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TOTALLY P-STABLE ABELIAN GROUPS

E. A. Palyutin* UDC 510.67:512.57

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We give a complete description of Abelian groups that are totally P-stable for the following four natural types of subgroups: arbitrary subgroups, pure subgroups, elementary subsystems, and algebraically closed subgroups.

INTRODUCTION

Origins and development of the notion of P-stability and bibliographic references are contained in [1]. For convenience, we cite some of the definitions and results from [1, 2], which will be used in what follows.

1. TERMINOLOGY, NOTATION, AND PRELIMINARY RESULTS

For relevant information on the theory of Abelian groups, we ask the reader to consult, e.g., [3; 4, Sec. 8.4]. Necessary information on model theory can be found, e.g., in [4, 5]. Below by a theory we mean an elementary theory in a language L. We recall some well-known concepts from the theory of Abelian groups. Throughout, by a group we always mean an Abelian group.

For a group A and for a natural number n, we denote by $A[n]$ the subgroup

$$\{a \mid a \in A, na = 0\},$$

and by nA the subgroup $\{na \mid a \in A\}$. The letter p will always stand for a prime number.

A group B is said to be *divisible* if $nB = B$ for any natural $n > 0$. A group B is *reduced* if it contains no nonzero divisible subgroups.

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Sobolev Institute of Mathematics, pr. Akad. Koptyuga 4, Novosibirsk, 630090 Russia. Novosibirsk State University, ul. Pirogova 2, Novosibirsk, 630090 Russia; palyutin@math.nsc.ru. Translated from *Algebra i Logika*, Vol. 54, No. 4, pp. 463-492, July-August, 2015. Original article submitted January 19, 2015.

A group all elements of which have order p^k for some natural k is called a *p-group*. A maximal p-subgroup of A is called the *p-component* of A and is denoted by A_p. An *elementary p-group* is a direct sum of cyclic groups of order p. An *elementary group* is an elementary p-group for some p. A group A is said to be *bounded* if there exists a natural number n such that $nA = 0$. For a group A and for a cardinal λ, $A^{<\lambda}$ denotes a direct sum of λ copies of the group A.

Let $\bar{\alpha} = \langle n_1, \dots, n_k \rangle$ be a tuple of integers. For a tuple $\mathbf{x} = \langle x_1, \dots, x_k \rangle$ of variables, $\bar{\alpha}\mathbf{x}$ denotes the term $n_1x_1 + \cdots + n_kx_k$; for a tuple $\mathbf{a} = \langle a_1, \dots, a_k \rangle$ of elements of A, $\bar{\alpha}\mathbf{a}$ stands for the element $n_1a_1 + \cdots + n_ka_k$.

A subgroup P of A is said to be *pure* if $nP = (P \cap nA)$ for any natural number n. This property is equivalent to the fact that $p^kP = (P \cap p^kA)$ for any prime number p and any natural number k.

A substructure B of a structure A is said to be *algebraically closed* if B contains every finite set $X \subseteq A$ definable in A by a formula $\Phi(x)$ with parameters in B. Below the term an *algebraically closed subgroup* will be understood in just this sense.

We will often use the following facts from the theory of Abelian groups.

If the p-component A_p of a group A is bounded then it is distinguished by a direct summand in A (see, e.g., [3, Chap. 5]).

A divisible subgroup of A is a direct summand of A. The greatest divisible subgroup of A is called the *divisible part* of A and is denoted by A_d. If $A = A_d \oplus C$, then C is called the *reduced part* of A. Using Szmielew's description [6] of elementarily equivalent Abelian groups (see also [4, Sec. 8.4]), we obtain the following two facts:

if the reduced part of the p-component A_p of a group A is unbounded, then, for any cardinal λ, the group A is elementarily equivalent to a group $A \oplus C_{p^\infty}^{<\lambda}$, where C_{p^∞} is a quasicyclic group;

if a group A is unbounded, then, for any cardinal λ, the group A is elementarily equivalent to a group $A \oplus Q^{<\lambda}$, where Q is the additive group of rational numbers.

For a group A, for prime numbers p, and for natural numbers m, the *Szmielew invariants* of A are the following cardinals:

$$\alpha_{p,m}(A) = \min\{\dim((p^mA)[p]/p^{(m+1)}A)[p]), \omega\},$$
$$\beta_p(A) = \min\{\inf\{\dim(p^nA)[p] \mid n \in \omega\}, \omega\},$$
$$\gamma_p(A) = \min\{\inf\{\dim((A/A[p^n])/p(A/A[p^n]))|n \in \omega\}, \omega\},$$
$$\varepsilon(A) \in \{0, 1\} \text{ and } (\varepsilon(A) = 0 \Leftrightarrow A \text{ is bounded}).$$

THEOREM 1 [6]. Abelian groups A and B are elementarily equivalent if and only if their Szmielew invariants coincide.

When we speak about *Szmielew invariants of a complete theory* T, we mean Szmielew invariants of its models.

If T is some complete theory of Abelian groups, then the group

$$\Sigma(T) = Q^{\varepsilon(T)} \oplus \bigoplus_{p,n} C_{p^{(n+1)}}^{<\alpha_{p,n}(T)} \oplus \bigoplus_{p} C_{p^\infty}^{<\beta_p(T)} \oplus \bigoplus_{p} R_p^{<\gamma_p(T)}$$

will have the same invariants as the theory T; hence it will be a model of T. We call $\Sigma(T)$ the *standard model* of T.

In this section, we fix a complete theory T in a language L. For convenience, to manipulate with models of T, we (as is conventional in modern model theory) fix some sufficiently saturated model C of T and assume that all T-models under consideration are elementary submodels of C. Such a T-model C is called a *monster model* of T.

The set of all tuples of elements of a set U is denoted by $U^{<\omega}$. For the length of a tuple $\mathbf{u}$, we write $l(\mathbf{u})$. For simplicity, instead of $\mathbf{u} \in U^{<\omega}$, we will write $\mathbf{u} \in U$. For the monster model C, tuples of elements and tuples of variables are denoted by lower-case bold letters from, respectively, the beginning and the end of the Latin alphabet: for example, $\mathbf{a}, \mathbf{b}, \ldots$ and $\ldots, \mathbf{x}, \mathbf{y}, \mathbf{z}$.

Let A be an L-structure and $\Phi(\mathbf{x})$ an L-formula with parameters in A; then we put $\Phi(A) = \{\mathbf{a} \mid A \models \Phi(\mathbf{a}), \mathbf{a} \in A\}$. For an L-structure A and its subset X, by $\mathrm{acl}_A(X)$ we denote the set

$$\bigcup \{\Phi(A) \mid |\Phi(A)| < \omega, \Phi(x) \text{ is an } L\text{-formula with parameters in } X\}.$$

If X is a subset of the monster model C, then we call X a set in the theory T. By $L(X)$ we denote a language which is obtained by adding to L the set X as a set of new constants. Denote by $T(X)$ the following set of formulas in the language $L(X)$:

$$\{\varphi(\mathbf{a}) \mid \mathbf{a} \in X, C \models \varphi(\mathbf{a}), \varphi(\mathbf{x}) \text{ is an } L\text{-formula without parameters}\}.$$

Clearly, $T(X)$ is a complete theory in $L(X)$ and is an extension of T.

Let a language L_P be obtained by adding a new unary predicate symbol P to L.

Definition 1. Let T be a complete L-theory, Δ a set of L_P-sentences, and X a set in T. Denote by $C_T(\Delta, X)$ the cardinality of a set of completions in a language $(L(X))_P$ of the set

$$T_\Delta(X) = (T(X) \cup \{P(a) \mid a \in X\} \cup \Delta).$$

A cardinal function assigning to a cardinal λ the supremum of a set of cardinals such as

$$\{C_T(\Delta, X) \mid X \text{ is a set in } T,\ |X| \leqslant \lambda\}$$

is called the *P_Δ-spectrum of the theory* T and is denoted by $S_T(P, \Delta)$.

A P_Δ-spectrum is said to be *maximal* if $S_T(P, \Delta)(\varkappa) = 2^\varkappa$ for every infinite cardinal $\varkappa$.

LEMMA 1 [2]. For every complete theory T in a finite or countable language L, for an arbitrary set Δ of sentences in the language L_P, and for any at most countable set X, the cardinal $C_T(\Delta, X)$ may assume only one of the following values: 2^ω, ω, and n, where $n \in \omega$.

Definition 2. Let Δ be some set of L_P-sentences.

(1) A complete theory T in L is *P_Δ-stable* if $S_T(P, \Delta)(\lambda) \leqslant \lambda$ for some infinite cardinal λ.

(2) A complete theory T in L is *P_Δ-superstable* if there exists a cardinal $\varkappa$ such that $S_T(P, \Delta)(\lambda) \leqslant \lambda$ for all cardinals $\lambda \geqslant \varkappa$.

(3) A complete theory T in L is *totally* P_Δ-*stable* if $S_T(P,\Delta)(\lambda) \leqslant \lambda$ for every infinite cardinal λ.

(4) A structure A is P_Δ-(*super*)*stable* (*totally* P_Δ-*stable*) if so is its theory.

Note that the P_Δ-stability concept is a particular case of the concept of E^*-stability introduced in [7].

Definition 3. Below we consider complete theories T of Abelian groups, taking as Δ the following sets Δ_i, $i \in \{s,p,e,a\}$, of sentences:

(1) Δ_s says that a predicate P defines an arbitrary subgroup;

(2) Δ_p says that a predicate P defines a pure subgroup;

(3) Δ_e says that a predicate P defines an elementary subsystem;

(4) Δ_a says that a predicate P defines an algebraically closed subgroup.

For $i \in \{s,p,e,a\}$, instead of the terms P_{Δ_i}-stability and P_{Δ_i}-spectrum, we will use the terms (P,i)-stability and (P,i)-spectrum.

In [1, 2], (P,i)-stable and (P,i)-superstable Abelian groups, as well as (P,i)-spectra for $i \in \{s,p,e,a\}$, were described completely. In the present paper, we give a description of totally (P,i)-stable Abelian groups for $i \in \{s,p,e,a\}$.

A *primitive formula* in a language L is a formula of the form

$$\exists x_1 \ldots \exists x_n \Phi,$$

where Φ is a conjunction of atomic formulas in the language L. A subset X of a structure A is said to be *primitive* if $X = \Phi(A)$ for some primitive formula $\Phi(\mathbf{x})$.

In what follows, L denotes a language of the theory of Abelian groups, consisting of a binary function symbol $+$, a unary function symbol $-$, and a constant symbol 0. By AG we denote the theory of all Abelian groups defined by ordinary axioms for Abelian groups in L.

The lemma below is well known. Its proof is contained, for instance, in [4, 8].

LEMMA 2. Let T be a complete theory of Abelian groups. Every formula in the language of Abelian groups is equivalent in T to a Boolean combination of primitive formulas.

In a similar way, we can prove the following:

LEMMA 3. Let A be an Abelian group and P its subgroup. Every L_P-formula is equivalent in $\mathrm{Th}(\langle A,P\rangle)$ to a Boolean combination of primitive L_P-formulas.

It is easy to verify that every primitive formula $\Phi(x)$ in the language L_P defines in the structure $\langle A,P\rangle$ a subgroup of A. If P_1 and P_2 are subgroups of A and $P_2 \subseteq P_1$, then $[P_1 : P_2]$ denotes the number $\min\{|P_1/P_2|, \omega\}$, which is called the *index of the subgroup* P_2 in the subgroup P_1.

The following proposition is proved in exactly the same way as its counterpart for modules [8].

PROPOSITION 1. Let A_1 and A_2 be Abelian groups and P_1 and P_2 be their subgroups. For structures $\langle A_1,P_1\rangle$ and $\langle A_2,P_2\rangle$ to be elementarily equivalent, it is necessary and sufficient that for any primitive L_P-formulas $\Phi(x)$ and $\Psi(x)$ with the condition $\vdash \Psi(x) \to \Phi(x)$, the following equality will hold:

$$[\Phi(\mathbf{A}_1) : \Psi(\mathbf{A}_1)] = [\Phi(\mathbf{A}_2) : \Psi(\mathbf{A}_2)],$$

where $\mathbf{A}_i = \langle A_i, P_i\rangle$.

The next lemma goes back to [6]; in the given form, it is contained, for instance, in [4, Lemma 8.4.7].

LEMMA 4. Every primitive formula $\Phi(x_1, \ldots, x_n)$ in the language of Abelian groups is equivalent in the theory AG to a conjunction of formulas like $\alpha_1 x_1 + \ldots + \alpha_n x_n = 0$ and $\exists y \alpha_1 x_1 + \ldots + \alpha_n x_n = p^k y$ for integers $\alpha_1, \ldots, \alpha_n$, prime numbers p, and natural numbers k, which are called *standard formulas* of, respectively, the *first* and *second kind*. In this case a prime p in a standard formula of the second kind is referred to as a *module* of that formula.

Instead of $\exists y \alpha_1 x_1 + \cdots + \alpha_n x_n = p^k y$, we will write $\alpha_1 x_1 + \cdots + \alpha_n x_n = 0 \,(\mathrm{mod}\, p^k)$.

Definition 4. A theory T in a countable language L is said to be P_Δ-*small* if the set $(T \cup \Delta)$ has an at most countable number of completions in the language L_P.

For complete theories of Abelian groups, for $i \in \{s, p, e, a\}$, instead of the term P_{Δ_i}-small, we use the term (P, i)-small.

2. TOTAL (P, p)-STABILITY

Lemma 4 gives rise to the following:

LEMMA 5. Let A be an Abelian group and P a pure group. For any primitive L-formula $\Phi(\mathbf{x})$ and any tuple $\mathbf{a} \in P$,

$$A \models \Phi(\mathbf{a}) \Leftrightarrow P \models \Phi(\mathbf{a}).$$

The proof of the next lemma is contained, for instance, in [2, proof of Lemma 8].

LEMMA 6. For any primitive L_P-formula $\Phi(\mathbf{x})$, there exists a primitive L-formula $\Phi^*(\mathbf{x})$ such that for every Abelian group A and its arbitrary pure subgroup P, the formula $\Phi^*(\mathbf{x})$ defines in A on the subgroup P the same predicate as is defined by the formula $\Phi(\mathbf{x})$ in the structure $\langle A, P\rangle$.

Lemmas 3 and 6 immediately imply the following:

THEOREM 2. For a complete theory T of Abelian groups, the following conditions are equivalent:

(1) T is totally (P, p)-stable;

(2) T is (P, p)-small.

LEMMA 7. Let P, B, and C be subgroups of A and $C \subseteq B$. Then

$$[B : C] = [(P \cap B) : (P \cap C] \cdot [(P + B) : (P + C)].$$

Proof. Let $X = \{a_i + P + C \mid i \in K\}$ be the set of all elements of a group $(P + B)/(P + C)$, with $a_i \in B$ and $(a_i - a_j) \notin (P + C)$ if $i \neq j$. Let $Y = \{b_j + (P \cap C) \mid j \in L\}$ be the set of all elements of a group $(P \cap B)/(P \cap C)$, with $b_j \in (P \cap B)$, $j \in L$, and $(b_i - b_j) \notin C$ if $i \neq j$. Consider a set $Z = \{a_i + b_j \mid i \in K, j \in L\}$ of elements. We show that the set Z is a complete system of representatives of C-classes in the group B. If $i_1 \neq i_2$, then, for any j_1 and j_2, we have

$a_{i_1} - a_{i_2} \notin (P + C)$ and $b_{j_1}, b_{j_2} \in P$; hence $(a_{i_1} + b_{j_1}) - (a_{i_2} + b_{j_2}) \notin (C + P)$. If $i_1 = i_2$ and $j_1 \neq j_2$, then $(a_{i_1} + b_{j_1}) - (a_{i_2} + b_{j_2}) = (b_{j_1} - b_{j_2}) \notin C$.

On the other hand, take an arbitrary element $a \in B$. Let $(a - a_i) \in (C + P)$. Clearly, $(a - a_i) \in (B \cap (C + P)$. Since $((a - a_i) + C) \subseteq B$, we have

$$(((a - a_i) + C) \cap P) \subseteq (P \cap B).$$

Then there is b_j for which

$$(b_j + (P \cap C)) \cap ((a - a_i) + C) \cap P \neq \varnothing.$$

If

$$e \in (b_j + (P \cap C)) \cap ((a - a_i) + C) \cap P,$$

then $(e + (P \cap C)) = (b_j + (P \cap C))$ and

$$(e + (P \cap C)) \subseteq ((a - a_i) + C).$$

Consequently, $(b_j + C) = ((a - a_i) + C)$; i.e., $(a - (a_i + b_j)) \in C$. $\square$

THEOREM 3. Let A_1 and A_2 be Abelian groups and P_1 and P_2 be their pure subgroups. For structures $\langle A_1, P_1 \rangle$ and $\langle A_2, P_2 \rangle$ to be elementarily equivalent, it is necessary and sufficient that the conditions $P_1 \equiv P_2$ and $A_1/P_1 \equiv A_2/P_2$ be satisfied.

Proof. The necessity is obvious. In view of Proposition 1, to prove the sufficiency, we need to show that in the case where $P_1 \equiv P_2$ and $A_1/P_1 \equiv A_2/P_2$, the equality

$$[\Phi(\mathbf{A}_1) : \Psi(\mathbf{A}_1)] = [\Phi(\mathbf{A}_2) : \Psi(\mathbf{A}_2)], \tag{1}$$

where $\mathbf{A}_i = \langle A_i, P_i \rangle$, holds for any primitive L_P-formulas $\Phi(x)$ and $\Psi(x)$ with the condition that $\vdash \Psi(x) \to \Phi(x)$. The required statement follows from Lemmas 5-7. $\square$

Definition 5. For every Abelian group A, put

$$Sh(A) = \{\langle p, n \rangle \mid \alpha_{p,n}(A) \neq 0, n < \omega, \} \cup \{p \mid \beta_p(A) \neq 0\} \cup \{p \mid \gamma_p(A) \neq 0\}.$$

Definition 6. An Abelian group A is *Szmielew bounded* if the set $Sh(A)$ is finite. As usual, a complete theory T of Abelian groups is *Szmielew bounded* if its models are Szmielew bounded.

PROPOSITION 2. For an Abelian group A to be Szmielew bounded, it is necessary and sufficient that there exist a finite set $\sigma(A)$ of primes with the following properties:

(1) every element of A is divisible by each prime number p not belonging to $\sigma(A)$;

(2) for every prime number p not belonging to $\sigma(A)$, the p-component A_p of A is zero;

(3) for every prime number p, the reduced part of the p-component A_p of A is bounded.

Proof. Let

$$X = \{p \mid \alpha_{p,n}(A) \neq 0 \text{ for some } n\},$$

$$Y = \{p \mid \beta_p(A) \neq 0\},$$
$$Z = \{p \mid \gamma_p(A) \neq 0\}.$$

Necessity. Let $Sh(A)$ be a finite set. As $\sigma(A)$ we take a finite set $(X \cup Y \cup Z)$. Property (2) follows from the inclusion $(X \cup Y) \subseteq \sigma(A)$. Property (1) follows from property (2) and the inclusion $Z \subseteq \sigma(A)$. We verify (3). Suppose that for some prime p, the reduced part H of the p-component A_p of A is unbounded. Clearly, property (3) is preserved under the passage to elementarily equivalent groups. If we look at the form of the standard model of $T = \mathrm{Th}\,(A)$ we see that the set $\{\langle p, n\rangle \mid \alpha_{p,n}(A) \neq 0\}$ is infinite, which is a contradiction with $Sh(A)$ being finite.

Sufficiency. Suppose that properties (1)-(3) hold for some finite set $\sigma(A)$. Property (1) implies finiteness of Z; property (2) entails finiteness of Y; finiteness of the set $\{\langle p, n\rangle \mid \alpha_{p,n}(A) \neq 0\}$ follows from properties (2) and (3). Thus $Sh(A)$ is finite. □

LEMMA 8. Let A be an Abelian p-group. For the reduced part of A to be bounded, it is necessary and sufficient that there exist a natural number n for which the following condition holds:

$(*)$ $p^n A$ is a divisible group.

Proof. We represent A as $C \oplus B$, where C is a divisible group and B a reduced group.

Necessity. If $p^n B = 0$ then $p^n A = p^n C = C$; i.e., condition $(*)$ holds.

Sufficiency. Suppose that $(*)$ is satisfied. We may assume that $p^n A \subseteq C$; in particular, $p^n B \subseteq C$. Since $(B \cap C)$ is a zero subgroup, $p^n B = 0$. □

LEMMA 9. Let p be a prime and D a divisible p-subgroup of A. Let the reduced part of the p-component A_p be unbounded. Then the reduced part of the p-component of a group A/D is unbounded.

Proof. We represent the p-component A_p as $B \oplus D \oplus E$, where $D \oplus E$ is the largest divisible subgroup of A_p. In order to prove that the reduced part of the p-component $(A/D)_p$ of A/D is unbounded, it suffices to show that the groups $B \oplus E$ and $(A/D)_p$ are isomorphic. We claim that the desired isomorphism will be a mapping φ assigning to an element $b + e \in B \oplus E$ an element $(b + e) + D$. That φ is injective follows from the fact that $(B + E) \cap D = 0$. Clearly, $\varphi(B \oplus E) \subseteq (A/D)_p$. It remains to show that if $a + D \in (A/D)_p$ then $a \in A_p$. The condition $a + D \in (A/D)_p$ implies $p^k a \in D$ for some k. In view of $D \subseteq A_p$, we have $a \in A_p$. □

LEMMA 10. Let an Abelian group A be Szmielew bounded. Then, for any pure subgroup P of A, the groups P and A/P are Szmielew bounded, in which case as $\sigma(P)$ and $\sigma(A/P)$ we can take the set $\sigma(A)$.

Proof. We show that the subgroup P is Szmielew bounded. Property (1) in Proposition 2 follows from the fact that P is pure. Property (2) for P is an immediate consequence of property (2) for A. Property (3) follows from Lemma 8 and the fact that P is a pure subgroup.

Now we argue to state that the subgroup A/P is Szmielew bounded. Property (1) is obtained based on the fact that the formula $\forall x \exists y py = x$ is positive and is therefore preserved under homomorphisms. Suppose that property (2) for A/P fails, i.e., there exist $p \notin \sigma(A)$ and $a \notin P$

with the condition $pa \in P$. The subgroup P is pure, and so there exists $b \in P$ with the condition $pb = pa$, i.e., $p(b-a) = 0$. Since $a \notin P$, we obtain $(b-a) \neq 0$, which clashes with the property that $A[p] = 0$.

We verify property (3) for A/P. Let D be the largest divisible subgroup of the p-component A_p of a group A. We show that the reduced part of the p-component of a group $A/(P+D)$ is bounded. By hypothesis, we have $p^k A_p \subseteq D$ for some k. Suppose that the reduced part of the p-component of $A/(P+D)$ is unbounded. Then there exists an element $a \in A$ such that $p^{(k+1)}a \in (P+D)$ and $p^k a \notin (P+D)$. The subgroup P is pure, and D is a divisible group; so $P+D$ is a pure subgroup of A. Consequently, there is an element $b \in (P+D)$ with the condition $p^{(k+1)}b = p^{(k+1)}a$. Then $p^{(k+1)}(a-b) = 0$; in particular, $(a-b) \in A_p$. By the choice of a number k, we have $p^k(a-b) \in D$, which clashes with the conditions $b \in (P+D)$ and $p^k a \notin (P+D)$.

Suppose that the reduced part of the p-component of A/P is unbounded. Obviously, the subgroup $D+P$ is a divisible subgroup of A/P. The groups $A/(P+D)$ and $(A/P)/(D+P)$ are isomorphic, and by Lemma 9, the reduced part of the p-component of $A/(P+D)$ is unbounded, which is a contradiction with the above. $\square$

PROPOSITION 3. There exist only countably many complete theories of Szmielew bounded Abelian groups.

Proof. The required statement follows immediately from the definition of Szmielew bounded Abelian groups and Theorem 1, since the Szmielew invariants each assumes at most countably many values. $\square$

Definition 7. For an Abelian group A, by $\overline{Sh}(A)$ we denote the set

$$\{\langle p,n\rangle \mid \alpha_{p,n}(A) \neq 0, n < \omega,\} \cup \{\langle p,\beta\rangle \mid \beta_p(A) \neq 0\} \cup \{\langle p,\gamma\rangle \mid \gamma_p(A) \neq 0\}.$$

THEOREM 4. For a theory of an Abelian group A to be (P,p)-small, it is necessary and sufficient that the group A be Szmielew bounded.

Proof. Necessity. Suppose that $Sh(A)$ is an infinite set. Then the set $\overline{Sh}(A)$ will also be infinite. For any subset $X \subseteq \overline{Sh}(A)$, we take a subgroup P_X of the standard model $\Sigma(\mathrm{Th}\,(A))$ generated by its direct summands

$$\left\{ C_{p^{(n+1)}}^{<\alpha_{p,n}(T)} \,\middle|\, \langle p,n\rangle \in X \right\},\ \left\{ C_{p^\infty}^{<\beta_p(T)} \,\middle|\, \langle p,\beta\rangle \in X \right\},\ \left\{ R_p^{<\gamma_p(T)} \,\middle|\, \langle p,\gamma\rangle \right\}.$$

The subgroup P_X is a direct summand of the group $\Sigma(\mathrm{Th}\,(A))$ and is therefore pure in $\Sigma(\mathrm{Th}\,(A))$. If X and Y are different subsets of $\overline{Sh}(A)$, then the groups P_X and P_Y will have different Szmielew invariants. Consequently, P_X and P_Y are not elementarily equivalent. The set $\overline{Sh}(A)$ is infinite, and the theory $\mathrm{Th}\,(A)$ is not (P,p)-small.

The sufficiency follows from Theorem 3, Lemma 10, and Prop. 3. $\square$

Theorems 2 and 4 can be combined to yield

THEOREM 5. For a theory of an Abelian group A to be totally (P,p)-stable, it is necessary and sufficient that the group A be Szmielew bounded.

3. TOTAL (P, e)-STABILITY

The proposition below is well known (see, e.g., [8]).

PROPOSITION 4. Let A be an Abelian group and P its subgroup. For the subgroup P to be an elementary subsystem of the group A, it is necessary and sufficient that P be a pure subgroup and the property $A \equiv P$ hold.

Clearly, elementary subsystems of A will be pure subgroups. As in the previous section, therefore, Lemma 6 implies the following:

THEOREM 6. For a complete theory T of Abelian groups, the following conditions are equivalent:

(1) T is totally (P, e)-stable;

(2) T is (P, e)-small.

Definition 8. For an Abelian group A, by $Sh_\omega(A)$ we denote the set

$$\{\langle p, n\rangle \mid \alpha_{p,n}(A) = \omega, n \in \omega\} \cup \{p \mid \beta_p(A) = \omega\} \cup \{p \mid \gamma_p(A) = \omega\}.$$

Definition 9. An Abelian group A is *Szmielew ω-bounded* if the set $Sh_\omega(A)$ is finite. As usual, a complete theory T of Abelian groups is *Szmielew ω-bounded* if its models are Szmielew ω-bounded.

LEMMA 11. Suppose that for an Abelian group A and for a prime number p, we have $\beta_p(A) < \omega$. Then the following conditions are equivalent:

(1) the reduced part of the p-component A_p of A is finite;

(2) $A[p]$ is a finite subgroup.

Proof. (1)⇒(2) Follows from the fact that $\beta_p(A)$ is a finite invariant.

(2)⇒(1) Let $A[p]$ be finite and $A_p = B \oplus D$, where D is a maximal divisible subgroup in A_p. By induction on n, it is easy to show that for any natural n, the subgroup $A[p^n]$ is finite. Suppose that B is an infinite subgroup. Then there exists a sequence $\{a_n \mid n \in \omega\}$ of elements of infinite p-height in B with the properties $pa_0 = 0$ and $pa_{(n+1)} = a_n$. Obviously, the subgroup generated by the set $\{a_n \mid n \in \omega\}$ is isomorphic to a group C_{p^∞}, which is a contradiction with B being reduced. □

Remark 1. It is not hard to verify that for any subgroup H of A, the following isomorphism holds:

$$(A/H)/p(A/H) \simeq A/(pA + H).$$

PROPOSITION 5. For an Abelian group A to be Szmielew ω-bounded, it is necessary and sufficient that there exist a finite set $\sigma_\omega(A)$ of primes with the following properties:

(1) for any prime p not belonging to $\sigma_\omega(A)$, $A[p]$ is a finite subgroup;

(2) for any prime p not belonging to $\sigma_\omega(A)$, A/pA is a finite group;

(3) for any prime p, $\{\langle p, n\rangle \mid \alpha_{p,n}(A) = \omega, n \in \omega\}$ is a finite set.

Proof. Consider the sets

$$X = \{p \mid \alpha_{p,n}(A) = \omega \text{ for some } n\},$$

$$Y = \{p \mid \beta_p(A) = \omega\},$$
$$Z = \{p \mid \gamma_p(A) = \omega\}.$$

Necessity. Let $Sh_\omega(A)$ be a finite set. Property (3) is obvious. As $\sigma_\omega(A)$ we take a finite set $(X \cup Y \cup Z)$. Suppose that property (1) fails, i.e., for some $p \notin (X \cup Y \cup Z)$, the subgroup $A[p]$ is infinite. By Lemma 11, the reduced part of the p-component A_p of A is infinite. In view of the fact that $p \notin X$, the group A contains a subgroup of the form $C_{p^k}^{<\omega}$, for any natural k. By the compactness theorem, there is a model of $\mathrm{Th}\,(A)$ having a subgroup of the form $C_{p^\infty}^{<\omega}$, which contradicts the condition that $p \notin Y$.

We verify property (2). Suppose that for some $p \notin (X \cup Y \cup Z)$, the group A/pA is infinite. Property (1) and the condition $p \notin Y$ imply that the subgroup $A[p^k]$ is finite for any natural number k. Consequently, the group $A/(pA + A[p^k])$ is infinite for any natural k. In view of Remark 1, we obtain $\gamma_p(A) = \omega$, which is a contradiction with the condition that $p \notin Z$.

Sufficiency. Suppose that properties (1) and (2) hold. Property (1) implies the inclusion $(X \cup Y) \subseteq \sigma(A)$. Properties (1) and (2), combined with Remark 1, yield the inclusion $Z \subseteq \sigma(A)$. In view of $(X \cup Y \cup Z)$ being finite and property (3), we conclude that $Sh_\omega(A)$ is finite. $\square$

LEMMA 12. Let A be an Abelian group and the group A/pA be finite for a prime p. Then $p(A/P) = A/P$ for any elementary subsystem P of A; in particular, $\gamma_p(A/P) = 0$.

Proof. Take any $a \in A$. There is only a finite number k of conjugacy classes with respect to a subgroup pA, and $P \equiv A$; so $|P/pP| = k$. Since P is an elementary subsystem, divisibility by p of elements of the subgroup P coincides in the group A and in the subgroup P. Thus there exists $b \in P \cap (a + pA)$, i.e., $a - b = pc$ for some $c \in A$. Hence $(a + P) \in p(A/P)$. $\square$

LEMMA 13. Suppose that the subgroup $A[p]$ of A is finite for a prime p. Then $(A/P)[p] = 0$ for any elementary subsystem P of A; in particular, the Szmielew invariants $\beta_p(A/P)$ and $\alpha_{p,n}(A/P)$, $n < \omega$, are equal to zero.

Proof. Suppose that $pa \in P$ for some $a \in A$. The set $X = \{b \mid pb = pa\}$ will be the conjugacy class of the subgroup $A[p]$. Consequently, X is finite. Since P is an elementary subsystem of A, we have $X \subseteq P$ $\square$.

LEMMA 14. Let the Szmielew invariant $\alpha_{p,n}(A)$ of a group A be finite for a prime number p and for a natural number n. Then the Szmielew invariant $\alpha_{p,n}(A/P)$ of a group A/P is equal to zero for any elementary subsystem P of the group A.

Proof. Suppose that the Szmielew invariant $\alpha_{p,n}(A/P)$ of A/P is not equal to zero. Hence, for some $a \in A$, the following conditions hold: $a \notin P$, $pa \in P$, $(a + P) \in p^n(A/P)$, and $(a + P) \notin p^{(n+1)}(A/P)$. Replacing, if necessary, the element a by some element of $a + P$, we may assume that $p^n b = a$ for some b. Since $P \preccurlyeq A$, there is an element $e \in P$ with $p^{(n+1)}e = pa$. Consequently, for an element $f = (a - p^n e)$, the conditions $f \in A[p]$ and $f \in p^n A$ hold. Since $\alpha_{p,n}(A)$ is finite and $P \preccurlyeq A$, there is an element $h \in (p^n P)[p]$ with $(f - h) \in p^{(n+1)}A$. Hence $(a + P) \in p^{(n+1)}(A/P)$, which contradicts the hypothesis. $\square$

Proposition 5 and Lemmas 12-14 entail

PROPOSITION 6. Let an Abelian group A be Szmielew ω-bounded. Then for elementary subsystems P there are at most countably many elementary theories for groups A/P.

THEOREM 7. For a theory of an Abelian group A to be (P, e)-small, it is necessary and sufficient that the group A be Szmielew ω-bounded.

Proof. The sufficiency follows from Theorem 3 and Prop. 6.

Necessity. Suppose that a theory of an Abelian group A is not Szmielew ω-bounded. Then a standard model S of $\mathrm{Th}\,(A)$ has the form

$$S = E \oplus \bigoplus_{B \in X} B^{<\omega}, \tag{2}$$

where the set X consists of infinitely many groups B like C_{p^n}, C_{p^∞}, and R_p. Consider an arbitrary subset $Y \subseteq X$. Then a group S can be represented as

$$S = P_Y \oplus \bigoplus_{B \in Y} B, \tag{3}$$

where the subgroup P_Y is isomorphic to A and is a direct summand of A. Direct summands are pure subgroups, and by Proposition 4, $P_Y \preccurlyeq A$. If $Y_1 \neq Y_2$, then A/P_{Y_1} and A/P_{Y_2} will not be elementarily equivalent. Thus the group A is not (P, e)-small. $\square$

Theorems 6 and 7 give rise to

THEOREM 8. For a complete theory T of Abelian groups, the following conditions are equivalent:

(1) T is totally (P, e)-stable;

(2) T is Szmielew ω-bounded.

4. TOTAL (P, a)-STABILITY

An elementary subsystem is algebraically closed, and Theorem 7 entails the following property:

(A) if an Abelian group A is (P, a)-small, then the group A is Szmielew ω-bounded.

Furthermore, if an Abelian group A is totally (P, a)-stable (in particular, (P, a)-stable), then, by [1, Thm. 3], the following properties hold:

(B) for any prime p, the subgroup $(pA)[p]$ of A is finite;

(C) for any prime p, either the subgroup $A[p]$ of A is finite or its Szmielew invariant $\gamma_p(A)$ is finite.

Remark 2. Using property (B) and induction on k, we can readily show that $(pA)[p^k]$ is a finite subgroup for any k (see also [1, p. 415, property (3)]).

Remark 3. Remark 2 implies that for every algebraically closed subgroup P of A, the following property holds:

(B^*) $(P \cap p^{(k+1)}A) \subseteq p^k P$ for any natural k; moreover, if $p^{(k+1)}a \in P$ for $a \in A$, then $pa \in P$ (see also [1, Lemma 9]).

LEMMA 15. Suppose that $A = P \oplus B$ and the group B has no primitive $\varnothing$-definable finite nonzero subgroups. Then P is an algebraically closed subgroup of A.

The **proof** follows from [1, Lemma 5] and the fact that primitive formulas are filtering. □

PROPOSITION 7. If a theory of an Abelian group A is (P, a)-small, then the following property holds:

(D) the set $W = \{p \mid \gamma_p(A) \neq 0\}$ is finite.

Proof. In a group, which is a direct sum of groups of the form R_p for prime p, there are no nonzero finite subgroups. Using Lemma 15, we see that for any subset $X \subseteq W$, a standard model C of $\mathrm{Th}\,(A)$ can be represented as

$$C = P_X \oplus \bigoplus_{p \in X} R_p,$$

where P_X is some algebraically closed subgroup of C. If $X_1 \neq X_2$, then the groups C/P_{X_1} and C/P_{X_2} will not be elementarily equivalent. Consequently, for an infinite set W, $\mathrm{Th}\,(A)$ is not (P, a)-small. □

Definition 10. Suppose that $\gamma_p(A)$ is a finite Szmielew invariant. By [2, Lemma 15], pA/p^2A is a finite group.

Let X and P be algebraically closed subgroups of A and $X \subseteq P$. In what follows, the subgroups P will be changed, while the subgroup X will be kept fixed.

Consider a group such as

$$G_P = (pA \cap P)/((pA \cap X) + pP).$$

By property (B^*), we have $P \cap p^2A \subseteq pP$. Consequently,

$$|G_P| \leqslant |(pA \cap P)/((pA \cap X) + (P \cap p^2A)|.$$

Then

$$|G_P| \leqslant |(pA \cap P)/(P \cap p^2A)| \leqslant |pA/p^2A|.$$

Therefore, the group G_P is finite. Obviously, $pG_P = 0$.

Take elements $a_1^P, \ldots, a_n^P \in (pA \cap P)$ such that the elements

$$a_1^P + ((pA \cap X) + pP), \ldots, a_n^P + ((pA \cap X) + pP) \tag{4}$$

will form a basis in the group G_P treated as a vector space over a field consisting of p elements. Consider a finite set like

$$W^P = \{k_1 a_1^P + \cdots k_n a_n^P \mid k_1 < p, \ldots, k_n < p\}.$$

In the subgroup $(pA \cap X)$, take elements $b_1, \dots, b_m$, one from each conjugacy p^2A-class having nonempty intersection with $(X \cap pA)$.

Denote by U the set $\{b_1, \dots, b_m\}$.

Remark 4. The choice of elements $b_1, \dots, b_m$ does not depend on the subgroup P; therefore, we choose them equal for different subgroups P.

Definition 11. Elements $a_1^P, \dots, a_n^P$ are called *outer parameters*, while $b_1, \dots, b_m$ are called *inner parameters*.

Definition 12. An equivalence whose classes consist of conjugacy classes of the subgroup p^2A in the group pA is denoted by ε_p.

For a tuple $\mathbf{c} = \langle c_1, \dots, c_s \rangle$, we denote by $x \in \mathbf{c}$ the formula

$$(z = c_1 \vee \dots \vee z = c_s).$$

Similarly, for a finite set C, by $x \in C$ we denote the formula

$$\bigvee_{a \in C} x = a.$$

LEMMA 16 [12, Lemma 19]. Let $\mathrm{Th}\,(A)$ have a nonmaximal (P, a)-spectrum and let the Szmielew invariant $\gamma_p(A)$ be equal to zero. Then each ε_p-class has an element of a finite subgroup $(pA)[p^n]$ for some natural n.

LEMMA 17 [2, Lemma 14]. (1) Let some algebraically closed subgroup P of A be bounded. Then the p-component A_p of A is bounded for any prime p, i.e., $p^n A_p = 0$ for some n.

(2) Let $p^n A_p = 0$ for a prime p and some n. By Remark 2, the subgroup $(pA)[p^n]$ is finite and, consequently, is contained in any algebraically closed subgroup P. Suppose also that

$$\Phi_p(z) = \exists x \exists y (x \in (pA)[p^n] \wedge (z - x) = py).$$

Then any bounded algebraically closed subgroup P and every element $a \in P$ satisfy the property

$$A \models \exists y a = py \Leftrightarrow P \models \Phi_p(a).$$

(3) Assume that for a prime number p and for a natural number n, we have $p^n A_p = 0$, and

$$\Phi_{p^k}(x) = \exists z (\Phi_p(z) \wedge x = p^{(k-1)} z)$$

for $k > 1$, where the formula $\Phi_p(z)$ is as in item (2). Then an arbitrary bounded algebraically closed subgroup P, any number $k > 1$, and every element $a \in P$ enjoy the property

$$A \models \exists y a = p^k y \Leftrightarrow P \models \Phi_{p^k}(a).$$

LEMMA 18. Suppose that for an Abelian group A and for a prime number p, the Szmielew invariant $\gamma_p(A)$ is finite. Then, for any algebraically closed subgroup P and any natural number $k \geqslant 1$, there exists a formula $\Psi^P_{p^k}(x)$ with parameters in the set

$$\{a_1^P, \ldots, a_n^P\} \cup U,$$

for which the following properties hold:

(1) for any $a \in P$,

$$P \models \Psi^P_{p^k}(a) \Leftrightarrow a \in p^k A;$$

(2) this property derives from (1) if in $\Psi^P_{p^k}(x)$ we replace any outer parameter $a_i^P \in \{a_1^P, \ldots, a_n^P\}$ by any element of the class $a_i^P + pP$;

(3) if the Szmielew parameter $\gamma_p(A)$ is equal to zero, then all parameters of the formula $\Psi^P_{p^k}(x)$ with property (1) can be taken from the set $\{b_1, \ldots, b_m\}$ of inner parameters.

Proof. Case 1. Let $k = 1$. As $\Psi^P_p(x)$ we may take the formula

$$2x \in pP \vee \exists z(z \in U \wedge (x - z) \in pP) \vee \exists z \exists u(z \in W^P \wedge u \in U \wedge (x - z - u) \in pP).$$

Since $(W^P \cup U) \subseteq pA$, $P \models \Psi^P_p(a)$ implies $a \in pA$.

Let $a \in (pA \cap P)$. If $a \in pP$, then the formula $\Psi^P_p(a)$ is obviously true in P.

Let $a \in ((pA \cap X) + pP)$. Consider an element $c \in (pA \cap X)$ for some $a \in (c + pP)$. By the choice of elements $b_1, \ldots, b_m$, there is $b \in U$ such that $(c - b) \in p^2A$. Since $P \cap p^2A \subseteq pP$, we have $(a - b) \in pP$. Hence $P \models \Psi^P_p(a)$.

It remains to look into the situation where

$$a \in ((pA \cap P) \setminus ((pA \cap X) + pP).$$

By the choice of elements $a_1^P, \ldots, a_n^P$, there is $b \in W^P$ such that $(a - b) \in ((pA \cap X) + pP)$. Once we consider the situation where $a \in ((pA \cap X) + pP)$, we will see that there exists an element $c \in U$ such that $(a - b - c) \in pP$. Therefore, $P \models \Psi^P_p(a)$.

Case 2. Let $k > 1$. In virtue of property (B), as the formula $\Psi^P_{p^k}(x)$ we can take a formula of the form

$$\exists z(x = p^{(k-1)}z \wedge \Psi^P_p(z)).$$

Property (2) follows from the form of $\Psi^P_{p^k}(x)$.

Lemma 16, the algebraic closedness of a subgroup X, and property (2) can be combined to produce property (3). $\square$

PROPOSITION 8. If A is a totally (P, a)-stable group, then it can be represented as

$$A = B \oplus C,$$

where B is a direct sum of finitely many infinite elementary groups, and for any prime p, the p-components C_p of the group C are finite.

Proof. In view of property (B), for any prime p, the p-component A_p is bounded and, consequently, is distinguished by a direct summand in A. Furthermore, (B) implies that the group A_p is a direct sum of an elementary group and a finite group. Thus, with property (A) in mind, we obtain the desired decomposition of A. □

LEMMA 19 [2, Lemma 17]. If $A[p]$ is a finite subgroup, then any standard formula of the form $\bar{\alpha}\mathbf{x} = 0 \,(\mathrm{mod}\, p^k)$ and any tuple $\mathbf{a} \in P$ satisfy

$$A \models \bar{\alpha}\mathbf{a} = 0 \,(\mathrm{mod}\, p^k) \Leftrightarrow P \models \bar{\alpha}\mathbf{a} = 0 \,(\mathrm{mod}\, p^k).$$

THEOREM 9. For a complete theory of an Abelian group A to be totally (P, a)-stable, it is necessary and sufficient that the group A be (P, a)-small and properties (B) and (C) hold.

Proof. The necessity of the conditions above was noticed at the beginning of the section.

Sufficiency. Assume the contrary. This means that there exist an Abelian group A satisfying properties (B) and (C), which is (P, a)-small (in particular, properties (A) and (D) hold), and a set X in $T = \mathrm{Th}\,(A)$ such that the structures

$$\langle A_\lambda, P_\lambda \rangle, \ \lambda < 2^\omega,$$

satisfy the following conditions:

(i) $X \subseteq P_\lambda$ for every $\lambda < 2^\omega$;

(ii) if $\lambda < \mu < 2^\omega$, then $\langle A_\lambda, a \rangle_{a \in X} \equiv \langle A_\mu, a \rangle_{a \in X}$;

(iii) if $\lambda < \mu < 2^\omega$, then $\mathrm{Th}\,(\langle A_\lambda, P_\lambda, a \rangle_{a \in X}) \neq \mathrm{Th}\,(\langle A, P_\mu, a \rangle_{a \in X})$.

We may assume that statements (1)-(4) (see below) hold.

(1) Abelian groups A_λ, $\lambda < 2^\omega$, coincide. In fact, in view of the extension theorem, we may assume that the groups have equal cardinality $\varkappa$. By virtue of condition (ii), it follows by the Keisler–Shelah theorem that there exists an ultrafilter U such that ultrapowers A_λ^U, $\lambda < 2^\omega$, are isomorphic. The isomorphic groups A_λ, $\lambda < 2^\omega$, are denoted by A.

(2) For subgroups P_λ, $\lambda < 2^\omega$, and for primes p for which $A[p]$ is an infinite subgroup, the following sets do not depend on λ:

(α) $\{\varepsilon_p(a) \mid a \in (pA \cap P_\lambda)\}$;

(β) $\{\varepsilon_p(a) \mid a \in pP_\lambda\}$;

(γ) $\{\varepsilon_p(a) \mid a \in (X \cap pP_\lambda)\}$.

Indeed, Lemma 16, the property $X \subseteq P_\alpha$, $\alpha < 2^\omega$, and property (D) imply that for almost all primes p, the sets (α), (β), and (γ) coincide with the set of all ε_p-classes. For other p (finite in number by property (D)), these sets are finite in view of (C). Therefore, among P_α, $\alpha < 2^\omega$, there are continuum many subgroups for which statement (2) holds.

(3) For subgroups P_λ, $\lambda < 2^\omega$, and for primes p for which $\gamma_p(A)$ is a finite Szmielew invariant not equal to zero, the subgroups of G_P in Definition 10 have equal cardinality. This follows from the property of G_P being finite, property (D), and Prop. 7.

Statement (3) gives rise to the following:

Remark 5. In the case where the invariant $\gamma_p(A)$ is finite and is not equal to zero, there exists a natural number n such that for any $\lambda < 2^\omega$, the subgroup P_λ contains elements $a_1^\lambda, \ldots, a_n^\lambda$ satisfying the conditions of Definition 10 for $P = P_\lambda$. In this event we may assume that for any $\alpha < \beta < 2^\omega$ and for every $i \in \{1, \ldots, n\}$, the elements a_i^α and a_i^β sit in a same ε_p-class, and that class contains no elements of the set X.

(4) If $\lambda < \mu < 2^\omega$, then $\langle A, P_\lambda\rangle \equiv \langle A, P_\mu\rangle$.

The statement follows from the fact that A is (P, a)-small.

Remark 6. We may assume that a primitive L_P-formula $\Phi(\mathbf{x})$ has the form

$$\exists y_1 \cdots \exists y_k (P(y_1) \wedge \cdots \wedge P(y_k) \wedge \Psi),$$

where Ψ is a primitive L-formula. Indeed, if Φ contains a subformula of the form $P(t)$ for some term t, then Φ is equivalent to a formula $\exists y(P(y) \wedge \Phi^*)$, where the variable y does not occur in Φ, while the formula Φ^* is obtained from Φ by replacing $P(t)$ by $y = t$.

In view of Lemma 3, any L_P-formula $\varphi(\mathbf{x})$ is equivalent in the theory T^* to a Boolean combination of primitive formulas. If now we show that every primitive L_P-formula $\Phi(\mathbf{x})$ defines in models $\langle A, P_\alpha\rangle$, $\alpha < 2^\omega$, the sole predicate on a set X, then we will arrive at a contradiction with condition (iii).

Thus, to prove Theorem 9, it suffices to show the validity of the following:

LEMMA 20. Every primitive L_P-formula $\Phi(\mathbf{x})$ defines in models $\langle A, P_\alpha\rangle$, $\alpha < 2^\omega$, the sole predicate on a set X.

Proof. Let $\Phi(x_1, \ldots, x_n)$ be a primitive L_P-formula. By Remark 6, we may assume that it has the form

$$\exists y_1 \cdots \exists y_k (P(y_1) \wedge \cdots \wedge P(y_k) \wedge \Psi),$$

where $\Psi(x_1, \ldots, x_n; y_1, \ldots, y_k)$ is a primitive L-formula. By virtue of Lemma 4, the formula $\Psi(x_1, \ldots, x_n; y_1, \ldots, y_k)$ is equivalent in AG to a conjunction of standard formulas. Truth of standard formulas of the first kind on elements of the subgroup P_α in the group A is equivalent to their being true in P_α.

By statements (2) and (3), Lemma 19, and the construction of $\Psi_{p^k}^P(x)$, it follows from Lemma 18 and property (iv) that there exists a formula $\Theta(x_1, \ldots, x_n; y_1, \ldots, y_k; z_1, \ldots, z_s)$, not depending on α, such that for every $\alpha < 2^\omega$, there are parameters $a_1^\alpha, \ldots, a_s^\alpha$ in $(P_\alpha \cap pA)$ such that for any $c_1, \ldots, c_n; b_1, \ldots, b_k \in P_\alpha$,

$$A \models \Psi(c_1, \ldots, c_n; b_1, \ldots, b_k) \Leftrightarrow P_\alpha \models \Theta(c_1, \ldots, c_n; b_1, \ldots, b_k; a_1^\alpha, \ldots, a_s^\alpha). \tag{5}$$

The parameters $a_1^\alpha, \ldots, a_s^\alpha$ in order of their appearance in Lemma 18 will be related to a respective prime p. By Lemma 19, we may assume that if the subgroup $A[p]$ is finite, then there are no parameters among $a_1^\alpha, \ldots, a_s^\alpha$ relating to a prime p. In addition, in view of Lemma 17, we will assume that all parameters $a_1^\alpha, \ldots, a_s^\alpha$ are inner if the subgroup P_α is bounded. By Lemma 16, we can suppose that if the Szmielew invariant $\gamma_p(A)$ is equal to zero, then all parameters of the formula relating to p are inner.

Lemma 18(2) implies the following: if we replace an outer parameter a_i^α corresponding to a prime p by any element of the set $a_i^\alpha + pP_\alpha$, then equivalence (5) will be preserved.

Equivalence (5) and the form of a formula Φ yield the equivalence

$$\langle A, P_\alpha\rangle \models \Phi(c_1, \dots, c_n) \Leftrightarrow P_\alpha \models \exists y_1 \cdots \exists y_k \Theta(c_1, \dots, c_n; y_1, \dots, y_k; a_1^\alpha, \dots, a_s^\alpha) \tag{6}$$

for any elements $c_1, \dots, c_n \in P_\alpha$.

Looking over the form of formulas $\Psi_{p^k}^P(x)$ in Lemma 18 and the form of a formula $\Phi_{p^k}^P(x)$ in Lemma 17, we conclude that the formula $\Theta(x_1, \dots, x_n; y_1, \dots, y_k; a_1^\alpha, \dots, a_s^\alpha)$ is a disjunction of primitive formulas. Moreover, if we replace its outer parameter relating to a prime p by an ε_p-equivalent element of the subgroup P_α, then every formula among the members of this disjunction will be equivalent in P_α to the initial formula. The existential quantifier distributes over disjunction. Therefore, in view of the form of formulas $\Psi_{p^k}^P(x)$ in Lemma 18 and the form of a formula $\Phi_{p^k}^P(x)$ in Lemma 17, and also statement (2), in order to prove the lemma, it suffices to show that the following statement holds.

(**) Let $\Delta(\mathbf{x}; \mathbf{y}; \mathbf{z})$ be a primitive L-formula without parameters. Let $\mathbf{b}$ be a tuple of inner parameters. For every $\alpha < 2^\omega$, we choose tuples $\mathbf{a}^\alpha$ of outer parameters such that every element a_i^α of $\mathbf{a}^\alpha$ is assigned some prime number p not depending on α, with $a_i^\alpha \in (pA \cap P_\alpha)$. Moreover, for any α and any β, the elements a_i^α and a_i^β sit in one ε_p-class, and that class contains no elements of the subgroup X; in particular, it contains no elements of the subgroups $pA[p^n]$ for any n. Let $\Delta(\mathbf{x}; \mathbf{b}; \mathbf{a}^\alpha)$ be a formula in which any parameter of $\mathbf{a}^\alpha$ relating to a prime p is replaced by an ε_p-equivalent element of the subgroup P_α, and let it be equivalent in P_α to the initial formula. Then the predicates

$$\{(\Delta(P_\alpha; \mathbf{b}; \mathbf{a}^\alpha) \cap X) \mid \alpha < 2^\omega\}$$

coincide.

Let $\mathbf{a}^\alpha = \langle a_1^\alpha, \dots, a_s^\alpha\rangle$ and $\bar{\mathbf{z}} = \langle z_1, \dots, z_s\rangle$. If, for any $\alpha < 2^\omega$, the formula $\Delta(\mathbf{x}; \mathbf{b}; \mathbf{a}^\alpha)$ in the group P_α on the set X defines an empty predicate, then there is nothing to prove. Below we assume that there exists α such that

$$(\Delta(P_\alpha; \mathbf{b}; \mathbf{a}^\alpha) \cap X) \neq \varnothing. \tag{7}$$

By virtue of Szmielew's theorem, we may assume that the formula $\Delta(\mathbf{x}; \mathbf{y}; \mathbf{z})$ is a conjunction of standard formulas. Consider all possible types of conjunctive terms appearing in $\Delta(\mathbf{x}; \mathbf{y}; \mathbf{z})$.

(1) Take a standard formula $\Phi_0(\mathbf{x}; \mathbf{b})$ without outer parameters, which is a conjunctive term of the formula $\Delta(\mathbf{x}; \mathbf{b}; \mathbf{z})$.

If Φ_0 is a standard formula of the first kind, then its truth in P_α on tuples of X is equivalent to its being true in A; therefore, this truth does not depend on α.

Let Φ_0 be a standard formula of the second kind modulo p. If $A[p]$ is a finite subgroup, then the fact that truth of Φ_0 on tuples of X does not depend on α follows from Lemma 19.

Let $A[p]$ be an infinite subgroup. Then the Szmielew invariant $\gamma_p(A)$ is finite. In view of statement (2)(γ), the set $X \cap pP_\alpha$ does not depend on α. Property (B^*) for $P = X$ with any $k > 1$ entails the following:

$$a \in (X \cap p^k P_\alpha) \Leftrightarrow \exists b(b \in X \wedge b^{(k-1)} = a \wedge b \in pP_\alpha).$$

Therefore, since the set $X \cap pP_\alpha$ is independent of α, it follows that $X \cap p^k P_\alpha$ is independent of α for any k.

(2) Consider a formula $\Phi_1(\mathbf{x}; \mathbf{b}; \mathbf{a}^\alpha) = \bar{n}\mathbf{b} + \bar{m}\mathbf{a}^\alpha + \bar{s}\mathbf{x} = 0$, which is a conjunctive term of the formula $\Delta(\mathbf{x}; \mathbf{b}; \mathbf{a}^\alpha)$ and is true in P_α on some tuple of X.

In view of case (1), we may assume that the tuple $\bar{m}$ consists of nonzero elements. Suppose that tuples $\mathbf{b}^\alpha$ and $\bar{k}$ are obtained from $\mathbf{a}^\alpha$ and $\bar{m}$ by deleting the first element. Then

$$\Phi_1(\mathbf{x}; \mathbf{b}; \mathbf{a}^\alpha) = \bar{n}\mathbf{b} + \bar{k}\mathbf{b}^\alpha + m_1 a_1^\alpha + \bar{s}\mathbf{x} = 0,$$

where a_1^α relates to a prime number p. Take an arbitrary element $d \in p^2 P_\alpha$. If $m_1 d \neq 0$, then the formulas $\bar{n}\mathbf{b} + \bar{k}\mathbf{b}^\alpha + m_1 a_1 + \bar{s}\bar{\mathbf{x}} = 0$ and $\bar{n}\mathbf{b} + \bar{k}\mathbf{b}^\alpha + m_1(a_1 + d) + \bar{s}\bar{\mathbf{x}} = 0$ have no common solutions in P_α. This, by virtue of property (7), clashes with the fact that the elements a_1 and $(a_1 + d)$ are ε_p-equivalent, and the truth of $\Delta(\mathbf{x}; a_1^\alpha, \ldots, a_s^\alpha)$ is preserved under the replacement of parameters relating to a prime p by ε_p-equivalent parameters. Thus $m_1 p^2 P_\alpha = 0$. By statement (4), the structures $\langle A, P_\alpha \rangle$, $\alpha < 2^\omega$, are elementarily equivalent, and so all subgroups P_α, $\alpha < 2^\omega$, are bounded. In view of Lemma 17, the formula $\Phi_1(\mathbf{x}; \mathbf{b}; \mathbf{a}^\alpha)$ is equivalent to a formula without outer parameters, and we can now use case (1).

(3) Consider a conjunctive term of $\Delta(\mathbf{x}; \mathbf{b}; \mathbf{a}^\alpha)$ of the form $\Phi_2(\mathbf{x}) = \bar{k}\mathbf{b}^\alpha + na + \bar{m}\mathbf{x} = 0 \,(\mathrm{mod}\, p^k)$, where the parameter a relates to a number $q \neq p$. We may assume that the number n is not divisible by p^k. Let $\Phi_2(\mathbf{x})$ be true in P_α on some tuple of X.

Take an arbitrary element $d \in q^2 P_\alpha$. Suppose $nd \neq 0 \,(\mathrm{mod}\, p^k)$. Then the formulas $\bar{k}\mathbf{b}^\alpha + na + \bar{m}\mathbf{x} = 0 \,(\mathrm{mod}\, p^k)$ and $\bar{k}\mathbf{b}^\alpha + n(a + d) + \bar{m}\mathbf{x} = 0 \,(\mathrm{mod}\, p^k)$ have no common solutions in P_α. Indeed, if we had $\bar{k}\mathbf{b}^\alpha + na + \bar{m}\mathbf{c} = 0 \,(\mathrm{mod}\, p^k)$ and $\bar{k}\mathbf{b}^\alpha + n(a + d) + \bar{m}\mathbf{c} = 0 \,(\mathrm{mod}\, p^k)$ for some tuple $\mathbf{c} \in P_\alpha$, then, choosing an appropriate remainder, we would obtain $nd = 0 \,(\mathrm{mod}\, p^k)$, which is impossible by assumption. On the other hand, solutions for the formula $\Delta(\mathbf{x}; \mathbf{b}; \mathbf{a}^\alpha)$ with a replaced by $(a+d)$ should coincide since these elements are ε_q-equivalent. Thus $nq^2 P_\alpha \subseteq p^k P_\alpha$. The number q^2 is not divisible by p, so $nP_\alpha \subseteq p^k P_\alpha$. In view of statement (4), this is true for any $\alpha < 2^\omega$. Therefore, $na = 0 \,(\mathrm{mod}\, p^k)$. Consequently, the formula $\Phi_2(\mathbf{x})$ is equivalent in P_α to a formula $\bar{k}\mathbf{b}^\alpha + \bar{m}\mathbf{x} = 0 \,(\mathrm{mod}\, p^k)$, which is true for any $\alpha < 2^\omega$.

It remains to consider formulas of the form $\bar{n}\mathbf{b} + \bar{s}\mathbf{a}^\alpha + \bar{m}\mathbf{x} = 0 \,(\mathrm{mod}\, p^k)$, where all elements of the tuples $\mathbf{b}$ and $\mathbf{a}^\alpha$ relate to a number p.

(4) Let a conjunctive term of $\Delta(\mathbf{x}; \mathbf{b}; \mathbf{a}^\alpha)$ be of the form

$$\Phi_3(\mathbf{x}) = \bar{n}\mathbf{b} + \bar{s}\mathbf{a}^\alpha + \bar{m}\mathbf{x} = 0 \,(\mathrm{mod}\, p^k), \tag{8}$$

where all elements of the tuples $\mathbf{b}$ and $\mathbf{a}^\alpha$ relate to a number p.

We will assume that there is a tuple $\mathbf{c} \in X$, which is a solution for the formula $\Phi_3(\mathbf{x})$ in some subgroup P_α. We can suppose that with decreasing k, the tuple $\mathbf{c}$ is no longer a solution for the thus obtained formula. We may also assume that each of the coefficients of $\Phi_3(\mathbf{x})$ is not divisible by p^k. In view of case (1), the tuple $\bar{s}$ has nonzero length and consists of nonzero elements.

The parameters of $\mathbf{a}^\alpha$ and $\mathbf{b}$ in order of their appearance in Lemma 18 will belong to the subgroup pA. Since $\mathbf{c}$ is a solution in P for the formula $\Phi_3(\mathbf{x})$, the element $\bar{m}\mathbf{c}$ belongs to the subgroup pA.

We can suppose that all numbers in the tuple $\bar{s}$ are of the form $p^{(k-1)}l_i$, where l_i is not divisible by p. Otherwise, by the truth-preserving property, if we replaced the parameter a_i by any other element of the set $a_i + pP_\alpha$, then we would have $p^{(k-r)}mP_\alpha \subseteq p^k P_\alpha$, where m is not divisible by p and $r > 0$. The number m is not divisible by p, and so $p^{(k-r)}P_\alpha \subseteq p^k P_\alpha$, which is a contradiction with k being minimal. The number l_i is not divisible by p, and we may assume that $l_i < p$.

Let $\bar{l}$ be a tuple of the same length as that of $\bar{s}$, composed of the multipliers l_i. Suppose that for some tuples $\mathbf{c} \in X$ and $e \in P_\alpha$,

$$\bar{n}\mathbf{b} + p^{(k-1)}\bar{l}\mathbf{a}^\alpha + \bar{m}\mathbf{c} = p^k e. \tag{9}$$

Elements of the tuple $\mathbf{a}^\alpha$ belong to the subgroup pA, and for some tuple $\mathbf{f}^\alpha \in A$, $p\mathbf{f}^\alpha = \mathbf{a}^\alpha$. From (9), we derive

$$p^k(\bar{l}\mathbf{f}^\alpha + e) \in X. \tag{10}$$

Using Remark 3, we obtain $p(\bar{l}\mathbf{f}^\alpha + e) \in X$. Consequently, $\bar{l}\mathbf{a}^\alpha + pe \in X$. We have arrived at a contradiction with the fact that sequence (4) in Definition 10 forms a basis in the group G_P treated as a vector space over a field consisting of p elements. □

We formulate the remaining question in terms of the following:

Conjecture. For a complete theory of an Abelian group A to be totally (P, a)-stable, it is necessary and sufficient that the group A be Szmielew ω-bounded and properties (B), (C), and (D) hold.

5. TOTAL (P, s)-STABILITY

THEOREM 10. For an Abelian group A, the following conditions are equivalent:

(1) a theory of A is totally (P, s)-stable;

(2) a theory of A is (P, s)-stable;

(3) a group A is a direct sum of a finite group and finitely many elementary Abelian groups.

Proof. That (2) and (3) are equivalent was proved in [1].

The implication (1)⇒(2) is obvious.

We show that (2)⇒(1). It follows from [2, Thm. 8] that if a theory T of a group A is (P, s)-stable then, for any cardinal λ, the value of a cardinal function $S_T(P, s)(\lambda)$ does not exceed ω, i.e., condition (1) holds. □

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