supsetneqq V_H(K_2) \supsetneqq V_H(K_3) \supsetneqq \ldots,$$

a contradiction. □

COROLLARY 3. Every finitely generated metabelian group is equationally Noetherian.

Proof. Let A be a finitely generated metabelian group and $\mathbf{V} = Var_A(A)$ be the variety of A-groups generated by A. It is easy to see that every A-group in $\mathbf{V}$ is also metabelian. In particular, the A-group $F_{\mathbf{V}}(X)$ is metabelian and is generated by a set $A \cup X$ (i.e., $F_{\mathbf{V}}(X)$ is finitely generated). By a well-known theorem of Ph. Hall, the A-group $F_{\mathbf{V}}(X)$ has the maximal property for normal groups. Therefore, $F_{\mathbf{V}}(X)$ satisfies the maximal condition for A-ideals. By the above theorem, every A-group of $\mathbf{V}$ (including A) is equationally Noetherian. □

The above result on metabelian groups is not new. We know of a theorem of Remeslennikov which says that every finitely generated metabelian group admits a faithful representation by matrices over a ring which is isomorphic to a direct product of finitely many fields. Furthermore, by Hilbert's basis theorem, every finitely generated metabelian group is equationally Noetherian. Our proof does not employ the result of Remeslennikov; however, we make implicit use of Hilbert's basis theorem because it is intended in the proof of Ph. Hall's.

Recall that a variety is *finitely based* if it can be defined by a finite set of identities. A variety has finite *axiomatic rank* if it can be defined by a finite number of variables.

COROLLARY 4. Let $\mathbf{V}$ be a variety of A-groups which has finite axiomatic rank. If all elements of $\mathbf{V}$ are equationally Noetherian, then $\mathbf{V}$ is finitely based.

Proof. Let $X = \{x_1, \ldots, x_n\}$ be a finite set of variables that we need to define $\mathbf{V}$, and let $R = \mathrm{Id}_X(\mathbf{V})$. It is clear that

$$R = \mathrm{Rad}(\overline{x}_1, \ldots, \overline{x}_n),$$

and so R is finitely generated as an A-ideal of the A-group $F_{\mathbf{V}}(X)$ (by the above theorem, $F_{\mathbf{V}}(X)$ is equationally Noetherian). Consequently, $\mathbf{V}$ is finitely based. □

In the next theorem, we give a sufficient condition under which an A-group A is equationally Noetherian. An A-group G is said to be *finitely cogenerated* if, for any family $\{K_i\}_{i\in I}$ of A-ideals, the condition $\bigcap\limits_{i\in I} K_i = 1$ implies that there exists a finite index set $I_0 \subseteq I$ with the property $\bigcap\limits_{i\in I_0} K_i = 1$.

THEOREM 5. Let $\mathbf{V} = \mathrm{Var}_A(A)$ be a variety generated by an A-group A. Suppose that for all $m \geq 1$, all finitely generated subgroups of A^m have the maximal property for their normal subgroups. Assume also that for every finite set X, the A-group $F_{\mathbf{V}}(X)$ is finitely cogenerated. Then the A-group A is equationally Noetherian.

Proof. For any finite set X, we have

$$R_{\mathbf{V}}(X) = \bigcap \left\{ R \trianglelefteq A[X] : A \cap R = 1,\ \frac{A[X]}{R} \hookrightarrow A \right\}$$

(the symbol $\hookrightarrow$ denotes embedding) and

$$F_{\mathbf{V}}(X) = \frac{A[X]}{R_{\mathbf{V}}(X)}.$$

Suppose

$$\mathcal{C} = \left\{ R \trianglelefteq A[X] : A \cap R = 1,\ \frac{A[X]}{R} \hookrightarrow A \right\}.$$

Let $R \in \mathcal{C}$. Then $R/R_{\mathbf{V}}(X) \trianglelefteq F_{\mathbf{V}}(X)$ and $R/R_{\mathbf{V}}(X)$ is an A-ideal since $A \cap R = 1$. We have

$$\bigcap_{R \in \mathcal{C}} \frac{R}{R_{\mathbf{V}}(X)} = 1,$$

and since the A-group $F_{\mathbf{V}}(X)$ is finitely cogenerated, there are finitely many elements $R_1, \ldots, R_m \in \mathcal{C}$ for which

$$R_{\mathbf{V}}(X) = \bigcap_{i=1}^{m} R_i.$$

Hence

$$F_{\mathbf{V}}(X) = \frac{A[X]}{\bigcap\limits_{i=1}^{m} R_i} \hookrightarrow \prod_{i=1}^{m} \frac{A[X]}{R_i} \hookrightarrow A^m.$$

By assumption, $F_{\mathbf{V}}(X)$ has the maximal property for normal groups, and so it satisfies the maximal condition for A-ideals. Consequently, by the above theorem, every element of $\mathbf{V}$ (in particular, A) is equationally Noetherian. $\square$

If A is a locally finite group, then all finitely generated subgroups of A^m automatically satisfy the conditions of Theorem 5. We are therefore led to

COROLLARY 5. Let A be a locally finite group and $\mathbf{V} = \mathrm{Var}_A(A)$ be the variety of A-groups generated by A. Suppose also that for every finite set X, the A-group $F_{\mathbf{V}}(X)$ is finitely cogenerated. Then A is equationally Noetherian.

5. EQUATIONALLY ARTINIAN ALGEBRAS

We say that an $\mathcal{L}$-algebra A is *equationally Artinian* if every strictly ascending chain of algebraic sets over A terminates at a finite step.

5.1. A-radicals. We have

THEOREM 6. For an arbitrary $\mathcal{L}$-algebra A, the following conditions are equivalent:

(1) for any natural n and for $E \subseteq A^n$, there exists a finite subset $E_0 \subseteq E$ such that

$$\mathrm{Rad}(E) = \mathrm{Rad}(E_0);$$

(2) every strictly descending chain of A-radical ideals terminates at a finite step;

(3) A is equationally Artinian.

Proof. (1)⇔(2) Suppose that an $\mathcal{L}$-algebra A satisfies (1). Let

$$\mathrm{Rad}(E_1) \supseteq \mathrm{Rad}(E_2) \supseteq \mathrm{Rad}(E_3) \supseteq \dots$$

be a descending chain of A-radicals for the sets $E_i \subseteq A^n$. Put

$$E = \bigcup_{i=1}^{\infty} V_A(\mathrm{Rad}(E_i)).$$

By condition (1), there exists a finite subset $E_0 \subseteq E$ with the property $\mathrm{Rad}(E) = \mathrm{Rad}(E_0)$. Since E_0 is finite, there is a number $k \geq 1$ such that

$$E_0 \subseteq V_A(\mathrm{Rad}(E_k)).$$

Consequently,

$$\mathrm{Rad}(E) = \mathrm{Rad}(E_0) \supseteq \mathrm{Rad}(E_k).$$

On the other hand,

$$\mathrm{Rad}(E) = \bigcap_{i=1}^{\infty} \mathrm{Rad}(E_i) \subseteq \mathrm{Rad}(E_k),$$

and the chain of A-radicals terminates at step k. Now suppose that an $\mathcal{L}$-algebra A satisfies condition (2), and let $E \subseteq A^n$. Choose an arbitrary element $c_1 \in E$. If $\mathrm{Rad}(E) = \mathrm{Rad}(\{c_1\})$, then (1) is satisfied. Thus $\mathrm{Rad}(E) \subsetneq \mathrm{Rad}(\{c_1\})$. Consequently, there is an $\mathcal{L}$-equation $p \approx q$ with the property

$$(p, q) \in \mathrm{Rad}(\{c_1\}) \setminus \mathrm{Rad}(E).$$

Hence there is an element $c_2 \in E$ with $p^A(c_2) \neq q^A(c_2)$. If $\mathrm{Rad}(E) = \mathrm{Rad}(\{c_1, c_2\})$, then (1) is true; if the given process is infinite, then we obtain an infinite strictly descending chain of A-radicals, a contradiction with (2).

(2)⇒(3) Suppose

$$Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq \dots$$

is an infinite chain of algebraic sets in A^n and $Y_i = V_A(S_i)$ for some systems S_i of $\mathcal{L}$-equations. Now

$$\mathrm{Rad}(Y_1) \supseteq \mathrm{Rad}(Y_2) \supseteq \mathrm{Rad}(Y_3) \supseteq \dots$$

is a descending chain of A-radicals and so it terminates, i.e.,

$$\mathrm{Rad}(Y_k) = \mathrm{Rad}(Y_{k+1}) = \dots$$

for some natural number k. Consequently,

$$V_A(\mathrm{Rad}(Y_k)) = V_A(\mathrm{Rad}(Y_{k+1})) = \dots,$$

and so

$$Y_k = Y_{k+1} = \dots,$$

which proves that the $\mathcal{L}$-algebra A is equationally Artinian.

(3)⇒(2) Let

$$\mathrm{Rad}(E_1) \supseteq \mathrm{Rad}(E_2) \supseteq \mathrm{Rad}(E_3) \supseteq \dots$$

be a descending chain of A-radicals with $E_i \subseteq A^n$. For algebraic sets, we have

$$E_1^{ac} \subseteq E_2^{ac} \subseteq E_3^{ac} \subseteq \dots,$$

and by condition (3), this chain terminates. Then the chain of A-radicals also terminates at a finite step. □

Now we describe the class of equationally Artinian equational domains using the following topological notion. A subset of a topological space is *contracompact* if each of its coverings by closed sets has a finite subcovering. (In literature, this type of compactness is also referred to as *strong S-closedness*).

THEOREM 7. An equational domain A is equationally Artinian, if and only if every subset of A^n is contracompact for any natural n.

Proof. Let an equational domain A be equationally Artinian. Consider a subset $C \subseteq A^n$ and its covering

$$C \subseteq \bigcup_{i \in I} C_i$$

by closed subsets. If $C \subseteq C_{i_0}$ for some $i_i \in I$, then we automatically obtain a finite covering. Otherwise, there is $i_1 \in I$ such that C_{i_1} is not contained in C_{i_0} and $C \cap C_{i_1} \neq \varnothing$. Continuing this process, we arrive at a chain

$$C_{i_0} \subsetneq C_{i_0} \cup C_{i_1} \subsetneq C_{i_0} \cup C_{i_1} \cup C_{i_2} \subsetneq \dots.$$

Since an $\mathcal{L}$-algebra A is an equational domain, all terms of the chain are algebraic sets, which contradicts the assumption that A is equationally Artinian.

Suppose now that every subset of A^n is contracompact. Consider some subset $E \subseteq A^n$. In view of E being contracompact, there is a finite subset $E_0 \subseteq E$, which is dense in E (i.e., $\overline{E}_0 = E$). Since an $\mathcal{L}$-algebra A is an equational domain, we have $E_0^{ac} = E$. Hence

$$\mathrm{Rad}(E_0) = \mathrm{Rad}(E_0^{ac}) = \mathrm{Rad}(E).$$

By the previous theorem, the $\mathcal{L}$-algebra A is equationally Artinian. □

As an application of the last theorem, we consider the following property of equationally Artinian groups. Let G be a group and $B \subseteq G^n$. We say that the set B is a *basis of identities* for G if the fact that all elements of B satisfy some equality $w \approx 1$ ($w \in F(x_1, x_2, \dots, x_n)$) implies that $w \approx 1$ is an identity in G.

COROLLARY 6. Every equationally Artinian group has a finite basis of identities.

Proof. We have

$$\mathrm{Id}_G(x_1, \dots, x_n) = \mathrm{Rad}(G^n).$$

Since G is equationally Artinian, there is a finite set B such that $B \subseteq G^n$ and $\mathrm{Rad}(G^n) = \mathrm{Rad}(B)$. Therefore, the set B is a basis of identities for G. □

5.2. V-systems of equations. Let $\mathbf{V}$ be a prevariety of A-algebras in the language $\mathcal{L}(A)$ and X be a finite set of variables. In Sec. 3.1, the free algebra $F_{\mathbf{V}}(X)$ of the variety $\mathbf{V}$ was defined. We denote an arbitrary element of $F_{\mathbf{V}}(X)$ by $\overline{t}$, where t is an $\mathcal{L}(A)$-term.

Let $B \in \mathbf{V}$ and $(b_1, \dots, b_n) \in B^n$. We know that there exists a homomorphism $\varphi : F_{\mathbf{V}}(X) \to B$ with the property

$$\varphi(\overline{p}) = p^B(b_1, \dots, b_n).$$

Therefore, if $\overline{p}_1 = \overline{p}_2$, then $p_1^B(b_1, \dots, b_n) = p_2^B(b_1, \dots, b_n)$ for arbitrary $\mathcal{L}(A)$-terms p_1 and p_2. Thus the following definition is sound.

Definition 5. A $\mathbf{V}$-*equation* is an expression of the form $\overline{p} \approx \overline{q}$, where p and q are terms in the language $\mathcal{L}(A)$. Suppose B is an A-algebra. A point $(b_1, \dots, b_n) \in B^n$ is a *solution for an equation* $\overline{p} \approx \overline{q}$ over an A-algebra B if $p^B(b_1, \dots, b_n) = q^B(b_1, \dots, b_n)$.

A solution for a system S of $\mathbf{V}$-equations is denoted by $V_B^{\mathbf{V}}(S)$. The given set is algebraic in the conventional sense by virtue of the following observation. Let S' be the set of all equations $p \approx q$ such that $\overline{p} \approx \overline{q} \in S$. It is straightforward to verify that

$$V_B^{\mathbf{V}}(S) = V_B(S').$$

Below, therefore, the algebraic set $V_B^{\mathbf{V}}(S)$ will be denoted by $V_B(S)$. Moreover, the Zariski topology arising from algebraic sets relative to the prevariety $\mathbf{V}$ is just the ordinary Zariski topology. Let $Y \subseteq B^n$; put

$$\mathrm{Rad}_B^{\mathbf{V}}(Y) = \{\overline{p} \approx \overline{q} : \forall \overline{b} \in Y \ p^B(b_1, \dots, b_n) = q^B(b_1, \dots, b_n)\}.$$

The quotient algebra

$$\Gamma_{\mathbf{V}}(Y) = \frac{F_{\mathbf{V}}(X)}{\mathrm{Rad}_B^{\mathbf{V}}(Y)}$$

is the $\mathbf{V}$-*coordinate algebra* of Y. It is easy to show that $\Gamma_{\mathbf{V}}(Y) \cong \Gamma(Y)$.

5.3. Ideals of V-free algebras. An algebra B is said to be *Noetherian* (*Artinian*) if any strictly ascending (descending) chain of ideals in B terminates. For the case where B is an A-algebra, the condition of being Noetherian (Artinian) is formulated for chains of A-ideals. A congruence R on B is called an A-*ideal* if the condition $(a_1, a_2) \in R$ implies $a_1 = a_2$ for any pair of elements $a_1, a_2 \in A$.

THEOREM 8 [20]. Let $\mathfrak{Y}$ be a variety of $\mathfrak{L}$-algebras and let an algebra $A \in \mathfrak{Y}$ contain a trivial subalgebra. Denote by $\mathbf{V} = \mathfrak{Y}_A$ the set of all elements of $\mathfrak{Y}$ that are A-algebras. For any finite set X, the free algebra $F_{\mathbf{V}}(X)$ is Noetherian if and only if every algebra $B \in \mathbf{V}$ is A-equationally Noetherian.

We prove an analog of the above theorem for Artinian algebras.

THEOREM 9. Let $\mathfrak{Y}$ be a variety of $\mathfrak{L}$-algebras and let $A \in \mathfrak{Y}$ contain a trivial subalgebra. Denote by $\mathbf{V} = \mathfrak{Y}_A$ the set of all elements of $\mathfrak{Y}$ that are A-algebras. For any finite set X, the free algebra $F_{\mathbf{V}}(X)$ is Artinian if and only if every algebra $B \in \mathbf{V}$ is A-equationally Artinian.

Proof. The main idea of the proof is the same as in Theorem 4.

Necessity. Let the free algebra $F_{\mathbf{V}}(X)$ be Artinian, $B \in \mathbf{V}$, and

$$Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq \dots$$

be a chain of algebraic sets in B^n. Then we have

$$\mathrm{Rad}(Y_1) \supseteq \mathrm{Rad}(Y_2) \supseteq \mathrm{Rad}(Y_3) \supseteq \dots,$$

which is a chain of A-ideals in $F_{\mathbf{V}}(X)$. By assumption, the latter chain terminates, i.e., there is m such that

$$\mathrm{Rad}(Y_m) = \mathrm{Rad}(Y_{m+1}) = \dots.$$

Consequently, $Y_m = Y_{m=1} = \dots$, and therefore B is equationally Artinian.

Sufficiency. Let an arbitrary algebra $B \in \mathbf{V}$ be equationally Artinian. For every A-ideal R in $F_{\mathbf{V}}(X)$, we put $B(R) = F_{\mathbf{V}}(X)/R$. Clearly, the algebra $B(R)$ belongs to $\mathbf{V}$ and $(\overline{p}, \overline{q}) \in R$ iff

$$(\overline{x}_1/R, \dots, \overline{x}_n/R) \in V_{B(R)}(\overline{p} \approx \overline{q}).$$

Suppose

$$R_1 \supsetneq R_2 \supsetneq R_3 \supsetneq \dots$$

is an infinite chain of A-ideals in $F_{\mathbf{V}}(X)$ and every A-ideal R_i is treated as a system of $\mathbf{V}$-equations. Let $(\overline{p}_i, \overline{q}_i) \in R_i \setminus R_{i+1}$. Consider an A-ideal T_i generated by a set $R_{i+1} \cup (\overline{p}_i, \overline{q}_i)$. Then

$$R_{i+1} \subsetneq T_i \subsetneq R_i.$$

Let $B_i = B(R_i)$ and $B = \prod\limits_i B_i$; then $B \in \mathbf{V}$. We prove that the algebra B is not equationally Artinian (which will contradict the assumption). Let

$$U = (\overline{x}_1/R_{i+1}, \dots, \overline{x}_n/R_{i+1}) \in V_{B_{i+1}}(R_{i+1}) \subseteq V_B(R_{i+1}).$$

The point U does not belong to $V_B(T_i)$, since otherwise $U \in V_B(\overline{p}_i \approx \overline{q}_i)$ and $(\overline{p}_i, \overline{q}_i) \in R_{i+1}$.

Thus we have an infinite strictly ascending chain

$$V_B(R_i) \subseteq V_B(T_i) \subsetneq V_B(R_{i+1}),$$

and so the algebra B is not equationally Artinian. $\square$

5.4. Hilbert's basis theorem. In [20], the statement of Hilbert's basis theorem was formulated for algebras in an arbitrary functional language $\mathcal{L}$. We consider several examples dealing with this problem. Suppose $\mathcal{L}$ is a functional language, $\mathfrak{Y}$ is a variety of $\mathcal{L}$-algebras, $A \in \mathfrak{Y}$, and $\mathbf{V} = \mathfrak{Y}_A$ is the family of all algebras of $\mathfrak{Y}$ that are A-algebras. In examples below, we discuss the following problem:

Does the fact that A satisfies the maximal property for ideals imply that $F_{\mathbf{V}}(X)$ is Noetherian? The given problem was formulated in [20] where analogs of Hilbert's basis theorem were considered for arbitrary algebras.

Example 2. Let $\mathcal{L} = (0, 1, +, \times)$ be the language of unital rings, $\mathfrak{Y}$ be the variety of all commutative rings in $\mathcal{L}$, $A \in \mathfrak{Y}$, and $\mathbf{V} = \mathfrak{Y}_A$. If $X = \{x_1, \ldots, x_n\}$, then $F_{\mathbf{V}}(X) = A[x_1, \ldots, x_n]$, and by Hilbert's basis theorem, the answer to the question posed above is affirmative.

Example 3. Let $\mathcal{L} = (e, ^{-1}, \cdot)$ be the language of groups, $\mathfrak{Y}$ be the variety of all groups, A be any group, and $\mathbf{V} = \mathfrak{Y}_A$. Then $F_{\mathbf{V}}(X) = A * F(X)$. We show that $F_{\mathbf{V}}(X)$ is not Noetherian even if A has the maximal property for normal subgroups. Consider the Baumslag–Solitar group

$$B_{m,n} = \langle a, t : ta^m t^{-1} = a^n \rangle,$$

where $m, n \geq 1$ and $m \neq n$. As shown in [1], the group $B_{m,n}$ is not equationally Noetherian. Then $B = A * B_{m,n}$ is not A-equationally Noetherian. Consequently, by Theorem 8, $A * F(X)$ is not Noetherian, and so the problem under consideration has a negative solution.

Example 4. Let $\mathfrak{Y}$ be the variety of Abelian groups, $A \in \mathfrak{Y}$ be finitely generated, and $\mathbf{V} = \mathfrak{Y}_A$. It is easy to verify that $F_{\mathbf{V}}(X) = A \oplus F_{ab}(X)$, where the group $F_{ab}(X)$ is a free Abelian group generated by X. Thus $F_{\mathbf{V}}(X) = A \oplus \mathbb{Z}^n$. The group $A \oplus \mathbb{Z}^n$ is Noetherian as a $\mathbb{Z}$-module, and so the above problem has a positive solution.

Let $\mathfrak{Y}$ be a variety of algebras, $A \in \mathfrak{Y}$, and $\mathbf{V} = \mathfrak{Y}_A$. If there exists an algebra $B \in \mathfrak{Y}$ which is not equationally Noetherian, then, by Theorem 8, $F_{\mathbf{V}}(X)$ is not Noetherian. In this case, therefore, our problem has a negative solution in $\mathfrak{Y}$.

Example 5. Let $\mathfrak{Y}$ be the variety of all nilpotent groups of class at most c, $A \in \mathfrak{Y}$, and $\mathbf{V} = \mathfrak{Y}_A$. In [5], it was shown that any group $B \in \mathfrak{Y}$ that is not finitely generated is not equationally Noetherian. By Theorem 8, the group $F_{\mathbf{V}}(X)$ is not equationally Noetherian as well.

5.5. Examples of equationally Artinian algebras. We give some examples of equationally Artinian algebras.

Example 6. Suppose A is an equationally Artinian algebra (for example, a finite algebra). We show that for any set I, the Cartesian power A^I is also equationally Artinian. Indeed, for any equation $p \approx q$, there is a natural bijection between the sets $V_A(p \approx q)^I$ and $V_{A^I}(p \approx q)$ given by

$$(a_i^1, \ldots, a_i^n)_{i \in I} \mapsto ((a_i^1)_{i \in I}, \ldots, (a_i^n)_{i \in I}).$$

Hence any chain of algebraic sets

$$V_{A^I}(S_1) \subset V_{A^I}(S_2) \subset V_{A^I}(S_3) \subset \dots$$

corresponds to the chain

$$V_A(S_1)^I \subset V_A(S_2)^I \subset V_A(S_3)^I \subset \dots,$$

which terminates because

$$V_A(S_1) \subset V_A(S_2) \subset V_A(S_3) \subset \dots$$

terminates in the algebra A.

Example 7. Let R be a Noetherian ring which is an Abelian group A relative to $+$. We prove that A is equationally Artinian.

Consider $p = a_1x_1 + \ldots + a_nx_n$ with $a_i \in \mathbb{Z}$. We can show that the set $V_A(p \approx 0)$ is a submodule of R^n. Hence a solution for any system S of equations is a submodule of R^n (n is the number of variables in S). Since the space R^n is Noetherian, the group A is equationally Artinian.

Example 8. Let R be a Noetherian ring and $\Lambda = (\lambda_1, \dots, \lambda_m)$ be a tuple of elements of R with the property $\sum\limits_i \lambda_i = 1$. Define an m-ary operation

$$p_\Lambda(x_1, \dots, x_m) = \sum_i \lambda_i x_i.$$

Denote by A the set of elements of the ring R relative to p_Λ. Using essentially the same argument as in the previous example, we can show that A is equationally Artinian. If we assume that R is a Noetherian but not Artinian ring, then A will be an example of an equationally Artinian algebra that is not equationally Noetherian.

Example 9. We can show that only finite fields are equationally Artinian in the class of all rings in the language $\mathcal{L} = (+, \cdot, 0, 1)$.

6. OTHER TYPES OF COMPACTNESS

Let A be an $\mathcal{L}$-algebra. A system S of $\mathcal{L}$-equations is said to be A-*independent* if for any finite subset $S_0 \subseteq S$ there exists an equation $(p \approx q) \in S_0$ with the property $V_A(S_0) \subsetneq V_A(S_0 \setminus p \approx q)$.

THEOREM 10. For any $\mathcal{L}$-algebra A and any system S of $\mathcal{L}$-equations, there exists an A-independent subsystem $S' \subseteq S$ equivalent to S over A.

Proof. Suppose $\mathcal{F}$ is the family of all A-independent subsystems of S. Consider an ascending chain $\{T_\alpha\}_\alpha \subseteq \mathcal{F}$ and put $T = \bigcup T_\alpha$. Let T^0 be an arbitrary finite subset of T; then $T^0 \subseteq T_\alpha$ for some α. Hence there exists an equation $(p \approx q) \in T^0$ such that

$$V_A(T^0) \subsetneq V_A(T^0 \setminus p \approx q).$$

Thus the system T is A-independent, and so the chain $\{T_\alpha\}_\alpha$ has a maximal element in $\mathcal{F}$. By Zorn's lemma, the entire set $\mathcal{F}$ has a maximal element S'.

Suppose $V_A(S) \subsetneq V_A(S')$. Then there exist a point $\overline{a} \in V_A(S') \setminus V_A(S)$ and an equation $(p_1 \approx q_1) \in S$ such that $p_1(\overline{a}) \neq q_1(\overline{a})$. Letting $S'' = S' \cup (p_1 \approx q_1)$, we show that $S'' \in \mathcal{F}$. To do this, consider a finite subsystem $T_0 \subseteq S''$. If $T_0 \subseteq S'$, then there is an equation $(p \approx q) \in T_0$ with the property

$$V_A(T_0) \subsetneq V_A(T_0 \setminus p \approx q).$$

If $(p_1 \approx q_1) \in T_0$, then $T_0 \setminus p_1 \approx q_1 \subseteq S'$, whence

$$\overline{a} \in V_A(T_0 \setminus p_1 \approx q_1).$$

Since $p_1(\overline{a}) \neq q_1(\overline{a})$, the point $\overline{a}$ does not belong to $V_A(T_0)$. Therefore,

$$V_A(T_0) \subsetneq V_A(T_0 \setminus p_1 \approx q_1),$$

and hence $S'' \in \mathcal{F}$, which contradicts the choice of a system S'. Thus $V_A(S) = V_A(S')$. $\square$

A topological space M is *ω-cocompact* if for any countable open covering $M = \bigcup\limits_{i=1}^{\infty} C_i$ there exists $m \geq 1$ such that for all $j_1, j_2 \geq m$, we have

$$\bigcup_{i=j_1}^{\infty} C_i = \bigcup_{i=j_2}^{\infty} C_i.$$

LEMMA 2. A topological space M is Artinian if and only if every subset of M is ω-cocompact.

Proof. Necessity. We show that every subset $N \subseteq M$ is also Artinian. Suppose

$$Y_1' \subseteq Y_2' \subseteq Y_3' \subseteq \ldots$$

is a chain of closed subsets in N. There exist sets Y_i closed in M for which $Y_i' = Y_i \cap N$. Clearly, sets $V_i = Y_1 \cup \ldots \cup Y_i$ likewise are closed in M, and we have

$$V_1 \subseteq V_2 \subseteq V_3 \subseteq \ldots.$$

By assumption, there is a natural number m such that $V_m = V_{m+1} = V_{m+2} = \ldots$. Consequently,

$$Y_1 \cup \ldots \cup Y_m = Y_1 \cup \ldots \cup Y_m \cup Y_{m+1} = \ldots,$$

whence

$$Y_1' \cup \ldots \cup Y_m' = Y_1' \cup \ldots \cup Y_m' \cup Y_{m+1}' = \ldots,$$

which yields $Y_m' = Y_{m+1}' = \ldots$. Thus the space N is Artinian.

Consider a countable open covering $M = \bigcup\limits_{i=1}^{\infty} C_i$ of M, where $C_i = M \setminus D_i$ and the sets D_i are closed. Then $\bigcap\limits_{i=1}^{\infty} D_i = \varnothing$. If we put

$$Y_m = \bigcap_{i=m}^{\infty} D_i$$

we obtain a chain of closed sets $Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq \ldots$. By assumption, there exists a natural m such that $Y_m = Y_{m+1} = \ldots$. Therefore, for arbitrary $j_1, j_2 \geq m$,

$$\bigcap_{i=j_1}^{\infty} D_i = \bigcap_{i=j_2}^{\infty} D_i.$$

Consequently,

$$\bigcup_{i=j_1}^{\infty} C_i = \bigcup_{i=j_2}^{\infty} C_i,$$

which proves that the space M is ω-cocompact.

Sufficiency. Suppose M is ω-cocompact and

$$Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq \ldots$$

is a chain of closed subsets of M. Then

$$M = \bigcup_{i=1}^{\infty} Y_i^c \cup M,$$

where the superscript ^{c} denotes the complement. By assumption, there exists a natural number m such that for all $j_1, j_2 \geq m$,

$$\bigcup_{i=j_1}^{\infty} Y_i^c = \bigcup_{i=j_2}^{\infty} Y_i^c.$$

Consequently,

$$\bigcap_{i=m}^{\infty} Y_i = \bigcap_{i=m+1}^{\infty} Y_i = \ldots,$$

whence

$$Y_m = \bigcap_{i=m+1}^{\infty} Y_i = \bigcap_{i=m+2}^{\infty} Y_i,$$

$$Y_{m+1} = \bigcap_{i=m+2}^{\infty} Y_i = \ldots.$$

Thus $Y_m = Y_{m+1} = \ldots$, and so the space M is Artinian. $\square$

By Lemma 2, an equational domain A is equationally Artinian iff any subset of A^n is ω-cocompact for all n.

We say that a system S of $\mathcal{L}$-equations is A-*stable* (here A is some $\mathcal{L}$-algebra) if $V_A(S) = V_A(S \setminus S')$ for any proper subset $S' \subseteq S$.

THEOREM 11. Let an $\mathcal{L}$-algebra A be a weakly Noetherian equational domain. Suppose that for any system S of $\mathcal{L}$-equations, there exists a finite subsystem S_0 such that $S \setminus S_0$ is A-stable. Then the $\mathcal{L}$-algebra A is equationally Artinian. The converse is true if, in addition, the language $\mathcal{L}$ is countable.

Proof. We show that A^n is ω-cocompact. Let $A = \bigcup\limits_{i=1}^{\infty} C_i$ be an open covering. We have $C_i = A^n \setminus V_A(S_i)$ for some finite S_i. Hence $\bigcap\limits_{i=1}^{\infty} V_A(S_i) = \varnothing$, and so $V_A\left(\bigcup\limits_{i=1}^{\infty} S_i\right) = \varnothing$. Suppose $S = \bigcup\limits_{i=1}^{\infty} S_i$. Then there exists a finite subset $S_0 \subseteq S$ such that the system $S \setminus S_0$ is A-stable. Since S_0 is finite, there exists m such that the set $V_A\left(\bigcup\limits_{i=j}^{\infty} S_i\right)$ is the same for all $j \geq m$. In other words, for all $j_1, j_2 \geq m$,

$$\bigcap_{i=j_1}^{\infty} V_A(S_i) = \bigcap_{i=j_2}^{\infty} V_A(S_i),$$

which is equivalent to

$$\bigcup_{i=j_1}^{\infty} C_i = \bigcup_{i=j_2}^{\infty} C_i,$$

and so A^n is ω-cocompact.

Assume now that $\mathcal{L}$ is a countable language, an $\mathcal{L}$-algebra A is equationally Artinian, and S is an arbitrary system of equations. The set of equations of S being countable is representable as $S = \{p_i \approx q_i\}_{i=1}^{\infty}$. For $C = A^n \setminus V_A(S)$, we have

$$C = \bigcup_{i=1}^{\infty} (A^n \setminus V_A(p_i \approx q_i)).$$

Since the $\mathcal{L}$-algebra A is ω-cocompact, there exists a natural m such that

$$\bigcup_{i=j_1}^{\infty} (A^n \setminus V_A(p_i \approx q_i)) = \bigcup_{i=j_2}^{\infty} (A^n \setminus V_A(p_i \approx q_i))$$

for all $j_1, j_2 \geq m$. Put $S_0 = \{p_1 \approx q_1, \ldots, p_{m-1} \approx q_{m-1}\}$. Then

$$V_A(S \setminus S_0) = V_A(S \setminus (S_0 + p_m \approx q_m + \ldots + p_{m+j} \approx q_{m+j}))$$

for any j, and so $S \setminus S_0$ is A-stable. $\square$

Acknowledgments. We are grateful to A. Blass, A. Klyachko, G. Paseman, and B. Steinberg for useful comments and suggestions during a discussion in mathoverflow.net.

REFERENCES

1. G. Baumslag, A. Myasnikov, and V. N. Remeslennikov, "Algebraic geometry over groups. I. Algebraic sets and ideal theory," *J. Alg.*, **219**, No. 1, 16-79 (1999).

2. G. Baumslag, A. Myasnikov, and V. Remeslennikov, "Discriminating and co-discriminating groups," *J. Group Th.*, **3**, No. 4, 467-479 (2000).
3. G. Baumslag, A. Myasnikov, and V. Roman'kov, "Two theorems about equationally noetherian groups," *J. Alg.*, **194**, No. 2, 654-664 (1997).
4. O. Kharlampovich and A. Myasnikov, "Irreducible affine varieties over a free group. I: Irreducibility of quadratic equations and Nullstellensatz," *J. Alg.*, **200**, No. 2, 472-516 (1998).
5. A. Myasnikov and V. N. Remeslennikov, "Algebraic geometry over groups. II: Logical foundations," *J. Alg.*, **234**, No. 1, 225-276 (2000).
6. O. Kharlampovich and A. Myasnikov, "Tarski's problem about the elementary theory of free groups has a positive solution," *El. Res. Ann. AMS*, **4**, No. 14, 101-108 (1998).
7. Z. Sela, "Diophantine geometry over groups. I-X," Preprints, Arxiv.
8. O. Kharlampovich and A. Myasnikov, "Elementary theory of free non-abelian groups," *J. Alg.*, **302**, No. 2, 451-552 (2006).
9. E. Yu. Daniyarova and V. N. Remeslennikov, "Bounded algebraic geometry over free Lie algebras," *Algebra and Logic*, **44**, No. 3, 148-167 (2005).
10. P. V. Morar and A. N. Shevlyakov, "Algebraic geometry over the additive monoid of natural numbers: Systems of coefficient free equations," in *Combinatorial and Geometric Group Theory. Dortmund and Ottawa–Montreal Conf. (Trends Math.)*, O. Bogopolski et al. (Eds.), Birkhauser, Basel (2010), pp. 261-278.
11. A. N. Shevlyakov, "Algebraic geometry over natural numbers. The classification of coordinate monoids," *Groups, Complex. Cryptol.*, **2**, No. 1, 91-111 (2010).
12. E. Daniyarova, A. Myasnikov, and V. Remeslennikov, "Unification theorems in algebraic geometry," in *Aspects of Infinite Groups, Algebra Discr. Math. (Hackensack)*, **1**, World Sci., Hackensack, NJ (2008), pp. 80-111.
13. E. Daniyarova, A. Myasnikov, and V. Remeslennikov, "Algebraic geometry over algebraic structures. II. Foundations," *Fundam. Prikl. Mat.*, **17**, No. 1, 65-106 (2011/2012).
14. E. Daniyarova, A. Myasnikov, and V. Remeslennikov, "Algebraic geometry over algebraic structures. III: Equationally Noetherian property and compactness," *South. Asian Bull. Math.*. **35**, No. 1, 35-68 (2011).
15. E. Daniyarova, A. Myasnikov, and V. Remeslennikov, "Algebraic geometry over algebraic structures. IV. Equational domains and codomains," *Algebra and Logic*, **49**, No. 6, pp. 483-508.
16. S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Springer-Verlag, New York (1981).

17. V. A. Gorbunov, *Algebraic Theory of Quasivarieties, Sib. School Alg. Log.* [in Russian], Nauch. Kniga, Novosibirsk (1999).
18. A. I. Mal'tsev, *Algebraic Systems* [in Russian], Nauka, Moscow (1970).
19. S. I. Adyan, "Infinite irreducible systems of group identities," *Izv. Akad. Nauk SSSR, Ser. Matem.*, **34**, No. 4, 715-734 (1970).
20. M. Shahryari, "Equational noetherian algebras and chain conditions." *J. Sib. Fed. Univ. Math. Phys.*, **6**, No. 4, 521-526 (2013).