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THE COORDINATE GROUP OF AN AFFINE SPACE OVER A RIGID METABELIAN PRO-p-GROUP

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For a finitely generated 2-step solvable 2-graded rigid pro-p-group G, the coordinate group of an affine space G^m is found and the space G^m is stated to be irreducible in the Zariski topology.

In [1], rigid metabelian graded pro-p-groups were studied. Recall that a metabelian pro-p-group G is *rigid* if it has a normal series of the form

$$G = G_1 \geqslant G_2 \geqslant G_3 = 1 \tag{1}$$

with Abelian factors, in which case the factor group $A = G/G_2$ is torsion-free and G_2 is torsion-free as a $\mathbb{Z}_pA$-module (here $\mathbb{Z}_p$ is the ring of p-adic integers). In the case where a group is exactly 2-step solvable, the specified series (if it exists) is defined by the group uniquely; a nontrivial Abelian pro-p-group is rigid if it is torsion-free, and so we can put either $G_2 = 1$ or $G_1 = G_2$. Depending on these three options, a metabelian rigid pro-p-group is assigned a respective grading $(1, 1)$, $(1, 0)$, or $(0, 1)$. A *morphism* between 2-graded rigid pro-p-groups with corresponding series of form (1) is a homomorphism $\varphi : G \to H$ such that $G_i\varphi \leqslant H_i$ $(i = 1, 2, 3)$. In [1], it was proved that in the category of 2-graded rigid pro-p-groups, a coproduct operation exists, and its properties were explored. Denote by $G \circ H$ a coproduct of two 2-graded rigid pro-p-groups G and H.

In [2], by analogy with abstract groups [3, 4], definitions were couched and some fundamental facts proved concerning algebraic geometry over profinite groups. If the definitions mentioned are brought to bear on pro-p-groups, then an equation in $x_1, \ldots, x_n$ over a given pro-p-group G is an expression of the form $v = 1$, where v is an element of the free pro-p-product $G * \langle x_1, \ldots, x_n\rangle$ of the group G and the free pro-p-group $\langle x_1, \ldots, x_n\rangle$. A set $S \subseteq G^n$ of solutions to some system

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$\{v_i(x_1, \dots, x_n) = 1 \mid i \in I\}$ of equations is called an *algebraic subset* of an affine space G^n. The factor group of $G * \langle x_1, \dots, x_n\rangle$ with respect to an annihilator S is called the *coordinate group* of an algebraic set S. On G^n, the Zariski topology is defined by taking all algebraic subsets to be a subbasis of a system of closed sets.

The main result of the paper is the following:

THEOREM 1. Let G be a finitely generated 2-step solvable 2-graded rigid pro-p-group, and let $F = \langle x_1, \dots, x_n\rangle$ be a free metabelian pro-p-group of rank n which is treated with grading $(1, 0)$ for $n = 1$ and with grading $(1, 1)$ for all other values of n. Then $G \circ F$ is G-discriminated by G.

From Theorem 1, using [2, Thm. 6], we deduce

THEOREM 2. Under the hypotheses of Theorem 1, the group $G \circ F$ is the coordinate group of an affine space G^n and the space G^n is irreducible.

The last theorem is similar to a corresponding statement proved for metabelian rigid abstract groups in [5] and for rigid abstract groups of any derived length in [6].

Proof of Theorem 1. The problem reduces to the case where F has rank 1. In fact, in view of [1, Cor. 1], $F = E_2 \circ E_1$, where $E_2 = \langle x_2, \dots, x_n\rangle$ and $E_1 = \langle x_1\rangle$. We have $G \circ F = (G \circ E_2) \circ E_1$. By induction, assume that $G \circ E_2$ is G-discriminated by G. To prove the theorem, it suffices to show that $(G \circ E_2) \circ E_1$ is $(G \circ E_2)$-discriminated by $G \circ E_2$.

Suppose that $F = \langle x\rangle$ has rank 1. Let G_i and F_i $(i = 1, 2, 3)$ be respective members of series of form (1) for pro-p-groups G and F (note that $F_2 = 1$). Denote by $H = G \circ F$ a 2-rigid product of groups G and F with a series $H = H_1 > H_2 > H_3 = 1$. Let $A = G/G_2$ be a nontrivial free Abelian pro-p-group of finite rank and $a_1, \dots, a_m$ its free generators. For images of elements of F in F/F_2, we keep the same notation.

Free splittings of groups G and F (see [1]) over G_2 and F_2 have respective forms $\begin{pmatrix} A & 0 \\ D(G) & 1 \end{pmatrix}$ and $\begin{pmatrix} \langle x\rangle & 0 \\ t \cdot \mathbb{Z}_p[[y]] & 1 \end{pmatrix}$, where $x - 1 = y$ and $D(G)$ is a $\mathbb{Z}_pA$-module. Let $C = A \times \langle x\rangle$. It follows from [1] that a free splitting of the group H over H_2 has the form $\begin{pmatrix} C & 0 \\ D(H) & 1 \end{pmatrix}$, where the $\mathbb{Z}_pC$-module $D(H)$ equals

$$D(G) \bigotimes_{\mathbb{Z}_pA} \mathbb{Z}_pC \oplus t \cdot \mathbb{Z}_pC.$$

We need to prove that H is G-discriminated by G. Let $h_1, \dots, h_s$ be a tuple of nontrivial elements of H, which we want to discriminate. We may assume that $h_i \in H_2$ $(i = 1, \dots, s)$. Indeed, if $h_i \notin H_2$, then h_i can be replaced by $[h_i, h]$, where $h = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$ is some nontrivial element of G_2.

Thus let $h_i = \begin{pmatrix} 1 & 0 \\ v_i & 1 \end{pmatrix} \in H_2$ $(i = 1, \dots, s)$. In essence, we must discriminate a tuple of nonzero elements $V = \{v_1, \dots, v_s\}$ in the module $D(H)$. It follows from [1] that every element $v \in D(H)$

is uniquely representable as $v = \sum_{k=0}^{\infty} u_k y^k + t\alpha$, where $u_k \in D(G)$ and $\alpha \in \mathbb{Z}_p C$. We partition the set V into two subsets V_1 and V_2, where V_1 consists of those v for which $\alpha = 0$ and V_2 consists of the other v. We search for a required homomorphism in the form

$$\tau_g : G \xrightarrow{id} G,\ x \mapsto g = \begin{pmatrix} gG_2 & 0 \\ u & 1 \end{pmatrix} \in G,$$

where $gG_2 = a_1^{\beta}$. First we find a suitable value for β and then correct g by means of an element of G_2.

The algebra $\mathbb{Z}_p A$ coincides with an algebra $\mathbb{Z}_p[[z_1, \ldots, z_m]]$ of formal power series in variables $z_i = a_i - 1$ $(1 \le i \le m)$. It is known [1] that a finitely generated torsion-free $\mathbb{Z}_p A$-module $D(G)$ embeds in a free module T of finite rank, and we let $t_1, \ldots, t_l$ be its free generators. We claim that $D(G) \bigotimes_{\mathbb{Z}_p A} \mathbb{Z}_p C$ embeds in $T \bigotimes_{\mathbb{Z}_p A} \mathbb{Z}_p C = t_1 \cdot \mathbb{Z}_p C \oplus \ldots \oplus t_l \cdot \mathbb{Z}_p C$. Consider a canonical homomorphism $\varphi : D(G) \bigotimes_{\mathbb{Z}_p A} \mathbb{Z}_p C \to T \bigotimes_{\mathbb{Z}_p A} \mathbb{Z}_p C$. Take a nonzero element $w \in D(G) \bigotimes_{\mathbb{Z}_p A} \mathbb{Z}_p C$. According to [1], the element w is uniquely representable as $\sum u_k y^k$, where $u_k \in D(G)$. Choose a minimal k_0 for which $u_{k_0} \neq 0$. The element u_{k_0} is representable as $t_1\gamma_1 + \ldots + t_l\gamma_l$ in the module T, where γ_i are not all equal to zero. Then the image of w in $T \bigotimes_{\mathbb{Z}_p A} \mathbb{Z}_p C$ is distinct from zero since every element of $T \bigotimes_{\mathbb{Z}_p A} \mathbb{Z}_p C$ is also uniquely representable as $\sum u_k y^k$, where $u_k \in T$.

Clearly, $V_1 \tau_g$ is completely defined by the value $x\tau_g$ modulo G_2, i.e., by the choice of $\beta \in \mathbb{Z}_p$. Now we expand $v \in V_1$ in terms of the basis elements $t_1, \ldots, t_l$ of the module $T \bigotimes_{\mathbb{Z}_p A} \mathbb{Z}_p C : v = t_1 w_1 + \ldots + t_l w_l$, where $w_i \in \mathbb{Z}_p[[z_1, \ldots, z_m, y]]$, and one of these coefficients should be other than zero. Choose a value for β so that for any $v \in V_1$, the image of a corresponding coefficient in $\mathbb{Z}_p A$ will not be zero, and if $v \in V_2$ and $v = \sum_k u_k y^k + t\alpha$, then the image of α, denoted $\overline{\alpha}$, will be other than zero as well.

The existence of an appropriate value for β follows from the following simple fact. Let W be a finite set of nonzero elements of $\mathbb{Z}_p C$. Then there exists $\beta \in \mathbb{Z}_p$ such that under the homomorphism $\overline{\tau} : \mathbb{Z}_p C \to \mathbb{Z}_p A$ defined by the mapping $A \xrightarrow{id} A$, $x \mapsto a_1^{\beta}$, the images of all elements in W are distinct from zero.

Choose an arbitrary element $v = \sum_{k=0}^{\infty} u_k y^k + t\alpha \in V_2$. For some $g = \begin{pmatrix} gG_2 & 0 \\ u & 1 \end{pmatrix} \in G$, where $gG_2 = a_1^{\beta}$, consider the image of v under the homomorphism τ_g. We have chosen β so that $\overline{\alpha} \neq 0$. Let $g' = gg_2$, where $g_2 = \begin{pmatrix} 1 & 0 \\ u' & 1 \end{pmatrix} \in G_2$. We have $v\tau_{g'} = v\tau_g + u'\overline{\alpha}$. Clearly, we can select a $g_2 \in G_2$ for which the image of every element $v \in V_2$ under the homomorphism $\tau_{g'}$ is not equal to zero. Theorem 1 is proved.

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