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RIGID METABELIAN PRO-p-GROUPS

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A metabelian pro-p-group G is rigid if it has a normal series of the form $G = G_1 \geqslant G_2 \geqslant G_3 = 1$ such that the factor group $A = G/G_2$ is torsion-free Abelian and $C = G_2$ is torsion-free as a $\mathbb{Z}_pA$-module. If G is a non-Abelian group, then the subgroup G_2, as well as the given series, is uniquely defined by the properties mentioned. An Abelian pro-p-group is rigid if it is torsion-free, and as G_2 we can take either the trivial subgroup or the entire group. We prove that all rigid 2-step solvable pro-p-groups are mutually universally equivalent. Rigid metabelian pro-p-groups can be treated as 2-graded groups with possible gradings $(1,1)$, $(1,0)$, and $(0,1)$. If a group is 2-step solvable, then its grading is $(1,1)$. For an Abelian group, there are two options: namely, grading $(1,0)$, if $G_2 = 1$, and grading $(0,1)$ if $G_2 = G$. A morphism between 2-graded rigid pro-p-groups is a homomorphism $\varphi : G \to H$ such that $G_i\varphi \leqslant H_i$. It is shown that in the category of 2-graded rigid pro-p-groups, a coproduct operation exists, and we establish its properties.

INTRODUCTION

In [1-7], rigid solvable groups were defined and explored, and many aspects of algebraic geometry over such groups were studied. Important examples of rigid groups are free solvable groups. In [8], by analogy with abstract groups [9, 10], foundations of algebraic geometry over profinite groups, in particular, over pro-p-groups, were laid and a number of general facts were proved, which will be used below. Relevant information on profinite groups can be found in [11].

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An extension of the concept of a rigid group to pro-p-groups involves some difficulties. Therefore, we confine ourselves to the case of metabelian pro-p-groups where these problems do not arise. Thus we say that a metabelian pro-p-group G is *rigid* if it has a normal series of the form

$$G = G_1 \geqslant G_2 \geqslant G_3 = 1 \tag{1}$$

such that the factor group $A = G/G_2$ is torsion-free Abelian and $C = G_2$ is torsion-free as a $\mathbb{Z}_pA$-module. Recall that the group algebra $\mathbb{Z}_pA$ is an algebra of power series in some (converging to zero) set of commuting variables. If G is a non-Abelian group, then the subgroup C, as well as series (1), is uniquely defined by these properties, since C coincides with the centralizer of any nontrivial commutator of two elements of G. Consequently, C is a characteristic subgroup. An Abelian pro-p-group is rigid if it is torsion-free, and as G_2 in it we can take either the trivial subgroup or the entire group. That free metabelian pro-p-groups are rigid can be derived from the construction of the Magnus embedding (see [12, 13]).

The objective of the present paper is to carry over some important facts on abstract rigid groups to metabelian rigid pro-p-groups. In [14], it was proved that (abstract) metabelian groups that are universally equivalent to a free metabelian group are exactly (in our terminology) rigid 2-step solvable groups. We show that for pro-p-groups, the following holds:

THEOREM 1. All rigid 2-step solvable pro-p-groups are mutually universally equivalent.

In connection with Theorem 1, it is worth observing that the concept of a term and also the concept of a universal theory in profinite groups are defined in a slightly different manner compared to how these are defined in abstract groups (see [8]). As distinct from the abstract case, we do not know whether a pro-p-group universally equivalent to a rigid 2-step solvable pro-p-group will be rigid itself.

By analogy with [4], rigid metabelian pro-p-groups can be treated as 2-graded groups with possible gradings (1,1), (1,0), and (0,1). If a group is 2-step solvable, then its grading is (1,1). For an Abelian group, there are two options depending on the choice of series (1): grading (1,0), if $G_2 = 1$, and grading (0,1) if $G_2 = G$. A *morphism* between 2-graded rigid pro-p-groups with respective series of form (1) is a homomorphism $\varphi : G \to H$ such that $G_i\varphi \leqslant H_i$ $(i = 1, 2, 3)$. We will prove that in the category of 2-graded rigid pro-p-groups, a coproduct operation exists. Theorem 2 below and Theorem 1 (on abstract graded rigid groups) in [4] have similar formulations.

THEOREM 2. Let G and H be two 2-graded rigid pro-p-groups. Then there exists a 2-graded rigid pro-p-group $G \circ H$, which is called a 2-rigid product of G and H, satisfying the following conditions:

(1) G and H embed in $G \circ H$ and generate this group;

(2) arbitrary homomorphisms

$$\gamma_1 : G \to L,\ \gamma_2 : H \to L$$

of 2-graded rigid pro-p-groups extend to a homomorphism of the form

$$\gamma : G \circ H \to L.$$

COROLLARY 1. (1) The group $G \circ H$ is defined by conditions (1) and (2) uniquely up to isomorphism between 2-graded rigid pro-p-groups.

(2) The operation $\circ$, if treated as a coproduct operation, is commutative and associative.

(3) Let $F_1, \dots, F_n$ be free one-generated pro-p-groups with grading $(1, 0)$. Then their 2-rigid product $F_1 \circ \dots \circ F_n$ is a free metabelian pro-p-group of rank n.

Proof. We verify item (3) only. It suffices to note that any collection of homomorphisms F_i into an arbitrary 2-rigid pro-p-group G, in particular, into a free metabelian pro-p-group, extends to a homomorphism $F_1 \circ \dots \circ F_n \to G$.

1. AUXILIARY DEFINITIONS AND FACTS

1.1. Assume that a metabelian pro-p-group G has a normal Abelian subgroup C and $\overline{G} = G/C$ is an Abelian group. Set $\overline{g} = gC$ for $g \in G$. The group G acts by conjugations $x \to x^g = g^{-1}xg$ on C. Clearly, in fact, $\overline{G}$ acts and C can be treated as a right topological $\mathbb{Z}_p\overline{G}$-module. Suppose also that there is a pro-p-group which decomposes into a semidirect product of its subgroup $\overline{G}$ and some normal Abelian subgroup $D(G)$, which has the following matrix representation: $\begin{pmatrix} \overline{G} & 0 \\ D(G) & 1 \end{pmatrix}$. We call the last group a *splitting of* G *over* C if an embedding of G in it is specified so that $g = \begin{pmatrix} \overline{g} & 0 \\ d(g) & 1 \end{pmatrix}$, and $D(G)$ is generated as a $\mathbb{Z}_p\overline{G}$-module by elements $d(g)$, $g \in G$.

The splitting $\begin{pmatrix} \overline{G} & 0 \\ D(G) & 1 \end{pmatrix}$ is said to be *free* if, for any epimorphism $\gamma : G \to H$, where the group H has a normal Abelian subgroup L and $C\gamma \leqslant L$, and for any splitting $\begin{pmatrix} \overline{H} & 0 \\ D(H) & 1 \end{pmatrix}$ of the group H over L, the mapping $d(g) \to d(g\gamma)$ determines a module epimorphism $D(G) \to D(H)$, which agrees with a ring epimorphism $\mathbb{Z}_p\overline{G} \to \mathbb{Z}_p\overline{H}$. Clearly, this gives rise to the splitting epimorphism

$$\begin{pmatrix} \overline{G} & 0 \\ D(G) & 1 \end{pmatrix} \to \begin{pmatrix} \overline{H} & 0 \\ D(H) & 1 \end{pmatrix},$$

whose restriction to G coincides with γ. General considerations imply that if a free splitting exists then it is defined uniquely up to isomorphism. Hence, for two free splittings

$$\begin{pmatrix} \overline{G} & 0 \\ D(G) & 1 \end{pmatrix}, \begin{pmatrix} \overline{G} & 0 \\ D_1(G) & 1 \end{pmatrix},$$

the mapping $d(g) \to d_1(g)$ determines a module isomorphism $D(G) \to D_1(G)$, which in turn yields a group isomorphism

$$\begin{pmatrix} \overline{G} & 0 \\ D(G) & 1 \end{pmatrix} \to \begin{pmatrix} \overline{G} & 0 \\ D_1(G) & 1 \end{pmatrix}.$$

We say that the *splitting*

$$\begin{pmatrix} \overline{G} & 0 \\ D(G) & 1 \end{pmatrix}$$

has a differential if the mapping $d(g) \to \overline{g} - 1$ determines an epimorphism (a differential) δ of the module $D(G)$ onto the difference ideal $(\overline{G} - 1) \cdot \mathbb{Z}_p\overline{G}$ of a group ring $\mathbb{Z}_p\overline{G}$, and the kernel of this epimorphism is C (here C is naturally identified with a submodule of $D(G)$).

Based on the Magnus embedding [12, 13], we construct a particular splitting with differential, which will be called the *Magnus splitting*. To do this, we represent a group G as the factor group of a free pro-p-group F with basis $\{x_i \mid i \in I\}$ (converging to one). Let $\varphi_1 : F \to G$ and $\varphi_2 : F \to \overline{G}$ be canonical epimorphisms and $g_i = x_i\varphi_1$, $i \in I$. Denote by T a right free topological $\mathbb{Z}_p\overline{G}$-module with basis $\{t_i \mid i \in I\}$ (converging to zero). Consider a module epimorphism $\psi : T \to (\overline{G} - 1) \cdot \mathbb{Z}_p\overline{G}$, which is defined by a formula $(\sum t_i u_i)\psi = \sum(\overline{g}_i - 1)u_i$. Also consider the pro-$p$-group Magnus homomorphism

$$\tau : F \to \begin{pmatrix} \overline{G} & 0 \\ T & 1 \end{pmatrix}$$

defined by the mapping

$$x_i \to \begin{pmatrix} \overline{g}_i & 0 \\ t_i & 1 \end{pmatrix}, \; i \in I.$$

In view of the properties of the Magnus embedding, $\ker \tau \leqslant \ker \varphi_1 \leqslant \ker \varphi_2$ and

$$(\ker \varphi_2)\tau = \begin{pmatrix} 1 & 0 \\ U & 1 \end{pmatrix},$$

where $U = \ker \psi$. Consequently,

$$(\ker \varphi_1)\tau = \begin{pmatrix} 1 & 0 \\ U_1 & 1 \end{pmatrix},$$

where U_1 is some submodule of U. By construction, if we put $D(G) = T/U_1$ then the group G embeds in $\begin{pmatrix} \overline{G} & 0 \\ D(G) & 1 \end{pmatrix}$. Under this embedding, the image of C equals $\begin{pmatrix} 1 & 0 \\ U/U_1 & 1 \end{pmatrix}$ and can be identified with a module U/U_1. A homomorphism $\delta : D(G) \to (\overline{G} - 1) \cdot \mathbb{Z}_p\overline{G}$ is defined via ψ, and the kernel of δ is C. By construction, $d(g_i)\delta = \overline{g}_i - 1$ holds for generating elements g_i of the group G, and so $d(g)\delta = \overline{g} - 1$ for all $g \in G$.

Proofs for the two splitting lemmas below repeat verbatim the proofs of appropriate statements for abstract groups, given in [5].

LEMMA 1. For a given splitting $\begin{pmatrix} \overline{G} & 0 \\ D(G) & 1 \end{pmatrix}$ of a pro-p-group G over C, the following conditions are equivalent:

(1) the splitting is free;

(2) the splitting has a differential;

(3) the splitting is isomorphic to the Magnus splitting.

LEMMA 2. A free splitting of any subgroup $H \leqslant G$ over $H \cap C$ is induced by a free splitting of G over C. Hence, if $\begin{pmatrix} \overline{G} & 0 \\ D(G) & 1 \end{pmatrix}$ is a free splitting of G and $D(H)$ is a $\mathbb{Z}_p\overline{H}$-submodule of $D(G)$ generated by elements $d(h)$, $h \in H$, then $\begin{pmatrix} \overline{H} & 0 \\ D(H) & 1 \end{pmatrix}$ is a free splitting of H.

COROLLARY 2. Let G be a rigid metabelian pro-p-group with a respective series of form (1), $C = G_2$, and $\begin{pmatrix} \overline{G} & 0 \\ D(G) & 1 \end{pmatrix}$ be a free splitting of G over C. Then the module $D(G)$ is $\mathbb{Z}_p\overline{G}$-torsion-free; i.e., the splitting is also a rigid metabelian pro-p-group.

In fact, $D(G)$ is an extension of the torsion-free module C by the torsion-free module $(\overline{G} - 1) \cdot \mathbb{Z}_p\overline{G}$.

1.2. We need to augment a group ring over which a given torsion-free module will be treated. To do this, we use the following:

LEMMA 3. Let E be a torsion-free pro-p-module over a ring $\mathbb{Z}_p[[X]]$ of formal power series in a set X (converging to zero) of commuting variables, $\mathbb{Z}_p[[X, Y]]$ be a ring of formal power series in a set $X \cup Y$ (converging to zero) of commuting variables, and $X \cap Y = \varnothing$. Consider a topological tensor product such as

$$E \bigotimes_{\mathbb{Z}_p[[X]]} \mathbb{Z}_p[[X, Y]] = E'.$$

Then E', being a $\mathbb{Z}_p[[X, Y]]$-module, is also torsion-free. If $\{M_k \mid k \in K\}$ is the set of all monomials in Y, then every element of E' is uniquely representable as $\sum\limits_k v_k M_k$, where $v_k \in E$. In particular, E embeds in E'.

Proof. In a standard manner, the argument reduces to the case where X and Y are finite sets. Clearly,

$$\mathbb{Z}_p[[X, Y]] = \mathbb{Z}_p[[X]] \bigotimes_{\mathbb{Z}_p} \mathbb{Z}_p[[Y]],$$

and so

$$E \bigotimes_{\mathbb{Z}_p[[X]]} \mathbb{Z}_p[[X, Y]] = E \bigotimes_{\mathbb{Z}_p} \mathbb{Z}_p[[Y]].$$

The module E, being a $\mathbb{Z}_p$-module, is free, with basis $\{e_i \mid i \in I\}$ (converging to zero). The totality $\{M_k \mid k \in k\}$ of all monomials in Y form a basis for $\mathbb{Z}_p[[Y]]$ treated as a $\mathbb{Z}_p$-module. The monomials will be ordered lexicographically. We may assert that every element v of E' is uniquely representable as

$$v = \sum_{i,k} e_i M_k \alpha_{i,k} = \sum_k v_k M_k,$$

where

$$\alpha_{i,k} \in \mathbb{Z}_p,\ v_k = \sum_i e_i \alpha_{i,k} \in E.$$

Let

$$v \neq 0,\ 0 \neq \beta = \sum_k \beta_k M_k \in \mathbb{Z}_p[[X,Y]],\ \beta_k \in \mathbb{Z}_p[[X]].$$

Suppose that M_{k_1} and M_{k_2} are minimal monomials occurring in the decompositions of v and β, respectively, with nonzero coefficients v_{k_1} and β_{k_2}. Then the minimal monomial $M_{k_1}M_{k_2}$ will occur in the decomposition of $v\beta$ with a nonzero coefficient $v_{k_1}\beta_{k_2}$. Therefore, $v\beta \neq 0$. The lemma is proved.

2. UNIVERSAL EQUIVALENCE OF RIGID METABELIAN PRO-p-GROUPS

2.1. Let $A = \langle a_1, \ldots, a_m \rangle$ be a finitely generated Abelian pro-p-group or rank m, and let

$$T = t_1 \cdot \mathbb{Z}_p A + \ldots + t_n \cdot \mathbb{Z}_p A$$

be a finitely generated free $\mathbb{Z}_p A$-module of rank n. Denote by $W_{n,m}$ a group of matrices $\begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}$.

PROPOSITION 1. Every finitely generated 2-graded rigid pro-p-group embeds in a group of the form $W_{n,m}$.

Proof. Let G be a finitely generated 2-graded rigid pro-p-group. If G is an Abelian group with grading (1,0) or (0,1), then everything is obvious: we embed G in A in the former case and embed G in the additive group of the module T in the latter case. Let G be a 2-step solvable group, and let C be a normal Abelian subgroup such as in the definition of rigidity. Consider a free splitting $\begin{pmatrix} \overline{G} & 0 \\ D(G) & 1 \end{pmatrix}$ of G over C. Set $A = \overline{G}$, which is a free Abelian pro-p-group of finite rank. We know that $D(G)$ is a finitely generated torsion-free $\mathbb{Z}_p A$-module. Therefore, it suffices to embed $D(G)$ in a free module of finite rank.

Now we represent $D(G)$ as a factor module T/U, where

$$T = t_1 \cdot \mathbb{Z}_p A + \ldots + t_n \cdot \mathbb{Z}_p A$$

is a free $\mathbb{Z}_p A$-module of rank n and U is an isolated submodule. Let $\{u_1, \ldots, u_r\}$ be a maximal system of elements of U that is linearly independent over $\mathbb{Z}_p A$. Using elementary transformations over $\mathbb{Z}_p A$ of the form $u_i \mapsto u_i\alpha + u_j\beta$, where $\alpha \neq 0$, and renaming $t_i \leftrightarrow t_j$, we can bring the given system into the form

$$\begin{cases} u_1' = t_1\beta + 0 + \ldots + 0 - t_{r+1}\beta_{1,r+1} - \ldots - t_n\beta_{1,n}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ u_r' = 0 + \ldots + 0 + t_r\beta - t_{r+1}\beta_{r,r+1} - \ldots - t_n\beta_{r,n}, \end{cases}$$

where $\beta \neq 0$. Consequently, the following relations hold in T/U:

$$\begin{cases} t_1\beta = t_{r+1}\beta_{1,r+1} + \ldots + t_n\beta_{1,n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ t_r\beta = t_{r+1}\beta_{r,r+1} + \ldots + t_n\beta_{r,n}. \end{cases}$$

Now we embed T in the vector space

$$\overline{T} = t_1 \cdot Q(A) + \ldots + t_n \cdot Q(A)$$

over the field of fractions, $Q(A)$, of a ring $\mathbb{Z}_pA$. Let $\overline{U}$ be a subspace generated by U, and also by elements $u'_1, \ldots, u'_r$. Since U is an isolated submodule, $T \cap \overline{U} = U$. Clearly, $\dim(\overline{T}/\overline{U}) = n - r$ and images of elements $t_{r+1}, \ldots, t_n$ constitute a basis for $\overline{T}/\overline{U}$. The images of these elements in $\overline{T}/\overline{U}$ will likewise be denoted by $t_1, \ldots, t_n$. We have

$$\begin{cases} t_1 = (t_{r+1}\beta^{-1})\beta_{1,r+1} + \ldots + (t_n\beta^{-1})\beta_{1,n}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ t_r = (t_{r+1}\beta^{-1})\beta_{r,r+1} + \ldots + (t_n\beta^{-1})\beta_{r,n}. \end{cases}$$

Therefore, the module T/U embeds in a free module with generators $t_{r+1}\beta^{-1}, \ldots, t_n\beta^{-1}$. The proposition is proved.

We need the following simple fact.

LEMMA 4. Let F be a field, B an infinite subset in F, and

$$v_1 = (v_{11}, \ \ldots, \ v_{1k}), \ \ldots, \ v_q = (v_{q1}, \ldots, v_{qk})$$

nonzero rows in F^k. Then there exists a tuple $(\beta_1, \ldots, \beta_k) \in B^k$ such that all linear combinations $\beta_1 v_{i1} + \ldots + \beta_k v_{ik}$ are distinct from zero, where $1 \leq i \leq q$.

PROPOSITION 2. A group $W_{n,m}$ is discriminated by a group $W_{1,1}$.

Proof. First we show that the group $W_{n,m}$ is discriminated by a group $W_{n,1} = \begin{pmatrix} \langle a \rangle & 0 \\ S & 1 \end{pmatrix}$, where $S = s_1 \cdot \mathbb{Z}_p\langle a \rangle + \ldots + s_n \cdot \mathbb{Z}_p\langle a \rangle$ is a free $\mathbb{Z}_p\langle a \rangle$-module.

The algebra $\mathbb{Z}_pA$ coincides with an algebra $\mathbb{Z}_p[[y_1, \ldots, y_m]]$ of formal power series in variables $y_i = a_i - 1$, and the algebra $\mathbb{Z}_p\langle a \rangle$ coincides with an algebra $\mathbb{Z}_p[[y]]$ of series in a variable $y = a - 1$.

Consider a set of q nonidentity elements of the group $W_{n,m}$, which we want to discriminate: i.e.,

$$\left\{ w_i = \begin{pmatrix} z_i & 0 \\ t_1 u_{i1} + \ldots + t_n u_{in} & 1 \end{pmatrix} \neq 1 \ \middle| \ u_{ij} \in \mathbb{Z}_p[[y_1, \ldots, y_m]], \ 1 \leqslant i \leqslant q, \ 1 \leqslant j \leqslant n \right\}.$$

The following system of disjunctions holds:

$$\begin{cases} u_{11} \neq 0 \ \vee \ \ldots \ \vee \ u_{1n} \neq 0 \ \vee \ z_1 - 1 \neq 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ u_{q1} \neq 0 \ \vee \ \ldots \ \vee \ u_{qn} \neq 0 \ \vee \ z_q - 1 \neq 0. \end{cases} \tag{2}$$

The inequality $z_i - 1 \neq 0$ can be treated as $u_{i,n+1} \neq 0$. Since system (2) is satisfied, there exist nonzero members among elements of each of the sets $U_i = \{u_{i1}, u_{i2}, \ldots, u_{in}, u_{i,n+1}\}$. In these, we choose minimal nonzero homogeneous components.

Every mapping $(1+y_i) \mapsto (1+y)^{\alpha_i} = 1+\alpha_i y + \ldots$, where $\alpha_i \in \mathbb{Z}_p$, $1 \le i \le m$, determines a homomorphism $\langle a_1, \ldots, a_m\rangle \to \langle a\rangle$ between pro-p-groups and a homomorphism $\varphi : \mathbb{Z}_p[[y_1, \ldots, y_m]] \to \mathbb{Z}_p[[y]]$ between algebras of formal power series, and then a group homomorphism $W_{n,m} \to W_{n,1}$ arises under a mapping $t_j \mapsto s_j$, $1 \le j \le n$. Denote by v_{ij} the image of an element u_{ij} under the homomorphism φ.

Let $f(y_1, \ldots, y_m)$ be a homogeneous polynomial of degree l. Its image $f(y_1, \ldots, y_m)\varphi$ has a minimal homogeneous component $f(\alpha_1, \ldots, \alpha_m)y^l$. Obviously, there are values $\alpha_1, \ldots, \alpha_m \in \mathbb{Z}_p$ such that $f(\alpha_1, \ldots, \alpha_m) \neq 0$. If as the polynomial $f(y_1, \ldots, y_m)$ we take a product of homogeneous components of least degrees of nonzero elements u_{ij}, then, under an appropriate homomorphism φ, the image v_{ij} of a nonzero element u_{ij} will be nonzero.

From (2), we derive

$$\begin{cases} v_{11} \neq 0 \quad \vee \quad \ldots \quad \vee \quad v_{1n} \neq 0 \quad \vee \quad v_{1,n+1} \neq 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ v_{q1} \neq 0 \quad \vee \quad \ldots \quad \vee \quad v_{qn} \neq 0 \quad \vee \quad v_{q,n+1} \neq 0. \end{cases}$$

Consequently, the images of all elements of the set $\{w_i \mid 1 \leqslant i \leqslant q\}$ are distinct from 1. Hence the group $W_{n,m}$ is discriminated by the group $W_{n,1}$.

It remains to prove that $W_{n,1}$ is discriminated by the group

$$W_{1,1} = \begin{pmatrix} \langle a\rangle & 0 \\ t \cdot \mathbb{Z}_p\langle a\rangle & 1 \end{pmatrix}.$$

Consider q nonidentity elements of $W_{n,1}$,

$$\left\{ \begin{pmatrix} z_i & 0 \\ s_1 v_{i1} + \ldots + s_n v_{in} & 1 \end{pmatrix} \middle| \; v_{ij} \in \mathbb{Z}_p[[y]], \; 1 \leqslant i \leqslant q, \; 1 \leqslant j \leqslant n \right\}.$$

We arrive at the following system of disjunctions:

$$\begin{cases} s_1 v_{11} \quad + \quad \ldots \quad + \quad s_n v_{1n} \quad \neq \quad 0 \quad \vee \quad z_1 \neq 1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ s_1 v_{q1} \quad + \quad \ldots \quad + \quad s_n v_{qn} \quad \neq \quad 0 \quad \vee \quad z_q \neq 1. \end{cases} \tag{3}$$

A mapping given by the rule $s_k \mapsto \beta_k t$, $a \mapsto a$, where $\beta_k \in \mathbb{Z}_p$, $1 \le k \le n$, extends naturally to a homomorphism ψ of the group $W_{n,1}$ into $W_{1,1}$.

If the inequality $z_i \neq 1$ holds, then the image of the element

$$\begin{pmatrix} z_i & 0 \\ s_1 v_{i1} + \ldots + s_n v_{in} & 1 \end{pmatrix}$$

under the homomorphism ψ will be a nonidentity element. Therefore, we assume that the first of the inequalities holds in each row in (2). Under this mapping, the nonzero elements of the module

S are sent to elements

$$\begin{cases} t\beta_1 v_{11} \quad + \quad \ldots \quad + \quad t\beta_n v_{1n}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ t\beta_1 v_{q1} \quad + \quad \ldots \quad + \quad t\beta_n v_{qn}. \end{cases} \tag{4}$$

By Lemma 4, there exist $\beta_1, \ldots, \beta_n \in \mathbb{Z}_p$ such that the elements in (4) will be nonzero. Hence the group $W_{n,1}$ is discriminated by the group $W_{1,1}$. The proposition is proved.

2.2. We recall the following definition from [8]. Let G be a pro-p-group. A *universal theory* for G is a set of formulas true on G having the form $\forall x_1, \ldots, x_n \Phi(x)$, where $\Phi(x)$ is a disjunction of a finite system of equalities and inequalities of the form $v(x) = 1$ or $v(x) \neq 1$, where $v(x)$ is an element of a free pro-p-group $\langle x_1, \ldots, x_n \rangle$.

Now we are in a position to prove Theorem 1. Propositions 1 and 2 imply that a finitely generated rigid 2-step solvable pro-p-group G is discriminated by the group $W_{1,1}$. Conversely, if in G we take a pair of elements $a \in G \setminus G_2$, $1 \neq b \in G_2$, then these generate a subgroup isomorphic to $W_{1,1}$. Consequently, the group $W_{1,1}$ is discriminated by the group G. If we consider two rigid 2-step soluble pro-p-groups, then each of them is locally discriminated by the other and the two have equal universal theories. Theorem 1 is proved.

3. COPRODUCT OF RIGID METABELIAN PRO-p-GROUPS

The proof of Theorem 2 is partially the same as in [4, proof of Thm. 1].

(1) Under the conditions of the theorem, let G_i and H_i ($i = 1, 2, 3$) be members of respective series of form (1), $A = \overline{G} = G/G_2$, and $B = \overline{H} = H/H_2$. We construct a free splitting (see Lemma 1 above) for the desired group $E = G \circ H$ given free splittings $\begin{pmatrix} A & 0 \\ D(G) & 1 \end{pmatrix}$ and $\begin{pmatrix} B & 0 \\ D(H) & 1 \end{pmatrix}$ of groups G and H over G_2 and H_2, respectively. Set $C = A \times B$. Let $\{a_i \mid i \in I\}$ be a basis for the free Abelian pro-p-group A and $\{b_j \mid j \in J\}$ be one for B; then $\{a_i, b_j \mid i \in I, j \in J\}$ is a basis of C. We have

$$\mathbb{Z}_p A = \mathbb{Z}_p[[x_i \mid i \in I]],\ \mathbb{Z}_p B = \mathbb{Z}_p[[y_j \mid j \in J]],\ \mathbb{Z}_p C = \mathbb{Z}_p[[x_i, y_j \mid i \in I, j \in J]],$$

where $a_i = 1 + x_i$ and $b_j = 1 + y_j$ ($i \in I$, $j \in J$). Consider the right $\mathbb{Z}_p C$-module

$$T = D(G) \bigotimes_{\mathbb{Z}_p A} \mathbb{Z}_p C \oplus D(H) \bigotimes_{\mathbb{Z}_p B} \mathbb{Z}_p C.$$

By Corollary 2, this module is torsion-free, and $D(G)$ and $D(H)$ embed in T. Consequently, $M = \begin{pmatrix} C & 0 \\ T & 1 \end{pmatrix}$ will be a metabelian rigid pro-p-group. Clearly, the differentials

$$\delta_G : D(G) \to (A - 1)\mathbb{Z}_p(A),\ \delta_H : D(H) \to (B - 1)\mathbb{Z}_p(B)$$

extend to a $\mathbb{Z}_pC$-module homomorphism $\delta : T \to (C-1)\mathbb{Z}_pC$. The homomorphism is surjective since the image contains sets $(A-1)\mathbb{Z}_p(A)$ and $(B-1)\mathbb{Z}_p(B)$, which generate a difference ideal $(C-1)\mathbb{Z}_p(C)$ over $\mathbb{Z}_pC$. Elements of a pro-p-group M of the form $\begin{pmatrix} c & 0 \\ t & 1 \end{pmatrix}$, where $t\delta = c-1$, constitute a subgroup denoted $G \circ H$. By construction, $G \circ H$ contains G and H as subgroups.

We prove that $G \circ H$ is generated by its subgroups G and H. It suffices to state that the submodule $\ker \delta$ of T, which is identified with some normal subgroup of M, is contained in the subgroup $\langle G, H \rangle$. The mutual commutant $[G,H]$ lies in $\ker \delta$ and is also a normal subgroup of M. Let

$$g = \begin{pmatrix} \overline{g} & 0 \\ d(g) & 1 \end{pmatrix} \in G,\ h = \begin{pmatrix} \overline{h} & 0 \\ d(h) & 1 \end{pmatrix} \in H,\ \overline{g} = a,\ \overline{h} = b.$$

Then the commutator $[g,h]$ is identified with $d(g)(b-1) - d(h)(a-1)$; so the element $d(h)(a-1)$ is comparable with $d(g)(b-1)$ modulo $[G,H]$. A $\mathbb{Z}_pC$-module $D(H) \otimes_{\mathbb{Z}_pB} \mathbb{Z}_pC$ is generated by elements of the form $d(h)$; hence it lies in $D(G) \bigotimes_{\mathbb{Z}_pA} \mathbb{Z}_pC + D(H) + [G,H]$. Consequently, an element $t \in \ker \delta$ can be represented as $t_1 + t_2 + t_3$, where $t_1 \in D(G) \bigotimes_{\mathbb{Z}_pA} \mathbb{Z}_pC$, $t_2 \in D(H)$, and $t_3 \in [G,H]$. We have $0 = t_1\delta + t_2\delta$, $t_1\delta \in (A-1)\mathbb{Z}_pC$, and $t_2\delta \in (B-1)\mathbb{Z}_pB$. Since $(A-1)\mathbb{Z}_pC \cap \mathbb{Z}_pB = 0$, it follows that $t_1\delta = 0$ and $t_2\delta = 0$. The definition of a free splitting implies that t_2 is identified with an element of H. Let $\{M_k \mid k \in k\}$ be the set of all monomials in y_i. In view of Lemma 3, each element of $D(G) \bigotimes_{\mathbb{Z}_pA} \mathbb{Z}_pC$ is uniquely representable as $\sum_k v_k M_k$, where $v_k \in D(G)$. Since $t_1\delta = 0$, all $v_k\delta$ are equal to 0. Therefore, v_k is identified with an element of G, and t_1 with an element of $\langle G, H \rangle$. Statement (1) of Theorem 2 is proved. Moreover, we can assert that the pro-p-group M constructed is a free splitting of $G \circ H$ over a normal Abelian subgroup, which is identified with $\ker \delta$.

(2) There is no loss of generality in assuming that a pro-p-group L is generated by its subgroups $G\gamma_1$ and $H\gamma_2$. Consider a free splitting $\begin{pmatrix} \overline{L} & 0 \\ D(L) & 1 \end{pmatrix}$ of the pro-p-group L over L_2, where L_2 stands for a respective member of a series of form (1). By Lemma 2, we may suppose that the free splittings

$$\begin{pmatrix} \overline{G\gamma_1} & 0 \\ D(G\gamma_1) & 1 \end{pmatrix},\ \begin{pmatrix} \overline{H\gamma_2} & 0 \\ D(H\gamma_2) & 1 \end{pmatrix}$$

of groups $G\gamma_1$ and $H\gamma_2$ over $G\gamma_1 \cap L_2$ and $H\gamma_2 \cap L_2$, respectively, are contained in $\begin{pmatrix} \overline{L} & 0 \\ D(L) & 1 \end{pmatrix}$. The epimorphisms

$$\gamma_1 : G \to G\gamma_1,\ \gamma_2 : H \to H\gamma_2$$

determine splitting epimorphisms such as

$$\begin{pmatrix} \overline{G} & 0 \\ D(G) & 1 \end{pmatrix} \to \begin{pmatrix} \overline{G\gamma_1} & 0 \\ D(G\gamma_1) & 1 \end{pmatrix},\ \begin{pmatrix} \overline{H} & 0 \\ D(H) & 1 \end{pmatrix} \to \begin{pmatrix} \overline{H\gamma_2} & 0 \\ D(H\gamma_2) & 1 \end{pmatrix},$$

which we denote by the same symbols γ_1 and γ_2.

Obviously, there exists an epimorphism

$$\overline{\gamma} : C = A \times B = \overline{G} \times \overline{H} \to \overline{L},$$

which extends

$$\overline{G} \to \overline{G\gamma_1},\ \overline{H} \to \overline{H\gamma_2}.$$

We lift it up to an epimorphism $\mathbb{Z}_p C \to \mathbb{Z}_p \overline{L}$ between group rings. By construction, a splitting of the group $M = G \circ H$ has the form $\begin{pmatrix} C & 0 \\ D(M) & 1 \end{pmatrix}$, where

$$D(M) = D(G) \bigotimes_{\mathbb{Z}_p \overline{G}} \mathbb{Z}_p C \oplus D(H) \bigotimes_{\mathbb{Z}_p \overline{H}} \mathbb{Z}C.$$

Since the group L is generated by $G\gamma_1$ and $H\gamma_2$, we have

$$D(L) = D(G\gamma_1) \cdot \mathbb{Z}_p \overline{L} + D(H\gamma_2) \cdot \mathbb{Z}_p \overline{L}.$$

The epimorphisms

$$D(G) \to D(G\gamma_1),\ D(H) \to D(H\gamma_2),\ \mathbb{Z}_p C \to \mathbb{Z}_p \overline{L}$$

yield a module epimorphism $D(M) \to D(L)$, which together with $\overline{\gamma}$ determines a pro-p-group epimorphism

$$\begin{pmatrix} C & 0 \\ D(M) & 1 \end{pmatrix} \to \begin{pmatrix} \overline{L} & 0 \\ D(L) & 1 \end{pmatrix}.$$

The last-mentioned epimorphism restricted to $G \circ H$ will be the desired homomorphism γ. Theorem 2 is proved.

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