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P-STABLE ABELIAN GROUPS

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(P, a)-stable and (P, s)-stable Abelian groups are described. It is also proved that every Abelian group is (P, p)-stable. In particular, results due to M. A. Rusaleev [6] *and T. A. Nurmagambetov* [7] *derive from these.*

INTRODUCTION

The concept of P-stability is a particular case of the concepts of T^*-stability in [1] and of E^*-stability in [2]. These, in turn, are generalizations of the classical notion of stability going back to M. Morley [3] and S. Shelah [4]. Research into P-stability deals also with the model theory for elementary pairs with which many mathematicians, such as B. Poiazat, E. Bouscaren, T. Mustafin, T. Nurmagambetov, A. Nurtazin, and others, have been concerned since 1980s. In our case the study focuses in essence on a theory for pairs of models without the requirement of being elementary for a P-submodel. In [5], it was shown that if no conditions are imposed on a predicate P then the condition of being P-stable for a theory T is equivalent to T being definably equivalent to some theory whose language consists only of unary predicate symbols. In [6], it was proved that if T is a theory for a torsion-free Abelian group then T is P-stable whenever P defines an algebraically closed subgroup. In [7], a result was announced which implies that under the generalized continuum hypothesis, a theory for any Abelian group is P-stable if P defines an elementary subsystem.

In Sec. 2, we prove that a theory for any Abelian group is P-stable if P defines a pure subgroup. This generalizes the above-mentioned results due to T. Nurmagambetov and M. Rusaleev. In Sec. 3, we describe theories for Abelian groups that are P-stable if P defines an algebraically closed subgroup. The main results of these two sections were announced in [8]. In Sec. 4, we characterize theories for Abelian groups that are P-stable if P defines an arbitrary subgroup.

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1. TERMINOLOGY, NOTATION, AND PRELIMINARY RESULTS

In what follows, by a group we always mean an Abelian group. For a group A and for a natural number n, $A[n]$ denotes a subgroup like

$$\{a \mid a \in A, na = 0\}.$$

The letter p denotes a prime number. We recall some well-known concepts from Abelian group theory. A *p-group* is one whose elements are all of order p^k for some natural k. A maximal p-subgroup of A is called a *p-component* of A and is denoted by A_p. A cyclic group of order n is denoted by C_n. An *elementary p-group* is a direct sum of cyclic groups of order p. An *elementary group* is an elementary p-group for some p. For a group A and for a cardinal λ, by $A^{<\lambda}$ we denote a direct sum of λ copies of the group A. A group A is said to be *bounded* if there exists a natural number n such that the order of any element of the group A does not exceed the given number n. We often use the following facts from Abelian group theory.

(a) If the reduced part of a p-component A_p of a group A is bounded, then it is distinguished by a direct summand in A and has the form $B \oplus C_{p^\infty}^{<\lambda}$, where B is a bounded p-group, C_{p^∞} is a quasicyclic group, and λ is some cardinal (see, e.g., [9, Chap. 5]). Below are two results that are derived using Szmielew's description of elementarily equivalent Abelian groups in [10] (see also [11, Sec. 8.4]).

(b) If the reduced part of a p-component A_p of a group A is unbounded, then the group A is elementarily equivalent to a group $A \oplus C_{p^\infty}^{<\lambda}$ for any cardinal λ.

(c) If a group A is unbounded, then the group A is elementarily equivalent to a group $A \oplus Q^{<\lambda}$, where Q is the additive group of rational numbers, for any cardinal λ.

In this section, we fix a complete theory T in a language L. For convenience, to manipulate with models of T, we also fix some sufficiently saturated model C of T and assume that all T-models under consideration are elementary submodels of C. Such a T-model C is called a *monster model* of T.

Finite sequences are called *tuples*, and we denote the set of all tuples of elements of a set U by $U^{<\omega}$. The length of a tuple $\mathbf{u}$ is denoted by $l(\mathbf{u})$. For simplicity, instead of $\mathbf{u} \in U^{<\omega}$ and $\mathbf{D} \subseteq U^{<\omega}$, we often use $\mathbf{u} \in U$ and $\mathbf{D} \subseteq U$, respectively. Tuples of elements and tuples of variables in a model are denoted by lower-case bold letters from, respectively, the beginning and the end of the Latin alphabet: for instance, $\mathbf{a}, \mathbf{b}, \ldots$ and $\ldots, \mathbf{x}, \mathbf{y}, \mathbf{z}$. If $\Phi(\mathbf{x})$ is an L-formula and A is an L-structure, then $\Phi(A)$ denotes the set $\{\mathbf{a} \mid A \models \Phi(\mathbf{a}), \mathbf{a} \in A\}$.

For an L-structure A and its subset X, by $\mathrm{acl}_A(X)$ we denote the set

$$\bigcup \{\Phi(A) \mid |\Phi(A)| < \omega,\ \Phi(x) \text{ is an } L\text{-formula with parameters in } X\}.$$

If $\mathrm{acl}_A(X) = X$, then we say that X is an algebraically closed substructure of A.

If X is a subset in the monster model C, then we call X a *set in the theory* T. By $L(X)$ we denote a language which is obtained by adding to L the set X as a set of new constants. Denote by $T(X)$ the following set of formulas in the language $L(X)$:

$$\{\varphi(\mathbf{a}) \mid \mathbf{a} \in X,\ C \models \varphi(\mathbf{a}), \varphi(\mathbf{x}) \text{ is an } L\text{-formula}\}.$$

Clearly, $T(X)$ is a complete theory in the language $L(X)$.

The concept of E^*-stability was introduced in [2]. A particular case of this concept is the concept P-stability.

Definition. Let a language L_P be obtained by adding a new unary predicate symbol P to L. Suppose Δ is some set of sentences in L_P. A theory T is said to be P_Δ-*stable in cardinality* λ if, for any set X of cardinality at most λ in T, the set

$$T_\Delta(X) = (T(X) \cup \{P(a) \mid a \in X\} \cup \Delta) \tag{1}$$

has at most λ completions in the language $(L(X))_P$.

Definition. A complete theory T is P_Δ-*stable* if it is P_Δ-stable in some infinite cardinality λ.

Definition. Let U be some set in a theory T and $\mathbf{X}$ and $\mathbf{Y}$ some sets of tuples of elements of length n in U.

(1) We say that a pair $\langle \mathbf{X}, \mathbf{Y} \rangle$ is *separable in a theory* $T(U)$ if there exists a formula $\Phi(\mathbf{z})$ in a language $L(U)$ such that $l(\mathbf{z}) = n$ and the following condition holds:

$$(\{\Phi(\mathbf{a}) \mid \mathbf{a} \in \mathbf{X}\} \cup \{\neg\Phi(\mathbf{a}) \mid \mathbf{a} \in \mathbf{Y}\}) \subseteq T(U).$$

In this event we say that $\Phi(\mathbf{z})$ *separates* $\mathbf{X}$ and $\mathbf{Y}$ (or separates the pair $\langle \mathbf{X}, \mathbf{Y} \rangle$) in T.

(2) Let Δ be some set of sentences in L_P. We say that a pair $\langle \mathbf{X}, \mathbf{Y} \rangle$ *is separable in a theory* $T_\Delta(U)$ if there exists a formula $\Phi(\mathbf{z})$ in a language $(L(U))_P$ such that $l(\mathbf{z}) = n$ and the set

$$(T_\Delta(U) \cup (\{\Phi(\mathbf{a}) \mid \mathbf{a} \in \mathbf{X}\} \cup \{\neg\Phi(\mathbf{a}) \mid \mathbf{a} \in \mathbf{Y}\}))$$

is compatible. In this case we say that $\Phi(\mathbf{z})$ *separates* $\mathbf{X}$ and $\mathbf{Y}$ (or separates the pair $\langle \mathbf{X}, \mathbf{Y} \rangle$) in $T_\Delta(U)$.

We cite the main theorem from [2] adapted to our case of E^*-presentations.

THEOREM 1. Let T be a complete theory in a language L and Δ some set of sentences in a language L_P. Then the following conditions are equivalent:

(a) T is P_Δ-stable;

(b) for any set U in T, each pair $\langle \mathbf{X}, \mathbf{Y} \rangle$ of sets of tuples of elements of equal length in U that is separable in $T_\Delta(U)$ is separable in a theory $T(U)$.

In [2], note, instead of a set in a theory T, we used a set of variables realizing some complete type for a language L. Clearly, these concepts are equivalent.

We consider several important cases of P_Δ-stability.

(1) P_{Δ_1}-stability for $\Delta_1 = \varnothing$ is called $(P, 1)$-*stability.*

(2) A theory T is said to be (P, s)-*stable* if T is P_{Δ_s}-stable for a set Δ_s consisting of sentences expressing the fact that a predicate P is closed with respect to functions definable by function symbols of L; i.e., P is a substructure.

(3) A theory T is said to be (P, a)-*stable* if T is P_{Δ_a}-stable for a set Δ_a consisting of sentences expressing the fact that a predicate P is an algebraically closed set; i.e. it contains all finite sets definable in the structure C by L-formulas with parameters in the predicate P.

(4) A theory T is said to be (P, e)-*stable* if T is P_{Δ_e}-stable for a set Δ_e consisting of sentences expressing the fact that a predicate P is an elementary substructure.

Clearly, the following implications hold:

$(P, 1)$-stability $\Rightarrow$ (P, s)-stability $\Rightarrow$ (P, a)-stability $\Rightarrow$ (P, e)-stability.

In [5], it was proved that a complete theory T is $(P, 1)$-stable iff T is definably equivalent to some theory whose language consists of unary predicate symbols only. Obviously, a theory for any infinite Abelian group cannot be interpreted within the theory of unary predicates. Below we will show that any complete theory of Abelian groups is (P, e)-stable. Out of the above four types of P-stability for Abelian groups, only (P, s)- and (P, a)-stabilities may be nontrivial. In the next sections, we give a complete description of Abelian groups with these P-stability properties.

A *primitive formula* in a language L is a formula of the form

$$\exists x_1 \ldots \exists x_n \Phi,$$

where Φ is a conjunction of atomic formulas in the language L. A set X of tuples in a structure A is *primitive* if $X = \Phi(A)$ for some primitive formula $\Phi(\mathbf{x})$.

In what follows, L denotes a language of the theory of Abelian groups, consisting of a binary function symbol $+$, a unary function symbol $-$, and a constant symbol 0. As usual, for terms t and q in L, $(t - q)$ stands for the term $(t + (-q))$. By AG we denote the theory of all Abelian groups defined by ordinary axioms for Abelian groups in L.

The following lemma is proved in exactly the same way as the corresponding lemma for modules (see, e.g., [11, 12]).

LEMMA 1. Let A be an Abelian group, P its subgroup, $\Phi(\mathbf{x}; \mathbf{y})$ a primitive L_P-formula, $l(\mathbf{x}) = n$, $l(\mathbf{y}) = m$, $\mathbf{b} \in A$, $l(\mathbf{b}) = m$, and $\mathbf{0}$ an m-tuple consisting of zeros in the group A. Then a formula $\Phi(A; \mathbf{0})$ defines a subgroup of Cartesian power $\langle A, P\rangle^n$, while a formula $\Phi(\mathbf{x}; \mathbf{b})$ defines either the empty set or a conjugacy class with respect to the subgroup $\Phi(A; \mathbf{0})$.

This immediately implies the following:

LEMMA 2. Let A be an Abelian group, P its subgroup, and $\Phi(\mathbf{x})$ a primitive L_P-formula with parameters in A. Suppose also that $\langle A, P\rangle \models \Phi(\mathbf{a})$ holds for some tuple $\mathbf{a}$. Then there exists a primitive L_P-formula $\Psi(\mathbf{x})$ with parameters in $\mathbf{a}$ which defines in $\langle A, P\rangle$ the same predicate as is defined by $\Phi(\mathbf{x})$.

In view of Lemma 1, a proof for the next lemma does not differ in essence from the proof of the corresponding lemma for complete theories of Abelian groups or modules (see [12, Thm. 1.1; 11, Lemma 8.4.5]).

LEMMA 3. Let A be an Abelian group and P its subgroup. In a theory $\mathrm{Th}(\langle A; P\rangle)$, every L_P-formula is equivalent to a Boolean combination of primitive L_P-formulas.

If, in the previous three lemmas, as P we take a zero subgroup, then we obtain statements for Abelian groups in L without a predicate P. Below, without further comment, we will use these particular formulations of the given lemmas.

A useful consequence of Theorem 1 is

COROLLARY 1. Let T be a complete theory of Abelian groups and Δ some set of sentences in L_P. Then the following conditions are equivalent:

(1) T is P_Δ-stable.

(2) For any model $M = \langle A, P\rangle$ of a theory $T^* = (T \cup \Delta)$ and for an arbitrary primitive L_P-formula $\Phi(\mathbf{x})$, there exists an L-formula $\Psi(\mathbf{x})$ with parameters in $P(M)$ that defines the same predicate on the set $P(M)$ in the structure M as is defined by $\Phi(\mathbf{x})$ in the structure $M = \langle A, P\rangle$.

(3) For any model $M = \langle A, P\rangle$ of a theory $T^* = (T \cup \Delta)$ and for an arbitrary L_P-formula $\Phi(\mathbf{x})$, there exists an L-formula $\Psi(\mathbf{x})$ with parameters in $P(M)$ that defines the same predicate on the set $P(M)$ in the structure M as is defined by $\Phi(\mathbf{x})$ in the structure $M = \langle A, P\rangle$.

Proof. (3)$\Rightarrow$(2) Is trivial.

(2)$\Rightarrow$(3) Follows from Lemma 3.

It remains to show that condition (b) in Theorem 1 is equivalent to condition (3) in the corollary.

(b)$\Rightarrow$(3). Consider an arbitrary model M of T^* and a formula $\Phi(\mathbf{x})$ in L_P. As a set U we take $P(M)$ for which

$$T(U) = \{\varphi(\mathbf{a}) \mid M \models \varphi(\mathbf{a}),\ \varphi(\mathbf{x}) \text{ is an } L\text{-formula},\ \mathbf{a} \in U\}.$$

As sets X and Y of tuples we consider $\{\mathbf{a} \mid \langle A, P\rangle \models \Phi(\mathbf{a}), \mathbf{a} \in U\}$ and $\{\mathbf{a} \mid \langle A, P\rangle \models \neg\Phi(\mathbf{a}), \mathbf{a} \in U\}$, respectively. By property (b), there exists an L-formula $\Psi(\mathbf{x})$ with parameters in U that defines a pair $\langle X, Y\rangle$ in T. Clearly, this formula will define the same predicate on $P(M)$ in M as is defined by $\Phi(\mathbf{x})$.

(3)$\Rightarrow$(b). Let a set U in T and a pair $\langle \mathbf{X}, \mathbf{Y}\rangle$ of sets of tuples of elements in U be given. Suppose that $\Phi(\mathbf{x}; \mathbf{y})$ is an L_P-formula, $\mathbf{b}$ is a tuple of elements in U, $l(\mathbf{b}) = l(\mathbf{y})$, and X and Y are separated by a formula $\Phi(\mathbf{x}; \mathbf{b})$ in a theory $T_\Delta(A)$. By the definition of separability in $T_\Delta(A)$, the set

$$W = (T_\Delta(U) \cup (\{\Phi(\mathbf{a}; \mathbf{b}) \mid \mathbf{a} \in \mathbf{X}\} \cup \{\neg\Phi(\mathbf{a}; \mathbf{b}) \mid \mathbf{a} \in \mathbf{Y}\}))$$

is compatible. Take an arbitrary model M of the set W. By property (3), there exists an L-formula $\Psi(\mathbf{x}; \mathbf{y})$ with parameters in $P(M)$ that defines the same predicate on the set $P(M)$ in M as is defined by $\Phi(\mathbf{x}; \mathbf{y})$. By the definition of a theory $T_\Delta(U)$, we have $U \subseteq P(M)$, and hence $\mathbf{b} \in P(M)$. Obviously, a formula $\Psi(\mathbf{x}; \mathbf{b})$ will separate X and Y in T.

The formula $\Psi(\mathbf{x};\mathbf{y})$ may have parameters that do not enter the set U. We show that there exists a formula $\Theta(\mathbf{x})$ with parameters in U that separates X and Y in T. By Lemma 3, we may assume that the formula $\Psi(\mathbf{x};\mathbf{b})$ is a disjunction of formulas $\varphi_1(\mathbf{x}),\ldots,\varphi_k(\mathbf{x})$ each of which is a conjunction of primitive formulas and their negations. A conjunction of primitive formulas is equivalent to one primitive formula. Therefore, we may assume that formulas $\varphi_i(\mathbf{x})$ have the form

$$(\Phi(\mathbf{x}) \wedge \neg\Psi_1(\mathbf{x}) \wedge \ldots \wedge \neg\Psi_n(\mathbf{x})),$$

where $\Phi(\mathbf{x}),\Psi_1(\mathbf{x}),\ldots,\Psi_n(\mathbf{x})$ are primitive L-formulas (with parameters). In view of Lemma 2, it suffices to show that each of the formulas $\Phi(\mathbf{x}),\Psi_1(\mathbf{x}),\ldots,\Psi_n(\mathbf{x})$ has a solution in the set $(X\cup Y)$. If $\Phi(\mathbf{x})$ has no solution in X, then the disjunctive term $\varphi_i(\mathbf{x})$ of the formula $\Psi(\mathbf{x};\mathbf{b})$ can be removed from the latter formula without violating the separability of X and Y via the thus obtained new formula. If $\Psi_j(\mathbf{x})$ has no solution in Y, then the formula $\neg\Psi_j(\mathbf{x};\mathbf{b})$ can be removed from $\varphi_i(\mathbf{x})$ without violating the separability of X and Y via the thus obtained new formula. □

Note: The reader unacquainted with [2] may well take the readily grasped property (3) in Corollary 1 to be the definition of P_Δ-stability for a theory T.

2. (P,p)-STABILITY OF ABELIAN GROUPS

The next lemma goes back to [10]; in the given form, it is contained in [11, Lemma 8.4.7].

LEMMA 4. Any primitive formula $\Phi(x_1,\ldots,x_n)$ in the language of Abelian groups is equivalent in the theory AG to a conjunction $\Psi(x_1,\ldots,x_n)$ of formulas like $\alpha_1x_1+\ldots+\alpha_nx_n=0$ and $\exists y\alpha_1x_1+\ldots+\alpha_nx_n=p^ky$ for integers $\alpha_1,\ldots,\alpha_n$, prime numbers p, and natural numbers k, which are called *standard formulas* of, respectively, the *first* and *second kind*. In this case a prime p in a standard formula of the second kind is referred to as a *module* of that formula.

As is known, a subgroup P of a group A is said to be *pure* if $nP=(P\cap nA)$ for any natural number n. This property is equivalent to the fact that $p^kP=(P\cap p^kA)$ for any prime p and for an arbitrary natural k.

COROLLARY 2. Let A be an Abelian group, P its pure subgroup, and $\Phi(x_1,\ldots,x_n)$ a primitive formula in the language of Abelian groups. Then, for any $a_1,\ldots,a_n\in P$, the following condition holds:

$$A\models\Phi(a_1,\ldots,a_n)\Leftrightarrow P\models\Phi(a_1,\ldots,a_n). \tag{2}$$

Proof. By virtue of Lemma 4, we may assume that the formula Φ is a conjunction of standard formulas. Therefore, we will think of Φ as a standard formula. If Φ is a standard formula of the first kind, then condition (2) is satisfied, since Φ is a quantifier-free formula. If Φ is a standard formula of the second kind, then condition (2) is satisfied, since the subgroup P is pure in A. □

Remark 1. We may assume that a primitive L_P-formula $\Phi(\mathbf{x})$ has the form

$$\exists y_1\ldots\exists y_k(P(y_1)\wedge\ldots\wedge P(y_k)\wedge\Psi),$$

where Ψ is a primitive L-formula. Indeed, if Φ contains a subformula of the form $P(t)$ for some term t, then Φ is equivalent to a formula $\exists y(P(y) \wedge \Phi^*)$, where the variable y does not occur in Φ, while Φ^* is obtained by replacing $P(t)$ with $y = t$ in Φ.

Definition. A theory T is said to be (P,p)-*stable* if T is P_{Δ_p} stable for a set Δ_p consisting of sentences expressing the fact that P is a pure subgroup.

In [8], note, (P,p)-stability as defined above was called (P,s)-stability.

THEOREM 2. Every complete theory T of Abelian groups is (P,p)-stable.

Proof. Let $\Phi(x_1,\ldots,x_n)$ be a primitive L_P-formula. By Remark 1, we may assume that $\Phi(x_1,\ldots,x_n)$ has the form

$$\exists y_1 \ldots \exists y_k (P(y_1) \wedge \ldots \wedge P(y_k) \wedge \Psi),$$

where $\Psi(x_1,\ldots,x_n;y_1,\ldots,y_k)$ is a primitive L-formula.

Let A be a model of a theory T and P a pure subgroup of A. In view of Corollary 2, any elements $a_1,\ldots,a_n,b_1,\ldots,b_k \in P$ satisfy the property

$$A \models \Psi(a_1,\ldots,a_n;b_1,\ldots,b_k) \Leftrightarrow P \models \Psi(a_1,\ldots,a_n;b_1,\ldots,b_k). \tag{3}$$

This yields the equivalence

$$\langle A,P\rangle \models \Phi(a_1,\ldots,a_n) \Leftrightarrow P \models \exists y_1 \ldots \exists y_k \Psi(a_1,\ldots,a_n;y_1,\ldots,y_k) \tag{4}$$

for all elements $a_1,\ldots,a_n \in P$. By Corollary 2, any elements $a_1,\ldots,a_n \in P$ satisfy the property

$$\begin{aligned} &P \models \exists y_1 \ldots \exists y_k \Psi(a_1,\ldots,a_n;y_1,\ldots,y_k) \\ \Leftrightarrow\ &A \models \exists y_1 \ldots \exists y_k \Psi(a_1,\ldots,a_n;y_1,\ldots,y_k). \end{aligned} \tag{5}$$

In view of (4) and (5), an L-formula $\exists y_1 \ldots \exists y_k \Psi(x_1,\ldots,x_n;y_1,\ldots,y_k)$ in the group A defines on $P(A)$ the same predicate as is defined by the formula $\Phi(x_1,\ldots,x_n)$ in the structure $\langle A;P\rangle$. Now the required result follows from Corollary 1. □

COROLLARY 3 [6]. A torsion-free Abelian group is (P,a)-stable.

Proof. In a torsion-free Abelian group A, for any element $a \in A$ and for an arbitrary natural n, there exists at most one $b \in A$ with $nb = a$. Therefore, every algebraically closed torsion-free subgroup of A is pure, and we can apply Theorem 2. □

3. (P,a)-STABLE ABELIAN GROUPS

Recall that a substructure B of a structure A is said to be *algebraically closed* if B contains every finite set $X \subseteq A$ definable in A by a formula $\Phi(x)$ with parameters in the substructure B. Below the term an *algebraically closed subgroup* will be understood in just this sense.

LEMMA 5. For any group A and for a set $X \subseteq A$, the set $\mathrm{acl}_A(X)$ is a union of finite primitive sets definable by primitive formulas with parameters in the set X.

Proof. Lemma 3 implies that $\mathrm{acl}_A(X)$ is a union of finite sets U definable over X via a conjunction of a primitive formula $\Phi(x)$ and negations of primitive formulas $\Psi_1(x), \ldots, \Psi_k(x)$. Thus the set $\Phi(A)$ is covered by sets $\Psi_1(A), \ldots, \Psi_k(A)$ and finitely many singletons. Recall that nonempty primitive sets are conjugacy classes with respect to some subgroup. If the set $\Phi(A)$ were infinite, then singleton sets would have infinite index in $\Phi(A)$, and by Neumann's lemma, would be inessential in this cover; i.e., the set U would turn out to be empty. □

An upper bound for element orders of a group A of the form p^k for a natural number k is called the *p-height* of the group A. If there is no such bound, then we say that the p-height of A is infinite.

In proofs of the next two lemmas, use is made of the following:

Remark 2. Let A be a group and X and Y its infinite disjoint subsets; moreover, every permutation of a set $Z = (X \cup Y)$ will extend to an automorphism of the group A. Then a pair $\langle X, Y \rangle$ is inseparable in a theory $T(Z)$, where $T = \mathrm{Th}(A)$.

Indeed, let $\varphi(x)$ be an L-formula with parameters in a finite set $U \subseteq Z$ and let $A \models \varphi(a)$ for some $a \in (Z \setminus U)$. Then $A \models \varphi(b)$ for any $b \in (Z \setminus U)$, if we choose an automorphism of A that translates a into b and keeps elements of the set U fixed.

LEMMA 6. If the subgroup $A(p) = (A[p] \cap pA)$ of A is infinite, then the theory $T = \mathrm{Th}(A)$ is not (P, a)-stable. Moreover, for any infinite cardinality λ, there exists a set X of cardinality λ in T for which the set $T_{\Delta_a}(X)$ has 2^λ completions in the language L_P.

Proof. Let λ be an infinite cardinal. Consider some λ-saturated T-model of A. Suppose that for some prime p, $A(p) = (A[p] \cap pA)$ is an infinite subgroup.

If A is a group and p is a prime, then we say that a *Szmielew invariant* $\beta_p(A)$ *is finite* if there exists a natural number k such that $(p^k A)[p])$ is a finite group. Otherwise, we say that $\beta_p(A)$ *is infinite.*

Case 1. Let the Szmielew invariant $\beta_p(A)$ be finite.

Assume that k^* is a maximal natural number for which there exists a subgroup B of A that is a direct summand of the group A and is isomorphic to a group $C_{p^{k^*}}^{<\lambda}$. Since $\beta_p(A)$ is finite, then the reduced part of a p-component A_p is bounded. The subgroup $(A[p] \cap pA)$ is infinite, the structure A is λ-saturated, and $\beta_p(A)$ is finite; therefore, such a number k^* exists and $k^* > 1$.

We choose one generator in each direct summand in the decomposition of the group B and use these to form a set S. Let $X = p^{(k^*-1)}S$. Partition the set X into two arbitrary subsets V and W. Let $Y = V$ and $Y^* = \{p^{(k^*-2)}d \mid d \in S, p^{(k^*-1)}d \in W\}$. We show that the subgroup $P_V = \mathrm{acl}_A(Y \cup Y^*)$ satisfies the condition $(Y \cap pP_V) = \varnothing$. Assume the contrary. By Lemma 5, there exists a finite primitive set U containing some element $a^* \in P_V$ for which $pa^* = a$ with some $a \in Y$. In view of Lemma 4, the set U is defined by a conjunction Θ of standard formulas with parameters in $(Y \cup Y^*)$.

Let $\Phi(x, a_1, \ldots, a_n; d_1^*, \ldots, d_m^*)$ be a conjunctive term of the formula Θ, which is a standard formula of the first kind, where $a_1, \ldots, a_n \in Y$ and $d_1^*, \ldots, d_m^* \in Y^*$. Let Φ be of the form

$$nx = m_1 a_1 + \ldots + m_n a_n + k_1 d_1^* + \ldots + k_m d_m^*.$$

The formula Φ is realized by the element a^*. Therefore,

$$na^* = m_1a_1 + \ldots + m_na_n + k_1d_1^* + \ldots + k_md_m^*.$$

Multiplying the above equality by p yields

$$na = k_1pd_1^* + \ldots + k_mpd_m^*. \tag{6}$$

The elements $a, pd_1^*, \ldots, pd_m^*$ are contained in different direct summands, and (6) implies that each of the numbers $n, k_1, \ldots, k_m$ is divisible by p. Let $n^* = n/p$. Thus Φ has the form

$$n^*px = m_1a_1 + \ldots + m_na_n + k_1d_1^* + \ldots + k_md_m^*.$$

Therefore, all elements of the set $(a^* + A[p])$ realize the formula Φ.

Suppose $\Phi(x, a_1, \ldots, a_n; d_1^*, \ldots, d_m^*)$ is a conjunctive term of the formula Θ, which is a standard formula of the second kind. We may assume that Θ has the form

$$nx \equiv^{q^t} m_1a_1 + \ldots + m_na_n + k_1d_1^* + \ldots + k_sd_s^*,$$

where $a_1, \ldots, a_n \in Y$, $d_1^*, \ldots, d_m^* \in Y^*$, t is a natural number, and q is a prime. If $q \neq p$, then all elements of the subgroup B are divisible by q^t. Therefore, all elements of the set $a^* + B$ realize the formula Φ.

Let Φ be of the form

$$nx \equiv^{p^t} m_1a_1 + \ldots + m_na_n + k_1d_1^* + \ldots + k_sd_s^*.$$

The formula Φ is realized by a^*, so

$$na^* \equiv^{p^t} m_1a_1 + \ldots + m_na_n + k_1d_1^* + \ldots + k_sd_s^*. \tag{7}$$

Case 1(a). Let $t < k^*$.

The subgroup B is a direct sum of cyclic group of order p^{k^*}, hence all elements $b \in B[p]$ are divisible by $p^{(k^*-1)}$, and so all elements of the set $(a^* + B[p])$ realize Φ.

Case 1(b) Let $t \geqslant k^*$.

Multiplying equality (5) by p yields the equivalence

$$na \equiv^{p^t} k_1pd_1^* + \ldots + k_mpd_m^*. \tag{8}$$

The elements $a, pd_1^*, \ldots, pd_m^*$ are contained in different direct summands and are not divisible by p^t. This, together with equivalence (8), implies each of the numbers $n, k_1, \ldots, k_m$ is divisible by p. Let $n^* = n/p$. Thus Φ has the form

$$n^*px \equiv^{p^t} m_1a_1 + \ldots + m_na_n + k_1d_1^* + \ldots + k_md_m^*.$$

Therefore, all elements of the set $(a^* + B[p])$ realize Φ. In this way each conjunctive term of the formula Θ is realized by all elements of $(a^* + B[p])$. Since $B[p]$ is an infinite set, we are led to a contradiction with U being finite.

Case 2. Let the Szmielew invariant $\beta_p(A)$ be infinite.

Since the group A is λ-saturated, A contains a subgroup B of the form $C_{p^\infty}^{<\lambda}$, where C_{p^∞} is a quasicyclic p-group. The subgroup B is divisible and is distinguished by a direct summand in the group A. Let

$$A = B \oplus A^*.$$

Choose one element of order p^2 in each direct summand in the decomposition of B and use these to form a set S. Let $X = pS$. Partition X into two arbitrary sets V and W. Suppose $Y = V$ and $Y^* = \{d \mid pd \in W\}$. We show that the subgroup $P_V = \mathrm{acl}_A(Y \cup Y^*)$ satisfies the condition $(Y \cap pP_V) = \varnothing$. Assume the contrary. By Lemma 5, there exists a finite primitive set U containing some element $a^* \in P_V$ for which $pa^* = a$ with some $a \in Y$. In view of Lemma 4, the set U is defined by a conjunction Θ of standard formulas with parameters in $(Y \cup Y^*)$. Since B is a divisible group, all elements of the set $a^* + B$ realize standard formulas of the second kind that are realized by the element a^*.

As in Case 1, we may show that if Φ is a conjunctive term of the formula Θ, which is a standard formula of the first kind, then all elements of the set $(a^* + B[p])$ realize Θ. Hence in the present case, too, we arrive at a contradiction with U being finite.

Thus $(Y \cap pP_V) = \varnothing$. By virtue of $Y^* \subseteq P_V$, the formula $\exists y(P(y) \wedge py = x)$ is true for all elements of the set $W = pY^*$ and is false for all elements of the set $V = Y$. For $V_1 \neq V_2$, therefore, it is true that $\mathrm{Th}(\langle A, P_{V_1}\rangle) \neq \mathrm{Th}(\langle A, P_{V_2}\rangle)$. Hence the set $T_{\Delta_a}(X)$ has 2^λ completions in the language L_P. □

Definition. For a group A and for a prime number p, we say that a Szmielew invariant $\gamma_p(A)$ is finite if there exists a natural number k such that $(A/A[p^k])/p(A/A[p^k])$ is a finite group. Otherwise, we say that $\gamma_p(A)$ is infinite.

LEMMA 7. Suppose that for some prime p, $A[p]$ is an infinite subgroup of A and the Szmielew invariant $\gamma_p(A)$ is infinite. Then the theory $\mathrm{Th}(A)$ is not (P, a)-stable. Moreover, for any infinite cardinality λ, there exists a set X of cardinality λ in T for which $T_{\Delta_a}(X)$ has 2^λ completions in the language L_P.

Proof. In view of the previous lemma, we may assume that $A(p) = (A[p] \cap pA)$ is a finite subgroup. Then the reduced part of a p-component A_p of A is bounded. Since divisible subgroups and bounded pure subgroups are direct summands, the p-component A_p of A is distinguished by a direct summand.

Let λ be an infinite cardinal. Consider

$$B = A \oplus R_p^{<\lambda},$$

where R_p is a subgroup of rational numbers with denominators not divisible by p. The Szmielew invariant $\gamma_p(A)$ is infinite, and so the group B has the same Szmielew invariants as the group A. Consequently, B is a model of the theory $T = \mathrm{Th}(A)$. If we replace A with B we may assume that A contains as a direct summand the following group:

$$H = A_p \oplus R_p^{<\lambda}.$$

In each direct summand R_p in the decomposition of H, we choose an element representing an identity in that summand and use these to form a set U. Let $X = \{pa \mid a \in U\}$. Partition the set X into two arbitrary sets V and W. Suppose $Y = V$ and $Y^* = \{a \mid a \in U, pa \in W\}$. We show that the subgroup $P_V = \mathrm{acl}_A(Y \cup Y^*)$ satisfies the condition $(Y \cap pP_V) = \varnothing$. Assume the contrary. By Lemma 5, there exists a finite primitive set S containing some element $e \in P_V$ for which $pe = a$ with some $a \in Y$. In view of Lemma 4, the set S is defined by a conjunction Θ of standard formulas with parameters in $(Y \cup Y^*)$.

Let $\Phi(x, b_1, \ldots, b_k; d_1^*, \ldots, d_m^*)$ be a conjunctive term of the formula Θ, where $b_1, \ldots, b_k \in Y$, $d_1^*, \ldots, d_m^* \in Y^*$, and Φ has the form

$$nx = m_1 b_1 + \ldots + m_k b_k + k_1 d_1^* + \ldots + k_m d_m^*. \tag{9}$$

The formula Φ is realized by the element e, so

$$ne = m_1 b_1 + \ldots + m_k b_k + k_1 d_1^* + \ldots + k_m d_m^*.$$

If we multiply this equality by p we obtain

$$na = m_1 p b_1 + \ldots + m_k p b_k + k_1 p d_1^* + \ldots + k_m p d_m^*. \tag{10}$$

The elements $b_1, \ldots, b_k, d_1^*, \ldots, d_m^*$ are contained in different direct summands and $a \in Y$. This, together with equality (10), implies that $k_1 = \ldots = k_m = 0$, and among the numbers $m_1, \ldots, m_k$, only one number m_i is distinct from zero. Thus (10) has the form $na = m_i pa$. Since this equality is satisfied in a group isomorphic to R_p, and it is a torsion-free group, we obtain $n = m_i p$. Therefore, n is divisible by p, and the set of solutions for Φ contains $(e + A[p])$.

Let $\Phi(x, b_1, \ldots, b_k; d_1^*, \ldots, d_m^*)$ be a conjunctive term of the formula Θ, which is a standard formula of the second kind. We may assume that Θ has the form

$$nx \equiv^{q^t} m_1 b_1 + \ldots + m_k b_k + k_1 d_1^* + \ldots + k_m d_m^*,$$

where $b_1, \ldots, b_k \in Y$, $d_1^*, \ldots, d_m^* \in Y^*$, t is a natural number, and q is a prime. If $q \neq p$, then all elements of the subgroup $A[p]$ are divisible by q^t, and so all elements of the set $e + A[p]$ realize the formula Φ.

Let Φ be of the form

$$nx \equiv^{p^t} m_1 b_1 + \ldots + m_k b_k + k_1 d_1^* + \ldots + k_m d_m^*.$$

The element e realizes Φ, so

$$ne \equiv^{p^t} m_1 b_1 + \ldots + m_k b_k + k_1 d_1^* + \ldots + k_m d_m^*. \tag{11}$$

In particular,

$$ne \equiv^{p} m_1 b_1 + \ldots + m_k b_k + k_1 d_1^* + \ldots + k_m d_m^*. \tag{12}$$

The elements $b_1, \ldots, b_k$ are divisible by p, hence

$$ne \equiv^{p} k_1 d_1^* + \ldots + k_m d_m^*. \tag{13}$$

Since $pe \in Y$, we have $e = b + c$, where $b \in (U \setminus Y^*)$ and $c \in A[p]$. Thus the elements e and $k_1 d_1^* + \ldots + k_m d_m^*$ are contained in different direct summands, and ne is divisible by p. The elements b and c likewise belong to different direct summands, i.e., nb is divisible by p. The element b is not divisible by p, hence the number n should be divisible by p. Consequently, the set of solutions for Φ contains a set $(e + A[p])$.

In this way each conjunctive term of the formula Θ is realized by all elements of the set $(e+A[p])$. Since $A[p]$ is an infinite set, we are led to a contradiction with S being finite.

Thus $(Y \cap pP_V) = \varnothing$. By virtue of $Y^* \subseteq P_V$, the formula $\exists y(P(y) \wedge py = x)$ is true for all elements of the set $W = pY^*$ and is false for all elements of the set $V = Y$. For $V_1 \neq V_2$, therefore, it is true that $\mathrm{Th}(\langle A, P_{V_1}\rangle) \neq \mathrm{Th}(\langle A, P_{V_2}\rangle)$. Hence the set $T_{\Delta_a}(X)$ has 2^λ completions in the language L_P. $\square$

Our next goal is to show that a group A not satisfying the hypotheses of Lemmas 6 and 7 has the property of being (P, a)-stable. In what follows, we assume that A enjoys the following two properties.

(1) For any prime number p, the subgroup $(A[p] \cap pA)$ equal to a subgroup $(pA)[p]$ is finite.

(2) For any prime number p, either $A[p]$ is a finite subgroup or there exists a natural number n^* such that $(A/A[p^{n^*}])/p(A/A[p^{n^*}])$ is a finite elementary Abelian p-group.

Below we assume that P is an algebraically closed subgroup of A.

Property (1) entails the following property.

(3) For any prime p and for an arbitrary natural $k \geqslant 1$, $(pA)[p^k]$ is a finite subgroup and hence is contained in P

We prove (3) by induction on k. For $k = 1$, this is exactly property (1). Let $(pA)[p^{(k-1)}]$ be a finite subgroup. On the set $(pA)[p^k]$, we define the following equivalence:

$$\alpha = \{\langle a, b\rangle \mid a, b \in (pA)[p^k], pa = pb\}.$$

For any $a \in (pA)[p^k]$, we have $pa \in (pA)[p^{(k-1)}]$, and so the number of α-classes does not exceed the cardinality of the set $(pA)[p^{(k-1)}]$, which is finite by the inductive assumption. If elements $a, b \in (pA)[p^k]$ are contained in one α-class, then the element $(a - b)$ belongs to the subgroup $(pA)[p]$. By property (1), $(pA)[p]$ is finite; hence each α-class is finite. In this way $(pA)[p^k]$ is finite.

LEMMA 8. If $A[p]$ is a finite subgroup, then

$$A \models \Phi(\mathbf{a}) \Leftrightarrow P \models \Phi(\mathbf{a})$$

for any standard formula $\Phi(\mathbf{x})$ of the second kind with module p and for an arbitrary tuple $\mathbf{a} \in P$ of length $l(\mathbf{x})$.

Proof. Let $A[p]$ be finite. Using induction on k, we show that $A[p^k]$ is a finite subgroup for any k. Let $A[p^{(k-1)}]$ be finite. On the set $A[p^k]$, we define the following equivalence:

$$\alpha = \{\langle a, b\rangle \mid a, b \in A[p^k], pa = pb\}.$$

For any $a \in A[p^k]$, we have $pa \in A[p^{(k-1)}]$, and so the number of α-classes does not exceed the cardinality of the set $A[p^{(k-1)}]$, which is finite by the inductive assumption. If elements $a, b \in A[p^k]$ are contained in one α-class, then the element $(a-b)$ belongs to the subgroup $A[p]$. By hypothesis, $A[p]$ is finite; hence each α-class is finite. In this way $A[p^k]$ is finite.

For any element $a \in P$, the cardinality of the set

$$a/p^k = \{d \mid d \in A, p^k d = a\}$$

does not exceed the cardinality of $A[p^k]$; hence it is finite. The subgroup P is algebraically closed and the set a/p^k is defined in A by a formula with parameter a; so a/p^k is contained in P. Thus a is divisible by p^k in A iff a is divisible by p^k in P. □

LEMMA 9. Let p be a prime. If an element $a \in P$ is divisible by p^k, $k > 1$, in the group A, then a is divisible by $p^{(k-1)}$ in the subgroup P. Moreover, if $p^k b = a$ then $pb \in P$.

Proof. Take an arbitrary element b for which $p^k b = a$. The set

$$\{d \mid p^{(k-1)} d = a, d \in pA\}$$

is definable via a parameter a and coincides with a set $(pb + (pA)[p^{(k-1)}]$, which is finite in view of property (3) and hence is contained in P. In particular, $pb \in P$. □

LEMMA 10. Let $A[p]$ be an infinite subgroup. Then there exists a finite set $Z_p \subseteq P$ of elements divisible by p in the group A, in which case if an element $a \in P$ is divisible by p in A, then, for some $b \in Z_p$ and some $c \in (pA)[p^{n^*}]$, an element $((a-b)+c)$ is divisible by p^2 in A. In particular (by Lemma 9), the element $(a-b)+c$ is divisible by p in the subgroup P. The number n^* is as in property (2).

Proof. Suppose that there exists an infinite set $U \subseteq P$ of elements divisible by p in the group A, and for any distinct elements $a, b \in U$ and for arbitrary $c \in (pA)[p^{n^*}])$, the element $(a-b)+c$ is not divisible by p^2 in the group A. For each $a \in U$, we choose a^* for which $pa^* = a$. By property (2), there exist different $a, b \in U$, $e \in A[p^{n^*}]$, and $d \in A$ such that $(a^* - b^*) + e = pd$. Multiplying by p, we obtain the equality $(a-b)+pe = p^2 d$, which contradicts the assumption since $pe \in (pA)[p^{n^*}])$. □

For a finite set $X = \{a_1, \ldots, a_n\} \subseteq A$, $z \in X$ denotes a formula of the form $(z = a_1 \vee \ldots \vee z = a_n)$. In view of property (3), $(pA)[p^{n^*}]$ is a finite set and hence is contained in P.

Remark 3. Lemma 10 implies that there exists an L-formula $\Psi(x)$ with parameters in the subgroup P such that for any element $a \in P$, the condition $P \models \Psi(a)$ is equivalent to $a \in pA$.

In fact, as $\Psi(x)$ we may take the formula

$$\exists z \exists u \exists v (z \in Z_p \wedge u \in (pA)[p^{n^*}] \wedge (z - x + u) = pv),$$

where Z_p is as in Lemma 10. For an element a of P that is divisible by p in A, the condition $P \models \Phi(a)$ is derived from Lemma 10. On the other hand, if $P \models \Phi(a)$, then, for some $b \in Z_p$ and some $c \in (pA)[p^{n^*}]$, an element $d = (b - a) + c$ is divisible by p in P. Since b and c are divisible by p in A, a is divisible by p in A.

LEMMA 11. Let $\Phi(x_1, \dots, x_n)$ be a formula in the language of Abelian groups.

(a) There exists a formula $\Phi^*(x_1, \dots, x_n)$ in the language of Abelian groups, with parameters in P such that any $a_1, \dots, a_n \in P$ satisfy the condition

$$A \models \Phi(a_1, \dots, a_n) \Leftrightarrow P \models \Phi^*(a_1, \dots, a_n).$$

(b) There exists a formula $\Phi^\diamondsuit(x_1, \dots, x_n)$ in the language of Abelian groups, with parameters in P such that any $a_1, \dots, a_n \in P$ satisfy the condition

$$P \models \Phi(a_1, \dots, a_n) \Leftrightarrow A \models \Phi^\diamondsuit(a_1, \dots, a_n).$$

Proof. By virtue of Lemmas 3 and 4, we may assume that the formula Φ is a Boolean combination of standard formulas. Therefore, we can think of Φ as a standard formula. If Φ is a standard formula of the first kind then as Φ^* and $\Phi^\diamondsuit$ we can take the formula Φ. Let Φ be a standard formula of the second kind; i.e., Φ has the form $\exists y t = p^k y$, where t is a linear combination of variables $x_1, \dots, x_n$. Instead of $t(x_1, \dots, x_n)$ and $t(a_1, \dots, a_n)$, we will use $t(\mathbf{x})$ and $t(\mathbf{a})$, respectively. If $A[p]$ is a finite subgroup, then (in view of Lemma 8) as Φ^* and $\Phi^\diamondsuit$ we can take the formula Φ. Let $A[p]$ be an infinite subgroup and Z_p a finite set such as in Lemma 10.

(a) If $k = 1$, then as Φ^* we can take a formula $\Psi(t)$, where $\Psi(x)$ is as in Remark 3.

If $k > 1$, then (in view of Lemma 9) as Φ^* we may take a formula of the form

$$\exists y (t = p^{(k-1)} y \wedge \Psi(y)),$$

where Ψ is as above.

(b) Consider a formula such as

$$\Delta(x) = \exists z \exists w \exists y (z \in Z_p \wedge (z - x + w = p^2 y \wedge w \in (pA)[p^{n^*}] \wedge ((z + w) \notin pP)).$$

Since Z_p and $(pA)[p^{n^*}]$ are finite sets, the condition $(z + w) \notin pP)$ is expressed via a Boolean combination of formulas like $z = a$ and $w = b$, where $a \in Z_p$ and $w \in (pA)[p^{n^*}]$. Indeed, let

$$W = \{\langle a, b \rangle \mid a \in Z_p, b \in (pA)[p^{n^*}], (a + b) \notin pP\}.$$

The set W is finite, and as a formula $(z+w) \notin pP$ we can take

$$\bigvee\{(z = a \wedge w = b) \mid \langle a, b\rangle \in W\}.$$

Let $A \models \Delta(a)$ for an element $a \in P$. Then there exist elements $d \in Z_p$, $b \in (pA)[p^{n^*}]$, and $c \in A$ for which $(d - a + b) = p^2c$ and $(d + b)$ is not divisible by p in P. The elements d, b, and $(d - a + b)$ are divisible by p in A, and so a is divisible by p in A. On the other hand, $(d - a + b)$ is divisible by p in P in view of Lemma 9. If a were divisible by p in P, then $(d + b)$ would belong to a subgroup pP, which contradicts the condition that $(d + b) \notin pP$. Hence $a \notin pP$.

Let $a \in pA$ and $a \notin pP$ for $a \in P$. By virtue of Lemma 10, there exist $d \in Z_p$, $b \in (pA)[p^{n^*}]$, and $c \in A$ such that $(d - a + b) = p^2c$. By Lemma 9, we have $(d - a + b) \in pP$, and in view of the condition $a \notin pP$, we obtain $(d + b) \notin pP$.

In this way, for any element $a \in P$, the condition $A \models \Delta(a)$ is equivalent to the fact that a is divisible by p in the group A but is not divisible by p in the subgroup P. Clearly, as $\Phi^{\Diamond}$ we can take a formula of the form

$$\exists y(t = p^k y \wedge \neg\Delta(py)). \ \square$$

THEOREM 3. For an Abelian group A, its theory $T = \mathrm{Th}(A)$ is (P, a)-stable if and only if the group A satisfies the following conditions:

(1) for any prime p, $(pA)[p]$ is a finite subgroup of A;

(2) for any prime p, either $A[p]$ is a finite subgroup of A or its Szmielew invariant $\gamma_p(A)$ is finite.

Proof. If none of the conditions (1) or (2) is satisfied, then it follows by Lemmas 6 or 7 that T will not be (P, a)-stable.

Suppose that (1) and (2) are both satisfied and $\Phi(x_1, \ldots, x_n)$ is a primitive L_P-formula. In view of Remark 1, we may assume that Φ has the form

$$\exists y_1 \ldots \exists y_s(P(y_1) \wedge \ldots \wedge P(y_s) \wedge \Psi(y_1, \ldots, y_s; x_1, \ldots, x_n)),$$

where Ψ is a primitive L-formula. By Lemma 11(a), there exists an L-formula $\Psi^*(y_1, \ldots, y_s; x_1, \ldots, x_n)$ with parameters in P such that for any elements $b_1, \ldots, b_s, a_1, \ldots, a_n \in P$,

$$A \models \Psi(b_1, \ldots, b_s; a_1, \ldots, a_n) \Leftrightarrow P \models \Psi^*(b_1, \ldots, b_s; a_1, \ldots, a_n).$$

Let $\Theta(x_1, \ldots, x_n; z_1, \ldots, z_s)$ be an L-formula, $c_1, \ldots, c_s \in P$, and

$$\Theta(x_1, \ldots, x_n; c_1, \ldots, c_s) = \exists y_1 \ldots \exists y_s \Psi^*(y_1, \ldots, y_s; x_1, \ldots, x_n).$$

Suppose also that for any $a_1, \ldots, a_n \in P$, the following condition holds:

$$\langle A, P\rangle \models \Phi(a_1, \ldots, a_n) \Leftrightarrow P \models \Theta(a_1, \ldots, a_n; c_1, \ldots, c_s).$$

By virtue of Lemma 11(b), for an L-formula $\Theta^{\diamondsuit}(x_1, \ldots, x_n; z_1, \ldots, z_s)$ and for any $a_1, \ldots, a_n \in P$,

$$P \models \Theta(a_1, \ldots, a_n; c_1, \ldots, c_s) \Leftrightarrow A \models \Theta^{\diamondsuit}(a_1, \ldots, a_; c_1, \ldots, c_s).$$

Thus an L-formula $\Theta^{\diamondsuit}(x_1, \ldots, x_n; c_1, \ldots, c_s)$ with parameters in P will define the same predicate on $P(A)$ in the group A as is defined by $\Phi(x_1, \ldots, x_n)$ in the structure $\langle A; P\rangle$. It remains to apply Corollary 1. □

4. (P, s)-STABLE ABELIAN GROUPS

Recall that a direct sum of cyclic groups of order p, where p is some fixed prime, is called an *elementary Abelian group*.

THEOREM 4. A group A is (P, s)-stable if and only if A is a direct sum of finitely many elementary groups and a finite group.

Proof. First we show that a (P, s)-stable group A is bounded. Suppose the contrary. Then a model of $\mathrm{Th}(A)$ will be the group $B = (A \oplus M \oplus N)$, where M and N are isomorphic to a group $Q^{<\omega}$, and Q is the additive group of rational numbers. In the subgroup M, we take a set X consisting of identities of each of its direct summands isomorphic to Q. In the subgroup N, we take a similar set Y. Clearly, every permutation of the set $Z = (X \cup Y)$ extends to an automorphism of the group B. By Remark 2, the sets X and Y are inseparable in the theory $T(Z)$. In the structure $\langle A, P\rangle$, where P is generated by a set $(M \cup Y)$, however, X and Y are separated, for instance, by a formula $\exists y(P(y) \wedge 2y = x)$.

Thus a (P, s)-stable group A is bounded; i.e., it is a finite direct sum of bounded p-groups for some primes p. If some of these direct summands were not a direct sum of an elementary group and a finite group, then $(pA)[p]$ would be an infinite subgroup for some prime p. By Lemma 6, A is not (P, a)-stable; moreover, it will not be (P, s)-stable.

Now we show that $\mathrm{Th}(A)$, where A is a direct sum of finitely many elementary groups and a finite group, is a (P, s)-stable theory.

In any p-component A_p of A, $p^k A_p$ is a finite subgroup for any natural number $k \geqslant 1$. The group A is a direct sum of its p-components for distinct prime p. For any prime p, therefore, the subgroup $p^k A$ consists of elements of the form $(a + b)$, where $a \in p^k A_p$ and b has order coprime to p.

Let P be an arbitrary subgroup of A. As in the proof of Theorem 3, it suffices to show that a pair $\langle A; P\rangle$ of groups satisfies properties (a) and (b) in Lemma 11. As in the proof of Lemma 11, we need only assume that $\Phi = \exists y t = p^k y$, where t is a linear combination of variables $x_1, \ldots, x_n$.

(a) Let $X = (P \cap p^k A_p) = \{a_1 \ldots a_n\}$. Denote by $z \in X$ a formula $(z = a_1 \vee \ldots \vee z = a_n)$. Suppose that n^* is the least common multiple of the element orders of A, p^{k^*} is the greatest order of elements of a p-component A_p of A, and $m^* = n^*/p^{k^*}$. Obviously, m^* is coprime to p. Denote by $\Theta(x)$ a formula $m^* x = 0$. Clearly, every element of the set $\Theta(A)$ is divisible by p^k in A, for any natural k.

We show that as Φ^* we can take a formula of the form

$$\exists z \exists w (t(\mathbf{x}) = (z + w) \wedge z \in X \wedge \Theta(w)).$$

Indeed, if $P \models \Phi^*(\mathbf{a})$ for a tuple $\mathbf{a} \in P$, then, as noted, the element $t(\mathbf{a})$ is divisible by p^k in A, i.e., $A \models \Phi(\mathbf{a})$.

Let $t(\mathbf{a})$ be divisible by p^k in A; i.e., $t(\mathbf{a}) = e + b$, where $e \in p^k A_p$ and b has order coprime to p. The order of the element b is a divisor of the number m^*; hence $m^* t(\mathbf{a}) = m^* e$. The order of e is coprime to m^*; so there exists an integer l such that $lm^* e = e$. In this way $lm^* t(\mathbf{a}) = e$, which, in view of the condition $t(\mathbf{a}) \in P$, yields $e \in X$. Consequently, $b \in P$, and hence $P \models \Phi^*(\mathbf{a})$.

(b) Let $Y = p^k P_p$. Since $Y \subseteq X$, the set Y is finite, and so there exists a formula with parameters in P expressing the condition that $z \in Y$. As a formula $\Phi^\diamondsuit(\mathbf{x})$ we take

$$\exists z \exists w (t(\mathbf{x}) = (z + w) \wedge z \in Y \wedge \Theta(w)).$$

Suppose $P \models \Phi(\mathbf{a})$, i.e., $t(\mathbf{a}) = p^k d$ for some $d \in P$. The subgroup P is also a direct sum of its q-components for distinct prime q. Therefore, the subgroup $p^k P$ consists of elements of the form $(a + b)$, where $a \in p^k P_p$ and b has order coprime to p. Consequently, $A \models \Phi^\diamondsuit(\mathbf{a})$.

Assume that $A \models \Phi^\diamondsuit(\mathbf{a})$ for a tuple $\mathbf{a} \in P$; i.e., $t(\mathbf{a}) = e + b$ for elements $e \in p^k P_p$ and $b \in \Theta(A)$. Since $e, t(\mathbf{a}) \in P$, we have $b \in P$. The condition $e \in Y$ implies that there exists an element $d \in P$ for which $p^k d = e$. The order of b is coprime to p, and hence the element b is divisible by p^k in the subgroup generated by b; in particular, there exists an element $g \in P$ with $p^k g = b$. Thus $t(\mathbf{a}) = p^k h$ for $h = (d + g)$, i.e., $P \models \Phi(\mathbf{a})$. $\square$

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