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ALGEBRAIC SETS IN A FINITELY GENERATED 2-STEP SOLVABLE RIGID PRO-p-GROUP

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A 2-step solvable pro-p-group G is said to be rigid if it contains a normal series of the form $G = G_1 > G_2 > G_3 = 1$ such that the factor group $A = G/G_2$ is torsion-free Abelian, and the subgroup G_2 is also Abelian and is torsion-free as a $\mathbb{Z}_pA$-module, where $\mathbb{Z}_pA$ is the group algebra of the group A over the ring of p-adic integers. For instance, free metabelian pro-p-groups of rank ≥ 2 are rigid. We give a description of algebraic sets in an arbitrary finitely generated 2-step solvable rigid pro-p-group G, i.e., sets defined by systems of equations in one variable with coefficients in G.

INTRODUCTION

Abstract rigid solvable groups and algebraic geometry over them were defined and studied in [1-7]. In [8], the notion of a rigid metabelian pro-p-group was defined and some properties of such groups were explored. We give the definition.

A 2-step solvable pro-p-group G is said to be *rigid* if it contains a normal series of the form

$$G = G_1 > G_2 > G_3 = 1$$

such that the factor group $A = G/G_2$ is torsion-free Abelian, and the subgroup G_2 is also Abelian and is torsion-free as a $\mathbb{Z}_pA$-module, where $\mathbb{Z}_pA$ is the group algebra of the group A over the ring of p-adic integers. Such a series (if it exists whatsoever) is defined by the group uniquely. We call it a *rigid* series and denote it by $G_i = \rho_i(G)$. Important examples of 2-step solvable rigid pro-p-groups are free metabelian pro-p-groups of ranks ≥ 2. In [9], by analogy with abstract groups

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(see [10, 11]), elementary aspects of algebraic geometry over profinite groups, in particular, over pro-p-groups, were propounded. Note that in profinite groups, the concept of an equation differs from that used in abstract groups. Our objective is to receive information on algebraic sets in a finitely generated 2-step solvable rigid pro-p-group G (i.e., sets defined by systems of equations in one variable with coefficients in G), similar to one that was obtained in [12, 13] for abstract free metabelian groups and wreath products of two free Abelian groups. We will use the definitions and results in [9] concerning algebraic geometry over profinite groups, and in [8, 14] dealing directly with 2-step solvable rigid pro-p-groups. For information on profinite groups, we ask the reader to consult [15].

Thus let G be a finitely generated 2-step solvable rigid pro-p-group. That G is equationally Noetherian follows, for example, from the fact that the coordinate group of every affine space G^n is also a finitely generated 2-step solvable pro-p-group and, hence, satisfies the maximal condition for normal subgroups. Consequently, the Zariski topology on G^n is Noetherian and an arbitrary closed set in G^n is representable as a union of finitely many irreducible components, each of which is an algebraic set. In [14], it was shown that the entire space G^n is irreducible. Here we handle the case where $n = 1$. We formulate our basic results.

THEOREM 1. In a finitely generated 2-step solvable rigid pro-p-group G, irreducible algebraic sets are divided into the following three types:

(1) singleton sets and the whole group G;

(2) cosets with respect to a subgroup $\rho_2(G)$;

(3) irreducible algebraic sets S which are not listed above and are such that the projection $\overline{S}$ of a set S onto $A = G/\rho_2(G)$ is infinite, the mapping $S \to \overline{S}$ is bijective, and the set S is defined by canonical equation (2) (see below).

THEOREM 2. Let G be a finitely generated 2-step solvable rigid pro-p-group. Algebraic sets in G are exactly the following:

(1) the whole group G;

(2) an irreducible algebraic set of type (3) in Theorem 1 or a union of such a set and a finite number of cosets with respect to a subgroup $\rho_2(G)$;

(3) a set of the form

$$\{g_1, \ldots, g_l\} \cup g_{l+1} \cdot \rho_2(G) \cup \ldots \cup g_k \cdot \rho_2(G),$$

where $g_1, \ldots, g_l, g_{l+1}, \ldots, g_k$ are in different cosets with respect to $\rho_2(G)$.

1. IRREDUCIBLE ALGEBRAIC SETS IN G

1.1. We recall some facts from [8] concerning splittings of pro-p-groups over Abelian normal subgroups. Suppose that a metabelian pro-p-group G has an Abelian normal subgroup C and $\overline{G} = G/C$ is an Abelian group. Put $\overline{g} = gC$ for $g \in G$. The group G acts by conjugations

$x \to x^g = g^{-1}xg$ on C. Here, in fact, the group $\overline{G}$ acts and C can be treated as a right topological $\mathbb{Z}_p\overline{G}$-module. Assume that there is a pro-p-group that decomposes into a semidirect product of its subgroup $\overline{G}$ and some Abelian normal subgroup $D(G)$, which is represented in matrix form as follows: $\begin{pmatrix} \overline{G} & 0 \\ D(G) & 1 \end{pmatrix}$. We call the last group a *splitting of* G *over* C if an embedding of G in it is specified so that $g = \begin{pmatrix} \overline{g} & 0 \\ d(g) & 1 \end{pmatrix}$, and $D(G)$ is generated as a $\mathbb{Z}_p\overline{G}$-module by elements $d(g)$, $g \in G$. Among all splittings of G over C, we distinguish a *free* one. For the free splitting, the mapping $d(g) \to \overline{g} - 1$ determines an epimorphism (a differential) δ of the module $D(G)$ onto the difference ideal $(\overline{G} - 1) \cdot \mathbb{Z}_p\overline{G}$ of the group ring $\mathbb{Z}_p\overline{G}$, and the kernel of this epimorphism is C (here C is naturally identified with a submodule of $D(G)$).

1.2. Let G be a finitely generated 2-step solvable rigid pro-p-group. Put $A = G/\rho_2(G)$, which is a free Abelian pro-p-group. Let $\{a_1, \ldots, a_n\}$ be its basis. The group algebra $\mathbb{Z}_pA$ is an algebra of (commutative) formal power series over $\mathbb{Z}_p$ in $a_1 - 1, \ldots, a_n - 1$. Let $\begin{pmatrix} A & 0 \\ D(G) & 1 \end{pmatrix}$ be a free splitting of G over $\rho_2(G)$. In [8], it was mentioned that the module $D(G)$ is torsion-free, and so our splitting is also a 2-step solvable rigid pro-p-group. The group G is finitely generated, and hence the module $D(G)$ likewise is finitely generated.

Let $F = \langle x_1, \ldots, x_m \rangle$ be a free metabelian pro-p-group. It was noted in [8] that as a group of equations in variables $x_1, \ldots, x_m$ with coefficients in G we can treat the coordinate group $\Gamma(G^m)$ of an affine space G^m. In [8], this group is represented as a 2-graded product $G \circ F$, which is also a 2-step solvable rigid pro-p-group. We show how to construct this product; more exactly, we outline the construction of a free splitting of a group $H = G \circ F$ over $\rho_2(H)$ for the case where $F = \langle x \rangle$ has rank 1.

Let $B = \langle b \rangle$ be a free Abelian pro-p-group of rank 1. We introduce into consideration a group $C = A \times B$, a free one-generated $\mathbb{Z}_pC$-module $z \cdot \mathbb{Z}_pC$, and a $\mathbb{Z}_pC$-module such as

$$D(H) = D(G) \otimes_{\mathbb{Z}_pA} \mathbb{Z}_pC \oplus z \cdot \mathbb{Z}_pC.$$

In [8], it was mentioned that $D(H)$ is $\mathbb{Z}_pC$-torsion-free and $D(G)$ is embedded in $D(H)$. A free splitting of H over $\rho_2(H)$ is $\begin{pmatrix} C & 0 \\ D(H) & 1 \end{pmatrix}$, in which case x is identified with a matrix $\begin{pmatrix} b & 0 \\ z & 1 \end{pmatrix}$ and H is generated by G and F. Clearly, the group $G \leqslant \begin{pmatrix} A & 0 \\ D(G) & 1 \end{pmatrix}$ is embedded in $\begin{pmatrix} C & 0 \\ D(H) & 1 \end{pmatrix}$.

As noted, the $\mathbb{Z}_pA$-module $D(G)$ is finitely generated and is torsion-free. By virtue of [8, proof of Prop. 1], $D(G)$ is embedded in some free $\mathbb{Z}_pA$-module T_A with finite basis $\{t_1, \ldots, t_m\}$, in which case there exists a nonzero element $e \in \mathbb{Z}_pA$ such that $T_A \cdot e \leqslant D(G) \leqslant T_A$. This readily implies that $D(H)$ is embedded in a free $\mathbb{Z}_pC$-module such as

$$T_C = T_A \otimes_{\mathbb{Z}_pA} \mathbb{Z}_pC + z \cdot \mathbb{Z}_pC = t_1 \cdot \mathbb{Z}_pC + \ldots + t_m \cdot \mathbb{Z}_pC + z \cdot \mathbb{Z}_pC$$

with basis $\{t_1, \ldots, t_m, z\}$, and $T_C \cdot e \leqslant D(H) \leqslant T_C$. Let

$$\delta : D(H) \to (C-1) \cdot \mathbb{Z}_p C$$

be a differential associated with the splitting of H over $\rho_2(H)$, whose kernel is $\rho_2(H)$. Note that an arbitrary group A-epimorphism $C \to A$ determines a module epimorphism $T_C \to T_A + z \cdot \mathbb{Z}_p A$, which is consistent with a ring epimorphism $\mathbb{Z}_p C \to \mathbb{Z}_p A$ over which the modules in question are treated.

Below we use the notation in Sec. 1.2 without further comment.

1.3. Consider an arbitrary equation $f(x) = 1$, where $f \in H$. Represent f as the matrix

$$\begin{pmatrix} \overline{f}(b) & 0 \\ t_1 \cdot u_1(b) + \ldots + t_m \cdot u_m(b) + z \cdot u(b) & 1 \end{pmatrix},$$

where $\overline{f}(b)$ is the image of f in C. Clearly, the equation $f(x) = 1$ is equivalent to a system of the form

$$\overline{f}(b) = 1 \ \wedge \ t_1 \cdot u_1(b) + \ldots + t_m \cdot u_m(b) + z \cdot u(b) = 0. \tag{1}$$

The system depends on variables b and z, and we seek values for b and z in A and T_A, respectively, in which case the value $x = \begin{pmatrix} b & 0 \\ z & 1 \end{pmatrix}$ should belong to G.

LEMMA 1. Every equation $v(b) = 0$ in a variable b, where $0 \neq v(b) \in \mathbb{Z}_p C$, has at most finitely many roots in A.

Proof. Recall that $\mathbb{Z}_p C$ is a ring of formal power series over $\mathbb{Z}_p$ in $a_1 - 1, \ldots, a_n - 1, y = b - 1$. The pro-$p$-topology on $\mathbb{Z}_p C$ is defined by the powers of the maximal ideal which is generated by the elements $p, a_1 - 1, \ldots, a_n - 1, y$. We can represent $\mathbb{Z}_p C$ as a ring of formal power series in y with coefficients in $\mathbb{Z}_p A$. For any $a \in A$, the series $v(b) = v(1 + y)$ is expanded in the powers of $y - (a - 1)$. If $v(a) = 0$, then $v(b)$ is divisible by $y - (a - 1)$ in the ring $\mathbb{Z}_p C$. If, in addition, $v(a') = 0$ where $a \neq a' \in A$, then $v(b)$ is divisible by $(y - (a - 1))(y - (a' - 1))$. Now we choose an s such that $v(b)$ does not belong to the sth power of the ideal of $\mathbb{Z}_p C$ generated by the elements $a_1 - 1, \ldots, a_n - 1, y$. Then the equation $v(b) = 0$ cannot have s roots in A. The lemma is proved.

1.4. We pass to a classification of irreducible algebraic sets in G, i.e., to the proof of Theorem 1.

(1) Irreducibility of a singleton set is obvious. Irreducibility of the whole group G (as mentioned) was proved in [14].

(2) Notice that every coset $g \cdot \rho_2(G)$ is defined by an equation $[g^{-1}x, d] = 1$, where d is an arbitrary nontrivial element of $\rho_2(G)$. That $g \cdot \rho_2(G)$ is irreducible is shown in the following:

LEMMA 2. If some equation $f(x) = 1$ distinguishes in $S = g \cdot \rho_2(G)$ a nonempty proper subset, then this subset is a singleton.

Proof. We have noted that an equation of the form $f(x) = 1$ reduces to system (1). Suppose that this equation has a solution belonging to S. Then the value of a variable b should be equal to the image $\overline{g}$ of an element g in A. In view of this, the second equation in system (1) can be written in the form

$$t_1 \cdot u_1(\overline{g}) + \ldots + t_m \cdot u_m(\overline{g}) + z \cdot u(\overline{g}) = 0.$$

If $u(\overline{g}) \neq 0$, then z, and hence x, will be defined uniquely. If $u(\overline{g}) = 0$, then $u_i(\overline{g})$ must also equal zero, and S as a whole satisfies $f(x) = 1$, which contradicts the hypothesis. The lemma is proved.

The above argument implies

LEMMA 3. If an irreducible algebraic set in G is contained in a union of finitely many cosets with respect to $\rho_2(G)$, then either it is a singleton or coincides with one of the cosets.

(3) Now we consider an irreducible algebraic set S in G, which is not entered into items (1) and (2), and study its properties.

Lemma 3 implies that the projection $\overline{S}$ of a set S onto A is infinite.

Let $f(x) = 1$ be a nontrivial equation which complies with the set S. Take system (1) corresponding to this equation. The first equation $\overline{f}(b) = 1$ in the system considered should be trivial; otherwise, S is contained in some coset with respect to $\rho_2(G)$. Therefore,

$$f(x) = \begin{pmatrix} 1 & 0 \\ t_1 \cdot u_1(b) + \ldots + t_m \cdot u_m(b) + z \cdot u(b) & 1 \end{pmatrix} \in \rho_2(H).$$

Thus system (1) is left only with the second equation

$$t_1 \cdot u_1(b) + \ldots + t_m \cdot u_m(b) + z \cdot u(b) = 0. \tag{2}$$

Notice that S cannot satisfy two such noncollinear equations over $\mathbb{Z}_p C$; otherwise, these yield an equation in which $u(b) \equiv 0$, and S will satisfy a system of the form $u_1(b) = 0 \wedge \ldots \wedge u_m(b) = 0$, which, by Lemma 1, defines either the empty set or the union of finitely many cosets with respect to $\rho_2(G)$.

For $u_1(b), \ldots, u_m(b), u(b)$, we choose all common roots $b = d_1, \ldots, d_k$ (with due regard for multiplicity) belonging to A, if such exist, and put

$$v(b) = (y - (d_1 - 1)) \ldots (y - (d_k - 1)).$$

The elements $u_1(b), \ldots, u_m(b), u(b)$ are divisible by $v(b)$ in $\mathbb{Z}_p C$; we let $u_1'(b), \ldots, u_m'(b), u'(b)$ be the corresponding quotients. It may turn out that a matrix of the form

$$\begin{pmatrix} 1 & 0 \\ t_1 \cdot u_1'(b) + \ldots + t_m \cdot u_m'(b) + z \cdot u'(b) & 1 \end{pmatrix}$$

is not in $\rho_2(H)$. Recall that the conditions required for the matrix $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$, where $t \in T_C$, to belong to $\rho_2(H)$ are the inclusion $t \in D(H)$ and the equality $t\delta = 0$. These conditions are satisfied by the following matrix:

$$\begin{pmatrix} 1 & 0 \\ (t_1 \cdot u_1'(b) + \ldots + t_m \cdot u_m'(b) + z \cdot u'(b))e & 1 \end{pmatrix}.$$

Indeed, $te \in D(H)$ for any $t \in T_C$, while the quality

$$((t_1 \cdot u_1'(b) + \ldots + t_m \cdot u_m'(b) + z \cdot u'(b))e)\delta = 0$$

follows from

$$((t_1 \cdot u_1'(b) + \ldots + t_m \cdot u_m'(b) + z \cdot u'(b))v(b)e)\delta$$
$$= ((t_1 \cdot u_1(b) + \ldots + t_m \cdot u_m(b) + z \cdot u(b))e)\delta = 0.$$

Note that the equation $v(b) = 0$ corresponds to the group equation $\begin{pmatrix} 1 & 0 \\ t \cdot v(b) & 1 \end{pmatrix} = 1$, where $t \neq 0$ and $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \in \rho_2(G)$. We may assert that Eq. (2) is equivalent to a system of two equations: namely,

$$v(b) = 0 \ \wedge \ t_1 \cdot u_1'(b) \cdot e + \ldots + t_m \cdot u_m'(b) \cdot e + z \cdot u'(b) \cdot e = 0, \tag{3}$$

each of which (as is (2)) is realized in the form of a group equation. Clearly, our set S should satisfy the second of these equations, in which the coefficients at $t_1, \ldots, t_m, z$ do not any longer have common roots in A.

We say that Eq. (2) corresponding to the group equation is *canonical* if $u_1(b), \ldots, u_m(b), u(b)$ have no common roots in A.

LEMMA 4. An irreducible algebraic set S of type (3) in G satisfies some canonical equation and is exactly a set of solutions for such an equation.

Proof. Above we have stated that S satisfies some canonical equation. Assume that the set S' of solutions for this equation is larger than S. Then there exists another canonical equation which complies with S and truncates in S' a proper subset. Let $t(b, z) = t_1 \cdot u_1(b) + \ldots + t_m \cdot u_m(b) + z \cdot u(b)$ be the left part of the first equation, $t'(b, z)$ the left part of the second equation, and $g = \begin{pmatrix} \overline{g} & 0 \\ d(g) & 1 \end{pmatrix} \in S' \backslash S$. Since $t(b, z)$ and $t'(b, z)$ should be collinear over the ring $\mathbb{Z}_p C$, there are $v(b), w(b) \in \mathbb{Z}_p C$ that have no common roots in A and are such that $t(b, z) \cdot v(b) = t'(b, z) \cdot w(b)$. We have $t(\overline{g}, d(g)) = 0$ and $t'(\overline{g}, d(g)) \neq 0$, so $w(\overline{g}) = 0$. Hence the whole coset $g \cdot \rho_2(G)$, which is an irreducible algebraic set, satisfies an equation $t(b, z) \cdot v(b) = 0$. This coset may satisfy an equation $t(b, z) = 0$ only if $b = \overline{g}$ is a common root of all $u_1(b), \ldots, u_m(b), u(b)$, which contradicts the hypothesis. Therefore, $v(\overline{g}) = 0$, which clashes with the fact that $v(b)$ and $w(b)$ have no common roots in A. The lemma is proved.

LEMMA 5. If S is an irreducible algebraic set of type (3) in G, (2) is a canonical equation defining this set, and $a \in \overline{S}$, then $u(a) \neq 0$.

Proof. Otherwise, the element a would be a common root for $u_1(b), \ldots, u_m(b), u(b)$, which contradicts the definition of a canonical equation. The lemma is proved.

To complete the proof of Theorem 1, we need to state that the projection $S \to \overline{S} \leqslant A$ is an injective mapping. Indeed, if we fix the value $b = a \in \overline{S}$ and consider canonical equation (2) for S, then $u(a) \neq 0$ by Lemma 5. Therefore, the value z, and hence the value x, will be defined uniquely.

2. EXAMPLES

Notice that irreducible sets of type (3) in G exist—for instance, centralizers of elements of G not lying in $\rho_2(G)$.

We give a more complicated example of an irreducible algebraic set of type (3) in a free metabelian pro-p-group $G = \langle g_1, \ldots, g_n \rangle$ of rank $n \geq 3$. As a group of equations in x over G we take a free metabelian pro-p-group $H = \langle g_1, \ldots, g_n, x \rangle$. The splitting construction here reduces to the Magnus embedding (see [16, 17]). Let

$$A = G/G' = \langle a_1, \ldots, a_n \rangle, \ C = H/H' = \langle a_1, \ldots, a_n, b \rangle$$

be free Abelian pro-p-groups, and let

$$D(G) = T_A = t_1 \cdot \mathbb{Z}_p A + \ldots + t_m \cdot \mathbb{Z}_p A, \ D(H) = T_C = t_1 \cdot \mathbb{Z}_p C + \ldots + t_m \cdot \mathbb{Z}_p C$$

be free modules. Free splittings for G and H have the forms

$$\begin{pmatrix} A & 0 \\ T_A & 1 \end{pmatrix}, \begin{pmatrix} C & 0 \\ T_C & 1 \end{pmatrix}.$$

In this case

$$g_1 = \begin{pmatrix} a_1 & 0 \\ t_1 & 1 \end{pmatrix}, \quad \ldots, \quad g_n = \begin{pmatrix} a_n & 0 \\ t_n & 1 \end{pmatrix}, \quad x = \begin{pmatrix} b & 0 \\ z & 1 \end{pmatrix}.$$

We point out a differential such as

$$\delta : t_1 u_1 + \ldots + t_m u_m + zu \to (a_1 - 1)u_1 + \ldots + (a_m - 1)u_m + yu.$$

LEMMA 6. If S is an irreducible algebraic set of type (3) in a free metabelian pro-p-group G, (2) is a canonical equation defining this set, and the values $b = a \in A$ and $z = t \in T_A$ give a solution for the equation, then $\begin{pmatrix} a & 0 \\ t & 1 \end{pmatrix} \in G$.

Proof. By Lemma 5, $u(a) \neq 0$. Then the element $u(a)$ should divide all $u_1(a), \ldots, u_m(a)$ in the ring $\mathbb{Z}_p A$. Let $u_1'(a), \ldots, u_m'(a)$ be the corresponding quotients. We obtain $t = -t_1 u_1'(a) - \ldots - t_m u_m'(a)$. Since

$$(a_1 - 1)u_1(b) + \ldots + (a_m - 1)u_m(b) + yu(b) = 0,$$

we have

$$(a_1 - 1)u_1(a) + \ldots + (a_m - 1)u_m(a) + (a - 1)u(a) = 0.$$

The latter implies

$$-(a_1 - 1)u_1'(a) - \ldots - (a_m - 1)u_m'(b) = a - 1;$$

i.e., $t\delta = a - 1$. Then $\begin{pmatrix} a & 0 \\ t & 1 \end{pmatrix} \in G$ (see properties of the Magnus embedding in [16, 17]). The lemma is proved.

Consider an equation of the form

$$(t_1 y - z(a_1 - 1))y + (t_2(a_3 - 1) - t_3(a_2 - 1))(a - 1)^2 = 0. \tag{4}$$

The left part of the equation is in $\ker \delta$, so (4) corresponds to the group equation. The condition for roots which we imposed on a canonical equation is also satisfied.

Let S be the set of solutions for Eq. (4). Consider $\overline{S}$. For the value b in A to belong to $\overline{S}$, it is necessary and sufficient that the appropriate value of the coefficient at z in the left part of Eq. (4) will divide in $\mathbb{Z}_p A$ the values of the coefficients at t_1, t_2, t_3. This occurs exactly when the value y divides $a_1 - 1$, and conversely, $a_1 - 1$ divides the value y. It is straightforward to verify that such values b constitute the set $A_1 \setminus A_1^p$, where A_1 is a subgroup generated by the element a_1.

In [12, 13], irreducible algebraic sets were studied for two kinds of abstract metabelian rigid groups: wreath products of two nontrivial free Abelian groups and a free metabelian group of rank ≥ 2. The situation there is similar to the one considered here; however, it turns out that canonical equation (2) defining an irreducible algebraic set of type (3) possesses an additional property: the element $u(b)$ in canonical equation (2) does not actually depend on b. Based on this, we succeeded in completely describing projections of sets of type (3) in A. For the case of an abstract free metabelian group, these projections have the form

$$D = D_0 \sqcup d_1 \cdot A' \sqcup \ldots \sqcup d_s \cdot A';$$

here D_0 is a finite set, A' is a nontrivial cyclic subgroup of A, $s \geq 1$, D is contained in some coset $d \cdot A''$, and A'' is a cyclic subgroup of A containing A'. The last example given shows that for pro-p-groups, a similar result does not hold.

3. ALGEBRAIC SETS IN G

3.1. Every subset closed in the Zariski topology in a finitely generated 2-step solvable rigid pro-p-group G is a union of finitely many irreducible algebraic sets, which are characterized in Theorem 1. However, this union itself may not be an algebraic set. In Theorem 2, therefore, we describe which of the unions mentioned are algebraic sets. Properly speaking, the proof of Theorem 2 follows the line of argument of a similar theorem in [13], which treats algebraic sets in an abstract free metabelian group and in a wreath product of two free Abelian groups.

3.2. First we verify that the sets specified in the formulation of Theorem 2 are indeed algebraic. Consider a set such as in item (2), which has the form $S \cup g_1 \cdot \rho_2(G) \cup \ldots \cup g_k \cdot \rho_2(G)$, where S is an irreducible algebraic set of type (3). Let S be defined by canonical equation (2). Obviously, the equation

$$(t_1 \cdot u_1(b) + \ldots + t_m \cdot u_m(b) + z \cdot u(b))(\overline{g_1} - b) \ldots (\overline{g_k} - b) = 0$$

defines the required set.

We continue to consider a set such as in item (3) in Theorem 2: namely,

$$P = \{g_1, \ldots, g_l\} \cup g_{l+1} \cdot \rho_2(G) \cup \ldots \cup g_k \cdot \rho_2(G),$$

where $g_1, \ldots, g_l, g_{l+1}, \ldots, g_k$ are in different cosets with respect to $\rho_2(G)$. There is no loss of generality in assuming that these cosets are all distinct from $\rho_2(G)$. First assume that the finite part $\{g_1, \ldots, g_l\}$ is missing; i.e., the set has the form $g_1 \cdot \rho_2(G) \cup \ldots \cup g_k \cdot \rho_2(G)$. In this case it is defined by an equation $t \cdot (\overline{g_1} - b) \ldots (\overline{g_k} - b) = 0$, where $1 \neq \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \in \rho_2(G)$.

Suppose now that by induction, the set

$$P' = \{g_1, \ldots, g_{l-1}\} \cup g_l \cdot \rho_2(G) \cup g_{l+1} \cdot \rho_2(G) \cup \ldots \cup g_k \cdot \rho_2(G)$$

is algebraic. Denote by L the centralizer of g_l in G. The equation

$$[x, g_l]^{(g_1 - b) \ldots (g_{l-1} - b)(g_{l+1} - b) \ldots (g_k - b)} = 1$$

defines the set

$$P'' = L \cup g_1 \cdot \rho_2(G) \cup \ldots \cup g_{l-1} \cdot \rho_2(G) \cup g_{l+1} \cdot \rho_2(G) \cup \ldots \cup g_k \cdot \rho_2(G).$$

It remains to observe that $P = P' \cap P''$. Indeed, all components of P' but the coset $g_l \cdot \rho_2(G)$ are contained in P''. This coset has no elements in common with all sets constituting P'', except L. In view of $L \cap g_l \cdot \rho_2(G) = \{g_l\}$, we obtain the result required.

3.3. The next three lemmas show that Theorem 2 indeed describes all algebraic sets in G.

LEMMA 7. If P is a proper algebraic set in G and $P \supseteq S$, where S is an irreducible algebraic set of type (3) or a coset with respect to $\rho_2(G)$, then S is an irreducible component of the set P.

Proof. Clearly, S is contained in some irreducible component. If S is a coset with respect to $\rho_2(G)$, then it cannot be contained in another coset or in an irreducible algebraic set of type (3). If S is an irreducible algebraic set of type (3), then S is a set of solutions for any canonical equation of form (2) that it satisfies; so S cannot be included in a larger irreducible algebraic set of type (3), nor can it be contained in any coset with respect to $\rho_2(G)$. Hence, in either case S coincides with an appropriate irreducible component. The lemma is proved.

LEMMA 8. Let P be a proper algebraic set in G and S be an irreducible component of type (3) in P. Then the remaining components (if they exist) are cosets with respect to $\rho_2(G)$.

Proof. It suffices to show that if $g \in P \setminus S$, then the coset $g \cdot \rho_2(G)$ is entirely in P. It follows by Lemma 7 that this coset will be an irreducible component, and irreducible components of this sort, combined with S, will yield P.

Consider an arbitrary nontrivial equation $f(x) = 1$ which complies with P. By virtue of $f(S) = 1$,

$$f(x) = \begin{pmatrix} 1 & 0 \\ t_1 \cdot u_1(b) + \ldots + t_m \cdot u_m(b) + z \cdot u(b) & 1 \end{pmatrix} \in \rho_2(H).$$

Therefore, the equation $f(x) = 1$ is rewritten in form (2), and then in the form of system (3) (see Sec. 1.4). As noted, S is exactly the set of solutions for the second equation in system (3). The element g must not satisfy this equation; hence $v(\overline{g}) = 0$ and the whole coset $g \cdot \rho_2(G)$ satisfies $f(x) = 1$. The lemma is proved.

LEMMA 9. If an algebraic set P in G contains two elements of the coset $g \cdot \rho_2(G)$, then this coset is entirely contained in P.

Proof. Otherwise, the algebraic set $P \cap g \cdot \rho_2(G)$ contains more than two elements, which contradicts Lemma 2. The lemma is proved.

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