DOI 10.1007/s10469-016-9378-5
Algebra and Logic, Vol. 55, No. 1, March, 2016
(Russian Original Vol. 55, No. 1, January-February, 2016)

FINITE GROUPS WITH GIVEN PROPERTIES OF THEIR PRIME GRAPHS

A. S. Kondrat'ev*

UDC 512.542

Presented by the Program Committee of the Conference "Mal'tsev Readings"

1. PRELIMINARIES

Studying finite groups depending on their arithmetic properties (orders of elements and subgroups, cardinalities of conjugacy classes, different π-properties, degrees of irreducible characters, etc.) is an important direction in the theory of finite groups, having a rich history. The classification of finite simple groups allows this study to be reduced by and large to the case of *almost simple* groups, i.e., groups A with the property $Inn(P) \leq A \leq \mathrm{Aut}\,(P)$, where P is some finite non-Abelian simple group.

Let G be a finite group. The set of prime divisors of $|G|$ is denoted by $\pi(G)$. The *prime graph* (*Gruenberg–Kegel graph*) $\Gamma(G)$ of G is a graph whose vertex set is $\pi(G)$, in which two vertices p and q are adjacent iff G contains an element of order pq.

The concept of a prime graph appeared in research into cohomological questions associated with integer-valued representations of finite groups, and turned out to be very fruitful. The graph $\Gamma(G)$ is a fundamental arithmetic invariant of a group G. Note that $\Gamma(G)$ can be uniquely determined from the character table of G.

Of interest are various problems of recognizability—a characterization of a group by a certain set of its parameters up to isomorphism—for instance, problems of recognizability of finite groups by prime graph. We say that a finite group G is *recognizable by prime graph* if, for any finite group H, $\Gamma(G) = \Gamma(H)$ implies $H \cong G$.

*Supported by RFBR (project No. 13-01-00469), by the Complex Research Program of UrO RAN (project No. 15-16-1-5), and by the Program for State Aid of Leading RF Universities (Agreement No. 02.A03.21.0006 between the Ministry of Education and Science of the Russian Federation and the Ural Federal University, 27.08.2013).

Krasovskii Institute of Mathematics and Mechanics, Ural Branch, Russian Academy of Sciences, ul. S. Kovalevskoi 16, Yekaterinburg, 620990 Russia. El'tsyn Ural Federal University, ul. Mira 19, Yekaterinburg, 620002 Russia; a.s.kondratiev@imm.uran.ru. Translated from *Algebra i Logika*, Vol. 55, No. 1, pp. 113-120, January-February, 2016. Original article submitted September 3, 2015.

The problem of recognizing finite groups by prime graph is a particular case of the general problem of studying finite groups by the properties of their prime graphs. In the context of this general problem, our attention is drawn to the class of finite groups with disconnected prime graph. Finite almost simple groups with disconnected prime graph were described by Willians [1], Kondrat'ev [2], Iiyori and Yamaki [3, 4], and Lucido [5, 6]. A useful criterion for vertices to be adjacent in the prime graphs of finite simple groups was found by Vasil'ev and Vdovin [7, 8].

The present paper gives a review of recent results on finite groups with given properties of their prime graphs, which have been obtained by the author in collaboration with his pupils. Our notation and terminology are standard. A finite group G is *n-primary* if $|\pi(G)| = n$.

Within the framework of the problem of thoroughly studying the class of finite groups with disconnected prime graph, Kondrat'ev and Khramtsov [9-12] explored finite groups whose prime graph is disconnected and has three or four vertices. A refined description of chief factors of 4-primary finite groups with disconnected prime graph can be found in [13-16].

Subsequently, research on finite n-primary groups for small n has progressed. The main objective here is to describe chief factors of finite unsolvable non almost simple groups with disconnected Gruenberg–Kegel graph.

Finite almost simple 5-primary graphs together with their prime graphs were characterized in [17, 18]. As a consequence, a list of finite simple 5-primary groups obtained in [19, 20] has been considerably refined.

Using results of [9, 10] and computations in the GAP computer algebra system [21], Kolpakova and Kondrat'ev [22] furnished a description of chief factors of commutator subgroups of finite unsolvable 5-primary groups G with disconnected graph $\Gamma(G)$ for the case where $G/F(G)$ is an almost simple n-primary group for $n \leq 4$.

Finite almost simple 6-primary groups together with their prime graphs were characterized in [23]. As a consequence, a list of finite simple 6-primary groups made up by Jafarzadeh and Iranmanesh [19] has been essentially improved. In [19, Problem 3.12], the following question was posed: For which power primes q does q^2-1 have at most five different prime divisors? A particular instance of this problem, namely, the case with $|\pi(q^2 - 1)| \leq 2$, is well known (see, e.g., [24]): $|\pi(q^2 - 1)| \leq 2$ iff $q \in \{2, 3, 4, 5, 7, 8, 9, 17\}$.

The cases where $|\pi(q^2-1)|$ is equal to 3, 4, and 5 were considered by Kondrat'ev and Khramtsov [10], Kondrat'ev [18], and Kolpakova and Kondrat'ev [23], respectively. We have thus obtained a classification of power primes q such that $|\pi(q^2 - 1)| \leq 5$. Further refinement of this classification leads to Diophantine equations whose solution is difficult, for modern number theory as well. For instance, even the question on finiteness of the set of power primes q such that $|\pi(q^2 - 1)| = 3$ is equivalent to Shi's question [25, Question 13.65], which is still open.

In [26], it was proved that groups $E_7(2)$ and $E_7(3)$ are recognized by prime graph. As a consequence, we have obtained the ultimate positive solution to Mazurov's problem in [27] which maintains that every simple group whose prime graph has at least three connected components either is recognizable by spectrum or is isomorphic to A_6. In [28], it was proved that the group $^2E_6(2)$ is recognized by prime graph. Note that the prime graphs of the groups $^2E_6(2)$, $E_7(2)$, and $E_7(3)$ consist of, respectively, 8, 12, and 15 vertices.

Of interest is the problem of realizing an abstract finite graph as the prime graph of a finite group. We know of only a small amount of works dealing with this problem. Thus, in unpublished bachelor's thesis of I. N. Zharkov, Mazurov's student, it was proved that a chain is realized as the prime graph of a finite group iff its length does not exceed 4.

Of course, in the general case the above-formulated problem has a negative solution. For instance, it follows from [1-4] that a graph consisting of five pairwise nonadjacent vertices (a 5-coclique) cannot be the prime graph of a finite group. In [29], however, it was shown that for any graph having at most five vertices, except for a 5-coclique, the problem has a positive solution.

Lucido [30] furnished a description of finite simple groups G such that the connected components of $\Gamma(G)$ are trees, i.e, connected graphs with no cycles. Furthermore, there, it was shown how a finite group whose prime graph is a tree is structured.

O. A. Alekseeva and the author are dealing with a more general problem, that of describing the structure of a finite group G such that $\Gamma(G)$ has no triangles (3-cycles). It turns out that if G is a group with such a property, then either G is solvable or the factor group $G/S(G)$ of a group G with respect to its solvable radical $S(G)$ is almost simple.

In [31], we obtained a description of all finite almost simple groups G with the property mentioned. As a consequence, every connected component of $\Gamma(G)$ is a tree: if a group G is simple, then $\Gamma(G)$ is disconnected; $|\pi(G)| \leq 8$ and the equality obtains if $G \cong \mathrm{Aut}\,(Sz(2^9))$. Moreover, we have essentially refined a list of finite simple groups such that the connected components of $\Gamma(G)$ are trees, created in [30].

In [31], also, we pointed out isomorphic types of prime graphs and determined bounds on the Fitting length of finite solvable groups whose prime graphs have no triangles. Isomorphic types of prime graphs in such solvable groups were also found by Gruber, Keller, Lewis, Naughton, and Strasser [32] using the classification of prime graphs of finite solvable groups, and some combinatorial results. Our proof of this result is straightforward; it is very short and makes no use of deep combinatorics information.

Finite simple non-Abelian groups with the prime graphs all connected components of which are complete graphs (cliques) were found by Lucido and Moghaddamfar [33] and Vasil'ev and Vdovin [8]. Finite simple non-Abelian groups whose prime graphs coincide with the prime graphs of Frobenius or 2-Frobenius groups were described by Zinov'eva and Mazurov [34]. Finite almost simple groups with the prime graphs all connected components of which are cliques were classified by Zinov'eva and Kondrat'ev [35].

REFERENCES

1. J. S. Williams, "Prime graph components of finite groups," *J. Alg.*, **69**, No. 2, 487-513 (1981).

2. A. S. Kondratiev, “Prime graph components of finite simple groups,” *Mat. Sb.*, **180**, No. 6, 787-797 (1989).

3. N. Iiyori and H. Yamaki, “Prime graph components of the simple groups of Lie type over the field of even characteristic,” *J. Alg.*, **155**, No. 2, 335-343 (1993).

4. N. Iiyori and H. Yamaki, “Corrigendum to: Prime graph components of the simple groups of Lie type over the field of even characteristic,” *J. Alg.*, **181**, No. 2, 659 (1996).

5. M. S. Lucido, “Prime graph components of finite almost simple groups,” *Rend. Semin. Mat. Univ. Padova*, **102**, 1-22 (1999).

6. M. S. Lucido, Addendum to “Prime graph components of finite almost simple groups,” *Rend. Semin. Mat. Univ. Padova*, **107**, 189-190 (2002).

7. A. V. Vasiliev and E. P. Vdovin, “An adjacency criterion for the prime graph of a finite simple group,” *Algebra and Logic*, **44**, No. 6, 381-406 (2005).

8. A. V. Vasiliev and E. P. Vdovin, “Cocliques of maximal size in the prime graph of a finite simple group,” *Algebra and Logic*, **50**, No. 4, 291-322 (2011).

9. A. S. Kondrat’ev and I. V. Khramtsov, “Finite 3-primary groups,” *Trudy Inst. Mat. Mekh. UrO RAN*, **16**, No. 3, 150-158 (2010).

10. A. S. Kondrat’ev and I. V. Khramtsov, “Finite 4-primary groups,” *Trudy Inst. Mat. Mekh. UrO RAN*, **17**, No. 4, 142-159 (2011).

11. A. S. Kondrat’ev and I. V. Khramtsov, “Finite nonsimple 3-primary groups,” *Sib. El. Mat. Izv.*, **9**, 472-477 (2012); http://semr.math.nsc.ru/v9/p472-477.pdf.

12. A. S. Kondrat’ev and I. V. Khramtsov, “Complete reducibility of some $GF(2)A_7$-modules,” *Trudy Inst. Mat. Mekh. UrO RAN*, **18**, No. 3, 139-143 (2012).

13. I. V. Khramtsov, “Finite nonsimple 4-primary groups,” *Sib. El. Mat. Izv.*, **11**, 695-708 (2014); http://semr.math.nsc.ru/v11/p695-708.pdf.

14. A. S. Kondrat’ev and I. V. Khramtsov, “Finite graphs with disconnected prime graph and composition factor isomorphic to $L_3(17)$,” *Proc. Int. Conf. “Algebra and Mathematical Logic: Theory and Applications,”* Kazan State Univ., Kazan (2014), pp. 81/82.

15. A. S. Kondrat’ev, I. D. Suprunenko, and I. V. Khramtsov, “Modular representations of the group $L_3(17)$,” *Proc. Int. Conf. “Mal’tsev Readings,”* Institute of Mathematics, Novosibirsk State University, Novosibirsk (2014), p. 63.

16. A. S. Kondrat’ev and I. V. Khramtsov, “Finite graphs with disconnected prime graph and composition factor isomorphic to $L_2(81)$,” *Proc. Int. Conf. on Group Theory Dedicated to the 70th Anniversary of V. V. Kabanov*, Kabardino-Balkarian State Univ., Nalchik (2014), pp. 56-58.

17. A. S. Kondrat'ev, "Finite almost simple 5-primary groups and their Gruenberg–Kegel graphs," *Izv. Skorina Gomel State Univ.*, No. 3 (84), 58-60 (2014).

18. A. S. Kondrat'ev, "Finite almost simple 5-primary groups and their Gruenberg–Kegel graphs," *Sib. El. Mat. Izv.*, **11**, 634-674 (2014); http://semr.math.nsc.ru/v11/p634-674.pdf.

19. A. Jafarzadeh and A. Iranmanesh, "On simple K_n-groups for $n = 5, 6$," in *London Math. Soc. Lect. Note Ser.*, **340**, C. M. Campbell et al. (Eds.), Cambridge Univ. Press, Cambridge (2007), pp. 517-526.

20. L. Zhang, W. Shi, H. Lu, D. Yu, and S. Chen, "OD-characterization of finite simple K_5-groups," Preprint (2011).

21. The GAP Group, *GAP—Groups, Algorithms, Programming*, a System for Computational Discrete Algebra, Vers. 4.8.2 (2016); http://www.gap-system.org.

22. V. A. Kolpakova and A. S. Kondrat'ev, "Finite unsolvable 5-primary groups G with disconnected Gruenberg–Kegel graph such that $|\pi(G/F(G))| \leq 4$," *Proc. Int. Conf. "Mal'tsev Readings,"* Institute of Mathematics, Novosibirsk State University, Novosibirsk (2014), p. 64.

23. V. A. Kolpakova and A. S. Kondrat'ev, "Finite almost simple 6-primary groups and their Gruenberg–Kegel graphs," *Proc. Int. Conf. "Algebra and Applications" Dedicated to the 100th Anniversary of L. A. Kaluzhnin*, Kabardino-Balkarian State Univ., Nalchik (2014), pp. 63-66.

24. G. Higman, "Finite groups in which every element has prime power order," *J. London Math. Soc.*, **32**, 335-342 (1957).

25. *Unsolved Problems in Group Theory, The Kourovka Notebook*, 18th edn., Institute of Mathematics SO RAN, Novosibirsk (2014); http://www.math.nsc.ru/∼alglog/18kt.pdf.

26. A. S. Kondrat'ev, "Recognizability of $E_7(2)$ and $E_7(3)$ by prime graph," *Trudy Inst. Mat. Mekh. UrO RAN*, **20**, No. 2, 223-229 (2014).

27. V. D. Mazurov, "Groups with prescribed orders of elements," *Izv. Ural. Gos. Univ., Mat. Mekh.*, **36**, No. 7, 119-138 (2005).

28. A. S. Kondrat'ev, "Recognizability of ${}^2E_6(2)$ by prime graph," *Abelian Groups, Proc. Int. Symp. Dedicated to the 100th Anniversary of L. Ya. Kulikov*, MGPU, Moscow (2014), pp. 35-37.

29. A. L. Gavrilyuk, I. V. Khramtsov, A. S. Kondrat'ev, and N. V. Maslova, "On realizability of a graph as the prime graph of a finite group," *Sib. El. Mat. Izv.*, **11**, 246-257 (2014); http://semr.math.nsc.ru/v11/p246-257.pdf.

30. M. C. Lucido, "Groups in which the prime graph is a tree," *Boll. Unione Mat. Ital., Sez. B, Artic. Ric. Mat. (8)*, **5**, No. 1, 131-148 (2002).

31. O. A. Alekseeva and A. S. Kondrat'ev, "Finite almost simple groups whose prime graphs have no triangles. I," *Trudy Inst. Mat. Mekh. UrO RAN*, **21**, No. 3, 3-12 (2015).

32. A. Gruber, T. M. Keller, M. L. Lewis, K. Naughton, and B. Strasser, "A characterization of the prime graphs of solvable groups," *J. Alg.*, **442**, 397-422 (2015).

33. M. S. Lucido and A. R. Moghaddamfar, "Groups with complete prime graph connected components," *J. Group Theory*, **7**, No. 3, 373-384 (2004).

34. M. R. Zinov'eva and V. D. Mazurov, "Finite groups with disconnected prime graph," *Trudy Inst. Mat. Mekh. UrO RAN*, **18**, No. 3, 99-105 (2012).

35. M. R. Zinov'eva and A. S. Kondrat'ev, "Finite almost simple groups with prime graphs all of whose connected components are cliques," *Trudy Inst. Mat. Mekh. UrO RAN*, **21**, No. 3, 132-141 (2015).