DOI 10.1007/s10469-015-9349-2
Algebra and Logic, Vol. 54, No. 4, September, 2015
(Russian Original Vol. 54, No. 4, July-August, 2015)

FRIEDBERG NUMBERINGS IN THE ERSHOV HIERARCHY

S. S. Ospichev*

UDC 510.532+510.57

Keywords: *computable numbering, Friedberg numbering, Ershov hierarchy.*

A Friedberg numbering of the family of all sets for any given level of the Ershov hierarchy is constructed, and we also consider different consequences of this result.

INTRODUCTION

In the paper we deal with one-to-one computable numberings in the Ershov hierarchy. These numberings attract considerable interest by virtue of the fact that a computable one-to-one numbering is (up to equivalence) a minimal element of the semilattice of computable numberings for a given family of sets. The study of such numberings was pioneered by R. Friedberg, who showed the existence of a one-to-one computable numbering for the family of all computably enumerable sets. Later, the existence of a Σ_n^{-1}-computable Friedberg numbering was stated for the family of all Σ_n^{-1}-sets [2].

Here the given result is generalized to all constructive ordinals, and we also consider different properties of the numberings obtained.

1. BASIC DEFINITIONS

We embark directly on a description of the Ershov hierarchy. (General definitions and basic facts can be found in [3].) A representation for sets in the Ershov hierarchy that we use differs only slightly from the one in [4].

*Supported by RFBR (project No. 14-01-00376) and by the Grants Council (under RF President) for State Aid of Leading Scientific Schools, grant NSh-860.2014.1.

Sobolev Institute of Mathematics, pr. Akad. Koptyuga 4, Novosibirsk, 630090 Russia. Novosibirsk State University, ul. Pirogova 2, Novosibirsk, 630090 Russia; ospichev@gmail.com. Translated from *Algebra i Logika*, Vol. 54, No. 4, pp. 444-462, July-August, 2015. Original article submitted August 17, 2012; revised March 17, 2015.

Definition 1.1. For all $a \in \mathcal{O}$ (hereinafter, $\mathcal{O}$ is Kleene's system of ordinal notation), a set A is a Σ_a^{-1}-*set* (*belongs to the class* Σ_a^{-1} of the Ershov hierarchy) if there exist a computable function $f(x,s)$ and a partial computable function $g(x,s)$ such that for all $x \in \omega$, the following conditions hold:

(1) $A(x) = \lim_s f(x,s)$, $f(x,0) = 0$;

(2) $g(x,s)\downarrow \to g(x,s+1)\downarrow \leq_o g(x,s) <_o a$;

(3) $f(x,s) \neq f(x,s+1) \to g(x,s+1)\downarrow \neq g(x,s)$.

A pair $\langle f, g\rangle$ is called a Σ_a^{-1}-*approximation* to a set A.

Definition 1.2. A is a Π_a^{-1}-*set* if, under the conditions of the previous definition, the requirement that $f(x,0) = 0$ is replaced with $f(x,0) = 1$. A is a Δ_a^{-1}-*set* if $f(x,0) = 0$ is replaced with the requirement that $g(x,0)$ is defined.

Note that if A is Σ_a^{-1} then its complement $\overline{A}$ is Π_a^{-1}, while every Δ_a^{-1}-set is both Σ_a^{-1} and Π_a^{-1}.

The above defined Σ_a^{-1}-approximation $\langle f, g\rangle$ to A can be strengthened by adding some properties for the functions f and g.

We define a parity function $e(a)$ for $a \in \mathcal{O}$ as follows: $e(a) = 0$, if a is a notation for an even or limit ordinal, and $e(a) = 1$ if a a notation for an odd ordinal.

PROPOSITION 1.3. If a set A has a Σ_a^{-1}-approximation $\langle \phi, \psi\rangle$, then A has a Σ_a^{-1}-approximation $\langle f, g\rangle$ with the following properties:

correct parity, i.e.,

if $f(x,s) = 0$, then $e(g(x,s)) = e(a)$ or $g(x,s)$ is undefined,

if $f(x,s) = 1$, then $e(g(x,s)) = 1 - e(a)$;

slowing down, i.e.,

for any $s \in \omega$ and any $x > s$, $f(x,s) = 0$,

for any $s \in \omega$ and any $x > s$, $g(x,s)$ is undefined.

Proof. Correct parity. Let $f(x,s) = \phi(x,s)$ for all $x, s \in \omega$.

(1) If $\psi(x,s)$ is undefined, then $g(x,s)$ is undefined.

(2) If $\psi(x,s)$ is defined, then:

(2.1) if $\psi(x,s) = b$, where $a = 2^b$ ($|a|_{\mathcal{O}}$ is a successor of $|b|_{\mathcal{O}}$), then:

(2.1.1) if $\phi(x,s) = 0$, then $g(x,s)$ is undefined (i.e., no change has been made as yet);

(2.1.2) if $\phi(x,s) = 1$, then $g(x,s) = \psi(x,s)$;

(2.2) if $e(\psi(x,s)) \neq e(a)$ and $\psi(x,s) \neq b$, then:

(2.2.1) $\phi(x,s) = 0 \Rightarrow g(x,s) = c$, where $|c|_{\mathcal{O}}$ is a successor of $|\psi(x,s)|_{\mathcal{O}}$;

(2.2.2) $\phi(x,s) = 1 \Rightarrow g(x,s) = \psi(x,s)$;

(2.3) if $e(\psi(x,s)) = e(a)$ and $\psi(x,s) \neq b$, then:

(2.3.1) $\phi(x,s) = 1 \Rightarrow g(x,s) = c$, where $|c|_{\mathcal{O}}$ is a successor of $|\psi(x,s)|_{\mathcal{O}}$;

(2.3.2) $\phi(x,s) = 0 \Rightarrow g(x,s) = \psi(x,s)$.

Slowing down. For all $x, s \in \omega$, put:

(1) if $x \leq s$, then $f(x,s) = \phi(x,s)$ and $g(x,s) = \psi(x,s)$;

(2) if $x > s$, then $f(x,s) = 0$ and $g(x,s)$ is undefined. □

The correct parity property allows us to assert the following: if, in constructing any set, an element x at some step s experiences the maximum number of changes (i.e., $g(x,s) = 0$), then we know exactly whether or not x will enter a given set (more specifically, $f(x,s) = e(a)$). The slowing down property makes it possible to consider at each step s only changes of the first s elements. In what follows, all Σ_a^{-1}-approximations will be thought of as correct-parity Σ_a^{-1}-approximations with slowing down.

Definition 1.4. We say that a set A m-reduces to a set B (written $A \leq_m B$) if there exists a computable function f such that $x \in A \Leftrightarrow f(x) \in B$ for any $x \in \omega$.

Notice that for $A \leq_m B$, the fact that B belongs to some class of the Ershov hierarchy implies that A belongs to the same class of the Ershov hierarchy.

We proceed with the definition of a computable numbering. Use will be made of the notion of generalized computability, as in [5].

Definition 1.5. A numbering $\{\nu_n\}_{n\in\omega}$ (denoted hereinafter by ν) is Σ_a^{-1}-*computable* if there exist a computable function $f(n,x,s)$ and a partial computable function $g(n,x,s)$ such that for any x and any n, the following conditions hold:

(1) $\{\nu_n\}(x) = \lim_s f(n,x,s),\ f(n,x,0) = 0$;

(2) $g(n,x,s)\downarrow \to g(n,x,s+1)\downarrow \leqslant_o g(n,x,s) <_o a$;

(3) $f(n,x,s) \neq f(n,x,s+1) \to g(n,x,s+1)\downarrow \neq g(n,x,s)$.

By analogy with Definition 1.1, a pair $\langle f,g\rangle$ is called a Σ_a^{-1}-*approximation to a numbering* ν.

In a similar way, we introduce definitions of Π_a^{-1}- and Δ_a^{-1}-computable numberings.

Definition 1.6. We say that a numbering α *reduces* to a numbering β ($\alpha \leq \beta$) if there exists a computable function f such that $\alpha_n = \beta_{f(n)}$ for any n.

Definition 1.7. We call η a *Friedberg* numbering if $\eta_n \neq \eta_m$ for any $n \neq m$.

Definition 1.8. A numbering ν is said to be *positive* (*decidable*) if $\{\langle x,y\rangle | \nu_x = \nu_y\}$ is a computably enumerable (computable) set.

Note that every Friedberg numbering is decidable and hence positive.

2. MAIN RESULT

We proceed to a construction of a Friedberg numbering for the family of all Σ_a^{-1}-sets.

THEOREM 2.1 [2]. For any $n > 0$, there exists a Σ_n^{-1}-computable Friedberg numbering of the family of all Σ_n^{-1}-sets.

Hereinafter, for ordinals corresponding to natural numbers, we use their natural notations.

The above result can somehow be extended to all constructive ordinals.

Let A be a Σ_a^{-1}-set such that for $e(a) = 0$, A is not finite, and for $e(a) = 1$, A is not cofinite. Let $\langle \xi^1, \xi^2\rangle$ be its Σ_a^{-1}-approximation. Define a family $\mathcal{T}$ as follows:

(1) $\{A, A\bigcup\{0\}, A\bigcup\{0,1\}, \dots\}$ if $e(a) = 1$;

(2) $\{A, A \setminus \{0\}, A \setminus \{0,1\}, \dots\}$ if $e(a) = 0$.

In what follows, in proving the main results, we use a Friedberg numbering of some subfamily of $\mathcal{T}$ having additional properties. Now we show that such a numbering exists.

LEMMA 2.2. There exists a Σ_a^{-1}-computable Friedberg numbering γ of some subfamily of $\mathcal{T}$ such that:

for any $n_1 < n_2$,

$\{0, 1, \dots, n_1\} \subset \gamma_{n_1} \subset \gamma_{n_2}$ if $e(a) = 1$, and

$\gamma_{n_2} \subset \gamma_{n_1} \subset \overline{\{0, 1, \dots, n_1\}}$ if $e(a) = 0$;

for any n, there is a unique s' such that $\rho(n, x, s') \neq \rho(n, x, s' + 1)$ with any $x \leq n$ (more precisely, $\rho(n, x, s)$ is undefined for $s \leq s'$, and then $\rho(n, x, s) = 0$). Notice also that $s' \leq n$.

Proof. Let $\langle \xi^1, \xi^2 \rangle$ be a Σ_a^{-1}-approximation to A. We construct a Σ_a^{-1}-approximation $\langle \delta, \rho \rangle$ to a numbering γ.

Put $\delta(0, x, s) = \xi^1(x, s)$ and $\rho(0, x, s) = \xi^2(x, s)$ for any $x, s \in \omega$; $\delta(n, 0, 0) = 0$ and $\rho(n, 0, 0)$ is undefined for all $n > 0$. Below if no mention is made of certain combinations (n, x, s), then we put $\delta(n, x, s + 1) = \delta(n, x, s)$ and $\rho(n, x, s + 1) = \rho(n, x, s)$ for these.

By recursion on s, we define functions δ, ρ, and an auxiliary function k.

(1) Step $s = 1$. Put $k(1, 1) = 0$, $\delta(1, 0, 1) = e(a)$, and $\rho(1, 0, 1) = 0$, and for all $x > 0$, let $\delta(1, x, 1) = \xi^1(x, 1)$ and $\rho(1, x, 1) = \xi^2(x, 1)$.

(2) Step $s + 1$.

(2.1) For all $n \leq s$ in increasing order,

(2.1.1) if, for some $n' < n$, it is true that $\delta(n', x, s) = \delta(n, x, s)$ with all $x \leq s$, then we put $k(n, s + 1) = k(n, s) + 1$, and otherwise, put $k(n, s + 1) = k(n, s)$;

(2.1.2) for all $x < k(n, s + 1)$, put $\delta(n, x, s + 1) = e(a)$ and $\rho(n, x, s + 1) = 0$, and for $x \geq k(n, s + 1)$, let $\delta(n, x, s + 1) = \xi^1(x, s + 1)$ and $\rho(n, x, s + 1) = \xi^2(x, s + 1)$;

(2.2) put $k(s + 1, s + 1) = k(s, s + 1) + 1$.

Put $\delta(n, x, s+1) = e(a)$ and $\rho(n, x, s+1) = 0$ for all $x < k(s+1, s+1)$; $\delta(n, x, s+1) = \xi^1(x, s)$ and $\rho(n, x, s + 1) = \xi^2(x, s + 1)$ for $x \geq k(s + 1, s + 1)$.

Pass to the next step s.

The numbering γ is constructed. □

Finally, let $\mathcal{S}$ be a Σ_a^{-1}-computable family that includes a family $\mathcal{T}$ and a set ω if $e(a) = 1$, or the empty set if $e(a) = 0$.

THEOREM 2.3. For every notation of a nonzero ordinal $a \in \mathcal{O}$, there is a Σ_a^{-1}-computable Friedberg numbering of the family $\mathcal{S}$.

Proof. Let α be a Σ_a^{-1}-computable numbering for $\mathcal{S}$. There is no loss of generality in assuming that $\alpha_0 = \varnothing$, if a denotes an even ordinal, and $\alpha_0 = \omega$ if a denotes an odd ordinal. We will construct a Σ_a^{-1}-computable Friedberg numbering β of $\mathcal{S}$, and also a $\varnothing'$-partial computable function h (approximable by partial computable functions h_s) using a numbering γ such as in Lemma 2.2.

Let $\langle\phi,\psi\rangle$ and $\langle\delta,\rho\rangle$ be Σ_a^{-1}-approximations to numberings α and γ, respectively. We construct an approximation $\langle f,g\rangle$ to a numbering β.

Requirements for the construction.

(1) If $\alpha_n=\alpha_{n'}$ for some $n'<n$, then $h(n)$ is undefined.

(2) If $\alpha_n\neq\alpha_{n'}$ for all $n'<n$, then either $h(n)$ is defined and $\alpha_n=\beta_{h(n)}$, or $\alpha_n=\gamma_x$ for some x and there exists $m\in\omega\setminus\mathrm{range}(h)$ such that $\alpha_n=\beta_m$.

(3) Every set β_m, $m\notin\mathrm{range}(h)$, coincides with γ_y for some y.

(4) For any y, there is a unique m such that $\beta_m=\gamma_y$.

Construction of a numbering β.

Step $s=0$. Define

$f(n,x,0)=0$ and $g(n,x,0)\uparrow$ for all natural n and x;

$f(0,x,s)=e(a)$ and $g(0,x,s)=0$ for any x and for $s>0$;

$h(0)=h_0(0)=0$ and $h_0(n)\uparrow$ for any $n>0$.

Step $s+1$. For every $n\leq s$, we consecutively perform the following actions.

$(s+1.1)$ If $h_s(n)$ is defined, and for some number $n'<n$,

$$\phi(n',x,s)=\phi(n,x,s)\text{ for all }x\in[0,h_s(n)+1],$$

then we assume that $h_{s+1}(n)$ is undefined.

$(s+1.2)$ If $h_s(n)$ is defined, $n>0$, and for some numbers $s'<s$ and $m\in\mathrm{range}(h_{s'})\setminus\mathrm{range}(h_s)$,

$$f(m,x,s)=f(h_s(n),x,s)\text{ for all }x\in[0,h_s(n)+1],$$

then we assume that $h_{s+1}(n)$ is undefined.

$(s+1.3)$ If $h_s(n)$ is defined, while $h_{s+1}(n)$ becomes undefined as a result of actions $(s+1.1)$ and $(s+1.2)$, then, for every such number n (in increasing order), we put

$$f(h_s(n),x,s')=\delta(y,x,s'+y),\ \ g(h_s(n),x,s')=\rho(y,x,s'+y)$$

for all $s'>s$, where y is some natural number greater than is any number mentioned previously in the construction.

$(s+1.4)$ If $h_s(n)$ is undefined for $n\leqslant s$, then, for every such n (in increasing order), we set $h_{s+1}(n)$ equal to a least m which is not in $\bigcup\limits_{s'\leqslant s}\mathrm{range}(h_s)$ and differs from the value of $h_{s+1}(n')$ for all $n'<n$.

$(s+1.5)$ If $h_s(n)$ is defined, while $h_{s+1}(n)$ has not become undefined as a result of actions $(s+1.1)$ or $(s+1.2)$, then we put $h_{s+1}(n)=h_s(n)$.

$(s+1.6)$ If $h_{s+1}(n)$ is defined, then we set $f(h_{s+1}(n),x,s+1)=\phi(n,x,s+1)$ and $g(h_{s+1}(n),x,s+1)=\psi(n,x,s+1)$ for all $x\in\omega$.

Now we show that all the requirements stated above are fulfilled.

(1) In view of $(s+1.1)$, if $\alpha_n=\alpha_{n'}$ for some $n'<n$, then $h_s(n)$ is undefined for infinitely many steps s.

(2) If $\alpha_n \neq \alpha_{n'}$ for all $n' < n$, then $h(n)$ becomes undefined at the expense of $(s+1.1)$ not more than finitely many times. If $h(n)$ becomes undefined at the expense of $(s+1.2)$ infinitely often for a same number m, then $\alpha_n = \beta_m = \gamma_y$ for some y, as required. If we assume that $h(n)$ becomes undefined at the expense of $(s+1.2)$ infinitely often for infinitely many m, then $\phi(n,x,s) = f(h_s(n),x,s) = \delta(y,x,s)$ for ever larger y. Hence $\phi(n,x,s) = e(a)$ for any x; also $\alpha_n = \omega$, if $e(a) = 1$, and $\alpha_n = \varnothing$ if $e(a) = 0$. Then α_n coincides with α_0, and there is no need to search a place for α_n in the numbering β.

(3) Is fulfilled by virtue of $(s+1.4)$.

(4) Consider some y. In view of $(s+1.2)$ and $(s+1.4)$, there exists at most one m for which $\beta_m = \gamma_y$. We choose a least n such that $\alpha_n = \gamma_y$. Then either $h(n)$ is defined and $\beta_{h(n)} = \gamma_y$, or, as in (2), we can show that there is a number m such that $\beta_m = \gamma_y$.

All the requirements are thus fulfilled. We show that the resulting numbering is Σ_a^{-1}-computable.

(1) A function f on any tuple (n,x,s) is computed in finitely many steps; hence it is computable.

(2) For any natural m, β_m is constructed in the same way as α_n up to some step s; then, possibly, a transition to a numbering γ_y occurs for some natural n and y. If a transition does not take place, then $f(m,x,s) = \phi(n,x,s)$ and $g(m,x,s) = \psi(n,x,s)$; hence β is Σ_a^{-1}-computable. If a transition will occur, then $f(m,x,s) = \phi(n,x,s)$ and $g(m,x,s) = \psi(n,x,s)$, for all $s \leq s'$, and $f(m,x,s) = \delta(y,x,s+y)$ and $g(m,x,s) = \rho(y,x,s+y)$ for $s' < s$. The slowing down property and the correct parity property for a numbering α, as well as the choice of y and the properties of a numbering γ, imply that the following relations hold:

$$\phi(n,x,s') \neq \delta(y,x,s'+1+y) \Rightarrow \rho(y,x,s'+1+y) <_o \psi(n,x,s'),$$
$$\phi(n,x,s') = \delta(y,x,s'+1+y) \Rightarrow \rho(y,x,s'+1+y) \leq_o \psi(n,x,s');$$

hence the properties of a Σ_a^{-1}-approximation are also preserved. □

The family of all Σ_a^{-1}-sets satisfies the hypothesis of Theorem 2.3. (That the family of all Σ_a^{-1}-sets is Σ_a^{-1}-computable was proved, for instance, in [6]; as a family $\mathcal{T}$ we can take the family of all initial segments or the family of its complements.)

COROLLARY 2.4. For any notation of a nonzero ordinal $a \in \mathcal{O}$, the family of all Σ_a^{-1}-sets has a Σ_a^{-1}-computable Friedberg numbering.

We take a set Ξ_a^{-1} (an m-universal set for the class Σ_a^{-1}; for details, see [3]) for constructing a numbering γ, and also a Σ_a^{-1}-computable family S containing $\mathcal{T}$ such that every set $A \in S$ is Σ_a^{-1}-proper (i.e., $A \in \Sigma_a^{-1}$, but $A \notin \Sigma_b^{-1}$ for any $b <_{\mathcal{O}} a$). According to Theorem 2.3, the family $\mathcal{S}$ has a Σ_a^{-1}-Friedberg numbering but has no Σ_b^{-1}-computable numbering for any $b <_{\mathcal{O}} a$.

We now turn to other big families—the family of all Π_a^{-1}-sets and the family of all Δ_a^{-1}-sets. The family of all Π_a^{-1}-sets consists of complements of all elements of the class Σ_a^{-1}. Hence as a Π_a^{-1}-computable Friedberg numbering for the family of all Π_a^{-1}-sets we can take a numbering $\beta'_x = \overline{\beta_x}$, where β is a Σ_a^{-1}-computable Friedberg numbering for the family of all Σ_a^{-1}-sets.

For the family of all Δ_a^{-1}-sets, the situation is somewhat more complicated.

3. THE FAMILY OF ALL Δ_a^{-1}-SETS

In order to construct a Friedberg numbering for the family of all Δ_a^{-1}-sets using the construction given in Sec. 2, we need a computable numbering for the family of all Δ_a^{-1}-sets, and also a suitable additional family T. As T we can take a family constructed on the set ω, if $e(a) = 1$, or on the set $\varnothing$ if $e(a) = 0$.

In [7], it was proved that there does not exist a Δ_a^{-1}-computable numbering of the family of all Δ_a^{-1}-sets for any $a \in \mathcal{O}$. However, by analogy with the family of all computable sets (Δ_1^{-1}), for which there exists a computable (Σ_1^{-1}-computable) numbering, we have

PROPOSITION 3.1. For any ordinal notation of a nonzero ordinal $a \in \mathcal{O}$, there exists a Σ_a^{-1}-computable numbering of the family of all Δ_a^{-1}-sets.

Proof. In [8], it was shown that the class Δ_a^{-1} has an m-universal set. If a is a notation for a limit ordinal, then we can take $\bigoplus_{b<_o a} \Xi_b^{-1}$ to be such a set, and if $a = 2^b$ for some b, then we take $\Xi_b^{-1} \bigoplus \overline{\Xi_b^{-1}}$. If $a = 2^b$ for some b, then it is easy to see that $A \in \Delta_a^{-1}$ iff $A = A_1 \cap C \bigcup A_2 \cap \overline{C}$, where C is a computable set, $A_1 \in \Sigma_b^{-1}$, and $A_2 \in \Pi_b^{-1}$.

Let ν_n be a Δ_a^{-1}-computable numbering for all Σ_b^{-1}-sets and μ_n a Δ_a^{-1}-computable numbering for all Π_b^{-1}-sets. Let κ_n be a universal function in the class of unary partial computable functions and κ_n^t be the result of computing κ_n after t steps. With the help of κ_n^t, we can exhaustively search all computable sets. (That is, we will use exhaustive search of all partial computable functions, and if a function is neither increasing nor total, then we stop the computation, obtaining, for each n, either a finite set, or a computable set definable by an increasing computable function.) The resulting set will be involved in determining which of the numberings ν_n or μ_n should be used for a given n.

Our present goal is to construct a numbering $\alpha_{\langle l,m,n\rangle}$ for all Δ_a^{-1}-sets. To do this, we build up functions $f(\langle l,m,n\rangle, x, s)$ and $g(\langle l,m,n\rangle, x, s)$. Let functions $\phi_1(l,x,s)$ and $\psi_1(l,x,s)$ define a numbering ν_n, and let $\phi_2(m,x,s)$ and $\psi_2(m,x,s)$ define μ_n.

Construction.

For all $l,m,n,x,s \in \omega$, we define the pair

$$\langle f(\langle l,m,n\rangle, x, s), g(\langle l,m,n\rangle, x, s)\rangle.$$

Let k be a minimal natural number such that one of the following conditions holds:

(1) $\kappa_n^s(k)$ is undefined;

(2) $\kappa_n^s(k) > x$;

(3) $\kappa_n^s(k) = x$;

(4) $\kappa_n^s(k) < x$ and $\kappa_n^s(k+1) \le \kappa_n^s(k)$.

Depending on which of the conditions is satisfied, functions f and g are defined thus:

(1) $f(\langle l,m,n\rangle,x,s)=0$ and $g(\langle l,m,n\rangle,x,s)$ is undefined;

(2) an element x is not within the range of an increasing computable function, in which case our functions are defined in the same way as for the second numbering, i.e., $f(\langle l,m,n\rangle,x,s)=\phi_2(m,x,s)$ and $g(\langle l,m,n\rangle,x,s)=\psi_2(m,x,s)$;

(3) x belongs to the computable set under construction, in which case our functions are defined in the same way as for the first numbering, i.e., $f(\langle l,m,n\rangle,x,s)=\phi_1(l,x,s)$ and $g(\langle l,m,n\rangle,x,s)=\psi_1(l,x,s)$;

(4) since $\kappa_n^s(k+1)\le\kappa_n^s(k)$, the function κ_n is not increasing, and, as in the second case, x is in the complement of the computable set under construction; hence $f(\langle l,m,n\rangle,x,s)=\phi_2(m,x,s)$ and $g(\langle l,m,n\rangle,x,s)=\psi_2(m,x,s)$.

Let $f_1(\langle l,m,n\rangle,x,0)=0$ and $f_1(\langle l,m,n\rangle,x,s+1)=f(\langle l,m,n\rangle,x,s)$; $g_1(\langle l,m,n\rangle,x,0)$ is undefined and $g_1(\langle l,m,n\rangle,x,s+1)=g(\langle l,m,n\rangle,x,s)$. The numbering α definable by the functions f_1 and g_1 is Σ_a^{-1}-computable.

Let a be a notation for a limit ordinal; then coding the class of sets m-reducible to $\bigoplus\limits_{b<_oa}\Xi_b^{-1}$ is a simple matter. Let a be a Kleene notation for a limit ordinal, $a=3*5^e$, and $b=\kappa_e(m)$ for some sequence $b<_{\mathcal{O}}a$. For any $m\in\omega$, as ν^m we take a Σ_b^{-1}-computable numbering of the family of all Σ_b^{-1}-sets for one of b's in the specified sequence. Define a numbering such as

$$\nu_{\langle n,m\rangle}(x)=\begin{cases}\nu_n^{\kappa_e(m)}(x), \kappa_e(m)\downarrow;\\ \varnothing, \kappa_e(m)\uparrow.\end{cases}$$

Construct a Σ_a^{-1}-approximation $\langle f,g\rangle$ to a numbering ν using Σ_b^{-1}-approximations $\langle f_m,g_m\rangle$ to numberings ν^m. For any $s,n,m,x\in\omega$, put:

(1) if $\kappa_e^s(m)$ is defined, then $f(\langle n,m\rangle,x,s)=f_{\kappa_e(m)}(n,x,s)$ and $g(\langle n,m\rangle,x,s)=g_{\kappa_e(m)}(n,x,s)$;

(2) if $\kappa_e^s(m)$ is undefined, then $f(\langle n,m\rangle,x,s)=0$ and $g(\langle n,m\rangle,x,s)$ is undefined.

The resulting numbering will obviously be Σ_a^{-1}-computable. □

In a similar way, we can construct a Π_a^{-1}-computable numbering for the family of all Δ_a^{-1}-sets. Thus we have

COROLLARY 3.2. For every notation of a nonzero ordinal $a\in\mathcal{O}$, there exist Σ_a^{-1}- and Π_a^{-1}-computable Friedberg numberings of the family of all Δ_a^{-1}-sets.

4. CONSEQUENCES

In the final part of the paper, we give some consequences of the main result.

PROPOSITION 4.1. Let $\mathcal{S}$ be a Σ_a^{-1}-computable family and $\mathcal{T}$ be as in Lemma 2.2. Then the family $\mathcal{S}\bigcup\{\varnothing\}\setminus\mathcal{T}$ for $e(a)=0$ (or $\mathcal{S}\bigcup\{\omega\}\setminus\mathcal{T}$ for $e(a)=1$) has a Σ_a^{-1}-computable numbering.

Proof. Let α' be a Σ_a^{-1}-computable numbering for $\mathcal{S}$. As a numbering α we take the following:

(1) $\alpha_0=\varnothing$, if $e(a)=0$, and $\alpha_0=\omega$ if $e(a)=1$;

(2) $\alpha_{n+1} = \alpha'_n$.

We will use the numbering γ constructed in Lemma 2.2 (the application itself will be altered), and also a modification of the construction in Sec. 2.

We introduce changes in the construction.

At step $s+1$, actions $(s+1.2)$ and $(s+1.3)$ will be changed.

$(s+1.2)$ If $h_s(n)$ is defined, $n > 0$, and for some number $m < s$,

$$\delta(m, x, s) = f(h_s(n), a, s) \text{ for all } x \in [0, h_s(n)+1],$$

then we assume that $h_{s+1}(n)$ is undefined.

$(s+1.3)$ If $h_s(n)$ is defined, whereas $h_{s+1}(n)$ became undefined while taking substeps 1 and 2, then, for every such number n (in increasing order), we put

$$f(h_s(n), x, s') = e(a), \ g(h_s(n), x, s') = 0 \text{ for all } s' > s.$$

The construction proceeds as follows. First we construct β_n as some $\alpha_{n'}$ and pass to the construction of $\varnothing$ (or ω) if $\alpha_{n'}$ coincides with α_m, $m < n'$, or with some γ_y. By virtue of the correct parity property, no extra alterations will occur.

It is easy to see that the resulting numbering is positive, and the set of numbers of $\varnothing$ (or ω) in this numbering is computably enumerable, while all other sets occur only once. For our further reasoning, however, we need only Σ_a^{-1}-computability of a numbering β. $\square$

In [9], it was shown that for every finite level n of the Ershov hierarchy, each infinite computable family containing $\varnothing$ (for n even) or ω (for n odd) has infinitely many computable positive undecidable numberings, which are pairwise incomparable with respect to reducibility of numberings. (For $n = 1$, this was first proved in [10]). Subsequently, the result was generalized in [11] to all levels Σ_a^{-1} of the Ershov hierarchy, where a is a notation for any nonzero computable ordinal.

THEOREM 4.2 [9-11]. Let S be a Σ_a^{-1}-computable infinite family and $\varnothing \in S$, if $e(a) = 0$, or $\omega \in S$ if $e(a) = 1$. Then S has infinitely many Σ_a^{-1}-computable positive undecidable numberings, which are pairwise incomparable with respect to reducibility of numberings.

Note that in the numberings specified in Theorem 4.2, the set of numbers of $\varnothing$ (or ω) is computably enumerable, while all other sets occur only once.

Let $F = S \bigcup \{\varnothing\} \setminus T$ be an infinite Σ_a^{-1}-computable family such as in Proposition 4.1 (i.e., S has infinitely many elements that do not occur in T). According to Theorem 4.2, there exists a Σ_a^{-1}-computable positive undecidable numbering α of F. Let M be the set of numbers of $\varnothing$ (or ω).

Now we construct a Σ_a^{-1}-computable Friedberg numbering β for a family $F \bigcup T$ (construct $\langle f, g \rangle$, a Σ_a^{-1}-approximation to β), and also a computable function $h(s, n)$.

Let $\langle \phi, \psi \rangle$ be a Σ_a^{-1}-approximation with slowing down to α and $\langle \delta, \rho \rangle$ a Σ_a^{-1}-approximation to γ (according to Lemma 2.2). Let $\{M_s\}_{s\in\omega}$ be an increasing sequence of finite sets which, in the limit, gives M, $M_0 = \varnothing$, and $M_s \subseteq \bigcap\{0, 1, \ldots, s\}$.

Also consider an auxiliary numbering $\nu = \alpha \bigoplus \gamma$. Its Σ_a^{-1}-approximation $\langle \tau, \sigma \rangle$ is defined thus:

(1) $\tau(2n, x, s) = \phi(n, x, s)$ and $\sigma(2n, x, s) = \psi(n, x, s)$;

(2) $\tau(2n+1, x, s) = \delta(n+1, x, s)$, $\sigma(2n+1, x, s) = \rho(n+1, x, s)$, $\tau(1, x, s) = 0$, and $\sigma(1, x, s)$ is undefined.

Requirements for the construction.

(1) If $n \notin M$, then $\beta_{2n} = \alpha_n$.

(2) If $n \in M$, then $\beta_{2n} = \gamma_y$ for some y.

(3) For any n, $\beta_{2n+1} = \gamma_y$ with some y.

(4) For any n_1 and any n_2, $\beta_{n_1} \neq \beta_{n_2}$.

Construction for β.

Step $s = 0$. For any n and any x, assume $f(n, x, 0) = 0$ and $g(n, x, 0)$ is undefined. Let $y' = 0$, $k = 0$, $h(0, 2n) = 2n$, and $h(0, 2n+1) = 1$. Below if no mention is made of certain combinations (n, x, s), then we put $h(s+1, n) = h(s, n)$, $f(n, x, s+1) = f(n, x, s)$, and $g(n, x, s+1) = g(n, x, s)$ for these.

Step $s+1$. For all $n \leq s$, consider the following actions.

$(s+1.1)$ For all $n \in M_{s+1} \setminus M_s$ (in increasing order), put

$(s+1.1.1)$ $h(s+1, 2n) = 2y'' + 2$, where y'' is a natural number larger than y' and s;

$(s+1.1.2)$ for all $y' < y \leq y''$ (in increasing order), put $h(s+1, 2k+1) = 2y+1$ and $k := k+1$;

$(s+1.1.3)$ put $y' = y''$.

Pass to the next n in action $(s+1.1)$.

$(s+1.2)$ For all $n \notin M_{s+1}$, set $h(s+1, 2n) = h(s, 2n)$ and $h(s+1, 2n+1) = h(s, 2n+1)$.

$(s+1.s+1.3)$ For all n, set $f(n, x, s+1) = \tau(h(s+1, n), x, s+1)$ and $g(n, x, s+1) = \sigma(h(s+1, n), x, s+1)$.

The description of the construction is completed.

We verify whether the above requirements are fulfilled.

(1) If $n \notin M$, then the set α_n will always be constructed in β_{2n} at the expense of actions $(s+1.2)$ and $(s+1.3)$.

(2) If $n \in M$, then, starting with some s, the set γ_x is constructed in β_{2n} at the expense of action $(s+1.1)$. New errors will not appear due to the slowing down property.

(3) Is fulfilled since the set M is infinite and there is always at least one element between y' and $y''+1$.

(4) Is fulfilled in virtue of α being positive and γ being a Friedberg numbering.

We have thus constructed a Σ_a^{-1}-computable Friedberg numbering β for $F \bigcup T$ (more exactly, for the family $F \bigcup T \setminus \{\varnothing\}$, if $e(a) = 0$, and for the family $F \bigcup T \setminus \{\omega\}$ if $e(a) = 1$; however, either element can readily be added to the numbering constructed).

Let α^1 and α^2 be incomparable positive undecidable Σ_a^{-1}-computable numberings for F and β^1 and β^2 be their corresponding Σ_a^{-1}-computable numberings for $F \bigcup T$.

PROPOSITION 4.3. The numberings β^1 and β^2 are incomparable with respect to reducibility of numberings.

Proof. Assume $\beta^1 \leq \beta^2$. There exists a computable function f for which $\beta^1_n = \beta^2_{f(n)}$. Then $\alpha^1 \bigoplus \eta \leq \alpha^2 \bigoplus \eta$ via the same function f. Here $\eta_n = \varnothing$, if $e(a) = 0$, and $\eta_n = \omega$ if $e(a) = 1$ for any n. In view of $\eta \equiv \eta$, $\alpha^1 \leq \alpha^2$, a contradiction. □

THEOREM 4.4. The family of all Σ^{-1}_a-sets has infinitely many pairwise incomparable Σ^{-1}_a-computable Friedberg numberings for any notation of a nonzero ordinal $a \in \mathcal{O}$.

We consider yet another way of using the construction from Sec. 2, the idea behind which arose from the following:

THEOREM 4.5 [12]. Let $\mathcal{S}$ be a Σ^{-1}_1-computable family. Suppose that there exist two disjoint Σ^{-1}_1-computable subfamilies $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{S}$ such that $\mathcal{S}_1 \bigcup \mathcal{S}_2 = \mathcal{S}$, and

(1) $\mathcal{S}_2$ has a Σ^{-1}_1-computable Friedberg numbering;

(2) every finite subset of an element of $\mathcal{S}_1$ has infinitely many extensions to $\mathcal{S}_2$.

Then $\mathcal{S}$ possesses a Σ^{-1}_1-computable Friedberg numbering.

From the result cited above, we only use the partition of a family into two computable subfamilies, one of which has a Friedberg numbering. Our further reasoning seems interesting because there we attempt to avoid fixed complexity of families for obtaining a bigger class of Friedberg numberings.

PROPOSITION 4.6. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be infinite families of sets such that:

(1) $\mathcal{S}_1 \bigcap \mathcal{S}_2 = \varnothing$;

(2) $\mathcal{S}_1$ has a Σ^{-1}_a-computable numbering;

(3) $\mathcal{S}_2$ has a Σ^{-1}_b-computable Friedberg numbering, if $e(a) = 0$, and has a Π^{-1}_b-computable Friedberg numbering if $e(a) = 1$.

Then $\mathcal{S} = \mathcal{S}_1 \bigcup \mathcal{S}_2$ possesses a $\Sigma^{-1}_{b+_o a}$-computable Friedberg numbering.

Hereinafter, $+_\mathcal{O}$ is a partial computable function satisfying $|b +_\mathcal{O} a|_\mathcal{O} = |b|_\mathcal{O} + |a|_\mathcal{O}$ for $a, b \in \mathcal{O}$.

Proof. Again we appeal to the construction in Sec. 2. Let ν be a Σ^{-1}_a-computable numbering for the family $\mathcal{S}_1$. Put $\alpha = \nu \oplus \nu$, i.e., repeat the numbering ν twice in the numbering α. Initially, no restrictions on ν are imposed. Therefore, ν may have only finitely many repetitions, whereas every element in α occurs at least twice. In this way, we create an appropriate reserve of numbers onto which elements of S_2 will be put. As γ we use the corresponding Friedberg numbering for $\mathcal{S}_2$ by first changing its approximation $\langle \delta, \rho \rangle$ as follows:

for all $n, x \in \omega$ and for all s such that $\delta(n, x, 0) = \delta(n, x, 1) = \ldots = \delta(n, x, s) = 0$ (i.e., a first change has not yet occurred), put $\rho(n, x, s) = b$.

We describe changes to be entered into the construction.

Since $\mathcal{S}_1$ and $\mathcal{S}_2$ are disjoint, action $(s + 1.2)$ will not be needed, which makes it unnecessary to keep in α_0 the set ω (or $\varnothing$). At step 0, in addition, we define $y = 0$.

$(s + 1.3)$ If $h_s(n)$ is defined, while $h_{s+1}(n)$ has become undefined as a result of action $(s + 1.1)$, then, for every such number n (in increasing order), we put $f(h_s(n), x, s') = \delta(y, x, s')$ and

$g(h_s(n), x, s') = \rho(y, x, s')$ for all $s' > s$. Define $y = y + 1$. Pass to the next n. In this action, a position that has become vacant in the numbering β is filled by the next element of $\mathcal{S}_2$.

$(s + 1.6)$ If $h_{s+1}(n)$ is defined, then we put $f(h_{s+1}(n), x, s + 1) = \phi(n, x, s + 1)$ and $g(h_{s+1}(n), x, s + 1) = b +_{\mathcal{O}} \psi(n, x, s + 1)$ for all $x \in \omega$. In this action, we create an additional possibility for changes of elements in β, which might be needed in passing from the construction of an element of α to an element of γ.

We show that the resulting numbering is $\Sigma^{-1}_{b+_{\mathcal{O}}a}$-computable. First we construct a new element of the numbering (as a Σ^{-1}_a-set) while preserving the possibility for making another b changes. For this, action $(s + 1.6)$ is responsible. Next, if either $(s + 1.1)$ or $(s + 1.2)$ has been performed, then we start constructing Σ^{-1}_b- or Π^{-1}_b-sets, depending on $e(a)$.

If $e(a) = 0$, then, before performing action $(s+1.1)$, the minimum possible value of g is b, while the value of f is 0. Hence, after performing action $(s + 1.1)$, the construction of a Σ^{-1}_b-set starting with δ which takes on the value 0 will introduce no additional errors.

If $e(a) = 1$, then, for g minimal, the value of f is 1, while the construction of a Π^{-1}_b-set starts with δ which takes on the value 1. Additional errors will not appear. $\square$

Consider a family $\mathcal{S} = \{\{2x, 2x + 1\} \mid x \in A\} \bigcup \{\{2x\}, \{2x + 1\} \mid x \notin A\}$, where A is a computably enumerable uncomputable set. It is easy to see that this family is Σ^{-1}_1-computable and has no Σ^{-1}_1-computable Friedberg numberings (see, e.g., [13]). The family $\{\{2x\}, \{2x+1\} \mid x \notin A\}$ is Σ^{-1}_2-computable, while the family $\{\{2x, 2x + 1\} \mid x \in A\}$ has a Σ^{-1}_1-computable Friedberg numbering. Hence, by Proposition 4.6, $\mathcal{S}$ possesses a Σ^{-1}_3-computable Friedberg numbering.

REFERENCES

1. R. F. Friedberg, "Three theorems on recursive enumeration. I. Decomposition. II. Maximal set. III. Enumeration without duplication," *J. Symb. Log.*, **23**, No. 3, 309-316 (1958).
2. S. S. Goncharov, S. Lempp, and D. R. Solomon, "Friedberg numberings of families of n-computably enumerable sets," *Algebra and Logic*, **41**, No. 2, 81-86 (2002).
3. Yu. L. Ershov, "On a hierarchy of sets III," *Algebra and Logic*, **9**, No. 1, 20-31 (1970).
4. C. J. Ash and J. F. Knight, *Computable Structures and the Hyperarithmetical Hierarchy, Stud. Logic Found. Math.*, **144**, Elsevier, Amsterdam (2000).
5. S. S. Goncharov and A. Sorbi, "Generalized computable numerations and nontrivial Rogers semilattices," *Algebra and Logic*, **36**, No. 6, 359-369 (1997).
6. S. S. Ospichev, "Infinite family of Σ^{-1}_a-sets with only one computable numbering," *Vestnik NGU, Mat., Mekh., Inf.*, **11**, No. 2, 89-92 (2011).
7. S. S. Ospichev, "Some properties of numberings in various levels in Ershov's hierarchy," *Vestnik NGU, Mat., Mekh., Inf.*, **10**, No. 4, 125-132 (2010).

8. V. L. Selivanov, “Hierarchy of limiting computations,” *Sib. Math. J.*, **25**, No. 5, 798-806 (1984).

9. Zh. T. Talasbaeva, “Positive numberings of families of sets in the Ershov hierarchy,” *Algebra and Logic*, **42**, No. 6, 413-418 (2003).

10. S. A. Badaev, “Positive enumerations,” *Sib. Math. J.*, **18**, No. 3, 343-352 (1977).

11. M. Manat and A. Sorbi, “Positive undecidable numberings in the Ershov hierarchy,” *Algebra and Logic*, **50**, No. 6, 512-525 (2011).

12. M. Kummer, “Numberings of $\mathcal{R}_1 \cup \mathcal{F}$,” in *Lect. Notes Comput. Sci.*, **385**, Springer-Verlag, Berlin (1989), pp. 166-186.

13. M. B. Pour-El and W. A. Howard, “A structural criterion for recursive numeration without repetition,” *Z. Math. Logik Grundlagen Math.*, **10**, No. 2, 105-114 (1964).