*Algebra and Logic*, Vol. 51, No. 5, November, 2012
(Russian Original Vol. 51, No. 5, September-October, 2012)

# DOMINIONS IN ABELIAN SUBGROUPS OF METABELIAN GROUPS

**A. I. Budkin**

UDC 512.57

Keywords: *metabelian group, Abelian subgroup, dominion.*

*It is proved that a suitable free Abelian group of finite rank is not absolutely closed in the class $\mathcal{A}^2$ of metabelian groups. A condition is specified under which a torsion-free Abelian group is not absolutely closed in $\mathcal{A}^2$. Also we gain insight into the question when the dominion in $\mathcal{A}^2$ of the additive group of rational numbers coincides with this subgroup.*

## INTRODUCTION

The concept of a dominion was introduced in [1] for studying epimorphisms. In [2-5], dominions were treated for various classes of universal algebras (see also the bibliography in [3]). In particular, it was established that there exists a close relationship between dominions and amalgams (for details, see [2]). At present, dominion theory is being most intensively studied for groups.

Let $\mathcal{M}$ be an arbitrary quasivariety of groups. In this event, for any group $A$ in $\mathcal{M}$ and its subgroup $H$, the *dominion* $\mathrm{dom}_A^{\mathcal{M}}(H)$ of the subgroup $H$ in $A$ (in $\mathcal{M}$) is defined as follows:

$$\mathrm{dom}_A^{\mathcal{M}}(H) = \{a \in A \mid \forall M \in \mathcal{M} \, \forall f, g : A \to M, \text{ if } f\,|_H = g\,|_H, \text{ then } a^f = a^g\}.$$

Here, as usual, $f, g : A \to M$ denote homomorphisms of the group $A$ into the group $M$ and $f\,|_H$ stands for the restriction of $f$ to $H$.

Note that dominions were thoroughly investigated for quasivarieties of Abelian groups [6-9]. Dominions in the class of nilpotent groups were dealt with in a series of papers; we refer the reader to [5, 10, 11]. Recent trends are toward research on dominions in metabelian groups [12, 13].

A group $H$ is said to be *n-close* in a class $\mathcal{M}$ if $\mathrm{dom}_A^{\mathcal{M}}(H) = H$ for any group $A = \mathrm{gr}(H, a_1, \ldots, a_n)$ in $\mathcal{M}$ that contains $H$ and is generated modulo $H$ by suitable $n$ elements. A

Pavlovskii road, 60a-168, Barnaul, 656064 Russia; budkin@math.asu.ru. Translated from *Algebra i Logika*, Vol. 51, No. 5, pp. 608-622, September-October, 2012. Original article submitted June 18, 2012; revised September 28, 2012.

 

group $H$ is said to be *absolutely closed in a class* $\mathcal{M}$ if $H \leq A$ entails $\mathrm{dom}_A^{\mathcal{M}}(H) = H$ for any group $A$ in $\mathcal{M}$.

In [5, Cor. 2], it was shown that studying absolutely closed groups reduces to treating $n$-closed groups. This explains our interest in research into $n$-closed groups. In [5, Thm. 5], for instance, we described all 1-closed Abelian groups in each quasivariety of torsion-free nilpotent groups of class 2.

In the present paper, we prove that a suitable free Abelian group of finite rank is not absolutely closed in the class $\mathcal{A}^2$ of metabelian groups. A condition is specified under which a torsion-free Abelian group is not absolutely closed in $\mathcal{A}^2$. We work to gain insight into the question when the dominion in $\mathcal{A}^2$ of the additive group of rational numbers coincides with this subgroup. For basic notions in the theory of quasivarieties, the reader is referred to [14-17], and in group theory, to [18].

## 1. PRELIMINARIES

We recall certain of the notation and notions.

By writing $A \leq B$ we mean that $A$ is a subgroup of a group $B$. Denote by $\mathrm{gr}(S)$ a group generated by a set $S$, and by $(a)$ a cyclic group generated by an element $a$. $G'$ is the commutator subgroup of a group $G$ and $|a|$ is the order of an element $a$. Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ be the set of natural numbers, the set of integers, and the set of rational numbers, respectively. As usual, $[a, b] = a^{-1}b^{-1}ab$ and $a^b = b^{-1}ab$.

An *embedding* of a group $A$ in a group $B$ is any homomorphism $\varphi : A \to B$, which is an isomorphism of $A$ onto $A^\varphi$. If there exists an embedding of $A$ in $B$ then we say that $A$ is embeddable in $B$. A group $G$ is said to be *divisible* if for any integer $n > 0$ and any element $g \in G$ the equation $x^n = g$ has at least one solution in $G$. It is well known that every divisible Abelian group $G$ decomposes into a direct product of groups isomorphic to quasicyclic $p$-groups and to the additive group of rational numbers.

The cardinality of a maximal linearly independent system of elements of a torsion-free Abelian group is referred to as the *rank* of that group. We recall the definition of a direct wreath product of groups $A$ and $B$. Take a direct degree $\overline{A}$ of $A$, consisting of all functions $f : B \to A$ with finite support. For every $b \in B$, a map $\beta : f \to f^b$ is given by the rule $f^b(y) = f(yb^{-1})$ for all $y \in B$. The map $\beta$ is an automorphism of the group $\overline{A}$ and the set of all such automorphisms is a group isomorphic to $B$. An extension of $\overline{A}$ by this automorphism group is called a *direct wreath product* of groups $A$ and $B$ and is denoted $A \wr B$. The group $\overline{A}$ is called a *basic subgroup* of the wreath product.

Let $t_i(\overline{x})$, $t'_i(\overline{x})$ $(i \in I)$, $t(\overline{x})$, and $t'(\overline{x})$ be group words over an alphabet $\overline{x}$. We say that the equality $t(\overline{x}) = t'(\overline{x})$ is *deducible* in a quasivariety $\mathcal{M}$ from the set $\{t_i(\overline{x}) = t'_i(\overline{x}) \mid i \in I\}$ of equalities if the implication

$$(\forall \overline{x}) \left( \mathop{\&}_{i \in I} t_i(\overline{x}) = t'_i(\overline{x}) \to t(\overline{x}) = t'(\overline{x}) \right)$$

is true for every group in the quasivariety $\mathcal{M}$.

We will need the following:

**THEOREM** (Dyck's theorem; see [17, p. 281]). Let a group $G$ in a given variety $\mathcal{N}$ have the following representation:

$$G = \mathrm{gr}(\{x_i \mid i \in I\} \parallel \{r_j(x_{j_1}, \dots, x_{j_{l(j)}}) = 1 \mid j \in J\}).$$

Suppose that $H \in \mathcal{N}$ and the group $H$ contains a set $\{g_i \mid i \in I\}$ of elements such that the equality $r_j(g_{j_1}, \dots, g_{j_{l(j)}}) = 1$ is true in $H$ for every $j \in J$. Then a map $x_i \to g_i$ $(i \in I)$ extends to a homomorphism of $G$ into $H$.

We recall the definition of a free metabelian square of a group with amalgamated subgroup. Suppose that a group $G$ in $\mathcal{A}^2$ is represented as

$$G = \mathrm{gr}(\{x_i \mid i \in I\} \parallel \{r_j(\overline{x}) = r'_j(\overline{x}) \mid j \in J\}).$$

We take two groups $G_1$ and $G_2$ isomorphic to the group $G$ and fix their representations

$$G_1 = \mathrm{gr}(\{x_i \mid i \in I\} \parallel \{r_j(\overline{x}) = r'_j(\overline{x}) \mid j \in J\}),$$
$$G_2 = \mathrm{gr}(\{y_i \mid i \in I\} \parallel \{r_j(\overline{y}) = r'_j(\overline{y}) \mid j \in J\}).$$

Assume that $X = \{x_i \mid i \in I\}$ and $Y = \{y_i \mid i \in I\}$ have an empty intersection.

Let $H$ be a subgroup of $G$. Take an arbitrary set $\{h_l(\overline{x}) \mid l \in L\}$ of group words over an alphabet $X = \{x_i \mid i \in I\}$ whose set $\{h_l(\overline{x}) \mid l \in L\}$ of values generate $H$ in $G$. Consider a group $C$ which in $\mathcal{A}^2$ has the following representation:

$$C = \mathrm{gr}(X \cup Y \parallel \{r_j(\overline{x}) = r'_j(\overline{x}) \mid j \in J\} \cup \{r_j(\overline{y}) = r'_j(\overline{y}) \mid j \in J\}$$
$$\cup \{h_l(\overline{x}) = h_l(\overline{y}) \mid l \in L\}).$$

This group $C = G *_H^{\mathcal{A}^2} G$ is called the *free metabelian square of a group $G$ with an amalgamated subgroup $H$*. Maps $\lambda : G \to C$ and $\rho : G \to C$, where $x_i^\lambda = x_i$ and $x_i^\rho = y_i$ $(i \in I)$, are embeddings; subgroups $G^\lambda$, $G^\rho$, and $H^\lambda$ are again denoted by $G_1$, $G_2$, and $H$, respectively.

If $H = (1)$ then the resulting group $C$ is called a *free product of groups $G_1$ and $G_2$ in $\mathcal{A}^2$*. It is well known that $G_1 \cap G_2 = \mathrm{dom}_{G_1}^{\mathcal{A}^2}(H)$ (see, e.g., [2]).

## 2. ABSOLUTE CLOSEDNESS OF ABELIAN GROUPS

We know from [11, Thm. 3.11] that a free Abelian group is absolutely closed in the variety $\mathcal{N}_2$ of nilpotent groups of class at most 2 iff it is cyclic. In the present section, we deal with a similar problem within the class $\mathcal{A}^2$ of metabelian groups, showing that the dominion in $\mathcal{A}^2$ of an Abelian subgroup of a torsion-free finitely generated metabelian group may fail to coincide with this subgroup.

**LEMMA 1.** A free Abelian group of infinite rank is not absolutely closed in the class of metabelian groups.

**Proof.** Let $R$ be a free Abelian group of infinite rank. We represent $R$ as a direct product $R = H \times T$ of free Abelian groups, where $H$ is a group of countable rank. Take any finitely generated metabelian group $G$ such that its commutator subgroup contains a free Abelian subgroup of countable rank. Assume that this subgroup is freely generated by elements $x_i$ ($i \in P$), where $P$ is the set of all prime numbers. As $G$ we may take, for instance, a direct wreath product $Z \wr Z$ of two infinite cyclic groups. Since $x_i \in G'$, we conclude that $\mathrm{gr}(x_i^i \mid i \in P)$ is a free Abelian group of countable rank. Denote it also by $H$.

Consider a group $A = G \times T$. Suppose $R = H \times T$ is a subgroup of $A$. Let $C = A *_R^{\mathcal{A}^2} A$ be the free metabelian square of $A$ with amalgamated subgroup $R$. Images of elements $x_i$ under natural embeddings $\lambda : A \to A_1 \leq C$ and $\rho : A \to A_2 \leq C$ are denoted by $a_i$ and $b_i$, respectively. In particular, elements $a_i^i$ and $b_i^i$ are equal in $H \leq C$.

Since $a_p, b_p \in C'$ for $p \in P$, we have $[a_p, b_p] = 1$. On the other hand, $a_p^p = b_p^p$ entails $(a_p b_p^{-1})^p = 1$. $L = \mathrm{gr}(G^\lambda, G^\rho)$ is a finitely generated metabelian group and $a_p, b_p \in L$ for $p \in P$. It is well known that every finitely generated metabelian group satisfies the maximal condition for normal subgroups [19]. This implies that the commutator subgroup of $L$ has a finite set of element orders. Hence $a_p b_p^{-1} = 1$ for some $p \in P$, $a_p = b_p \in A^\rho \cap A^\lambda$, and $a_p \notin R$. Consequently, $\mathrm{dom}_A^{\mathcal{A}^2}(R) \neq R$. The lemma is proved.

If in the proof of Lemma 1 as $G$ we take the direct wreath product $Z \wr Z$ of two infinite cyclic groups then the group $A$ in Lemma 1 will be 2-generated modulo $R$. Thus we have

**COROLLARY 1.** A free Abelian group of infinite rank is not 2-closed in the class of metabelian groups.

**THEOREM 1.** There exists a free Abelian group of finite rank that is not absolutely closed in the class of metabelian groups.

**Proof.** Let groups $G$, $A$, and $C$ be as in Lemma 1. Suppose $T = (1)$, i.e., $G = A$. Let

$$A = \mathrm{gr}(x_1, \ldots, x_n \parallel \Sigma_1(\overline{x}))$$

be a representation of $A$ in $\mathcal{A}^2$ with generators $x_1, \ldots, x_n$. Assume that values for the set $\{t_i(\overline{x}) \mid i \in \mathbb{N}\}$ of group words freely generate a group $H$. Then the group $C = A *_H^{\mathcal{A}^2} A$ is represented in $\mathcal{A}^2$ as follows:

$$C = \mathrm{gr}(x_1, \ldots, x_n, y_1, \ldots, y_n \parallel \Sigma_1(\overline{x}), \Sigma_1(\overline{y}), \{t_i(\overline{x}) = t_i(\overline{y}) \mid i \in \mathbb{N}\}).$$

Take any $p \in P$ for which elements $a_p$ and $b_p$ such as in the proof of Lemma 1 are not contained in $H$. Fix a group word $t$ whose values $t(\overline{x}) \in A_1$ and $t(\overline{y}) \in A_2$ are equal in $C$ and coincide with elements $a_p \, (= t(\overline{x})) \in A_1$ and $b_p \, (= t(\overline{y})) \in A_2$.

By virtue of [16, Thm. 2.3.1], a well-known property of quasivarieties holds: if an infinite implication $(\forall \overline{x}) \left( \mathop{\&}\limits_{i \in I} \alpha_i(\overline{x}) \to \alpha(\overline{x}) \right)$ holds in an arbitrary quasivariety $\mathcal{K}$ then a quasi-identity

$(\forall \overline{x})\left(\underset{i\in F}{\&}\alpha_i(\overline{x}) \to \alpha(\overline{x})\right)$ holds in $\mathcal{K}$ for some finite subset $F \subseteq I$. A similar statement for $\mathcal{A}^2$ follows from the property that any subgroup of a finitely generated metabelian group is finitely generated (as a normal subgroup) [19].

The equality $t(x_1,\dots,x_n) = t(y_1,\dots,y_n)$ is deducible in $\mathcal{A}^2$ from the set $\Sigma = \Sigma_1(\overline{x}) \cup \Sigma_1(\overline{y}) \cup \{t_i(\overline{x}) = t_i(\overline{y}) \mid i \in \mathbb{N}\}$ of relations for a group $C$, and so it is deducible in $\mathcal{A}^2$ from some finite subset of this set. Hence there exists a natural number $k$ such that $t(x_1,\dots,x_n) = t(y_1,\dots,y_n)$ is a consequence of $\Sigma_0 = \Sigma_1(\overline{x}) \cup \Sigma_1(\overline{y}) \cup \{t_i(\overline{x}) = t_i(\overline{y}) \mid i = 1,\dots,k\}$ in $\mathcal{A}^2$.

Consider groups $C_1$ and $F$ which in $\mathcal{A}^2$ have the following representations:

$$C_1 = \mathrm{gr}(x_1,\dots,x_n,y_1,\dots,y_n \parallel \Sigma_1(\overline{x}), \Sigma_1(\overline{y}), \{t_i(\overline{x}) = t_i(\overline{y}) \mid i = 1,\dots,k\}),$$
$$F = \mathrm{gr}(x_1,\dots,x_n \parallel \Sigma_1(\overline{x})).$$

If $K$ is a subgroup of $F$ generated by elements $t_1(\overline{x}),\dots,t_k(\overline{x})$, then $K$ is a free Abelian group freely generated by these elements. Clearly, $C_1 = F *_K^{\mathcal{A}^2} F$ is the free metabelian square of the group $F$ with an amalgamated subgroup. In addition, the relation $t(x_1,\dots,x_n) = t(y_1,\dots,y_n)$ is deducible in $\mathcal{A}^2$ from the set $\Sigma_0$ of relations. Therefore, $a = b$ for values $a = t(x_1,\dots,x_n)$ and $b = t(y_1,\dots,y_n)$ of these words in the group $C_1$. This implies $a = b \in F^\rho \cap F^\lambda = \mathrm{dom}_F^{\mathcal{A}^2}(K)$.

By Dyck's theorem, there exists a natural homomorphism $\varphi : C_1 \to C$. Since $a^\varphi = a_p \notin H$, we have $a \notin K$. Thus $\mathrm{dom}_F^{\mathcal{A}^2}(K) \neq K$. The theorem is proved.

We may take a 2-generated group to be $G$, and so the proof of Theorem 1 entails

**COROLLARY 2.** There exists a free Abelian group of finite rank that is not 2-closed in the class of metabelian groups.

**Problem.** Is an infinite cyclic group absolutely closed in the class of metabelian groups?

**THEOREM 2.** If a free Abelian group of finite rank $k$ is not absolutely closed in the class of metabelian groups, then every torsion-free Abelian group of rank $k$ is not absolutely closed in $\mathcal{A}^2$.

**Proof.** First let $G$ be an arbitrary metabelian group, with $a \in G$ and $n \in \mathbb{N}$. We point out a method for constructing a group $G^{(n)}(a)$ in $\mathcal{A}^2$ containing $G$, in which an $n$th root is extracted of $a$. A similar argument was used in [20, proof of Lemma 3] (see also [21]).

Let $Z_n$ be a cyclic group of order $n$, $c$ its generator, and $K$ a basic subgroup of the wreath product $G \wr Z_n$ of groups $G$ and $Z_n$. For any $y \in G$, put $\overline{y}(x) = y$ for all $x \in Z_n$. Denote by $\varphi : G \to G \wr Z_n$ an embedding under which $y^\varphi = \overline{y}$ for each $y \in G$. Consider $C = \mathrm{gr}(cf, G^\varphi)$, where $f(1) = a$ and $f(x) = 1$ with $x \neq 1$. The element $c$ centralizes the subgroup $G^\varphi$. Therefore, the commutator subgroup $C'$ of $C$ is contained in the commutator subgroup $K'$ of $K$; hence $C$ is a metabelian group. Note that $(cf)^n = \overline{a}$. Identifying the subgroup $G^\varphi$ with the group $G$ (i.e., identifying every element $y \in G$ with $\overline{y}$) produces a group $C$ such that $G \leq C$ and an $n$th root is extracted of $a$. Put $G^{(n)}(a) = C$. The $n$th root of $a$ constructed is denoted by $\sqrt[n]{a}$.

Two important properties of the group $G^{(n)}(a)$ are the following:

(1) if $a, b \in G$ and $[a, b] = 1$, then elements $\sqrt[n]{a}$ and $b$ commute in $G^{(n)}(a)$;

(2) $\mathrm{gr}(\sqrt[n]{a}, a_2, \ldots, a_m) \cap G = \mathrm{gr}(a, a_2, \ldots, a_m)$, where $a, a_2, \ldots, a_m$ are arbitrary pairwise commuting elements of $G$.

Let $G$ be a given torsion-free Abelian group of rank $k$, $a_1, \ldots, a_k$ a maximal linearly independent system of elements of $G$, and

$$A_i = \{g \in G \mid (\exists n)(n \in \mathbb{N} \,\&\, n \neq 0 \,\&\, g^n \in (a_i))\}$$

the isolator of a subgroup $(a_i)$ in $G$. It is not hard to see that $G$ decomposes into a direct product of its subgroups $A_1, \ldots, A_k$ and each group $A_i$ has rank 1. (In particular, $A_i$ is a locally cyclic group.) We assume that $A_1, \ldots, A_l$ are not cyclic groups, whereas $A_{l+1}, \ldots, A_k$ are infinite cyclic ones. If $l = 0$, then $G$ is a free Abelian group of rank $k$, and by the hypothesis of the theorem, $G$ is not absolutely closed in $\mathcal{A}^2$. Suppose $l > 0$.

At the moment, we construct a sequence of groups $B_1, \ldots, B_l$ as follows. First, fix some representation of $A_1$: namely,

$$A_1 = \mathrm{gr}(x_1, x_2, x_3, \ldots \parallel x_i^{n_i} = x_{i-1},\ i = 2, 3, \ldots).$$

Take a group such as in Theorem 1 (denoted $A$) containing a free Abelian group $H = \mathrm{gr}(a_1, \ldots, a_k)$ of rank $k$ such that $D = \mathrm{dom}_A^{\mathcal{A}^2}(H) \neq H$. Put

$R_1 = A$, $b_1 = a_1$;

$R_2 = R_1^{(n_2)}(b_1)$, $b_2 = \sqrt[n_2]{b_1}$;

$R_{i+1} = R_i^{(n_{i+1})}(b_i)$, $b_{i+1} = \sqrt[n_{i+1}]{b_i}$ $(i = 2, 3, \ldots)$.

Then $R_1 \subset R_2 \subset R_3 \subset \ldots$. Let $B_1 = \bigcup\limits_{i \in \mathbb{N}} R_i$.

A group $B_j$ $(j = 2, \ldots, l)$ is constructed given the representation

$$A_j = \mathrm{gr}(x_1, x_2, x_3, \ldots \parallel x_i^{m_i} = x_{i-1},\ i = 2, 3, \ldots)$$

as follows. Consider sequences like

$$T_1 = B_{j-1},\ b_1 = a_j,$$
$$T_2 = T_1^{(m_2)}(b_1),\ b_2 = \sqrt[m_2]{b_1},$$
$$\ldots,$$
$$T_{r+1} = T_r^{(m_{r+1})}(b_r),\ b_{r+1} = \sqrt[m_{r+1}]{b_r},$$
$$\ldots.$$

Put $B_j = \bigcup\limits_{i \in \mathbb{N}} T_i$, $B = B_l$.

The above construction for $B$ shows that the subgroup $C_i$ generated by all roots of $a_i$ $(i \leq l)$ is isomorphic to the group $A_i$. Properties (1) and (2) for $G^{(n)}(a)$ imply that the group $M = \mathrm{gr}(C_1, \ldots, C_l, a_{l+1}, \ldots, a_k)$ decomposes into a direct product $M = C_1 \times \ldots \times C_l \times (a_{l+1}) \times \ldots \times (a_k)$, and so $M \cong G$. In addition, $M \cap A = H$.

Since $A \subseteq B$ and $H \subseteq M$, the definition of a dominion entails $D = \mathrm{dom}_A^{\mathcal{A}^2}(H) \subseteq \mathrm{dom}_B^{\mathcal{A}^2}(M)$. Hence $D \subseteq \mathrm{dom}_B^{\mathcal{A}^2}(M) \cap A$. If $\mathrm{dom}_B^{\mathcal{A}^2}(M) = M$, then $D \subseteq M \cap A = H$, which is false. Consequently, $\mathrm{dom}_B^{\mathcal{A}^2}(M) \neq M$. The theorem is proved.

It follows from the proof of Theorem 1 that as $A$ we can take a 2-generated group. Then the group $B$ constructed from $A$ is generated modulo $M$ by two elements, i.e., has the form $B = \mathrm{gr}(x, y, M)$. We derive the following:

**COROLLARY 3.** A torsion-free Abelian group of finite rank $k$ (where $k$ is as in the formulation of Theorem 2) is not 2-closed in the variety of metabelian groups.

**COROLLARY 4.** There exists a torsion-free divisible Abelian group of finite rank that is not 2-closed in $\mathcal{A}^2$.

## 3. DOMINION OF THE ADDITIVE GROUP OF RATIONAL NUMBERS

**THEOREM 3.** Let $G = \mathrm{gr}(A, H)$ be a metabelian group and $A$ and $H$ groups. In addition, let $H$ be isomorphic to the additive group of rational numbers. Suppose also that the normal closure $M = H^G$ of the subgroup $H$ in the group $G$ is a torsion-free Abelian group, $A' \cap M = (1)$, and $M \neq [A, M]$. Then $\mathrm{dom}_G^{\mathcal{A}^2}(H) = H$.

**Proof.** Fix an arbitrary nonidentity element $h$ in $H$. First we show that $H \cap [A, M] = (1)$. If $u \in A$ and $t \in M$ then $[u, t] = t^{-u}t$. This implies that $[A, M]$ is generated by elements of the form $h^q h^{-qa}$, $q \in \mathbb{Q}$, $a \in A$.

Next suppose that $H \cap [A, M] \neq (1)$. Then $h^n \in [A, M]$ for some integer $n$. Hence $h^n$ can be represented as

$$h^n = (h^{q_1} h^{-q_1 a_1}) \dots (h^{q_l} h^{-q_l a_l})$$

for suitable rational numbers $q_1, \dots, q_l$ and suitable elements $a_1, \dots, a_l \in A$. For each integer $r$, $r \neq 0$, consider an element of the form

$$g_r = (h^{r^{-1} q_1} h^{-r^{-1} q_1 a_1}) \dots (h^{r^{-1} q_l} h^{-r^{-1} q_l a_l}).$$

Clearly, $g_r \in [A, M]$ and $g_r^r = h^n$. Since the extraction of a root in a torsion-free Abelian group is unique, we see that $g_r \in H$, so $H \subseteq [A, M]$, and hence $M \subseteq [A, M]$, which is false. Thus $H \cap [A, M] = (1)$. Note that the equalities $A' \cap M = (1)$ and $H \cap [A, M] = (1)$ and the inclusion $H \subseteq M$ entail $H \cap A'[A, M] = (1)$.

Consider a natural homomorphism $\varphi : G \to G/A'[A, M]$. Since $H \cap A'[A, M] = (1)$, $\varphi$ maps $H$ onto $H^\varphi$ isomorphically. Furthermore, $G/A'[A, M] = H^\varphi A^\varphi$. For $H^\varphi$ is a divisible group, it is distinguished by a direct factor in the Abelian group $G^\varphi$, i.e., $G^\varphi = H^\varphi \times B$ for a suitable subgroup $B$ of $G^\varphi$. Let $\pi : G^\varphi \to H^\varphi$ be a projection and $\psi : H^{\varphi\pi} \to G$ an isomorphic embedding under which $h^{\varphi\pi\psi} = h$. Then a map $\varphi\pi\psi : G \to H$ is identical on $H$, and so $\mathrm{dom}_G^{\mathcal{A}^2}(H) = H$. The theorem is proved.

Given $G$, we construct a new group $\widetilde{G}$. Let $G = \mathrm{gr}(a, H)$, $H$ be a torsion-free Abelian group of rank 1 (i.e., every two elements in $H$ are linearly dependent), the normal closure $M = H^G = \prod_{l=0}^{k} H^{a^l}$ of $H$ in $G$ be a direct product of subgroups $H^{a^l}$ $(l = 0, \dots, k)$, and $(a) \cap M = (1)$.

Fix an arbitrary nonidentity element $h \in H$. An element $h^{a^{k+1}}$ belongs to $M = \prod_{l=0}^{k} H^{a^l}$, and so we can write it in the form

$$h^{a^{k+1}} = \prod_{i=0}^{k} h^{l_i a^i}.$$

Here $l_0, \dots, l_k$ are appropriate (uniquely determined) rational numbers.

Let $H_{ij} \cong H$, $\varphi_{ij} : H \to H_{ij}$ be an isomorphism, and $h^{ij} = h^{\varphi_{ij}}$. Assume that $H_{00} = H$, $\varphi_{00}$ is an identical map, and $R = \prod_{i,j=0}^{k} H_{ij}$. Define automorphisms $\alpha, \beta :\ R \to R$ that act on subgroups $\prod_{j=0}^{k} H_{ij}$ and $\prod_{i=0}^{k} H_{ij}$ as does the element $a$ on the group $M$. The automorphisms $\alpha$ and $\beta$ are defined by the formulas

$$\begin{aligned} (h^{ij})^{\alpha} &= h^{i,j+1} \text{ for } 0 \le j \le k-1,\ 0 \le i \le k, \\ (h^{ik})^{\alpha} &= \prod_{j=0}^{k} (h^{ij})^{l_j},\ 0 \le i \le k, \end{aligned} \tag{1}$$

$$\begin{aligned} (h^{ij})^{\beta} &= h^{i+1,j} \text{ for } 0 \le i \le k-1,\ 0 \le j \le k, \\ (h^{kj})^{\beta} &= \prod_{i=0}^{k} (h^{ij})^{l_i},\ 0 \le j \le k. \end{aligned} \tag{2}$$

We show that $\alpha\beta = \beta\alpha$. Clearly, $(h^{ij})^{\alpha\beta} = (h^{ij})^{\beta\alpha}$ for $i \ne k$ and $j \ne k$. We compute $(h^{ik})^{\alpha\beta}$ and $(h^{ik})^{\beta\alpha}$ for $i \ne k$ as follows:

$$(h^{ik})^{\alpha\beta} = \prod_{j=0}^{k} ((h^{i,j})^{l_j})^{\beta} = \prod_{j=0}^{k} (h^{i+1,j})^{l_j},$$

$$(h^{ik})^{\beta\alpha} = (h^{i+1,k})^{\alpha} = \prod_{j=0}^{k} (h^{i+1,j})^{l_j}.$$

Similarly, for $j \ne k$, we derive

$$(h^{kj})^{\alpha\beta} = (h^{k,j+1})^{\beta} = \prod_{i=0}^{k} (h^{i,j+1})^{l_i},$$

$$(h^{kj})^{\beta\alpha} = \prod_{i=0}^{k} ((h^{ij})^{l_i})^{\alpha} = \prod_{i=0}^{k} (h^{i,j+1})^{l_i}.$$

Now we find $(h^{kk})^{\alpha\beta}$ and $(h^{kk})^{\beta\alpha}$: namely,

$$(h^{kk})^{\alpha\beta} = \prod_{j=0}^{k} ((h^{kj})^{l_j})^{\beta} = \prod_{j=0}^{k} \left( \prod_{i=0}^{k} (h^{ij})^{l_i} \right)^{l_j} = \prod_{i,j=0}^{k} (h^{ij})^{l_i l_j},$$

$$(h^{kk})^{\beta\alpha} = \left(\prod_{i=0}^{k}(h^{ik})^{l_i}\right)^{\alpha} = \left(\prod_{i=0}^{k}\left(\prod_{j=0}^{k}(h^{ij}\right)^{l_j}\right)^{l_i} = \prod_{i,j=0}^{k}(h^{ij})^{l_i l_j}.$$

We see that $(h^{ij})^{\alpha\beta} = (h^{ij})^{\beta\alpha}$, i.e., $\alpha\beta = \beta\alpha$.

Consider a group $B = (b_1) \times (b_2)$, where $|b_1| = |b_2| = |a|$. Let $\gamma : B \to \mathrm{gr}(\alpha, \beta)$ be a homomorphism under which $b_1^\gamma = \alpha$, $b_2^\gamma = \beta$. This gives rise to a semidirect product $\widetilde{G} = R\lambda B$ of groups $R$ and $B$. Therefore, $\widetilde{G}$ is the desired metabelian group.

If $G = \mathrm{gr}(a, H)$, $H$ is a torsion-free Abelian group of rank 1, the normal closure $M = H^G = \prod_{l\in\mathbb{Z}} H^{a^l}$ is a direct product of subgroups $H^{a^l}$ ($l \in \mathbb{Z}$), and $(a) \cap M = (1)$, then the group $\widetilde{G}$ is constructed in a similar manner. In this event $R = \prod_{i,j\in\mathbb{Z}} H_{ij}$ and the definition of $\widetilde{G}$ defies equalities (1) and (2).

**THEOREM 4.** Let $G = \mathrm{gr}(a, H)$, where $H$ is isomorphic to the additive group of rational numbers. Suppose that the normal closure $M = H^G$ of $H$ in $G$ is a torsion-free Abelian group. Then $\mathrm{dom}_G^{\mathcal{A}^2}(H) = H$.

**Proof.** Note that if $G$ is an Abelian group then $\mathrm{dom}_G^{\mathcal{M}}(H) = H$ in view of [6, Thm. 1]. Below we assume that $G$ is non-Abelian.

We claim that $(a) \cap M = (1)$. Suppose $(a) \cap M \neq (1)$. Let $n$ be the least positive integer for which $a^n \neq 1$ and $a^n \in M$. Since $a^n \in M$ and $M$ is a divisible Abelian group, $a^n = v^n$ for some $v \in M$. This yields $(v^n)^a = v^n$. Keeping in mind that $M$ is an Abelian group and $v, v^a \in M$, we conclude that $(v^a v^{-1})^n = 1$. On the other hand, $M$ is torsion free, and hence $v^a v^{-1} = 1$. The equality $a^n = v^n$ entails $(av^{-1})^n = 1$. If $(av^{-1})^r \in M$ for some $r$, $0 < r < n$, then $a^r \in M$, which is a contradiction with $n$ being minimal. Thus $(av^{-1}) \cap M = (1)$. Obviously, $M$ is contained in the group $\mathrm{gr}(av^{-1}, H)$, whence $G = \mathrm{gr}(av^{-1}, H)$. If we take an element $av^{-1}$ in place of $a$ we face the case $(a) \cap M = (1)$, as claimed.

Thus we will assume that $(a) \cap M = (1)$. Consider $M$ as a vector space over a field $\mathbb{Q}$ of rational numbers. Obviously, $a$ induces a linear transformation of the vector space $M$ ($a$ acts on $M$ by conjugation). Fix an arbitrary nonidentity element $h \in H$. If $M$ is an infinite-dimensional vector space, then $\{h^{a^l} \mid l \in \mathbb{Z}\}$ is its basis. Hence $M = \prod_{l\in\mathbb{Z}} H^{a^l}$. If $M$ is a finite-dimensional vector space, then the set $\{h^{a^l} \mid l = 0, \ldots, k\}$ is a basis in $M$ for some $k$. Hence $M = \prod_{l=0}^{k} H^{a^l}$. Therefore, the above-described construction can be applied to $G$.

Consider the group $\widetilde{G}$. First let $M = \prod_{l=0}^{k} H^{a^l}$. It is not hard to see that

$$\mathrm{gr}(H_{00}, b_1) = \mathrm{gr}(H_{00}, H_{01}, \ldots, H_{0k}, b_1) \cong G,$$
$$\mathrm{gr}(H_{00}, b_2) = \mathrm{gr}(H_{00}, H_{10}, \ldots, H_{k0}, b_2) \cong G,$$
$$\mathrm{gr}(H_{00}, b_1) \cap \mathrm{gr}(H_{00}, b_2) = H_{00} = H.$$

This implies $\operatorname{dom}_G^{\mathcal{A}^2}(H) = H$. If $M = \prod_{l \in \mathbb{Z}} H^{a^l}$, then

$$\operatorname{gr}(H_{00}, b_1) \cong G,\ \operatorname{gr}(H_{00}, b_2) \cong G,\ \operatorname{gr}(H_{00}, b_1) \cap \operatorname{gr}(H_{00}, b_2) = H_{00} = H,$$

and so $\operatorname{dom}_G^{\mathcal{A}^2}(H) = H$. The theorem is proved.

## REFERENCES


1. J. R. Isbell, "Epimorphisms and dominions," *Proc. Conf. Cat. Alg., La Jolla 1965*, Springer-Verlag, New York (1966), pp. 232-246.
2. P. V. Higgins, "Epimorphisms and amalgams," *Coll. Math.*, **56**, No. 1, 1-17 (1988).
3. A. I. Budkin, "Dominions in quasivarieties of universal algebras," *Stud. Log.*, **78**, Nos. 1/2, 107-127 (2004).
4. A. I. Budkin, "Lattices of dominions of universal algebras," *Algebra Logika*, **46**, No. 1, 26-45 (2007).
5. A. I. Budkin, "Dominions of universal algebras and projective properties," *Algebra Logika*, **47**, No. 5, 541-557 (2008).
6. S. A. Shakhova, "Lattices of dominions in quasivarieties of Abelian groups," *Algebra Logika*, **44**, No. 2, 238-251 (2005).
7. S. A. Shakhova, "Distributivity conditions for lattices of dominions in quasivarieties of Abelian groups," *Algebra Logika*, **45**, No. 4, 484-499 (2006).
8. S. A. Shakhova, "A property of the intersection operation in lattices of dominions in quasivarieties of Abelian groups," *Izv. Altai Gos. Univ.*, No. 1-1(65), 41-43 (2010).
9. S. A. Shakhova, "The existence of a dominion lattice in quasivarieties of Abelian groups," *Izv. Altai Gos. Univ.*, No. 1-1(69), 31-33 (2011).
10. A. Magidin, "Dominions in varieties of nilpotent groups," *Comm. Alg.*, **28**, No. 3, 1241-1270 (2000).
11. A. Magidin, "Absolutely closed nil-2 groups," *Alg. Univ.*, **42**, Nos. 1/2, 61-77 (1999).
12. A. I. Budkin, "Dominions in quasivarieties of metabelian groups," *Sib. Mat. Zh.*, **51**, No. 3, 498-505 (2010).
13. A. I. Budkin, "The dominion of a divisible subgroup of a metabelian group," *Izv. Altai Gos. Univ.*, No. 1-2(65), 15-19 (2010).
14. A. I. Budkin, *Quasivarieties of Groups* [in Russian], Altai State University, Barnaul (2002).
15. A. I. Budkin and V. A. Gorbunov, "Toward a theory of quasivarieties of algebraic systems," *Algebra Logika*, **14**, No. 2, 123-142 (1975).

16. V. A. Gorbunov, *Algebraic Theory of Quasivarieties, Sib. School Alg. Log.* [in Russian], Nauch. Kniga, Novosibirsk (1999).

17. A. I. Mal'tsev, *Algebraic Systems* [in Russian], Nauka, Moscow (1970).

18. M. I. Kargapolov and Yu. I. Merzlyakov, *Fundamentals of Group Theory* [in Russian], Nauka, Moscow (1982).

19. P. Hall, "Finiteness conditions for soluble groups," *Proc. London Math. Soc.*, **4**, 419-436 (1954).

20. A. I. Budkin, "The axiomatic rank of a quasivariety containing a free solvable group," *Mat. Sb.*, **112**(**154**), No. 4(8), 647-655 (1980).

21. A. I. Budkin, *Q-Theories of Finitely Generated Groups. Quasi-Identities, Quasivarieties*, LAP, Saarbrücken, Germany (2012).