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UNIVERSAL FUNCTIONS OVER TREES

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An important generalization of the notion of computability is Σ-definability (generalized computability) in admissible sets. This generalization made it possible to study computability problems over arbitrary structures, for example, over the field of reals. The most significant results in computability theory for admissible sets, as well as their applications in theoretical computer science (semantic programming, dynamic logic, the theory of effective f-spaces, and so on), are collected in [1]. There, also, the importance of the following direction for future research was pointed out: For a better understanding of the general nature of computability (constructive cognoscibility), it is necessary to further develop (gain a better insight into) the notion of computability in admissible sets of the form $\mathbb{HF}(\mathfrak{A})$, a hereditarily finite superstructure over a structure $\mathfrak{A}$, where $\mathfrak{A}$ either is a model of a rather simple theory or is one of the classical objects, such as, for instance, the field $\mathbb{R}$ of real numbers [1, p. 12].

A fundamental result in classical computability theory is the existence of a universal partial computable function. We know from [1] that every admissible set of a finite signature contains a universal Σ-predicate, which is untrue for Σ-functions. A structure $\mathfrak{M}$, such that in the hereditarily finite superstructure $\mathbb{HF}(\mathfrak{M})$ there exists no universal Σ-function, was constructed in [2]. In this connection, it is of interest to find a condition which guarantees, for a structure $\mathfrak{M}$, the existence of a universal Σ-function in the hereditarily finite superstructure $\mathbb{HF}(\mathfrak{M})$.

A review of further results on this problem can be found in [3]. In [4], a torsion-free Abelian group A was constructed for which $\mathbb{HF}(A)$ has no universal Σ-function. In [5, 6], it was proved that a universal Σ-function exists in hereditarily finite superstructures over an Abelian p-group, a linear order, and an Ershov algebra. In [7, 8], the concept of a Σ-uniform structure was introduced and a condition was specified that is necessary and sufficient for a universal Σ-function to exist

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in hereditarily finite superstructures over such a structure; also examples of Σ-uniform Abelian groups and rings were given.

S. S. Goncharov came up with the question whether there exists a universal Σ-function in a hereditarily finite superstructure over a tree.

By an *admissible set* $\mathbb{A}$ we mean a KPU-structure in which the set $\operatorname{Ord}\mathbb{A}$ of all ordinals is well ordered. A function in $\mathbb{A}$ whose graph is defined by some Σ-formula in $\mathbb{A}$ is called a *Σ-function.*

A binary partial Σ-function $g(x,y) : A^2 \to A$ is said to be *universal* for a family of unary partial Σ-functions in an admissible set $\mathbb{A}$ if a family $\{\lambda y\ g(a,y) \mid a \in A\}$ consists of all unary partial Σ-functions

Let $\mathcal{P}_\omega(X)$ be the set of all finite subsets of a set X. A *hereditarily finite superstructure* $\mathbb{HF}(\mathfrak{M})$ *over a structure* $\mathfrak{M} = \langle M, \sigma_0\rangle$ is defined to be the structure $\langle M \cup HF(M), U, \in, \varnothing, \sigma_0\rangle$ of a signature $\sigma_1 = \langle U, \in, \varnothing, \sigma_0\rangle$, where $HF(M) = \bigcup\limits_{n\in\omega} HF_n(M)$, $HF_0(M) = \varnothing$, $HF_{n+1}(M) = \mathcal{P}_\omega(M \cup HF_n(M))$, the predicate U distinguishes the set of elements of $\mathfrak{M}$ (*urelements*), while the relation $\in$ and the constant $\varnothing$ are used in the usual set-theoretic sense.

A partially ordered set T is called a *tree* if, for any $x \in T$, the set of all elements smaller than x (*predecessors* of x in T) is well ordered and T contains a least element r, called the *root.* For every node $x \in T$, we denote by $\mathrm{level}_T(x)$ the order type of the set of all predecessors of x in T and call it the *level* of a node x in T. The *height* of T is defined by the equality $ht(T) = \sup\{\mathrm{level}_T(x) + 1 \mid x \in T\}$.

If $a, b \in T$, $a < b$ $(a \neq b)$, and among these, there are no elements of T, then b is called an *immediate successor* of a. Let T be a tree and $a \in T$. If the set of all immediate successors of a has cardinality α, then we say that *a α-ramifies.* If α is a finite cardinal, then we say that *a finitely ramifies.* We say that *a ramifies not more than β* if a α-ramifies and $\alpha \le \beta$.

Let $a \in T \setminus \{r\}$. A subtree $[a]_T = \{y \mid \exists z(z \le a \,\&\, \mathrm{level}_T(z) = 1 \,\&\, y \ge z)\} \cup \{r\}$ is called an *elementary* subtree containing a. Let T_0 and T_1 be trees with roots r_0 and r_1, respectively, and $T_0 \cap T_1 = \varnothing$. Then $T_0 \cup T_1$ denotes a tree obtained by joining T_0 and T_1 and identifying $r_0 = r_1 = r$. The join of finitely many elementary subtrees is called a *closed subtree.*

Let T be a tree of finite height. If there exists a number $N \in \omega$ for which any node in T, which is distinct from the root, ramifies not more than N, then T is called a *bounded branching tree.* A tree $T = F \cup I$ of finite height is *almost bounded branching* if F is a bounded branching tree, while I is the join of finitely many elementary trees with infinitely branching elements, and every successor of such elements does not have a successor.

A tree T is said to be *finitely branching* if every node, which is distinct from r, ramifies finitely. A finitely branching tree T is *unbounded branching of finite height* $h+1$ if the following hold:

(a) every maximal chain (i.e., a linearly ordered set) has height $h+1$;

(b) for any natural numbers m and n, there are n elementary subtrees every node in which ramifies at least m times.

Let $T = T_0 \cup T_1$ be a finitely branching tree of finite height $h+1$, $ht(T_0) < h+1$, and let T_1

be an unbounded branching tree. We call T an *almost unbounded branching tree of height* $h+1$.

Below is a necessary condition for a universal Σ-function to be missing in an admissible set $\mathbb{A}$, specified in [9]. With that condition in hand, the above-mentioned question of Goncharov was answered in the negative.

Proposition 1 [9]. Let $\mathbb{A} = \langle A, \sigma \rangle$ be an admissible set. Suppose also that for any element $a \in A$, there exists a Σ-function f in $\mathbb{A}$ such that for any element $b \in A$, there exist elements $c, d \in A$ and isomorphic embeddings $\varphi^\varepsilon : \mathbb{A} \to \mathbb{A}$, $\varepsilon < 2$, satisfying the following conditions:

(1) $f(c) = d$;

(2) $\varphi^\varepsilon(a) = a$, $\varepsilon < 2$, $\varphi^0(b) = \varphi^1(b)$, and $\varphi^0(c) = \varphi^1(c)$;

(3) $\varphi^0(d) \neq \varphi^1(d)$.

Then $\mathbb{A}$ does not contain a universal Σ-function.

THEOREM 1 [9]. There is a tree T of height $ht(T) = 4$ such that the hereditarily finite superstructure $\mathbb{HF}(T)$ over T has no universal Σ-function.

In what follows, unless otherwise stated, $\mathfrak{M}$ is a locally finite structure of a finite signature σ_0. By $\langle M_0 \rangle$ we denote a substructure generated by a set M_0.

A decidable, model complete, ω-categorical theory with a decidable set of complete formulas is said to be c-simple [3].

Let a structure $\mathfrak{M}$ be given. Suppose also that for every finite subset $M_0 \subseteq M$, there exists a uniquely defined finite subset $[M_0] \subseteq M$ such that $M_0 \subseteq [M_0]$ and $\big[[M_0]\big] = [M_0]$. The set $[M_0]$ is called the *closure* of a set M_0. If $[M_0] = M_0$, then M_0 is called a *closed set*.

Definition 1 [10]. Suppose that a structure $\mathfrak{M}$ and its substructure $\mathfrak{N} \leq \mathfrak{M}$ satisfy the following conditions:

(1) $\mathfrak{N}$ is a structure of a c-simple theory T and its universe N is a Σ-subset in $\mathbb{HF}(\mathfrak{M})$;

(2) $\mathbb{HF}(\mathfrak{N})$ contains Σ-formulas without parameters defining a Δ-predicate $\mathfrak{B} \subseteq \mathcal{P}_\omega(N) \times \mathcal{P}_\omega(N)$ for which $\mathbb{HF}(\mathfrak{N}) \models \mathfrak{B}(x, y \cup z) \,\&\, x \subseteq y,\ x \subseteq z \to \mathfrak{B}(x, y) \,\&\, \mathfrak{B}(x, z)$;

(3) $\mathfrak{M}$ is locally embeddable in $\mathfrak{N}$.

Let $A \leq \mathfrak{M}$, $B \leq \mathfrak{N}$, and $\alpha : A \to B$ be an isomorphism. Assume that the following conditions are satisfied:

(a) If A is closed, then, for any finite substructure $A^1 \geq A$, there exists an isomorphic embedding $\psi : A^1 \to \mathfrak{N}$ which extends α and is such that $\mathbb{HF}(\mathfrak{N}) \models \mathfrak{B}(B, \psi A^1)$.

(b) Let isomorphic embeddings $\varphi^\varepsilon : A^\varepsilon \to \mathfrak{N}$, $A^\varepsilon \geq A$, $\varepsilon < 2$, extend α and let $\mathbb{HF}(\mathfrak{N}) \models \mathfrak{B}(B, \varphi^\varepsilon A^\varepsilon)$. Then there exists an isomorphic embedding $\psi : \langle A^0 \cup A^1 \rangle \to \mathfrak{N}$ which extends α and is such that $\mathbb{HF}(\mathfrak{N}) \models \mathfrak{B}(B, \psi(\langle A^0 \cup A^1 \rangle))$.

(c) For any finite substructure $B^1 \geq B$ such that $\mathbb{HF}(\mathfrak{N}) \models \mathfrak{B}(B, B^1)$, there exists an isomorphic embedding $\psi : B^1 \to \mathfrak{M}$ extending α^{-1}.

We call $\mathfrak{M}$ with these properties an *almost c-simple* structure.

THEOREM 2 [10]. A universal Σ-function exists in the hereditarily finite superstructure $\mathbb{HF}(\mathfrak{M})$ over an almost c-simple structure $\mathfrak{M}$.

THEOREM 3 [10]. Let T be an almost bounded branching tree of finite height. Then there exists its constant expansion T', which is an almost c-simple structure.

COROLLARY 1 [10]. Let T be an almost bounded branching tree of finite height. Then a universal Σ-function exists in the hereditarily finite superstructure $\mathbb{HF}(T)$.

In [10, 11], note, we have also constructed families of almost c-simple equivalences and rings.

Definition 2 [5]. Suppose that a locally constructivizable structure $\mathfrak{M}$ and its finite subset M_0 satisfy the following conditions:

(1) The concept of a basis for any finite subset $X \subseteq M$ is defined. The predicate $\mathfrak{B}_0^{M_0}(X, Y) \rightleftharpoons$ "a finite sequence $Y \in M^{<\omega}$ is a basis for X" is a Δ-predicate of the signature $\sigma_1(M_0)$ in $\langle \mathbb{HF}(\mathfrak{M}), M_0\rangle$. If Y^0 and Y^1 are two bases for a subset X, then $X \subseteq \langle Y^\varepsilon\rangle$, $\varepsilon = 0, 1$, and either $\mathfrak{B}_0^{M_0}(sp\, Y^0, Y^1)$ or $\mathfrak{B}_0^{M_0}(sp\, Y^1, Y^0)$ is true. The sequence Y is called a *basis* if $\mathfrak{B}^{M_0}(Y) \rightleftharpoons \mathfrak{B}_0^{M_0}(sp\, Y, Y)$ is true.

(2) For every basis Y, the number $\chi^{M_0}(Y)$ (we call it the *characteristic* of the basis Y) is defined so that $\chi^{M_0}(Y)$ is a Σ-function of the signature $\sigma_1(M_0)$ in $\langle \mathbb{HF}(\mathfrak{M}), M_0\rangle$. The set Ξ^{M_0} of all characteristics is a computable subset of ω. There exists a Δ-predicate $Cor^{M_0}(z, Y, n)$ of the signature $\sigma_1(M_0)$ for which the following equivalence holds:

$$z \in \langle Y\rangle \Leftrightarrow \langle \mathbb{HF}(\mathfrak{M}), M_0\rangle \models \exists! n\big(n \neq 0 \ \&\ Cor^{M_0}(z, Y, n)\big).$$

The number n is called the *coordinate* of an element z with respect to a basis Y. If elements are not equal, then their characteristics are not equal either.

(3) Let bases Y^ε of equal characteristic χ and finite substructures $\mathfrak{M}^\varepsilon \supseteq \langle Y^\varepsilon\rangle$, $\varepsilon < 2$, be given. Then there exist a basis Y^2 and a substructure $\mathfrak{M}^2 \supseteq \langle Y^2\rangle$ for which the following conditions hold:

(a) $\chi = \chi(Y^2)$;

(b) there are isomorphic embeddings $\varphi_0^\varepsilon : \mathfrak{M}^\varepsilon \to \mathfrak{M}^2$ such that $\varphi^\varepsilon \restriction \langle M_0\rangle = \mathrm{id}$ and $\varphi^\varepsilon Y^\varepsilon = Y^2$, where the embeddings $\varphi^\varepsilon : \mathbb{HF}(\mathfrak{M}^\varepsilon) \to \mathbb{HF}(\mathfrak{M}^2)$ naturally extend φ_0^ε.

In particular, every two bases of the same characteristic are equal in length.

(4) For any partial function $f : \mathbb{HF}(\mathfrak{M}) \to \mathbb{HF}(\mathfrak{M})$ defined by a Σ-formula with parameters in M_0, it is true that if $u \in \mathbb{HF}(\mathfrak{M})$ and $u \in \delta f$, then there exists a basis Y for a subset $sp\, u$ such that $sp\, f(u) \subseteq \langle Y\rangle$.

We call such $\mathfrak{M}$ a Σ-*bounded structure with respect to* M_0. If, for any finite subset M_0, there exists a finite subset $M_0' \supseteq M_0$ such that $\mathfrak{M}$ is Σ-bounded with respect to M_0', then $\mathfrak{M}$ is called a Σ-*bounded structure*.

THEOREM 4 [5]. Let $\mathfrak{M}$ be a Σ-bounded structure. Then $\mathbb{HF}(\mathfrak{M})$ contains a universal Σ-function defined by a Σ-formula with parameter A if and only if, for any finite subset C with respect to which $\mathfrak{M}$ is Σ-bounded, there exists a finite subset C^1 such that for any finite subset X and for an arbitrary basis Y_X^C, there is a basis $Y_{X^*}^A$ for which $\langle Y_X^C\rangle \subseteq \langle Y_{X^*}^A\rangle$, where $X^* = C^1 \cup X$.

THEOREM 5. An unbounded branching tree T of finite height is Σ-bounded with respect to any finite closed subtree T_0.

COROLLARY 2. Let T be an unbounded branching tree of finite height. Then the hereditarily finite superstructure $\mathbb{HF}(T)$ contains a universal Σ-function defined by a Σ-formula without parameters.

Definition 3. Suppose that a structure $\mathfrak{M}$ and its substructure $\mathfrak{N}$ satisfy the following conditions:

(1) The concept of the closure of a finite subset $M_0 \subseteq M$ is defined; there are Σ-formulas without parameters defining a universe N and a Δ-predicate $z \in [N_0]$ in $\mathbb{HF}(\mathfrak{N})$.

(2) For any finite substructures $\mathfrak{M}_0 \leq \mathfrak{M}_1 \leq \mathfrak{M}$, where M_0 is closed, and for an isomorphic embedding $\varphi_0 : \mathfrak{M}_0 \to \mathfrak{N}$, there is an isomorphic embedding $\varphi_1 : \mathfrak{M}_1 \to \mathfrak{N}$ extending φ_0, with $\varphi_1 M_1 \cap [\varphi_0 M_0] = \varphi_0 M_0$.

(3) For any finite substructures $\mathfrak{M}_0 \leq \mathfrak{M}$ and $\mathfrak{N}_0 \leq \mathfrak{N}$ and for an isomorphic embedding $\varphi_0 : \mathfrak{M}_0 \to \mathfrak{N}$ such that M_0 is closed and $N_0 \cap [\varphi_0 M_0] = \varphi_0 M_0$, there is an isomorphic embedding $\psi_0 : \mathfrak{N}_0 \to \mathfrak{M}$ extending φ_0^{-1}.

(4) There exist Σ-formulas without parameters defining a Δ-predicate $\mathfrak{A}(x_0, x_1)$ such that for any finite closed substructure $\mathfrak{M}_0 \leq \mathfrak{M}$, there is an isomorphic embedding $\varphi_0 : \mathfrak{M}_0 \to \mathfrak{N}$ for which it is true that $\mathbb{HF}(\mathfrak{M}) \models \mathfrak{A}(x_0, M_0) \Leftrightarrow x_0 \notin M_0$ and $\mathbb{HF}(\mathfrak{N}) \models \mathfrak{A}(z_0, \varphi_0 M_0) \Leftrightarrow z_0 \notin [\varphi_0 M_0]$.

Such a structure $\mathfrak{N}$ is said to be *closed with respect to a structure* $\mathfrak{M}$.

THEOREM 6. Let $\mathfrak{N}$ be Σ-bounded and closed with respect to $\mathfrak{M}$. Then there exists a Σ-formula with no parameters that defines a universal Σ-function in $\mathbb{HF}(\mathfrak{M})$ if and only if there exists a Σ-formula with no parameters that defines a universal Σ-function in $\mathbb{HF}(\mathfrak{N})$.

THEOREM 7. Let $T = T_0 \cup T_1$ be an almost unbounded branching tree of finite height. Then the tree T_1 is closed with respect to T.

This, combined with Theorem 5 and Corollary 2, yields

COROLLARY 3. In a hereditarily finite superstructure over an almost unbounded branching tree of finite height, there exists a Σ-formula with no parameters that defines a universal Σ-function.

Every tree of height at most 3 either is an almost c-simple tree or is an almost unbounded branching tree. This, together with Theorem 2 and Corollary 3, entails

COROLLARY 4. In a hereditarily finite superstructure $\mathbb{HF}(T)$ over a tree T of height at most 3, there exists a universal Σ-function.

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