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LINEARLY MINIMAL JORDAN ALGEBRAS OF CHARACTERISTIC OTHER THAN 2

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It is proved that every nontrivial linearly minimal Jordan algebra of characteristic other than 2 *is a division algebra. Then Zel'manov's classification of Jordan division algebras is applied to show that such an algebra is a field.*

INTRODUCTION

An infinite algebraic structure M of a signature Σ is (*definably*) *minimal* if every subset of M definable by a first-order Σ-formula over M is finite or cofinite. In particular, an infinite ring $\langle R; +, \cdot, 0\rangle$ is minimal if every subset of R definable in a language $\{=, +, \cdot, R\}$ is finite or cofinite.

Research into nonassociative linearly minimal rings was initiated in [1] and then continued in [2], where a new kind of minimality for rings and algebras was introduced—namely, linear minimality (l-minimality).

We recall the definition of linearly minimal rings [2]. Let $\mathcal{R} = \langle R; +, \cdot, 0\rangle$ be a ring and $\mathrm{End}\,(\mathcal{R})$ the endomorphism ring of an Abelian group $\mathcal{R}^+ = \langle R; +, 0\rangle$. Then, for any element a of the ring, multiplication maps $l_a : x \mapsto a \cdot x$ and $r_a : x \mapsto x \cdot a$ are in $\mathrm{End}\,(\mathcal{R})$. Denote by $\mathrm{Mult}\,(\mathcal{R})$ a unital subring of $\mathrm{End}\,(\mathcal{R})$ generated by the set of all multiplication maps. We say that an infinite ring $\mathcal{R}$ is *linearly minimal* if every nonzero element of $\mathrm{Mult}\,(\mathcal{R})$ is a surjective map with finite kernel.

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Elementary examples of linearly minimal rings are (infinite) fields. It is easy to see that every minimal ring is linearly minimal. Every infinite nonalgebraically closed field of positive characteristic exemplifies a linearly minimal but not minimal ring [3].

Linearly minimal algebras can be defined similarly. Let $\mathfrak{A} = \langle A; +, \cdot, 0\rangle$ be an arbitrary algebra (over a field F). Then multiplication maps $L_a : x \mapsto a \cdot x$ and $R_a : x \mapsto x \cdot a$ are linear, for every element $a \in \mathfrak{A}$. Thus L_a and R_a are in $\mathfrak{E}(\mathfrak{A})$, where $\mathfrak{E}(\mathfrak{A})$ is an algebra of all linear transformations in $\mathfrak{A}$. In [4], the multiplication algebra $\mathfrak{T}(\mathfrak{A})$ is defined as an enveloping algebra of the set of all L_a and R_a, $a \in A$; i.e., $\mathfrak{T}(\mathfrak{A})$ is the unital subalgebra of $\mathfrak{E}(\mathfrak{A})$ generated by all L_a and R_a, $a \in A$.

Definition 1 [2]. We say that an infinite algebra $\mathfrak{A}$ is *linearly minimal* (*l-minimal* for short) if every nonzero linear map from the multiplication algebra $\mathfrak{T}(\mathfrak{A})$ is a surjective map with finite kernel.

The definition implies that every linearly minimal algebra is a linearly minimal ring. The theorem below shows that the converse is almost always true. Recall that a ring (or algebra) is trivial if $a \cdot b = 0$ for any elements a and b.

THEOREM 1 [2]. A nontrivial linearly minimal ring is a linearly minimal algebra over a field of its definable scalars. In addition, if the scalar field is infinite, then the ring itself is a field.

Thus, in what follows, we may deal with algebras only.

Obviously, every vector space can be transformed into a trivial algebra. Therefore, we will be interested in just nontrivial algebras, and unless otherwise stated, all algebras under consideration will be nontrivial.

Now we turn to Jordan rings and algebras of characteristic other than 2. Recall that $\mathcal{R} = \langle R; +, \cdot, 0\rangle$ is a *Jordan* ring if it satisfies the following identities:

$$x \cdot y = y \cdot x, \tag{C}$$

$$x^2 \cdot (y \cdot x) = (x^2 \cdot y) \cdot x, \tag{J}$$

where x^2 stands for $x \cdot x$.

Similarly, $\mathfrak{A} = \langle A; +, \cdot, 0\rangle$ is a *Jordan* algebra if it satisfies (C) and (J).

Remark. It is easy to see that every associative commutative algebra is a particular case of a Jordan algebra. Generally, Jordan algebras are thought of as being nonassociative. Except in the formulation of the main result, we also try to adhere to this convention, which ensures convenience of the investigation. Nontrivial associative linearly minimal rings and algebras have already been described: these are fields [5].

Below we denote the Jordan product of elements a and b by $a \bullet b$, keeping the notation $a \cdot b$ for products in adjacent associative algebras.

Let $\mathfrak{J} = \langle J; +, \bullet, 0\rangle$ be a Jordan algebra over a field F of characteristic not 2. Suppose that $a \in J$ is any (fixed) element and $R_a : x \mapsto a \bullet x$. The linear operator

$$U_a := 2R_a^2 - R_{a^2}$$

plays an important part in the theory of Jordan algebras. Since R_a, $R_{a^2} \in \mathfrak{T}(\mathfrak{J})$, it follows that $U_a \in \mathfrak{T}(\mathfrak{J})$. In particular, if the algebra is linearly minimal, then either $U_a \equiv 0$ or U_a is a surjective linear map with finite kernel. The following identity is the basic property of an operator U (called the *Macdonald identity* in [6] and referred to as the *fundamental formula* in [7]):

$$U_{U_x y} = U_x U_y U_x. \tag{M}$$

In Sec. 1, we prove that every linearly minimal Jordan algebra is a division algebra. In Sec. 2, we apply Zel'manov's description of Jordan division algebras to show that linearly minimal Jordan algebras are fields. In [1], it was stated (with no appeal to Zel'manov's theorem) that minimal Jordan rings are fields. Thus our main result strengthens this statement: the condition for minimality can be replaced by a weaker requirement for linear minimality.

1. LINEARLY MINIMAL JORDAN ALGEBRAS

We fix some Jordan algebra $\mathfrak{J} = \langle J; +, \bullet, 0 \rangle$ over a field F of characteristic $\neq 2$. In what follows, we assume that $\mathfrak{J}$ is linearly minimal and nonassociative.

Thus $\mathfrak{J}$ is a nontrivial algebra over a finite field F (which is even central in view of [2]). Let $b, c \in J$ so that $b \bullet c \neq 0$.

LEMMA 1. Every nilpotent element of $\mathfrak{J}$ is an annihilator.

Proof. Let $a \in J$ be an element distinct from zero such that $a^n = 0$ for some natural n. By virtue of [7, Thm. 5.2.2], we have $U_a^n = U_{a^n} = U_0 \equiv 0$. Hence U_a is not surjective, and linear minimality implies that $U_a \equiv 0$. In particular, $a^3 = U_a a = 0$, and also $(a^2)^2 = a^4 = a^3 \bullet a = 0$. Thus $R_{(a^2)^2} \equiv 0$. This yields

$$2(R_{a^2})^2 = 2(R_{a^2})^2 - R_{(a^2)^2} = U_{a^2} = U_a^2 = 0.$$

Hence R_{a^2} is not surjective, and again by linear minimality, we obtain $R_{a^2} \equiv 0$. Finally, $2R_a^2 = 2R_a^2 - R_{a^2} = U_a \equiv 0$ entails $R_a \equiv 0$; so a is an annihilator.

Definition 2. An element $x \in J$ is *invertible* if there exists $y \in J$ such that $U_x y = x$ and $U_x U_y = \mathrm{id}_J$. This y is called an *inverse* of x.

If $\mathfrak{J}$ is a unital algebra, then Definition 2 is equivalent to known classical definitions of an invertible element (see [6, 7]). We opt for this definition because it can be given for any Jordan algebra (without the unitality assumption).

LEMMA 2. Every nilpotent element of $\mathfrak{J}$ is invertible.

Proof. Let $a \in J$ be a nonnilpotent element. Since $U_a a = a^3 \neq 0$, the linear minimality condition implies that U_a is surjective and $\operatorname{Ker} U_a$ is finite. The map U_a^k is surjective and its kernel $\operatorname{Ker} U_a^k$ is finite for every $k = 0, 1, 2, \ldots$ since $|\operatorname{Ker} U_a^k| = |\operatorname{Ker} U_a|^k$ (here, of course, we put $U_a^0 = \mathrm{id}_J$).

In view of [7, Thm. 5.2.2], the subalgebra $\langle a\rangle$ generated by the element a is associative, and the set $\{a, a^2, a^3, \dots\}$ should be finite, for otherwise $\mathfrak{J}$ would be associative by [2, Lemma 5], which clashes with our assumption. Thus, for some natural numbers $m < n$, we have $a^m = a^n$, whence $U_{a^m} = U_a^m = U_a^n = U_{a^n}$ and $|\operatorname{Ker} U_a|^m = |\operatorname{Ker} U_a|^n$. Hence $\operatorname{Ker} U_a = \{0\}$; in other words, U_a is bijective and invertible. If we apply U_a^{-1} to the equality $U_a^m = U_a^n$ we obtain $U_{a^j} = U_a^j = \mathrm{id}_J$ for $j = n - m$. The least natural number j with this property is called the *order* of the operator U_a.

Now we note that $a \in J = U_a(J)$, and so there exists $y \in J$ such that $U_a y = a$. By virtue of identity (M), the following relation holds:

$$U_a = U_{U_a y} = U_a U_y U_a,$$

from which (applying U_a^{-1}) we derive $U_a U_y = \mathrm{id}_J$. Hence a is invertible. Lemma 2 is proved.

LEMMA 3. Every nonzero element of $\mathfrak{J}$ is invertible.

Proof. Let $b, c \in J$ be elements such that $b \bullet c \neq 0$. By Lemma 1, b is a nonnilpotent element. By Lemma 2, b is invertible and $\operatorname{Ker} U_b = \{0\}$. All annihilators of the algebra are contained in $\operatorname{Ker} U_b$, and so $\mathfrak{J}$ does not have nonzero annihilators. Again in view Lemma 2, every nonzero element is invertible.

THEOREM 2. $\mathfrak{J}$ is a division algebra.

Proof. Let $0 \neq x \in J$ be an arbitrary element. Then x is invertible as is U_x. Let k be the order of U_x; then $U_a = U_{x^k} = U_x^k = \mathrm{id}_J$ for $a = x^k$. Hence $a = U_a a = a^3$, and for $e = a^2$, we have

$$e = a^2 = a \bullet a = a^3 \bullet a = a^4 = e^2.$$

Consequently, $e = e^2$ and e is an idempotent. Notice that $U_e = U_{a^2} = U_a^2 = \mathrm{id}_J$. By virtue of [7, Prop. 5.2.4], e is the identity element of $\mathfrak{J}$. Thus $\mathfrak{J}$ is unital. Every nonzero element is invertible, so $\mathfrak{J}$ is a division algebra.

2. LINEAR MINIMALITY OF JORDAN DIVISION ALGEBRAS

A description of Jordan division algebras of characteristic not 2 was obtained by Zel'manov [8]. It turned out that these may be only one of the following:

(a) an exceptional Jordan algebra, finite-dimensional over its center;

(b) a Jordan algebra $H(D, {}^*)$ of symmetric elements of an associative division algebra D with respect to an involution *;

(c) an algebra $D^{(+)}$, where D is an associative division algebra;

(d) an $F \cdot 1 + M$-algebra of a symmetric bilinear form on a vector space M.

We look at each of these types of Jordan division algebras from the standpoint of linear minimality. First note that every element λ in the field of these algebras can be identified with an element $\lambda \bullet 1$. Therefore, we may assume that the algebras each contains its field (as part of its center; for definition of a center, see [7]). In addition, by virtue of [2], we may suppose that the field F of an algebra is finite and $\operatorname{char} F = p > 2$.

(A) In [2], it was proved that a nonassociative linearly minimal Jordan algebra can have only a finite center, and that it should be infinite-dimensional over its center. Thus exceptional Jordan division algebras that are finite-dimensional over their centers cannot be linearly minimal.

(B) Let D be an (infinite) associative division algebra of characteristic $p > 2$. Recall that a Jordan algebra $D^{(+)}$ is defined by Jordan multiplication

$$a \bullet b = \frac{1}{2}(a \cdot b + b \cdot a),$$

where $\cdot$ denotes multiplication in D. A linear transformation * of the algebra D is called an *involution* if

$$(a^*)^* = a \quad \text{and} \quad (a \cdot b)^* = b^* \cdot a^*$$

for all $a, b \in D$. By definition, $H(D, {}^*) = \{a \in D : a^* = a\}$. It is easy to see that $H(D, {}^*)$ is a subalgebra of $D^{(+)}$.

We may assume that D is not commutative; otherwise, the algebra $D^{(+)} = D$, as well as any of its subalgebras, would be associative, which is a contradiction with our initial assumption.

ASSERTION. $D^{(+)}$ cannot contain a linearly minimal nonassociative subalgebra.

Proof. Assume the contrary, letting J be a linearly minimal subalgebra of $D^{(+)}$ and letting J not be a field. If we had $a \cdot b = b \cdot a$ for all $a, b \in J$, then J would be a commutative subalgebra of D and, consequently, J would be a field, a contradiction. Therefore, there exist $a, b \in J$ for which $a \cdot b \neq b \cdot a$. We fix these a and b and consider the linear operator

$$\varphi := l_a - r_a,$$

where linear maps $l_a : x \mapsto a \cdot x$ and $r_a : x \mapsto x \cdot a$ are defined over D. Since l_a commutes with r_a and p is the characteristic of the algebra, we obtain

$$\varphi^p = (l_a - r_a)^p = l_a^p - r_a^p = l_{a^p} - r_{a^p}.$$

Moreover, for any natural k, we have

$$\varphi^{p^k} = (l_a - r_a)^{p^k} = l_{a^{p^k}} - r_{a^{p^k}}.$$

Recall that any 1-generated subalgebra of a Jordan algebra is associative. In particular, $\langle a \rangle$ is an associative subalgebra of J (and of $D^{(+)}$).

If $\langle a \rangle$ were infinite, then J would be associative, which contradicts our assumption. Thus the subalgebra $\langle a \rangle$ is finite, hence it is a field. Let $|\langle a \rangle| = p^k$, where k is a positive integer. Then

$$a^{p^k} = a.$$

An element has the same degree in D and $D^{(+)}$, so

$$\varphi^{p^k} = (l_a - r_a)^{p^k} = l_a - r_a = \varphi.$$

Since $\varphi(b) \neq 0$, we have $\varphi^{p^k}(b) = \varphi(b) \neq 0$. Then $\varphi^2(b) \neq 0$. An operator $\varphi^2 \restriction J$ belongs to the multiplication algebra for J: i.e.,

$$\varphi^2(x) = (l_a - r_a)^2(x) = (l_{a^2} + r_{a^2} - 2l_a r_a)(x) = 2(R_{a^2} - U_a)(x),$$

where $R_{a^2} : x \mapsto x \bullet a^2$. Consequently, $\varphi^2 \restriction J \in \mathfrak{T}(J)$ and $\varphi^2 \restriction J \not\equiv 0$. Thus, in view of J being linearly minimal, the operator $\varphi^2 \restriction J$ is surjective as is $\varphi^{p^k-1} \restriction J = (\varphi^2 \restriction J)^{\frac{p^k-1}{2}}$ (here use is made of the fact that $p^k - 1$ is an even number).

Let $c \in J$ so that $\varphi^{p^k-1}(c) = a$. Then $\varphi(c) = \varphi^{p^k}(c) = \varphi(a) = 0$, which is a contradiction with $\varphi^{p^k-1}(c) = \varphi^{p^k-2}(\varphi(c)) = a \neq 0$. The assertion is proved.

Thus the algebra $H(D, {}^*)$ in Zel'manov's list either is a field or is not linearly minimal.

(C) The assertion above implies that if D is an associative division algebra, then the algebra $D^{(+)}$ either is a field or is not linearly minimal.

(D) Recall that for an $F \cdot 1 + M$-algebra of a symmetric bilinear form $f(x, y)$ on a vector space M, the Jordan product is defined as follows:

$$(\alpha \cdot 1 + x) \bullet (\beta \cdot 1 + y) := (\alpha\beta + f(x, y)) \cdot 1 + (\beta x + \alpha y)$$

for $\alpha, \beta \in F$ and $x, y \in M$.

An element $e := 1 \cdot 1 + 0$ is unity in this algebra. An element a is invertible if there exists b for which $a \bullet b = e$ and $a^2 \bullet b = a$. That b is called an *inverse* of a.

If $f(x, x) = 0$ for some nonzero vector $x \in M$, then $a = 0 \cdot 1 + x$ is not invertible, since $a^2 = a \bullet a = 0$ and $a^2 \bullet b = 0 \neq a$ for any b.

As noted, checking our $F \cdot 1 + M$-algebra for linear minimality, we may assume that the field F is finite, $\operatorname{char} F = p > 2$, and the space M is infinite-dimensional over F. Chevalley's theorem [9] implies that the equation $f(x, x) = 0$ has a nontrivial solution in M (in fact, for this, it is sufficient that $\dim_F M \geq 3$), which yields a noninvertible element $0 \cdot 1 + x$ of the algebra.

Thus a Jordan division algebra of the form $F \cdot 1 + M$ cannot be linearly minimal.

The main result of the paper is the following:

COROLLARY. There does not exist a linearly minimal nonassociative Jordan algebra of characteristic other than 2.

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