DOI 10.1007/s10469-015-9344-7
Algebra and Logic, Vol. 54, No. 3, July, 2015
(Russian Original Vol. 54, No. 3, May-June, 2015)

COMPLEXITY OF QUASIVARIETY LATTICES

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Keywords: *computable set, lattice, quasivariety, Q-universality, undecidable problem, universal class, variety.*

If a quasivariety $\mathbf{A}$ *of algebraic systems of finite signature satisfies some generalization of a sufficient condition for Q-universality treated by M. E. Adams and W. A. Dziobiak, then, for any at most countable set* $\{\mathcal{S}_i \mid i \in I\}$ *of finite semilattices, the lattice* $\prod\limits_{i\in I} \mathrm{Sub}(\mathcal{S}_i)$ *is a homomorphic image of some sublattice of a quasivariety lattice* $\mathrm{Lq}(\mathbf{A})$*. Specifically, there exists a subclass* $\mathbf{K} \subseteq \mathbf{A}$ *such that the problem of embedding a finite lattice in a lattice* $\mathrm{Lq}(\mathbf{K})$ *of* $\mathbf{K}$*-quasivarieties is undecidable. This, in particular, implies a recent result of A. M. Nurakunov.*

1. AUXILIARY DEFINITIONS AND RESULTS

In the present paper, we study complexity of the structure of lattices for (relative) quasivarieties of algebraic systems. For an arbitrary fixed signature σ, $\mathbf{K}(\sigma)$ denotes the class of all σ-structures. A. M. Nurakunov proved the following:

THEOREM 1.1 [1, Thm. 1]. Let σ be a signature with at least one at least unary operation. Then there exists a quasivariety $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ such that a set of all (isomorphism types) of finite sublattices of a quasivariety lattice $\mathrm{Lq}\,(\mathbf{K})$ is uncomputable.

In [2], a similar result was proved for varieties of Abelian groups with a single constant.

Theorem 1.1 implies that there is no algorithm which, given a finite lattice, determines whether or not that lattice is embeddable in the lattice of quasivarieties under consideration. The existence of quasivarieties with this property reflects the fact that quasivariety lattices may be extremely

*The work is supported by Russian Science Foundation (project 14-21-00065).

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complex. Another measure of complexity is expressed in terms of the so-called concept of Q-*universality*, which was introduced by M. V. Sapir [3] and has been explored by many other authors. A lot of works dealt with the study of complexity of the structure of quasivariety lattices. Among these, it is worth mentioning [4-20]. Based on ideas in [1], we established a link between the two properties above. Namely, the following theorem was proved.

THEOREM 1.2 [21, Thm. 6.1.3]. A variety $\mathbf{K}(\sigma)$ of all σ-structures is Q-universal if and only if there exists a subclass $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ such that a set of all (isomorphism types) of finite sublattices of a quasivariety lattice $\mathrm{Lq}\,(\mathbf{K})$ is uncomputable.

It is well known that a class $\mathbf{K}(\sigma)$ is Q-universal iff σ contains either at least a binary predicate symbol or at least a unary function symbol, or σ is at least countable. This fact was mentioned in [22, Prop. 4.4] as a consequence of the results derived in [23] (see also [13]). With due regard for Theorems 1.1 and 1.2, combined with the fact that the variety of all unars, as well as the variety of all Abelian groups with one constant, is Q-universal, the following question was posed in [21]: Is it true that every Q-universal quasivariety $\mathbf{M}$ contains a subclass $\mathbf{K} \subseteq \mathbf{M}$ for which the set of all finite sublattices of $\mathrm{Lq}\,(\mathbf{K})$ is uncomputable? (See Problem 5.1 below).

A positive answer to this question will be given in Theorem 4.2 for almost all known Q-universal quasivarieties (see also discussion in Sec. 5). Namely, Theorem 4.2 states the following: if a quasivariety $\mathbf{M}$ satisfies some generalization of a known condition that is sufficient for being Q-universal (see [4, 22]), then the conclusion of Theorem 1.2 is also satisfied. The proof is underpinned by a structural result, according to which the fact that a quasivariety $\mathbf{M}$ satisfies the above sufficient condition for Q-universality implies that for every at most countable family $\{\mathcal{S}_i \mid i \in I\}$ of finite lower semilattices, there exists a subclass $\mathbf{K} \subseteq \mathbf{M}$ such that $\mathrm{Lq}\,(\mathbf{K}) \cong \prod_{i \in I} \mathrm{Sub}\,(\mathcal{S}_i)$. This result is the content of Theorem 3.4.

G. Birkhoff [24] and A. I. Mal'tsev [25] independently posed the question as to which lattices are isomorphic to quasivariety lattices. Nowadays, this question is often referred to as the Birkhoff–Mal'tsev problem (see [13, Chap. 5]). In view of Theorems 1.1, 1.2, and 4.2-4.4. the problem mentioned may turn out to be more complicated (if not altogether hopeless) than expected: finding a description of lattices isomorphic to quasivariety lattices, even for particular classes of algebraic systems, might well be extremely difficult.

The present paper is structured as follows. All basic definitions are couched in Sec. 2. For the concepts undefined there, we ask the reader to consult [13, 24, 25]. In Sec. 3, we present the main construction and prove auxiliary results, of which some are interesting in their own right (see, e.g., Thm. 3.4). In Sec. 4, basic results, Theorems 4.2-4.4, are proved. Finally, In Sec. 5, we discuss applications of the obtained results and some related issues.

We assume that all classes are *abstract*, i.e., are closed under isomorphisms. For instance, writing $\{\mathcal{A}_i \mid i \in I\}$ for some set I will always signify the set of isomorphism classes of systems in the set $\{\mathcal{A}_i \mid i \in I\}$.

2. BASIC DEFINITIONS

2.1. Semilattices. For a lower semilattice $\mathcal{S} = \langle S; \wedge \rangle$, let $\mathrm{Sub}\,(\mathcal{S})$ denote the lattice of all lower subsemilattices of $\mathcal{S}$. It is not hard to see that for any two subsemilattices $S_0, S_1 \in \mathrm{Sub}\,(\mathcal{S})$, the set $S_0 + S_1 = \{s_0 \wedge s_1 \mid s_0 \in S_0,\ s_1 \in S_1\}$ is a least lower subsemilattice of $\mathcal{S}$ containing $S_0 \cup S_1$, i.e., a lattice join of S_0 and S_1 in $\mathrm{Sub}\,(\mathcal{S})$.

2.2. Classes. Following [13], for a class $\mathbf{K} \subseteq \mathbf{K}(\sigma)$, by $\mathbf{Q}(\mathbf{K})$ we denote a least quasivariety containing $\mathbf{K}$. We also use the following notation: $\mathbf{H}(\mathbf{K})$ is the class of all systems in $\mathbf{K}(\sigma)$ that are homomorphic images of systems in $\mathbf{K}$; $\mathbf{P}(\mathbf{K})$ is the class of all systems in $\mathbf{K}(\sigma)$ that are isomorphic to Cartesian products of systems in $\mathbf{K}$; $\mathbf{P}_s(\mathbf{K})$ is the class of all systems in $\mathbf{K}(\sigma)$ that are isomorphic to subdirect products of systems in $\mathbf{K}$; $\mathbf{L}_s(\mathbf{K})$ is the class of all systems in $\mathbf{K}(\sigma)$ that are isomorphic to superdirect limits of systems in $\mathbf{K}$; $\mathbf{S}(\mathbf{K})$ is the class of all systems in $\mathbf{K}(\sigma)$ that are isomorphic to subsystems of systems in $\mathbf{K}$.

A system $\mathcal{A} \in \mathbf{K} \subseteq \mathbf{K}(\sigma)$ is said to be *l-projective* in $\mathbf{K}$ if, for any superdirect spectrum $\Lambda = \langle I, \mathcal{A}_i, \varphi_{ij} \rangle$ with $\varinjlim \Lambda \in \mathbf{K}$ in $\mathbf{K}$ and for every homomorphism $\varphi\colon \mathcal{A} \to \varinjlim \Lambda$, there exist an $i \in I$ and a homomorphism $\psi\colon \mathcal{A} \to \mathcal{A}_i$ such that $\varphi_{i\infty}\psi = \varphi$. In particular, if φ is an embedding then so is ψ. Thus if $\mathcal{A} \in \mathbf{K}$ is l-projective in $\mathbf{K}$ and $\mathcal{A}$ is embeddable in a system belonging to the class $\mathbf{L}_s(\mathbf{K})$, then $\mathcal{A} \in \mathbf{S}(\mathbf{K})$. Obviously, any finite system of a finite signature is l-projective in every quasivariety to which it belongs.

According to a result in [26],

$$\mathbf{Q}(\mathbf{K}) = \mathbf{L}_s\mathbf{SP}(\mathbf{K}) = \mathbf{L}_s\mathbf{P}_s(\mathbf{K})$$

(see also [13, Thm. 2.3.6]). Let $\mathbf{K}' \subseteq \mathbf{K} \subseteq \mathbf{K}(\sigma)$. We refer to $\mathbf{K}'$ as a $\mathbf{K}$-*quasiequational* class if $\mathbf{K}' = \mathbf{K} \cap \mathrm{Mod}\,(\Sigma)$ for some set Σ of quasi-identities of the signature σ. It is straightforward to verify the following:

LEMMA 2.1. A class $\mathbf{K}'$ is $\mathbf{K}$-quasiequational iff $\mathbf{K}' = \mathbf{K} \cap \mathbf{Q}(\mathbf{K}')$.

Let $\mathrm{Lq}\,(\mathbf{K})$ be the class of all $\mathbf{K}$-quasiequational subclasses in $\mathbf{K}$. If $\mathrm{Lq}\,(\mathbf{K})$ is ordered with respect to inclusion, then it forms a complete lattice, called the *lattice of* $\mathbf{K}$-*quasivarieties* (the *lattice of quasivarieties* for $\mathbf{K}$, or simply the *relative quasivariety lattice*).

According to [3], a quasivariety $\mathbf{K}$ of a finite signature is *Q-universal* if $\mathrm{Lq}\,(\mathbf{K}') \in \mathbf{HS}\big(\mathrm{Lq}\,(\mathbf{K})\big)$ for any quasivariety $\mathbf{K}'$ of a finite signature.

Let P be the set of all prime numbers. For a finite nonempty set $F = \{p_1, \ldots, p_n\} \subseteq P$ of primes, we use $[F]$, $[p \mid p \in F]$, or $[p_1, \ldots, p_n]$ to denote the least common multiple of numbers in F.

3. MAIN CONSTRUCTION

Let $\mathbf{A} = \{\mathcal{A}_X \mid X \in \mathcal{P}_{\mathrm{fin}}(\omega)\}$ be a class of systems. Consider the following properties:

(P_0) for any $X \in \mathcal{P}_{\text{fin}}(\omega)$, the system $\mathcal{A}_X$ is l-projective in $\mathbf{Q}(\mathbf{A})$, while the trivial congruence is a cocompact element in the lattice of relative congruences $\text{Con}_{\mathbf{Q}(\mathbf{A})}\mathcal{A}_X$;

(P_1) $\mathcal{A}_\varnothing$ is a trivial system;

(P_2) if $X = Y \cup Z$ in $\mathcal{P}_{\text{fin}}(\omega)$, then $\mathcal{A}_X \in \mathbf{Q}(\mathcal{A}_Y, \mathcal{A}_Z)$;

(P_3) if $\varnothing \neq X \in \mathcal{P}_{\text{fin}}(\omega)$ and $\mathcal{A}_X \in \mathbf{Q}(\mathcal{A}_Y)$, then $X = Y$;

(P_4) if $\mathcal{A}_X \leq \mathcal{B}_0 \times \mathcal{B}_1$ for some systems $\mathcal{B}_0, \mathcal{B}_1 \in \mathbf{Q}(\mathbf{A})$, then there exist $Y_0, Y_1 \in \mathcal{P}_{\text{fin}}(\omega)$ such that $\mathcal{A}_{Y_0} \in \mathbf{Q}(\mathcal{B}_0)$, $\mathcal{A}_{Y_1} \in \mathbf{Q}(\mathcal{B}_1)$, and $X = Y_0 \cup Y_1$.

It is not hard to see that if the class $\mathbf{A}$ consists of finite systems, then (P_4) is equivalent to the following condition:

(P'_4) if $\mathcal{A}_X \leq \mathcal{B}_0 \times \mathcal{B}_1$ for some finite systems $\mathcal{B}_0, \mathcal{B}_1 \in \mathbf{Q}(\mathbf{A})$, then there exist $Y_0, Y_1 \in \mathcal{P}_{\text{fin}}(\omega)$ such that $\mathcal{A}_{Y_0} \in \mathbf{Q}(\mathcal{B}_0)$, $\mathcal{A}_{Y_1} \in \mathbf{Q}(\mathcal{B}_1)$, and $X = Y_0 \cup Y_1$.

Properties (P_1)-(P_3) and (P'_4) of algebraic systems were introduced in [27]. There, also, it was shown that if a quasivariety $\mathbf{K}$ of a finite signature contains a class $\mathbf{A} = \{\mathcal{A}_X \mid X \in \mathcal{P}_{\text{fin}}(\omega)\}$ of finite systems that possesses properties (P_1)-(P_3) and (P'_4), then the lattice $\text{Lq}\,(\mathbf{K})$ does not satisfy any nontrivial lattice identity. A similar set of conditions was found in [23] (see also [13, Thm. 5.4.26]). In [4], properties (P_1)-(P_3) and (P'_4) were considered and the following was proved:

THEOREM 3.1 [4, Thm. 3.3]. If a quasivariety $\mathbf{K}$ of a finite signature contains a class $\mathbf{A} = \{\mathcal{A}_X \mid X \in \mathcal{P}_{\text{fin}}(\omega)\}$ of finite systems that possesses properties (P_1)-(P_3) and (P'_4), then $\mathbf{K}$ is Q-universal. Moreover, the ideal lattice of a free lattice of countable rank is embeddable in the lattice $\text{Lq}\,(\mathbf{K})$.

LEMMA 3.2. Suppose that a quasivariety $\mathbf{K}$ of a finite signature contains a class $\mathbf{A} = \{\mathcal{A}_X \mid X \in \mathcal{P}_{\text{fin}}(\omega)\}$ of systems that possesses properties (P_0)-(P_4), and $X, X_0, \ldots, X_n \in \mathcal{P}_{\text{fin}}(\omega)$. The inclusion $\mathcal{A}_X \in \mathbf{SP}(\mathcal{A}_{X_0}, \ldots, \mathcal{A}_{X_n})$ holds if and only if $X = X_0 \cup \ldots \cup X_n$.

Proof. If $X = X_0 \cup \ldots \cup X_n$, then applying property (P_2) n times yields

$$\mathcal{A}_X \in \mathbf{Q}(\mathcal{A}_{X_0}, \ldots, \mathcal{A}_{X_n}) = \mathbf{L}_s\mathbf{P}_s(\mathcal{A}_{X_0}, \ldots, \mathcal{A}_{X_n}) \subseteq \mathbf{Q}(\mathbf{A}).$$

The system $\mathcal{A}_X$ is l-projective in $\mathbf{Q}(\mathbf{A})$, and so

$$\mathcal{A}_X \in \mathbf{SP}(\mathcal{A}_{X_0}, \ldots, \mathcal{A}_{X_n}).$$

Now assume that $\mathcal{A}_X \in \mathbf{SP}(\mathcal{A}_{X_0}, \ldots, \mathcal{A}_{X_n})$. Then $\mathcal{A}_X \leq \mathcal{B}_0 \times \mathcal{B}_1$, where $\mathcal{B}_0 \in \mathbf{SP}(\mathcal{A}_{X_0}, \ldots, \mathcal{A}_{X_{n-1}})$ and $\mathcal{B}_1 = \mathcal{A}_{X_n}^I$ for some set I. Consequently, $\mathcal{B}_0, \mathcal{B}_1 \in \mathbf{Q}(\mathbf{A})$. According to ($P_4$), there are sets $Y_0, Y_1 \in \mathcal{P}_{\text{fin}}(\omega)$ for which $X = Y_0 \cup Y_1$, $\mathcal{A}_{Y_0} \in \mathbf{Q}(\mathcal{B}_0)$, and $\mathcal{A}_{Y_1} \in \mathbf{Q}(\mathcal{B}_1) \subseteq \mathbf{Q}(\mathcal{A}_{X_n})$. The last inclusion implies $Y_1 = X_n$ in view of (P_3), whence $X = Y_0 \cup Y_1 = Y_0 \cup X_n$. Moreover, $\mathcal{A}_{Y_0} \in \mathbf{Q}(\mathcal{B}_0) \subseteq \mathbf{Q}(\mathcal{A}_{X_0}, \ldots, \mathcal{A}_{X_{n-1}}) \subseteq \mathbf{Q}(\mathbf{A})$. Since $\mathcal{A}_{Y_0}$ is l-projective in $\mathbf{Q}(\mathbf{A})$, it is l-projective in $\mathbf{Q}(\mathcal{A}_{X_0}, \ldots, \mathcal{A}_{X_{n-1}})$, and hence $\mathcal{A}_{Y_0} \in \mathbf{SP}(\mathcal{A}_{X_0}, \ldots, \mathcal{A}_{X_{n-1}})$. Now we apply the same argument using property P_4 to the last inclusion. The result follows by induction. □

THEOREM 3.3. Let $\mathcal{S} = \langle S; \wedge \rangle$ be a finite lower semilattice and $\mathbf{M}$ be a quasivariety of a finite signature containing a class $\mathbf{A} = \{\mathcal{A}_X \mid X \in \mathcal{P}_{\text{fin}}(\omega)\}$ of systems possessing properties (P_0)-(P_4). Then there exists a class $\mathbf{K} \subseteq \mathbf{M}$ for which $\text{Lq}\,(\mathbf{K}) \cong \text{Sub}\,(\mathcal{S})$.

Proof. Suppose that $a \mapsto p_a$ realizes an injective mapping of the set S into the set P. For each $a \in S$, put $X(a) = \{p_b \in P \mid b \not\leq a\}$ and

$$\mathbf{K} = \{\mathcal{A}_{X(a)} \mid a \in S\}.$$

Define a mapping of the form

$$\varphi\colon \mathrm{Lq}\,(\mathbf{K}) \to \mathrm{Sub}\,(\mathcal{S}) \text{ by the rule } \mathbf{K}' \mapsto \{a \in S \mid \mathcal{A}_{X(a)} \in \mathbf{K}'\}.$$

Claim 3.3.1. The mapping φ is well defined.

Proof. We need to show that $\varphi(\mathbf{K}') \in \mathrm{Sub}\,(\mathcal{S})$ for any $\mathbf{K}' \in \mathrm{Lq}\,(\mathbf{K})$. Let $a, b \in \varphi(\mathbf{K}')$ and $c = a \wedge b$. Hence $\mathcal{A}_{X(a)}, \mathcal{A}_{X(b)} \in \mathbf{K}'$. By virtue of $X(c) = X(a) \cup X(b)$ and Lemma 8, we have $\mathcal{A}_{X(c)} \leq \mathbf{SP}(\mathcal{A}_{X(a)}, \mathcal{A}_{X(b)})$, whence $\mathcal{A}_{X(c)} \in \mathbf{Q}(\mathbf{K}') \cap \mathbf{K} = \mathbf{K}'$ and $c \in \varphi(\mathbf{K}')$. □

The mapping φ is obviously one-to-one and is meet-preserving.

Claim 3.3.2. The mapping φ is join-preserving.

Proof. It suffices to show that $\varphi(\mathbf{K}_0 \vee \mathbf{K}_1) \subseteq \varphi(\mathbf{K}_0) + \varphi(\mathbf{K}_1)$ for any $\mathbf{K}_0, \mathbf{K}_1 \in \mathrm{Lq}\,(\mathbf{K})$. Let $a \in \varphi(\mathbf{K}_0 \vee \mathbf{K}_1)$. Hence $\mathcal{A}_{X(a)} \in \mathbf{K}_0 \vee \mathbf{K}_1 = \mathbf{Q}(\mathbf{K}_0 \cup \mathbf{K}_1) \cap \mathbf{K}$. Thus there are $\mathcal{B}_0 \in \mathbf{K}_0$ and $\mathcal{B}_1 \in \mathbf{K}_1$ with $\mathcal{A}_{X(a)} \leq \mathcal{B}_0 \times \mathcal{B}_1$. In view of ($\mathrm{P}_4$), there exist $Y_0, Y_1 \in \mathcal{P}_{\mathrm{fin}}(\omega)$ such that $X(a) = Y_0 \cup Y_1$, $\mathcal{A}_{Y_0} \in \mathbf{Q}(\mathcal{B}_0) \subseteq \mathbf{Q}(\mathbf{K}_0) \subseteq \mathbf{Q}(\mathbf{A})$, and $\mathcal{A}_{Y_1} \in \mathbf{Q}(\mathcal{B}_1) \subseteq \mathbf{Q}(\mathbf{K}_1) \subseteq \mathbf{Q}(\mathbf{A})$.

Both of the systems $\mathcal{A}_{Y_0}$ and $\mathcal{A}_{Y_1}$ are l-projective in $\mathbf{Q}(\mathbf{A})$, so they are l-projective in $\mathbf{Q}(\mathbf{K}_0)$ and $\mathbf{Q}(\mathbf{K}_1)$, respectively. Consequently, $\mathcal{A}_{Y_0} \in \mathbf{SP}(\mathbf{K}_0)$ and $\mathcal{A}_{Y_1} \in \mathbf{SP}(\mathbf{K}_1)$. In other words, there are elements $a_0, \dots, a_k, b_0, \dots, b_n \in S$ for which $\mathcal{A}_{X(a_0)}, \dots, \mathcal{A}_{X(a_k)} \in \mathbf{K}_0$, $\mathcal{A}_{X(b_0)}, \dots, \mathcal{A}_{X(b_n)} \in \mathbf{K}_1$ and $\mathcal{A}_{Y_0} \in \mathbf{SP}(\mathcal{A}_{X(a_0)}, \dots, \mathcal{A}_{X(a_k)})$, $\mathcal{A}_{Y_1} \in \mathbf{SP}(\mathcal{A}_{X(b_0)}, \dots, \mathcal{A}_{X(b_n)})$. In view of Lemma 3.2,

$$\mathcal{A}_{X(a)} \in \mathbf{SP}(\mathcal{A}_{Y_0}, \mathcal{A}_{Y_1}) \subseteq \mathbf{SP}(\mathcal{A}_{X(a_0)}, \dots, \mathcal{A}_{X(a_k)}, \mathcal{A}_{X(b_0)}, \dots, \mathcal{A}_{X(b_n)}).$$

If again we apply Lemma 3.2 we obtain

$$X(a) = X(a_0) \cup \ldots \cup X(a_k) \cup X(b_0) \cup \ldots \cup X(b_n) = X(c_0) \cup X(c_1) = X(c_0 \wedge c_1),$$

where $c_0 = a_0 \wedge \ldots \wedge a_k \in \varphi(\mathbf{K}_0)$ and $c_1 = b_0 \wedge \ldots \wedge b_n \in \varphi(\mathbf{K}_1)$. This implies $a = c_0 \wedge c_1 \in \varphi(\mathbf{K}_0) + \varphi(\mathbf{K}_1)$. □

Claim 3.3.3. The mapping φ is surjective.

Proof. Let $S' \in \mathrm{Sub}\,(\mathcal{S})$ and $\mathbf{K}' = \{\mathcal{A}_{X(a)} \mid a \in S'\}$. Clearly, $\varphi(\mathbf{K}') = S'$. In order to prove that $\mathbf{K}' \in \mathrm{Lq}\,(\mathbf{K})$, it suffices to verify that $\mathbf{Q}(\mathbf{K}') \cap \mathbf{K} \subseteq \mathbf{K}'$. Indeed, suppose that $a \in S$ and $\mathcal{A}_{X(a)} \in \mathbf{Q}(\mathbf{K}') = \mathbf{L}_s\mathbf{P}_s(\mathbf{K}')$. The system $\mathcal{A}_{X(a)}$ is l-projective in $\mathbf{Q}(\mathbf{A})$, so it is l-projective in $\mathbf{Q}(\mathbf{K}')$. Consequently, $\mathcal{A}_{X(a)} \in \mathbf{SP}(\mathbf{K}')$. In other words, there are elements $a_0, \dots, a_k \in S'$ such that $\mathcal{A}_{X(a_0)}, \dots, \mathcal{A}_{X(a_k)} \in \mathbf{K}'$ and $\mathcal{A}_{X(a)} \in \mathbf{SP}(\mathcal{A}_{X(a_0)}, \dots, \mathcal{A}_{X(a_k)})$. By virtue of Lemma 3.2, we derive $X(a) = X(a_0) \cup \ldots \cup X(a_k) = X(a_0 \wedge \ldots \wedge a_k)$. Then $a = a_0 \wedge \ldots \wedge a_k \in S'$ and $\mathcal{A}_{X(a)} \in \mathbf{K}'$. □

The theorem now follows from Claims 3.3.1-3.3.3. □

THEOREM 3.4. Let $\mathbf{M}$ be a quasivariety of a finite signature containing a subclass $\mathbf{A} = \{\mathcal{A}_X \mid X \in \mathcal{P}_{\text{fin}}(\omega)\}$ of systems possessing properties (P_0)-(P_4), suppose $I \subseteq \omega$, and assume that $\mathcal{S}_i$ is an arbitrary finite nontrivial lower semilattice for any $i \in I$. Then there exists a subclass $\mathbf{K} \subseteq \mathbf{M}$ such that $\mathrm{Lq}\,(\mathbf{K})$ is a complete homomorphic image of some sublattice in $\mathrm{Lq}\,(\mathbf{M})$ and $\mathrm{Lq}\,(\mathbf{K}) \cong \prod_{i\in I} \mathrm{Sub}\,(\mathcal{S}_i)$.

Proof. Consider a partition $\bigcup_{i\in I} P_i \subseteq P$ of a subset of the set P of prime numbers such that $|P_i| = |S_i|$ for any $i \in I$. By virtue of Theorem 3.3 applied to a semilattice $\mathcal{S}_i$, $i \in I$, and a set P_i of primes, there is a subclass $\mathbf{K}_i \subseteq \mathbf{M}$ such that $\mathrm{Lq}\,(\mathbf{K}_i) \cong \mathrm{Sub}\,(\mathcal{S}_i)$. In addition, the proof of Theorem 3.3 and condition (P_3) imply that the class $\mathbf{K}_i \cap \mathbf{K}_j$ contains just a trivial system for $i \neq j$. Put

$$\mathbf{K} = \bigcup_{i\in I} \mathbf{K}_i \quad \text{and} \quad \mathbf{R} = \mathbf{Q}(\mathbf{K}).$$

Claim 3.4.1. A mapping of the form

$$\varphi\colon \mathrm{Lq}\,(\mathbf{R}) \to \mathrm{Lq}\,(\mathbf{K}) \text{ acting by the rule } \mathbf{R}' \mapsto \mathbf{R}' \cap \mathbf{K}$$

is a complete lattice homomorphism onto.

Proof. It is not hard to see that the mapping φ is well defined and is meet-preserving. In order to prove that φ preserves arbitrary joins, we assume that $\mathbf{R}_n \in \mathrm{Lq}\,(\mathbf{R})$ for all $n \in N$. Let $\mathbf{X} = \bigcup_{n\in N} \mathbf{R}_n$. It suffices to verify that $\mathbf{Q}(\mathbf{X}) \cap \mathbf{K} \subseteq \mathbf{Q}(\mathbf{X} \cap \mathbf{K})$. Indeed, let $\mathcal{A} \in \mathbf{Q}(\mathbf{X}) \cap \mathbf{K}$. Since $\mathcal{A} \in \mathbf{K}$, there are an index $i \in I$ and an element $a \in S_i$ such that

$$\mathcal{A} = \mathcal{A}_{X(a)} \in \mathbf{Q}(\mathbf{X}) \cap \mathbf{K}_i \subseteq \mathbf{Q}(\mathbf{X}) = \mathbf{L}_s\mathbf{P}_s(\mathbf{X}) \subseteq \mathbf{Q}(\mathbf{A}).$$

The system $\mathcal{A}_{X(a)}$ is l-projective in $\mathbf{Q}(\mathbf{A})$, so it is l-projective in $\mathbf{Q}(\mathbf{X})$. Thus $\mathcal{A}_{X(a)} \in \mathbf{SP}(\mathbf{X})$. In view of property (P_0), there exist systems $\mathcal{B}_0, \ldots, \mathcal{B}_k \in \mathbf{X} \subseteq \mathbf{Q}(\mathbf{A})$ such that $\mathcal{A}_{X(a)} \leq \mathcal{B}_0 \times \ldots \times \mathcal{B}_k$. By virtue of (P_4), there are sets $Y_0, Y_1 \in \mathcal{P}_{\text{fin}}(\omega)$ for which $X(a) = Y_0 \cup Y_1$, $\mathcal{A}_{Y_1} \in \mathbf{Q}(\mathcal{B}_0 \times \ldots \times \mathcal{B}_{k-1})$, and $\mathcal{A}_{Y_0} \in \mathbf{Q}(\mathcal{B}_k) \subseteq \mathbf{X}$. Since $\mathcal{A}_{Y_1}$ is l-projective in $\mathbf{Q}(\mathbf{A})$, it is l-projective in $\mathbf{Q}(\mathcal{B}_0 \times \ldots \times \mathcal{B}_{k-1})$. Thus $\mathcal{A}_{Y_1} \in \mathbf{SP}(\mathcal{B}_0 \times \ldots \times \mathcal{B}_{k-1})$.

The system $\mathcal{A}_{Y_1}$ is l-projective in $\mathbf{Q}(\mathcal{B}_0 \times \ldots \times \mathcal{B}_{k-1})$, so $\mathcal{A}_{Y_1} \leq \mathcal{B}_0^J \times \ldots \times \mathcal{B}_{k-1}^J$ for some set J. By (P_4) again, there exist sets $Y_2, Y_3 \in \mathcal{P}_{\text{fin}}(\omega)$ such that $Y_1 = Y_2 \cup Y_3$, $\mathcal{A}_{Y_3} \in \mathbf{Q}(\mathcal{B}_0^J \times \ldots \times \mathcal{B}_{k-2}^J) \subseteq \mathbf{Q}(\mathcal{B}_0 \times \ldots \times \mathcal{B}_{k-2})$, and $\mathcal{A}_{Y_2} \in \mathbf{Q}(\mathcal{B}_{k-1}) \subseteq \mathbf{X}$. Therefore, $X(a) = Y_0 \cup Y_2 \cup Y_3$. If we proceed by induction we see that there exist sets $Z_0, \ldots, Z_k \in \mathcal{P}_{\text{fin}}(\omega)$ for which $X(a) = Z_0 \cup \ldots \cup Z_k$ and $\mathcal{A}_{Z_j} \in \mathbf{Q}(\mathcal{B}_j) \subseteq \mathbf{X}$ with any $j \leqslant k$. By Lemma 3.2, $\mathcal{A}_{X(a)} \in \mathbf{SP}(\mathcal{A}_{Z_0}, \ldots, \mathcal{A}_{Z_k})$.

We have $\mathcal{A}_{Z_j} \in \mathbf{X} \subseteq \mathbf{Q}(\mathbf{K}) = \mathbf{L}_s\mathbf{P}_s(\mathbf{K})$ for any $j \leqslant k$. Since $\mathcal{A}_{Z_j}$ is l-projective in $\mathbf{Q}(\mathbf{A})$, it is l-projective in $\mathbf{Q}(\mathbf{K})$. Thus $\mathcal{A}_{Z_j} \in \mathbf{SP}(\mathbf{K})$. By virtue of (P_0), there are a finite set $\{a_{j,m} \mid m \in M(j)\} \subseteq \bigcup_{i\in I} S_i$ and systems $\mathcal{A}_{X(a_{j,m})}$, $m \in M(j)$, such that $\mathcal{A}_{Z_j} \leq \prod_{m\in M(j)} \mathcal{A}_{X(a_{j,m})}$. In view of

Lemma 3.2, therefore, $Z_j = \bigcup_{m\in M(j)} X(a_{j,m})$. Hence

$$X(a) = \bigcup_{m\in M(0)} X(a_{0,m}) \cup \ldots \cup \bigcup_{m\in M(k)} X(a_{k,m}).$$

We have $a \in S_i$, so $\bigcup_{j\leqslant k}\{a_{j,m} \mid m \in M(j)\} \subseteq S_i$. Thus, for any $j \leqslant k$, it is true that $Z_j = \bigcup_{m\in M(j)} X(a_{j,m}) = X(c_j)$, where $c_j = \bigwedge_{m\in M(j)} a_{j,m} \in S_i$, since the last set is a lower semilattice. Hence $\mathcal{A}_{Z_j} = \mathcal{A}_{X(c_j)} \in \mathbf{X}\cap\mathbf{K}_i$ and $\mathcal{A}_{X(a)} \in \mathbf{SP}(\mathcal{A}_{Z_0},\ldots,\mathcal{A}_{Z_k}) \subseteq \mathbf{Q}(\mathbf{X}\cap\mathbf{K})$, as desired.

Finally, the mapping φ is surjective by virtue of the fact that $\varphi\big(\mathbf{Q}(\mathbf{X})\big) = \mathbf{Q}(\mathbf{X})\cap\mathbf{K} = \mathbf{X}$ holds for any $\mathbf{X} \in \mathrm{Lq}\,(\mathbf{K})$. □

Claim 3.4.2. For every $i \in I$, a mapping of the form

$$\varphi_i\colon \mathrm{Lq}\,(\mathbf{K}) \to \mathrm{Lq}\,(\mathbf{K}_i) \text{ acting by the rule } \mathbf{K}' \mapsto \mathbf{K}'\cap\mathbf{K}_i$$

is a complete lattice homomorphism onto.

Proof. Obviously, the mapping φ_i is well defined and preserves arbitrary meets. In order to prove that φ_i preserves arbitrary joins, we assume that $\mathbf{X}_n \in \mathrm{Lq}\,(\mathbf{K})$ for all $n \in N$. Put $\mathbf{X} = \bigcup_{n\in N} \mathbf{X}_n$. It suffices to show that $\mathbf{Q}(\mathbf{X})\cap\mathbf{K}_i \subseteq \mathbf{Q}(\mathbf{X}\cap\mathbf{K}_i)$. Indeed, let an element $a \in S_i$ be such that

$$\mathcal{A}_{X(a)} \in \mathbf{Q}(\mathbf{X})\cap\mathbf{K}_i \subseteq \mathbf{Q}(\mathbf{X}) = \mathbf{L}_s\mathbf{P}_s(\mathbf{X}) \subseteq \mathbf{Q}(\mathbf{A}).$$

The system $\mathcal{A}_{X(a)}$ is l-projective in $\mathbf{Q}(\mathbf{A})$, so it is l-projective in $\mathbf{Q}(\mathbf{X})$. Thus $\mathcal{A}_{X(a)} \in \mathbf{SP}(\mathbf{X})$. In view of (P_0), there are systems $\mathcal{B}_0,\ldots,\mathcal{B}_k \in \mathbf{X}$ for which $\mathcal{A}_{X(a)} \leq \mathcal{B}_0\times\ldots\times\mathcal{B}_k$. Since $\mathbf{X}\subseteq\mathbf{K}$, we conclude that for every $j \leqslant k$, there are an index $i(j) \in I$ and an element $a_j \in S_{i(j)}$ such that $\mathcal{B}_j = \mathcal{A}_{X(a_j)} \in \mathbf{X}\cap\mathbf{K}_{i(j)}$; i.e., $\mathcal{A}_{X(a)} \leq \mathcal{A}_{X(a_0)}\times\ldots\times\mathcal{A}_{X(a_k)}$. In view of Lemma 3.2, we derive $X(a) = X(a_0)\cup\ldots\cup X(a_k)$. By virtue of the fact that $a \in S_i$, we have $a_0,\ldots,a_k \in S_i$. Therefore, $i(j) = i$ and $\mathcal{A}_{X(a_j)} \in \mathbf{X}\cap\mathbf{K}_i$ for all $j \leqslant k$. Hence $\mathcal{A}_{X(a)} \in \mathbf{Q}(\mathbf{X}\cap\mathbf{K}_i)$, as desired.

Finally, the mapping φ_i is surjective. Indeed, let $\mathbf{Y} \in \mathrm{Lq}\,(\mathbf{K}_i)$. Consider $\mathbf{X} = \mathbf{Q}(\mathbf{Y})\cap\mathbf{K}$. Then $\varphi_i(\mathbf{X}) = \mathbf{Q}(\mathbf{Y})\cap\mathbf{K}\cap\mathbf{K}_i = \mathbf{Q}(\mathbf{Y})\cap\mathbf{K}_i = \mathbf{Y}$. □

Claim 3.4.3. A mapping of the form

$$\varphi\colon \mathrm{Lq}\,(\mathbf{K}) \to \prod_{i\in I}\mathrm{Lq}\,(\mathbf{K}_i) \text{ acting by the rule } \mathbf{K}' \mapsto \big\langle \varphi_i(\mathbf{K}') \mid i \in I\big\rangle$$

is an isomorphism.

Proof. According to Claim 3.4.2, the mapping φ is a homomorphism. Obviously, φ is injective. We show that φ is surjective. Indeed, let $\mathbf{Y}_i \in \mathrm{Lq}\,(\mathbf{K}_i)$ for all $i \in I$ and let $\mathbf{Y} = \bigcup_{i\in I}\mathbf{Y}_i$. Consider a class $\mathbf{X} = \mathbf{Q}(\mathbf{Y})\cap\mathbf{K}$. We need to show that $\varphi_i(\mathbf{X}) = \mathbf{Y}_i$ for all $i \in I$. We have $\varphi_i(\mathbf{X}) = \mathbf{X}\cap\mathbf{K}_i = \mathbf{Q}(\mathbf{Y})\cap\mathbf{K}\cap\mathbf{K}_i = \mathbf{Q}(\mathbf{Y})\cap\mathbf{K}_i$. In view of $\mathbf{Y}_i \subseteq \mathbf{Q}(\mathbf{Y})\cap\mathbf{K}_i$, it suffices to verify that $\mathbf{Q}(\mathbf{Y})\cap\mathbf{K}_i \subseteq \mathbf{Y}_i$.

Suppose $a \in S_i$ and $\mathcal{A}_{X(a)} \in \mathbf{Q}(\mathbf{Y})$. The system $\mathcal{A}_{X(a)}$ is l-projective in $\mathbf{Q}(\mathbf{A})$, so it is l-projective in $\mathbf{Q}(\mathbf{Y})$. Thus $\mathcal{A}_{X(a)} \in \mathbf{SP}(\mathbf{Y})$. By virtue of (P_0), there are systems $\mathcal{B}_0,\ldots,\mathcal{B}_k \in \mathbf{Y}$

such that $\mathcal{A}_{X(a)} \leq \mathcal{B}_0 \times \ldots \times \mathcal{B}_k$. Since $\mathbf{Y} \subseteq \mathbf{K}$, we conclude that for any $j \leqslant k$, there exist an index $i(j) \in I$ and an element $a_j \in S_{i(j)}$ for which $\mathcal{B}_j = \mathcal{A}_{X(a_j)} \in \mathbf{Y} \cap \mathbf{K}_{i(j)} = \mathbf{Y}_{i(j)}$; i.e., $\mathcal{A}_{X(a)} \leq \mathcal{A}_{X(a_0)} \times \ldots \times \mathcal{A}_{X(a_k)}$. By Lemma 3.2, $X(a) = X(a_0) \cup \ldots \cup X(a_k)$. In view of the fact that $a \in A_i$, we have $a_0, \ldots, a_k \in S_i$. Therefore, $i(j) = i$ and $\mathcal{A}_{X(a_j)} \in \mathbf{Y}_i$ for all $j \leqslant k$. Hence $\mathcal{A}_{X(a)} \in \mathbf{Q}(\mathbf{Y}_i) \cap \mathbf{K}_i = \mathbf{Y}_i$. □

According to Theorem 3.3 and Claim 3.4.3,

$$\mathrm{Lq}\,(\mathbf{K}) \cong \prod_{i \in I} \mathrm{Lq}\,(\mathbf{K}_i) \cong \prod_{i \in I} \mathrm{Sub}\,(\mathcal{S}_i). \ \square$$

A generalized version of Theorem 3.1 is

COROLLARY 3.5. If a variety $\mathbf{K}$ of a finite signature contains a class $\mathbf{A} = \{\mathcal{A}_X \mid X \in \mathcal{P}_{\mathrm{fin}}(\omega)\}$ of systems that possesses properties (P_0)-(P_4), then $\mathbf{K}$ is Q-universal. Moreover, the ideal lattice of a free lattice of countable rank is embeddable in the lattice $\mathrm{Lq}\,(\mathbf{K})$.

Proof. We need only consider a set $\{\mathcal{B}_n \mid n > 0\}$, where $\mathcal{B}_n$ is the semilattice of all subsets of an n-element set under intersection, and then apply Theorem 3.4. □

4. THE PROPERTY OF BEING UNCOMPUTABLE

LEMMA 4.1 [1, Lemma 3]. Let $\mathbf{L}$ be an infinite computable set of mutually nonembeddable, subdirectly irreducible finite lattices containing at least three elements, $\mathcal{K}$ a lattice, and $\mathbf{L}_0 \subseteq \mathbf{L} \cap \mathbf{S}(\mathcal{K})$ a set such that $\mathcal{K} \leq_s \prod\{\mathcal{L} \mid \mathcal{L} \in \mathbf{L}_0\}$.

(i) If the set $\mathbf{L}_0$ is computably enumerable but not computable, then the set of all finite sublattices of $\mathcal{K}$ is also computably enumerable but not computable.

(ii) If the set $\mathbf{L}_0$ is not computably enumerable, then the set of all finite sublattices of $\mathcal{K}$ is not computably enumerable either.

THEOREM 4.2. Let $\mathbf{M}$ be a quasivariety of a finite signature containing a class $\mathbf{A} = \{\mathcal{A}_X \mid X \in \mathcal{P}_{\mathrm{fin}}(\omega)\}$ of systems possessing properties (P_0)-(P_4). Then there exists a subclass $\mathbf{K} \subseteq \mathbf{M}$ such that the set of isomorphism types of the class of all finite sublattices of $\mathrm{Lq}\,(\mathbf{K})$ is computably enumerable but not computable.

Proof. Let $N \subseteq \omega \setminus \{0, 1, 2\}$ be a computably enumerable but not computable set. By virtue of Theorem 3.4 applied to a class $\{\mathcal{K}_n \mid n \in N\}$, where $\mathcal{K}_n$ is the finite lower semilattice depicted in Fig. 1, there exists a subclass $\mathbf{K} \subseteq \mathbf{M}$ such that

$$\mathrm{Lq}\,(\mathbf{K}) \cong \prod_{n \in N} \mathrm{Sub}\,(\mathcal{K}_n).$$

According to [1, Lemma 17], the lattice $\mathrm{Sub}\,(\mathcal{K}_n)$ is subdirectly irreducible for any $n > 2$. By virtue of [1, Lemma 18], the lattice $\mathrm{Sub}\,(\mathcal{K}_m)$ is embeddable in $\mathrm{Sub}\,(\mathcal{K}_n)$ iff $m = n$. Let $\mathbf{L} = \{\mathrm{Sub}\,(\mathcal{K}_n) \mid n > 2\}$ and $\mathbf{L}_0 = \{\mathrm{Sub}\,(\mathcal{K}_n) \mid n \in N\}$. Then $\mathbf{L}_0 \subseteq \mathbf{L} \cap \mathbf{S}\big(\mathrm{Lq}\,(\mathbf{K})\big)$. Consequently, $\mathrm{Lq}\,(\mathbf{K})$ has the required property in view of Lemma 4.1(i). □

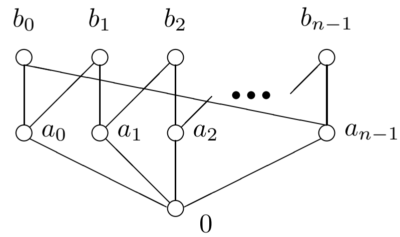

Fig. 1. The lower semilattice $\mathcal{K}_n$ (a crown).

THEOREM 4.3. Let $\mathbf{M}$ be a variety of a finite signature containing a class $\mathbf{A} = \{\mathcal{A}_X \mid X \in \mathcal{P}_{\mathrm{fin}}(\omega)\}$ of systems possessing properties (P_0)-(P_4). Then there exists a subclass $\mathbf{K} \subseteq \mathbf{M}$ such that the set of isomorphism types of the class of all finite sublattices of $\mathrm{Lq}\,(\mathbf{K})$ is not computably enumerable.

Proof. If in the proof of Theorem 4.2 we examine the set $N \subseteq \omega\backslash\{0,1,2\}$, which is not computably enumerable, and use Lemma 4.1(ii), then we obtain the result required. □

Applying the construction given in Theorem 4.2 to different sets N_0 and N_1 produces different classes $\mathbf{K}_0$ and $\mathbf{K}_1$. Since there exists an uncountable set of subsets in ω that are not computably enumerable, we derive the following:

THEOREM 4.4. Let $\mathbf{M}$ be a quasivariety of a finite signature containing a class $\mathbf{A} = \{\mathcal{A}_X \mid X \in \mathcal{P}_{\mathrm{fin}}(\omega)\}$ of systems possessing properties (P_0)-(P_4).

(i) There is a countable set of classes $\mathbf{K} \subseteq \mathbf{M}$ such that the set of isomorphism types of the class of all finite sublattices of $\mathrm{Lq}\,(\mathbf{K})$ is computably enumerable but not computable.

(ii) There is a set of classes $\mathbf{K} \subseteq \mathbf{M}$ having the cardinality of the continuum such that the set of isomorphism types of the class of all finite sublattices of $\mathrm{Lq}\,(\mathbf{K})$ is not computably enumerable.

5. DISCUSSION AND APPLICATIONS

Theorems 3.4 and 4.2-4.4 can be applied to many (almost all) known Q-universal quasivarieties, since in appropriate (and cited below) works it was stated that these contain a subclass of finite systems with (P_1)-(P_4) and are Q-universal. Among such classes, it is worth mentioning the following: varieties of commutative rings with unity and MV-algebras [4; 22, Prop. 3.5] (see also [12, 13]); varieties of De Morgan algebras and Kleene algebras [5]; varieties of distributive p-algebras and Heyting algebras [27]; varieties of p-semilattices [28], distributive lattices with a quantifier [6], and modular lattices [29]; some varieties of commutative semigroups [11]; some classes of graphs [14] (see also [21]); some varieties of bounded lattices [7, 10].

In [8], it was proved that every finite-to-finite universal quasivariety contains a class of finite systems possessing properties (P_1)-(P_4). Consequently, Theorems 3.4, 4.2, and 4.3 can be applied to such quasivarieties as well. Among these are the following: the quasivariety of all $(0,1)$-lattices (all posets) with n constants ($n \geqslant 2$) and the quasivariety of all distributive lattices with n constants

($n \geqslant 3$) [30]; the quasivariety of all left (resp., right) normal idempotent semigroups containing a regular involution as an additional operation [31]; the quasivariety of all directed graphs [32]. In [8], note, it was stated that the quasivariety of all undirected graphs likewise is universal (see also [16]).

A quasivariety for which Theorems 4.2-4.4 are valid was first exemplified in [1]. There, too, it was established that there exists a *subquasivariety* $\mathbf{K}$ of the variety of all unars (algebras whose signature contains a single unary function symbol) such that the set of all isomorphism types of the class of finite sublattices of $\mathrm{Lq}\,(\mathbf{K})$ is computably enumerable but not computable (not computably enumerable, resp.). The second example of a similar type (a quasivariety of Abelian groups with a single constant) was found in [2], where it was proved that the quasivariety in question is Q-universal. Other examples of such classes were obtained in [21, 33, 34]. In particular, these are a quasivariety of directed graphs, and also a class of differential groupoids. Moreover, as noted, it was shown in [21] that the class $\mathbf{K}(\sigma)$ of *all* systems of a signature σ is Q-universal iff there exists a class $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ such that the set of all isomorphism types of the class of finite sublattices of $\mathrm{Lq}\,(\mathbf{K})$ is computably enumerable but not computable (not computably enumerable, resp.). In this connection, the following problems arise.

Problem 5.1. Is it true that a Q-universal class $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ of systems of a signature σ contains a subclass $\mathbf{K}' \subseteq \mathbf{K}$ such that the set of all isomorphism types of the class of finite sublattices of $\mathrm{Lq}\,(\mathbf{K}')$ is computably enumerable but not computable? Does there exist a class $\mathbf{K}$ that is not Q-universal but nevertheless possesses the above-mentioned property?

Theorem 4.2 gives a positive answer to the first question for almost all known Q-universal quasivarieties (see the discussion above). However, the answer to the second question is still not known.

In [35], it was stated that varieties of Cantor algebras are Q-universal. In proving this fact, use was made of another sufficient condition for Q-universality, introduced in [13]. Although Theorem 3.1 cannot be applied to those varieties (since all systems in them are infinite), it follows from [35, proofs of Thm. 2.7 and Lemma 3.1] that in the varieties in question, there exist classes of systems possessing properties (P_0)-(P_4). Thus Theorems 4.2-4.4 turn out to be valid in this case too.

Problem 5.2. Is it true that a quasivariety (generated by a single semigroup), whose Q-universality was established in [3], contains a class $\mathbf{K}$ such that the set of all isomorphism types of the class of finite sublattices of $\mathrm{Lq}\,(\mathbf{K})$ is computably enumerable but not computable?

Issues related to the problems under consideration were also taken up in [3, 16-20, 22, 23, 33, 34, 36-42].

REFERENCES

1. A. M. Nurakunov, "Unreasonable lattices of quasivarieties," *Int. J. Alg. Comput.*, **22**, No. 3 (2012), Paper No. 1250006.

2. A. M. Nurakunov, “Quasivariety lattices of pointed Abelian groups,” *Algebra and Logic*, **53**, No. 3, 238-257 (2014).
3. M. Sapir, “The lattice of quasivarieties of semigroups,” *Alg. Univ.*, **21**, Nos. 2/3, 172-180 (1985).
4. M. E. Adams and W. Dziobiak, “Q-universal quasivarieties of algebras,” *Proc. Am. Math. Soc.*, **120**, No. 4, 1053-1059 (1994).
5. M. E. Adams and W. Dziobiak, “Lattice of quasivarieties of 3-element algebras,” *J. Alg.*, **166**, No. 1, 181-210 (1994).
6. M. E. Adams and W. Dziobiak, “Quasivarieties of distributive lattices with a quantifier,” *Discr. Math.*, **135**, Nos. 1-3, 15-28 (1994).
7. M. E. Adams and W. Dziobiak, “Joins of minimal quasivarieties,” *Stud. Log.*, **54**, No. 3, 371-389 (1995).
8. M. E. Adams and W. Dziobiak, “Finite-to-finite universal quasivarieties are $\mathcal{Q}$-universal, *Alg. Univ.*, **46**, Nos. 1/2, 253-283 (2001).
9. M. E. Adams and W. Dziobiak, “The lattice of quasivarieties of undirected graphs,” *Alg. Univ.*, **47**, No. 1, 7-11 (2002).
10. M. E. Adams and W. Dziobiak, “$\mathcal{Q}$-universal varieties of bounded lattices,” *Alg. Univ.*, **48**, No. 3, 333-356 (2002).
11. M. E. Adams and W. Dziobiak, “Quasivarieties of idempotent semigroups,” *Int. J. Alg. Comput.*, **13**, No. 6, 733-752 (2003).
12. V. A. Gorbunov, “Structure of lattices of varieties and lattices of quasivarieties: Similarity and difference. II,” *Algebra and Logic*, **34**, No. 4, 203-218 (1995).
13. V. A. Gorbunov, *Algebraic Theory of Quasivarieties, Sib. School Alg. Log.* [in Russian], Nauch. Kniga, Novosibirsk (1999).
14. A. V. Kravchenko, “The lattice complexity of quasivarieties of graphs and endographs,” *Algebra and Logic*, **36**, No. 3, 164-168 (1997).
15. A. V. Kravchenko, “Complexity of quasivariety lattices for varieties of unary algebras,” *Mat. Tr.*, **4**, No. 2, 113-127 (2001).
16. Kravchenko, “$\mathcal{Q}$-universal quasivarieties of graphs,” *Algebra and Logic*, **41**, No. 3, 173-181 (2002).
17. A. V. Kravchenko, “On the lattices of quasivarieties of differential groupoids,” *Comment. Math. Univ. Carol.*, **49**, No. 1, 11-17 (2008).
18. A. V. Kravchenko, “Complexity of quasivariety lattices for varieties of differential groupoids,” *Mat. Tr.*, **12**, No. 1, 26-39 (2009).

19. A. V. Kravchenko, “Complexity of quasivariety lattices for varieties of differential groupoids. II,” *Mat. Tr.*, **15**, No. 2, 89-99 (2012).

20. A. V. Kravchenko, “Minimal quasivarieties of differential groupoids with nonzero multiplication,” *Sib. El. Mat. Izv.*, **9**, 201-207 (2012); `http://semr.math.nsc.ru`.

21. M. V. Schwidefsky and A. Zamojska-Dzienio, “Lattices of subclasses. II,” *Int. J. Alg. Comput.*, **24**, No. 8, 1099-1126 (2014).

22. W. Dziobiak, “Selected topics in quasivarieties of algebraic systems,” manuscript (1997).

23. V. I. Tumanov, “Sufficient conditions for embedding a free lattice in a subquasivariety lattice,” Preprint No. 40, Sobolev Institute of Mathematics, Novosibirsk (1988).

24. G. Birkhoff, “Universal algebra,” *Proc. First Can. Math. Congress*, University of Toronto Press, Toronto (1946), pp. 310-326.

25. A. I. Mal’tsev, “Some borderline problems of algebra and logic,” in *Proc. Int. Congr. Math. (Moscow, 1966)*, Mir, Moscow (1968), pp. 217-231.

26. V. A. Gorbunov and V. I. Tumanov, “The structure of lattices of quasivarieties,” *Trudy Inst. Mat. SO AN SSSR*, **2**, 12-44 (1982).

27. W. Dziobiak, “On subquasivariety lattices of some varieties related with distributive p-algebras,” *Alg. Univ.*, **21**, No. 1, 62-67 (1985).

28. M. E. Adams, W. Dziobiak, M. Gould, and J. Schmid, “Quasivarieties of pseudocomplemented semilattices,” *Fund. Math.*, **146**, No. 3, 295-312 (1995).

29. W. Dziobiak, “On lattice identities satisfied in subquasivariety lattices of varieties of modular lattices,” *Alg. Univ.*, **22**, Nos. 2/3, 205-214 (1986).

30. M. E. Adams, V. Koubek, and J. Sichler, “Homomorphisms and endomorphisms of distributive lattices,” *Houston J. Math.*, **11**, 129-145 (1985).

31. V. Koubek, “Universal expansion of semigroup varieties by regular involution,” *Semigroup Forum*, **46**, No. 2, 152-159 (1993).

32. Z. Hedrlín and A. Pultr, “On full embeddings of categories of algebras,” *Ill. J. Math.*, **10**, 392-406 (1966).

33. M. V. Semenova and A. Zamojska-Dzienio, “Lattices of subclasses,” *Sib. Math. J.*, **53**, No. 5, 889-905 (2012).

34. A. M. Nurakunov, M. V. Semenova, and A. Zamojska-Dzienio, “On lattices connected with various types of classes of algebraic structures,” *Uch. Zap. Kazan State Univ., Ser. Fiz-Mat.*, **154**, No. 2, 167-179 (2012).

35. M. S. Sheremet, “Quasivarieties of Cantor algebras,” *Alg. Univ.*, **46**, Nos. 1/2, 193-201 (2001).

36. M. E. Adams and J. Sichler, "Homomorphisms of unary algebras with a given quotient," *Alg. Univ.*, **27**, No. 2, 194-219 (1990).

37. W. Dziobiak, "Quasivarieties of Sugihara semilattices with involution," *Algebra and Logic*, **39**, No. 1, 26-36 (2000).

38. P. Goralčik, V. Koubek, and J. Sichler, "Universal varieties of $(0, 1)$-lattices," *Can. J. Math.*, **42**, No. 3, 470-490 (1990).

39. V. K. Kartashov, "Lattices of quasivarieties of unars," *Sib. Math. J.*, **26**, No. 3, 346-357 (1985).

40. V. Koubek and J. Sichler, "Universal varieties of semigroups," *J. Aust. Math. Soc., Ser. A*, **36**, 143-152 (1984).

41. S. V. Sizyi, "Quasivarieties of graphs," *Sib. Math. J.*, **35**, No. 4, 783-794 (1994).

42. M. P. Tropin, "Embedding a free lattice in the lattice of quasivarieties of distributive lattices with pseudocomplementation," *Algebra and Logic*, **22**, No. 2, 113-119 (1983).