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INTERPOLATION AND THE PROJECTIVE BETH PROPERTY IN WELL-COMPOSED LOGICS

L. L. Maksimova* UDC 510.64

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We study the interpolation and Beth definability problems in propositional extensions of minimal logic J. Previously, all J-logics with the weak interpolation property (WIP) were described, and it was proved that WIP is decidable over J. In this paper, we deal with so-called well-composed J-logics, i.e., J-logics satisfying an axiom $(\perp \to A) \vee (A \to \perp)$. *Representation theorems are proved for well-composed logics possessing Craig's interpolation property (CIP) and the restricted interpolation property (IPR). As a consequence, we show that only finitely many well-composed logics share these properties and that IPR is equivalent to the projective Beth property (PBP) on the class of well-composed J-logics.*

INTRODUCTION

We deal with the interpolation and definability problems in propositional extensions of minimal logic J. The minimal logic introduced by Johansson [1] has the same positive fragment as the intuitionistic logic but has no special axioms for negation. As distinct from classical and intuitionistic logics, minimal logic admits nontrivial theories containing some proposition together with its negation.

The interpolation theorem proved by Craig [2] for classical first-order logic initiated a comprehensive study on the interpolation problem in classical and nonclassical theories [3, 4]. At present, interpolation, along with consistency, completeness, and so on, is recognized as a

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Sobolev Institute of Mathematics, Siberian Branch, Russian Academy of Sciences, pr. Akad. Koptyuga 4, Novosibirsk, 630090 Russia; Novosibirsk State University, ul. Pirogova 2, Novosibirsk, 630090 Russia; lmaksi@math.nsc.ru. Translated from *Algebra i Logika*, Vol. 51, No. 2, pp. 244-275, March-April, 2012. Original article submitted February 17, 2011; revised March 14, 2012.

standard property of logics. For intuitionistic predicate logic and for Johansson's minimal logic, an interpolation theorem was proved by Shutte [5]. A semantic proof for the interpolation theorem in intuitionistic predicate logic was obtained by Gabbay [6]. The interpolation property is closely connected with the Beth definability property [7], which is also widely used in the literature.

A family of extensions of Johansson's minimal logic J contains all superintuitionistic logics, for which the interpolation problem was resolved in [8]. It was proved that there are finitely many superintuitionistic logics with Craig's interpolation property (CIP) and that there is an algorithm for recognizing CIP in superintuitionistic calculi. All superintuitionistic logics with the projective Beth property (PBP) were described in [9]. It was stated that such logics are finite in number and that PBP is decidable over the intuitionistic logic Int [10]. Similar results were derived for positive logics containing a positive fragment of the intuitionistic logic Int^+ [11] and for negative J-logics.

The interpolation property admits of different versions, which are equivalent in classical logic but are not equivalent in other logics. It turned out that PBP, which follows from CIP, implies a restricted interpolation property (IPR) on the class of all J-logics. Moreover, IPR and PBP are equivalent on the classes of superintuitionistic, positive, and negative logics [12, 13].

In [14], we looked into the weak interpolation property (WIP) introduced in [15]. In [16], it was proved that in all extensions of minimal logic, WIP is equivalent to a weak version of Robinson's joint consistency. In [15], it was shown that all propositional superintuitionistic logics possess WIP. Since only finitely many propositional superintuitionistic logics have IPR, WIP and IPR are not equivalent over the intuitionistic logic. Moreover, WIP and IPR are not equivalent over the minimal logic J. Note that WIP is nontrivial in propositional extensions of minimal logic: a set of J-logics with WIP and a set of J-logics without WIP have the cardinality of the continuum [16].

In [16], an algebraic counterpart of WIP was found—namely, the weak amalgamation property. It was proved that the problem whether WIP is valid in J-logics reduces to treating extensions of a logic Gl, which is obtained by adding the law of excluded middle to the logic J. In [14], all logics with the weak interpolation property over Gl were described, logics with WIP over J were classified, and WIP was proved decidable over J. This means that there exists an algorithm which, given any finite set of axiom schemes, decides whether WIP is valid in a calculus obtained by adding these axiom schemes to Johansson's.

In [17], a complete description was found for logics over Gl possessing CIP, PBP, or IPR, and these properties were proved decidable over the logic Gl. Also it was stated that PBP and IPR are equivalent over Gl.

In this paper we are concerned with so-called well-composed J-logics, i.e., J-logics satisfying an additional axiom $(\bot \to A) \vee (A \to \bot)$. The study of such logics was initiated in [18]. Our present objective is to obtain representation theorems for well-composed logics possessing CIP, PBP, or IPR. As a consequence, we will prove that only finitely many well-composed logics share these properties and that PBP and IPR are equivalent on the class of well-composed J-logics.

1. INTERPOLATION AND DEFINABILITY

The language of the logic J contains &, $\vee$, $\to$, $\bot$, and $\top$ as primitive connectives; a negation is defined as an abbreviation $\neg A = A \to \bot$; $(A \leftrightarrow B) = (A \to B)\&(B \to A)$. A formula is said to be *positive* if it contains no occurrences of the constant $\bot$. The logic J can be axiomatized by a calculus which has the same axiom schemes as the positive intuitionistic calculus Int^+, and modus ponens $(A, A \to B \,/\, B)$ as the only rule of inference. More specifically, J is defined by the following axiom schemes:

(1) $A \to (B \to A)$;

(2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$;

(3) $A\&B \to A$;

(4) $A\&B \to B$;

(5) $A \to (B \to A\&B)$;

(6) $A \to A \vee B$;

(7) $B \to A \vee B$;

(8) $(A \to C) \to ((B \to C) \to (A \vee B \to C))$.

By a J-*logic* we mean any set of formulas that contains all axioms of the calculus J and is closed under modus ponens and substitution. Put

$$\mathrm{Int} = \mathrm{J} + (\bot \to p),\ \mathrm{Cl} = \mathrm{Int} + (p \vee \neg p),\ \mathrm{Neg} = \mathrm{J} + \bot,\ \mathrm{Gl} = \mathrm{J} + (p \vee \neg p),$$
$$\mathrm{JX} = \mathrm{J} + ((\bot \to p) \vee (p \to \bot)),\ \mathrm{For} = \mathrm{J} + p.$$

A logic is said to be *nontrivial* if it does not coincide with the set For of all formulas. A *superintuitionistic logic* is a J-logic that contains the intuitionistic logic Int. A *negative logic* is a J-logic that contains the logic Neg. A logic L is *paraconsistent* if it includes neither Int nor Neg. We can prove that a J-logic is negative iff it is not contained in Cl. For any J-logic L, $E(L)$ denotes a family of all J-logics containing L.

We write $\Gamma \vdash_L A$ if a formula A is deducible from $L \cup \{A\}$ via modus ponens. If $\mathbf{p}$ is a list of variables, then we denote by $A(\mathbf{p})$ a formula whose variables are all in $\mathbf{p}$, and by $\mathcal{F}(\mathbf{p})$ the set of all such formulas. Let L be a logic and $\vdash_L$ a deducibility relation in L. Suppose that $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{q}'$ are pairwise disjoint lists of variables not containing x and y, $\mathbf{q}$ and $\mathbf{q}'$ are equal in length, and $A(\mathbf{p}, \mathbf{q}, x)$ is a formula.

We say that L has the *projective Beth property* (PBP) if $A(\mathbf{p}, \mathbf{q}, x), A(\mathbf{p}, \mathbf{q}', y) \vdash_L (x \leftrightarrow y)$ implies $A(\mathbf{p}, \mathbf{q}, x) \vdash_L (x \leftrightarrow B(\mathbf{p}))$ for some formula $B(\mathbf{p})$. L has the *Beth property* (BP) if $A(\mathbf{p}, x), A(\mathbf{p}, y) \vdash_L (x \leftrightarrow y)$ implies $A(\mathbf{p}, x) \vdash_L (x \leftrightarrow B(\mathbf{p}))$ for a suitable formula $B(\mathbf{p})$.

A formula $B(\mathbf{p})$ is called an *explicit definition for* x.

PBP and BP were taken up, for instance, in [19] where they were denoted PB2 and B2. Obviously, B2 is a partial case of PB2. In [19], also, PB1 and B1, versions of respectively the projective Beth property and the Beth property, were defined. For the logics under consideration here, PB1 and B1 are equivalent to PB2 and B2, respectively. In addition, it was stated in [20] that all superintuitionistic logics possess BP. In a similar manner, we can prove the Beth property for all logics in this paper.

As in [2], the projective Beth property may be deduced from the Craig *interpolation property*

(CIP) if $\vdash_L A(\mathbf{p},\mathbf{q}) \to B(\mathbf{p},\mathbf{r})$, then there exists a formula $C(\mathbf{p})$ such that $\vdash_L A(\mathbf{p},\mathbf{q}) \to C(\mathbf{p})$ and $\vdash_L C(\mathbf{p}) \to B(\mathbf{p},\mathbf{r})$ ($\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ are pairwise disjoint).

A formula $C(\mathbf{p})$ is called an *interpolant*.

In view of the deduction theorem, CIP in J-logics is equivalent to the *deductive interpolation property*

(IPD) if $A(\mathbf{p},\mathbf{q}) \vdash_L B(\mathbf{p},\mathbf{r})$, then there exists a formula $C(\mathbf{p})$ such that $A(\mathbf{p},\mathbf{q}) \vdash_L C(\mathbf{p})$ and $C(\mathbf{p}) \vdash_L B(\mathbf{p},\mathbf{r})$.

In [21], we introduced the *restricted interpolation property*

(IPR) if $A(\mathbf{p},\mathbf{q}), B(\mathbf{p},\mathbf{r}) \vdash_L C(\mathbf{p})$, then there exists a formula $A'(\mathbf{p})$ such that $A(\mathbf{p},\mathbf{q}) \vdash_L A'(\mathbf{p})$ and $A'(\mathbf{p}), B(\mathbf{p},\mathbf{r}) \vdash_L C(\mathbf{p})$.

In [8], we brought up for consideration the *weak interpolation property*

(WIP) if $A(\mathbf{p},\mathbf{q}), B(\mathbf{p},\mathbf{r}) \vdash_L \bot$, then there exists a formula $A'(\mathbf{p})$ such that $A(\mathbf{p},\mathbf{q}) \vdash_L A'(\mathbf{p})$ and $A'(\mathbf{p}), B(\mathbf{p},\mathbf{r}) \vdash_L \bot$.

For all J-logics, the following holds:

$$\text{CIP} \iff \text{IPD} \Rightarrow \text{PBP} \Rightarrow \text{IPR} \Rightarrow \text{WIP}.$$

Furthermore, PBP does not imply IPD, while WIP does not imply IPR even on the class of superintuitionistic logics. For superintuitionistic, positive, and negative logics, IPR and PBP were proved equivalent. The question whether these properties are equivalent in J-logics is still not settled.

In [8], we obtained a description of all propositional superintuitionistic logics with the Craig interpolation property. There exist only finitely many superintuitionistic logics possessing CIP. All positive logics with CIP were described in [11] where, too, we initiated a study of this property in extensions of Johansson's minimal logic.

Below are several well-known facts on interpolation properties in extensions of minimal logic.

PROPOSITION 1.1 [9]. There exist exactly 16 superintuitionistic logics with the projective Beth property PBP, of which precisely 8 have CIP. These logics are all finitely axiomatixable and residually finite.

A list of superintuitionistic logics with CIP includes Int, LS, Cl, and a trivial logic For. The logic Cl is greatest among consistent superintuitionistic logics, while LS is greatest among consistent superintuitionistic logics distinct from Cl. The logic Cl is characterized by a two-element Boolean algebra B_0, and LS by a three-element linearly ordered Heyting algebra C_1.

PROPOSITION 1.2 [11]. There exist exactly 7 negative logics with the projective Beth property PBP, of which precisely 4 have CIP. These logics are all finitely axiomatizable and residually finite.

A list of negative logics with CIP includes the following:

$$\mathrm{Neg},\ \mathrm{NC} = \mathrm{Neg} + (p \to q) \vee (q \to p),\ \mathrm{NE} = \mathrm{Neg} + p \vee (p \to q),\ \mathrm{For} = \mathrm{Neg} + p.$$

The logic NC is characterized by linearly ordered negative algebras, and NE by a two-element negative algebra.

PROPOSITION 1.3. For all superintuitionistic and negative logics, IPR and PBP are equivalent.

Proof. For superintuitionistic logics, this was stated in [12]. For negative logics, the result follows immediately from the equivalence of these properties in positive logics, proved in [13]. □

All superintuitionistic and negative logics possess WIP. This, however, does not extend to all J-logics [14]. The problem whether WIP is valid in J-logics reduces to treating extensions of a logic $\mathrm{Gl} = \mathrm{J} + (p \vee (p \to \bot))$.

PROPOSITION 1.4 [16]. A J-logic L has WIP if and only if $L + \mathrm{Gl}$ has WIP.

All extensions of Gl and J with WIP were described in [14]. A set of logics with WIP over Gl and a set of logics without WIP have the cardinality of the continuum. A complete description of extensions of the logic Gl possessing CIP, IPR, or PBP is given in [17]. In particular, it was proved that there exist only finitely many such logics. Furthermore, IPR and PBP are equivalent over the logic Gl.

We say that a property P is decidable over a logic L if there exists an algorithm which, given any finite set of axiom schemes Ax, decides if the logic $L + Ax$ possesses the property P.

PROPOSITION 1.5 [14, 17]. WIP is decidable over J. CIP, PBP, and IPR are decidable over Gl.

2. ALGEBRAIC SEMANTICS

Algebraic semantics for extensions of minimal logic is constructed by using so-called J-*algebras*, i.e., algebras $\mathbf{A} = \langle A; \&, \vee, \to, \bot, \top \rangle$ satisfying the following conditions:

$\langle A; \&, \vee, \to, \top \rangle$ is an implicative lattice, i.e., a lattice with respect to & and $\vee$, with the greatest element $\top$;

$z \le x \to y \iff z \& x \le y$;

$\bot$ is an arbitrary element of A.

A J-algebra is called a *Heyting* (or *pseudo-Boolean*) algebra if $\bot$ is the least element of A. A J-algebra is called a *negative* algebra if $\bot$ is the greatest element of A. A one-element J-algebra $\mathbf{E}$ is said to be *unique* or *degenerate*; it is the only algebra that is simultaneously a negative algebra and a Heyting algebra.

A J-algebra $\mathbf{A}$ is *nondegenerate* if it contains at least two elements. A J-algebra $\mathbf{A}$ is *well connected* or *strongly compact* if

$$x \vee y = \top \Leftrightarrow (x = \top \text{ or } y = \top)$$

for all $x, y \in \mathbf{A}$. An element Ω of an algebra $\mathbf{A}$ is called an *opremum* of the algebra $\mathbf{A}$ if it is greatest among elements of $\mathbf{A}$ other than $\top$. We denote a two-element Boolean algebra by B_0, and a three-element Heyting algebra by C_1.

Recall that a nondegenerate algebra is *subdirectly irreducible* if it cannot be represented as a subdirect product of factors distinct from the algebra. An algebra is *finitely indecomposable* if it cannot be represented as a subdirect product of finitely many factors distinct from the algebra.

By a well-known theorem of Birkhoff (see, e.g., [22]), every variety is generated by a class of its subdirectly irreducible algebras and, hence, by a class of finitely indecomposable algebras.

The lemma below, which is known for Heyting algebras (see, e.g., [8]), extends readily to J-algebras.

LEMMA 2.1. For every J-algebra $\mathbf{A}$, the following conditions hold:

(a) $\mathbf{A}$ is finitely indecomposable if and only if a one-element filter $\nabla = \{\top\}$ is prime, i.e., $\mathbf{A}$ is well connected;

(b) $\mathbf{A}$ is subdirectly irreducible if and only if $\mathbf{A}$ has an opremum.

The next lemma is well known and easily proved.

LEMMA 2.2 (a) Every nondegenerate Heyting algebra contains B_0 as a subalgebra.

(b) If $\mathbf{A}$ is a subdirectly irreducible Heyting algebra and its opremum Ω is distinct from $\bot$, then the set $\{\bot, \Omega, \top\}$ forms a subalgebra of $\mathbf{A}$ isomorphic to C_1.

If $\mathbf{A}$ and $\mathbf{B}$ are partially ordered sets such that $\mathbf{A}$ has a greatest element and $\mathbf{B}$ has a least element, then we define a new set $\mathbf{C} = \mathbf{A} + \mathbf{B}$ as follows:

take a set $\mathbf{C} = \mathbf{A} \cup \mathbf{B}'$, where $\mathbf{B}'$ is isomorphic to $\mathbf{B}$ and its least element is glued to the greatest element of $\mathbf{A}$, while other elements do not enter $\mathbf{A}$, and moreover, $\mathbf{C}$ is partially ordered by the relation

$$x \leq_{\mathbf{C}} y \Leftrightarrow [(x \in \mathbf{A} \text{ and } y \in \mathbf{B}') \text{ or } (x, y \in \mathbf{A} \text{ and } x \leq_{\mathbf{A}} y) \\ \text{ or } (x, y \in \mathbf{B}' \text{ and } x \leq_{\mathbf{B}'} y)].$$

Thus $\mathbf{A}$ and $\mathbf{B}$ can be treated as intervals of the partially ordered set $\mathbf{C}$. By definition, $\mathbf{A}$ and $\mathbf{B}$ are sublattices of $\mathbf{C}$. If $\mathbf{A}$ and $\mathbf{B}$ are implicative lattices, then $\mathbf{C}$ likewise is an implicative lattice, and the operation $\rightarrow$ satisfies the following conditions:

$$x \rightarrow_{\mathbf{C}} y = \begin{cases} \top & \text{if } x \leq_{\mathbf{C}} y, \\ x \rightarrow_{\mathbf{A}} y & \text{if } x, y \in \mathbf{A}, x \not\leq_{\mathbf{A}} y, \\ x \rightarrow_{\mathbf{B}'} y & \text{if } x, y \in \mathbf{B}', \\ y & \text{if } x \in \mathbf{B}', y \in \mathbf{A} - \{\top_{\mathbf{A}}\}. \end{cases}$$

We recall a construction from [18]. If $\mathbf{A} = \langle A; \&, \vee, \to, \bot, \top\rangle$ is a negative algebra and $\mathbf{B} = \langle B; \&, \vee, \to, \bot, \top\rangle$ is a Heyting algebra, then we define a new J-algebra $\mathbf{C} = \mathbf{A}{\uparrow}\mathbf{B}$ to be a J-algebra with universe $\mathbf{A} + \mathbf{B}$, where $\bot_{\mathbf{C}} = \bot_{\mathbf{A}} = \top_{\mathbf{A}} = \bot_{\mathbf{B'}}$. Clearly, $\top_{\mathbf{C}} = \top_{\mathbf{B'}}$. In particular, every negative algebra $\mathbf{A}$ and every Heyting algebra $\mathbf{B}$ are representable as $\mathbf{A}{\uparrow}\mathbf{E}$ and $\mathbf{E}{\uparrow}\mathbf{B}$, respectively. We say that a J-algebra is *well composed* if it is of the form $\mathbf{A}{\uparrow}\mathbf{B}$ for a suitable negative algebra $\mathbf{A}$ and for a suitable Heyting algebra $\mathbf{B}$.

Of special importance in the paper are well-composed algebras like $\mathbf{A}{\uparrow}B_0$, where B_0 is a two-element Boolean algebra. For a negative algebra $\mathbf{A}$, we define

$$\mathbf{A}^{\Lambda} = \mathbf{A}{\uparrow}B_0.$$

Obviously, all J-algebras $\mathbf{A}^{\Lambda}$ are subdirectly irreducible and have $\bot$ as an opremum.

The definition readily entails

LEMMA 2.3. (1) An algebra $\mathbf{B}$ is isomorphic to a subalgebra of $\mathbf{C} = \mathbf{A}{\uparrow}\mathbf{B}$.

(2) An algebra $\mathbf{A}$ is a homomorphic image of the algebra $\mathbf{A}{\uparrow}\mathbf{B}$ under the homomorphism

$$f(z) = z\&\bot.$$

(3) $\mathbf{A}$ is a subalgebra of $\mathbf{A}{\uparrow}\mathbf{B}$ if and only if $\mathbf{B}$ is a degenerate algebra.

For any well-composed algebra $\mathbf{A}$, the following algebras are defined uniquely:

$$\mathbf{A}^l = \{x \in \mathbf{A} \mid x \leq \bot\} \text{ and } \mathbf{A}^u = \{x \in \mathbf{A} \mid x \geq \bot\},$$

with $\mathbf{A}^l$ a negative algebra, $\mathbf{A}^u$ a Heyting algebra, and $\mathbf{A} = (\mathbf{A}^l{\uparrow}\mathbf{A}^u)$.

LEMMA 2.4 [18]. Let $\mathbf{A}$ and $\mathbf{B}$ be well-composed algebras.

(1) A mapping $\alpha : \mathbf{A} \to \mathbf{B}$ is a monomorphism if and only if its restrictions α^l and α^u to $\mathbf{A}^l$ and $\mathbf{A}^u$, respectively, are monomorphisms of $\mathbf{A}^l$ into $\mathbf{B}^l$ and of $\mathbf{A}^u$ into $\mathbf{B}^u$.

(2) For any homomorphism $h : \mathbf{A} \to \mathbf{B}$, exactly one of the following conditions holds:

(a) $h(\bot) = \top_{\mathbf{B}}$ and the restriction h^l of a mapping h to $\mathbf{A}^l$ is a homomorphism of $\mathbf{A}^l$ into $\mathbf{B}$;

(b) $h(\bot) \neq \top_{\mathbf{B}}$, the restriction h^u of h to $\mathbf{A}^u$ is a homomorphism of $\mathbf{A}^u$ into $\mathbf{B}^u$, and the restriction h^l of h to $\mathbf{A}^l$ is a monomorphism of $\mathbf{A}^l$ into $\mathbf{B}^l$.

(3) Let $h_1 : \mathbf{A}^l \to \mathbf{B}$ be a homomorphism. Then $\mathbf{B}$ is a negative algebra, and a mapping defined by setting

$$h(x) = \begin{cases} \top & \text{if } x \in \mathbf{A}^u, \\ h^l(x) & \text{if } x \in \mathbf{A}^l \end{cases}$$

is a homomorphism of $\mathbf{A}$ into $\mathbf{B}$.

(4) Let $\alpha_1 : \mathbf{A}^l \to \mathbf{B}^l$ be a monomorphism and $h_2 : \mathbf{A}^u \to \mathbf{B}^u$ a homomorphism. Then a mapping defined by setting

$$h(x) = \begin{cases} h_2(x) & \text{if } x \in \mathbf{A}^u, \\ \alpha_1(x) & \text{if } x \in \mathbf{A}^l \end{cases}$$

is a homomorphism of $\mathbf{A}$ into $\mathbf{B}$.

It is well known that a family of J-algebras form a variety, and there is a one-to-one correspondence between logics containing J and varieties of J-algebras. If A is a formula and $\mathbf{A}$ is an algebra, then we say that the formula A is valid in the algebra $\mathbf{A}$, and write $\mathbf{A} \models A$, if $\mathbf{A}$ satisfies an identity $A = \top$. Instead of $(\forall A \in L)(\mathbf{A} \models A)$, we write $\mathbf{A} \models L$.

In correspondence with every logic $L \in E(\mathrm{J})$ is a variety of J-algebras such as

$$V(L) = \{\mathbf{A} \mid \mathbf{A} \models L\}.$$

Every logic is characterized by a variety $V(L)$. We say that a logic L *is generated* by some class of algebras if a variety $V(L)$ is generated by that class. If $V(L)$ is generated by an algebra $\mathbf{A}$, then sometimes we write $L = L\mathbf{A}$.

If $L \in E(\mathrm{Int})$, then $V(L)$ is a variety of Heyting algebras, and if $L \in E(\mathrm{Neg})$, then $V(L)$ is a variety of negative algebras. Clearly, the intersection of two J-logics is again a J-logic. An axiomatization of intersection can be easily found given an axiomatization for initial logics. For formulas A and B, $A \vee' B$ denotes a disjunction $A \vee B'$, where B' is obtained by replacing all variables in B with new variables not in A.

LEMMA 2.5. Let L be an intersection of two J-logics L_1 and L_2. Then:

(1) L is axiomatizable by formulas $A \vee' B$, where A is an axiom for L_1 and B is one for L_2;

(2) a finitely indecomposable algebra $\mathbf{A}$ belongs to $V(L)$ if and only if $\mathbf{A} \in (V(L_1) \cup V(L_2))$.

Proof. (1) An argument is similar to the proof of a theorem of Miura [23].

(2) The result follows from item (1) and Lemma 2.1. □

For $L_1 \in E(\mathrm{Neg})$ and $L_2 \in E(\mathrm{Int})$, we denote by $L_1{\uparrow}L_2$ a logic characterized by all algebras of the form $\mathbf{A}{\uparrow}\mathbf{B}$, where $\mathbf{A} \models L_1$ and $\mathbf{B} \models L_2$, and by $L_1{\Uparrow}L_2$ a logic characterized by a class of algebras of the form $\mathbf{A}{\uparrow}\mathbf{B}$, where $\mathbf{A}$ is a finitely indecomposable algebra in $V(L_1)$ and $\mathbf{B} \in V(L_2)$. In particular, if L_1 is the trivial logic For, then $L_1{\uparrow}L_2$ and $L_1{\Uparrow}L_2$ coincide with L_2. If L_2 is trivial, then $L_1{\uparrow}L_2$ and $L_1{\Uparrow}L_2$ coincide with L_1.

As an example we consider a logic $\mathrm{Gl} = \mathrm{J} + (p \vee \neg p)$.

PROPOSITION 2.6. A logic $\mathrm{Gl} = \mathrm{J} + (p \vee \neg p)$ coincides with $\mathrm{Neg}{\uparrow}\mathrm{Cl}$ and is generated by a class $\{\mathbf{A}^{\Lambda} \mid \mathbf{A}$ is a negative algebra$\}$.

An axiomatization for logics like $L_1{\uparrow}L_2$ and $L_1{\Uparrow}L_2$, where L_1 is a negative logic and L_2 is a superintuitionistic logic, was found in [18]. Following [24], we put

$$I(A(p_1, \dots, p_n)) = A(p_1 \vee \bot, \dots, p_n \vee \bot),$$
$$L_2 * L_1 = \mathrm{J} + \{(\bot \to A) \mid A \in L_1\} + \{I(A) \mid A \in L_2\}.$$

In [24], it was shown that if $L_1 = \mathrm{Neg} + Ax_1$ and $L_2 = \mathrm{Int} + Ax_2$, then

$$L_2 * L_1 = \mathrm{J} + \{(\bot \to A) \mid A \in Ax_1\} + \{I(A) \mid A \in Ax_2\}.$$

In addition, we set

$$\mathrm{JX} = \mathrm{J} + ((\bot \to p) \vee (p \to \bot)).$$

PROPOSITION 2.7 [14]. For any negative logic L_1 and any superintuitionistic logic L_2, the following equalities hold:

$$L_1{\uparrow}L_2 = \mathrm{JX} + (L_2 * L_1),$$
$$L{\Uparrow}L_2 = (L_2{\uparrow}L_1) + ((\bot \to p \vee q) \to (\bot \to p) \vee (\bot \to q)).$$

In particular, $\mathrm{JX} = \mathrm{Neg}{\uparrow}\mathrm{Int}$.

Analogously to Proposition 2.6, we have

PROPOSITION 2.8 [14]. For any negative logic L, the logic $L{\uparrow}\mathrm{Cl}$ is generated by a class of algebras $\mathbf{A}^{\Lambda}$, where $\mathbf{A} \in V(L)$, and the logic $L{\Uparrow}\mathrm{Cl}$ is generated by a class of algebras $\mathbf{A}^{\Lambda}$, where $\mathbf{A}$ is a finitely indecomposable algebra in $V(L)$.

3. INTERPOLATION, THE PROJECTIVE BETH PROPERTY, AND AMALGAMABILITY

Recall that a J-logic possesses the Craig interpolation property iff a variety $V(L)$ has the amalgamation property (AP) [11]. With J-algebras, AP is equivalent to the superamalgamation property (SAP). We recall relevant definitions.

Let V be a class of algebras closed under isomorphisms. A class V is *amalgamable* if, for any algebras $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ in V, the class V satisfies the condition

(AP) if $\mathbf{A}$ is a common subalgebra of $\mathbf{B}$ and $\mathbf{C}$, then there exist an algebra $\mathbf{D}$ in V and monomorphisms $\delta : \mathbf{B} \to \mathbf{D}$ and $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \varepsilon(x)$ for all $x \in \mathbf{A}$.

A triple $(\mathbf{D}, \delta, \varepsilon)$ is called an *amalgam* for $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. We say that a class V possesses SAP if any algebras $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ in V satisfy AP, and moreover, the following relations hold in $\mathbf{D}$:

$$\delta(x) \leq \varepsilon(y) \iff (\exists z \in \mathbf{A})(x \leq z \text{ and } z \leq y),$$
$$\delta(x) \geq \varepsilon(y) \iff (\exists z \in \mathbf{A})(x \geq z \text{ and } z \geq y).$$

A class V possesses a *restricted amalgamation property* [21] if V satisfies the condition

(RAP) For any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V$ such that $\mathbf{A}$ is a common subalgebra of $\mathbf{B}$ and $\mathbf{C}$, there exist an algebra $\mathbf{D}$ in V and homomorphisms $\delta : \mathbf{B} \to \mathbf{D}$ and $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \varepsilon(x)$ for all $x \in \mathbf{A}$ and the restriction δ' of δ to $\mathbf{A}$ is a monomorphism.

The concept of restricted amalgamability in [9, 11] is defined in a different way: namely, a class V has RAP* if AP is satisfied for any subdirectly irreducible algebras $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ having a common opremum.

A class V of algebras possesses *strong epimorphisms surjectivity* if V satisfies the condition

(SES) For any $\mathbf{A}$ and $\mathbf{B}$ in V such that $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and for an arbitrary $b \in \mathbf{B} - \mathbf{A}$, there exist an algebra $\mathbf{C} \in V$ and homomorphisms $g : \mathbf{B} \to \mathbf{C}$ and $h : \mathbf{B} \to \mathbf{C}$ such that $g(x) = h(x)$ for all $x \in \mathbf{A}$ and $g(b) \neq h(b)$.

We cite two theorems.

THEOREM 3.1 [11]. For any logic L in $E(\mathrm{J})$, the following conditions are equivalent:

(1) L has Craig's interpolation property;

(2) $V(L)$ is amalgamable;

(3) $V(L)$ has SAP;

(4) AP is satisfied for any well-connected algebras $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ in $V(L)$.

THEOREM 3.2 [11]. For any logic L in $E(\mathrm{J})$, the following conditions are equivalent:

(1) L has the projective Beth property;

(2) $V(L)$ has SES;

(3) $V(L)$ has RAP* and the class $FI(V(L))$ of finitely indecomposable algebras in $V(L)$ has SES.

In addition, it is worth mentioning that the descriptions of all superintuitionistic and negative logics with interpolation obtained in [8, 11] give rise to

THEOREM 3.3 [11]. For any logic L in $E(\mathrm{Int})$ or $E(\mathrm{Neg})$, the following conditions are equivalent:

(1) a variety $V(L)$ is amalgamable;

(2) a class of finitely indecomposable algebras in $V(L)$ is amalgamable.

We are unaware whether this statement is true for all extensions of minimal logic.

As regards the restricted interpolation property, we have

THEOREM 3.4 [18]. For any logic L in $E(\mathrm{J})$, the following conditions are equivalent:

(1) L has IPR;

(2) $V(L)$ has RAP;

(3) $V(L)$ has RAP*;

(4) for any subdirectly irreducible J-algebras $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ in $V(L)$ having a common opremum Ω, the fact that $\mathbf{A}$ is a common subalgebra of $\mathbf{B}$ and $\mathbf{C}$ implies that there exist a subdirectly irreducible algebra $\mathbf{D}$ in $V(L)$ and monomorphisms $\delta : \mathbf{B} \to \mathbf{D}$ and $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \varepsilon(x)$ for all $x \in \mathbf{A}$ and $\delta(\Omega)$ is an opremum in $\mathbf{D}$.

We see that for all varieties of J-algebras, RAP* follows from PBP. Therefore, we have

PROPOSITION 3.5. For all extensions of minimal logic, CIP implies PBP and PBP implies IPR.

An algebraic equivalent of the weak interpolation property in J-logics was found in [16].

For a class V of J-algebras, we define the *weak amalgamation property*

(WAPJ) for any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V$ and for monomorphisms $\beta : \mathbf{A} \to \mathbf{B}$ and $\gamma : \mathbf{A} \to \mathbf{C}$, there exist an algebra $\mathbf{D}$ in V and homomorphisms $\delta : \mathbf{B} \to \mathbf{D}$ and $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta\beta(x) = \varepsilon\gamma(x)$ for all $x \in \mathbf{A}$, with $\bot \neq \top$ in $\mathbf{D}$ if $\bot \neq \top$ in $\mathbf{A}$.

A variety of J-algebras is said to be *weakly amalgamable* if it has WAPJ.

Note that the definition above differs from the weak amalgamation property WAP as defined in [25]. WAP is a partial case of WAPJ.

Note that if a class V is closed under isomorphisms, then WAPJ is equivalent to the following condition:

for any $\mathbf{B}, \mathbf{C} \in V$ having a common subalgebra $\mathbf{A}$, there exist an algebra $\mathbf{D}$ in V and homomorphisms $\delta : \mathbf{B} \to \mathbf{D}$ and $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \varepsilon(x)$ for all $x \in \mathbf{A}$, with $\bot \neq \top$ in $\mathbf{D}$ if $\bot \neq \top$ in $\mathbf{A}$.

THEOREM 3.6 [16]. Let L be a J-logic. Then the following conditions are equivalent:

(1) L has WIP;

(2) $V(L)$ has WAPJ;

(3) $FG(V(L))$ has WAPJ.

For modal logics, an algebraic equivalent of WIP was obtained in [15].

In [16], it was proved that treating WIP in J-logics reduces to studying extensions of a logic $\mathrm{Gl} = \mathrm{J} + (p \vee \neg p)$.

For our further reasoning, we need two lemmas.

LEMMA 3.7. Let L be a J-logic, $V(L)$ an amalgamable variety, $\mathbf{A}$ a negative algebra, $\mathbf{B}$ a Heyting algebra, and $\mathbf{A}^{\Lambda}, \mathbf{B} \in V(L)$. Then $(\mathbf{A}\uparrow\mathbf{B}) \in V(L)$.

Proof. Note that a two-element Boolean algebra B_0 is a common subalgebra of $\mathbf{A}^{\Lambda}$ and $\mathbf{B}$. Since $V(L)$ is amalgamable, there exist an algebra $\mathbf{D} \in V(L)$ and monomorphisms $\delta : \mathbf{B} \to \mathbf{D}$ and $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \varepsilon(x)$ for $x \in \{\bot, \top\}$. We build a mapping α from $(\mathbf{A}\uparrow\mathbf{B})$ to $\mathbf{D}$ by setting

$$\alpha(x) = \begin{cases} \delta(x) & \text{if } x \in \mathbf{A}^{\Lambda}, \\ \varepsilon(x) & \text{if } x \in \mathbf{B}. \end{cases}$$

It is not hard to verify that α is a monomorphism. Obviously, α is one-to-one and respects lattice operations. In addition, $\alpha(x \to y) = \alpha(x) \to \alpha(y)$ in those cases where $x, y \in \mathbf{B}$, $x \leq y$, or $x, y \in \mathbf{A}^{\Lambda}$, $x \not\leq y$.

We are left to handle the case with $x \in \mathbf{B}$ and $y \in (\mathbf{A} - \{\bot\})$. In this event

$$\alpha(x \to y) = \alpha(y) = \delta(y),$$
$$\alpha(x) \to \alpha(y) = \varepsilon(x) \to \delta(y) \leq \varepsilon(\bot) \to \delta(y) = \bot \to \delta(y) = \delta(y).$$

Thus $(\mathbf{A}\uparrow\mathbf{B})$ embeds in $\mathbf{D}$ and, hence, belongs to $V(L)$. □

Denote by C_1 a three-element linearly ordered Heyting algebra with elements $\bot < a < \top$.

LEMMA 3.8. Let L be a J-logic, $V(L)$ a variety with RAP, $\mathbf{A}$ a negative algebra, $\mathbf{B}$ a subdirectly irreducible Heyting algebra, and $(\mathbf{A}\uparrow C_1), \mathbf{B} \in V(L)$. Then $(\mathbf{A}\uparrow\mathbf{B}) \in V(L)$.

Proof. If $\mathbf{B}$ is a two-element Boolean algebra, then $(\mathbf{A}\uparrow\mathbf{B})$ is a subalgebra of $(\mathbf{A}\uparrow C_1)$ and, hence, belongs to $V(L)$.

Suppose $\mathbf{B}$ contains at least three elements and Ω is its opremum. Then a mapping $\beta : C_1 \to \mathbf{B}$, where $\beta(a) = \Omega$, $\beta(\bot) = \bot$, and $\beta(\top) = \top$, is a monomorphism preserving the opremum. Furthermore, C_1 is a subalgebra of $(\mathbf{A}{\uparrow}C_1)$. In view of RAP, there exist an algebra $\mathbf{D} \in V(L)$ and monomorphisms $\delta : \mathbf{B} \to \mathbf{D}$ and $\varepsilon : (\mathbf{A}{\uparrow}C_1) \to \mathbf{D}$ such that $\delta(\Omega) = \varepsilon(a)$. We build a mapping α from $(\mathbf{A}{\uparrow}\mathbf{B})$ to $\mathbf{D}$ by setting

$$\alpha(x) = \begin{cases} \delta(x) & \text{if } x \in \mathbf{B}, \\ \varepsilon(x) & \text{if } x \in (\mathbf{A}{\uparrow}C_1). \end{cases}$$

Analogously to the proof of Lemma 3.7, it is not hard to verify that α is a monomorphism. □

4. DESCRIBING LOGICS WITH WIP

We recall the notation from Sec. 2. For any negative algebra $\mathbf{A}$, put

$$\mathbf{A}^{\Lambda} = (\mathbf{A}{\uparrow}B_0),$$

where B_0 is a two-element Boolean algebra. For a given J-logic L, define a class such as

$$\Lambda(L) = \{\mathbf{A}^{\Lambda} \mid \mathbf{A} \text{ is a negative algebra and } \mathbf{A}^{\Lambda} \in V(L)\}.$$

It is a simple matter to verify the following:

LEMMA 4.1. $\Lambda(L)$ is an empty class if and only if L is a negative logic.

Proposition 1.4 allows us to reduce treating WIP in J-logics to studying extensions of the logic Gl. Classes $\Lambda(L)$ play a large role in this study. The next proposition shows that these classes divide a family of Gl-logics into intervals and gives a useful regimentation of logics over Gl, which supplements the classification of J-logics in [24].

PROPOSITION 4.2 [16]. Let a J-logic L_0 be generated by a class $\Lambda(L_0)$. Then L_0 contains Gl, and for any $L \in E(\mathrm{Gl})$, the following equivalence holds:

$$\Lambda(L) = \Lambda(L_0) \iff \mathrm{Neg} \cap L_0 \subseteq L \subseteq L_0.$$

Now we handle extensions of the logic Gl of a special kind. An axiomatization for such logics $L{\uparrow}\mathrm{Cl}$ and $L{\Uparrow}\mathrm{Cl}$, where L is a negative logic, was pointed out in Prop. 2.7. A logic $\mathrm{Gl} = \mathrm{Neg}{\uparrow}\mathrm{Cl}$ is characterized by all algebras of the form $\mathbf{A}^{\Lambda}$, where $\mathbf{A}$ is a negative algebra (see Prop. 2.6).

Of special importance in describing J-logics with WIP [14] is the following list, SL, consisting of eight logics containing Gl:

$$\mathrm{For},\ \mathrm{Cl},\ (\mathrm{NE}{\uparrow}\mathrm{Cl}),\ (\mathrm{NC}{\uparrow}\mathrm{Cl}),\ (\mathrm{Neg}{\uparrow}\mathrm{Cl}),\ (\mathrm{NE}{\Uparrow}\mathrm{Cl}),\ (\mathrm{NC}{\Uparrow}\mathrm{Cl}),\ (\mathrm{Neg}{\Uparrow}\mathrm{Cl}).$$

Proposition 2.8 implies that each of these logics L is generated by a class $\Lambda(L)$. We have

PROPOSITION 4.3 [14]. Let L be any Gl-logic in the list SL. Then L has CIP and classes $V(L)$ and $\Lambda(L)$ are amalgamable.

In [14], all logics over Gl possessing the weak interpolation property were described and an effective criterion was found to verify WIP in all J-logics.

THEOREM 4.4 [14]. A logic L over Gl has WIP if and only if it is representable as $L = L_{neg} \cap L_0$, where $L_{neg} = L + \bot$ and $L_0 \in$ SL.

THEOREM 4.5 [14]. For any logic L in $E(\mathrm{J})$, the following conditions are equivalent:

(1) L has WIP;

(2) $\Lambda(L)$ is an amalgamable class;

(3) $\Lambda(L) = \Lambda(L_0)$ for some logic L_0 in the list SL.

Proof. That (1) is equivalent to (2) and (1) is equivalent to (3) was proved in [16, Thm. 6.2] and [14], respectively. □

Thus WIP is nontrivial in propositional extensions of minimal logic. A set of J-logics with WIP and a set of J-logics without WIP have the cardinality of the continuum. The former set contains all superintuitionistic logics, i.e., a family of cardinality continuum. The latter has as a minimum the same cardinality as a set of negative logics other than Neg, NC, NE, and For, and the set of negative logics is also of the cardinality of the continuum. As already noted in Proposition 1.5, WIP is decidable over J; i.e, there is an algorithm which, given any finite set of axiom schemes Ax, decides if the logic $\mathrm{J} + Ax$ has WIP.

Theorem 4.4 yields a convenient representation for Gl-logics with WIP. A similar representation was found for Gl-logics with CIP, IPR, and PBP.

THEOREM 4.6 [14]. Let L be an extension of the logic Gl, with $\mathrm{SL} = \{\mathrm{For}, \mathrm{Cl}\} \cup \{(L_1{\uparrow}\mathrm{Cl}), (L_1{\Uparrow}\mathrm{Cl}) \mid L_1 \in \{\mathrm{Neg}, \mathrm{NC}, \mathrm{NE}\}\}$.

(1) L has CIP if and only if $L = L_{neg} \cap L_0$, where $L_{neg} = L + \bot$ has CIP and $L_0 \in$ SL.

(2) L has IPR if and only if $L = L_{neg} \cap L_0$, where L_{neg} is a logic with IPR and $L_0 \in$ SL.

(3) L has PBP if and only if $L = L_{neg} \cap L_0$, where L_{neg} is a logic with PBP and $L_0 \in$ SL.

Theorem 4.6 entails

COROLLARY 4.7 [14]. (1) IPR and PBP are equivalent over Gl.

(2) There exist only finitely many logics with IPR over Gl.

5. CLASSIFYING WELL-COMPOSED J-LOGICS WITH CIP

In this section we consider extensions of a logic of the form

$$\mathrm{JX} = \mathrm{J} + (p \to \bot) \vee (\bot \to p).$$

J-logics that contain JX and their corresponding varieties are referred to as *well composed*. In [18], we dealt with logics of a special kind, containing JX. The definition of such logics $L_1{\uparrow}L_2$ and $L_1{\Uparrow}L_2$ is given in Sec. 2. We lay out a number of results.

PROPOSITION 5.1 [18]. Let L be any of the logics $L_1{\uparrow}L_2$ and $L_1{\Uparrow}L_2$, where L_1 is a negative logic and L_2 is a consistent superintuitionistic one. Then:

(1) L has CIP $\iff$ L_1 and L_2 have CIP;

(2) L has IPR $\iff$ L_1 has CIP and L_2 has IPR;

(3) L has PBP $\iff$ L_1 has CIP and L_2 has PBP.

Proof. The required result follows immediately from [18, Thms. 5.1, 5.2]. □

With Proposition 1.3 in mind, we derive

COROLLARY 5.2 [13]. IPR and PBP are equivalent for any logic of a special kind such as in Prop. 5.1.

In correspondence with every J-logic L are its negative and superintuitionistic fragments [24]: namely,

$$L_{neg} = L + \bot, \ L_{int} = L + (\bot \to p).$$

It follows from Proposition 2.7 that for any negative logic L_1 and any superintuitionistic logic L_2, $(L_1{\uparrow}L_2)_{neg} = (L_1{\Uparrow}L_2)_{neg} = L_1$ and $(L_1{\uparrow}L_2)_{int} = (L_1{\Uparrow}L_2)_{int} = L_2$.

LEMMA 5.3 [13]. If a J-logic L has CIP, IPR, or PBP, then L_{neg} and L_{int} possess a same property.

LEMMA 5.4. Let L contain JX and $\mathbf{A}$ be a finitely indecomposable algebra in $V(L)$. Then $\mathbf{A} = \mathbf{A}^l{\uparrow}\mathbf{A}^u$, where $\mathbf{A}^l \in V(L_{neg})$ and $\mathbf{A}^u \in V(L_{int})$. If a formula $(\bot \to p \vee q) \to (\bot \to p) \vee (\bot \to q)$ is valid in $\mathbf{A}$, then $\mathbf{A}^l$ is finitely indecomposable.

Proof. Let a be any element of $\mathbf{A}$. By finite indecomposability, we have $\bot \to a = \top$ or $a \to \bot = \top$, i.e., $\bot \le a$ or $a \le \bot$. In this event $\mathbf{A}^l = \{x \in \mathbf{A} \mid x \le \bot\}$ is a negative algebra, and since $\mathbf{A}^l$ is a homomorphic image of $\mathbf{A}$, we obtain $\mathbf{A}^l \in V(L_{neg})$. Furthermore, $\mathbf{A}^u = \{x \in \mathbf{A} \mid \bot \le x\}$ is a Heyting algebra and is a subalgebra of $\mathbf{A}$, so $\mathbf{A}^u \in V(L_{int})$.

If $(\bot \to p \vee q) \to (\bot \to p) \vee (\bot \to q)$ is valid in $\mathbf{A}$, then $\mathbf{A}^l$ is finitely indecomposable in view of [18, Lemma 3.1(4)]. □

Denote by G_1 a three-element J-algebra with an opremum $\bot$.

LEMMA 5.5. For every J-logic L,

$$\Lambda(L) \subseteq \Lambda(\mathrm{Cl}) \iff G_1 \notin V(L).$$

Proof. Let $\Lambda(L) \not\subseteq \Lambda(\mathrm{Cl})$. The class $\Lambda(\mathrm{Cl})$ contains only a two-element Boolean algebra B_0. Therefore, there exists a nondegenerate negative algebra $\mathbf{A}$ such that $\mathbf{A}^\Lambda \in \Lambda(L)$. For any $a \in \mathbf{A}$, $a \neq \bot$, the set $\{a, \bot, \top\}$ forms a subalgebra of $\mathbf{A}^\Lambda$ isomorphic to G_1, and so $G_1 \in V(L)$. The converse is obvious. □

Our next goal is to find a representation for J-logics with CIP similar to Theorem 4.6.

PROPOSITION 5.6 [17]. Let L_1 be a negative logic and L_2 any extension of the logic J. If L_1 and L_2 have CIP, then $L_1 \cap L_2$ has CIP.

The following simple lemma holds.

LEMMA 5.7. Every J-logic L is representable as $L = L_{neg} \cap L_1$ for a suitable J-logic L_1. If $L = L_{neg} \cap L_1$, then every negative algebra in $V(L_1)$ belongs to the variety $V(L_{neg})$.

THEOREM 5.8. Let a logic L contain JX. Then L has CIP if and only if L coincides with one of the following logics:

(1) $L_1 \cap L_2$, where $L_1 = L_{neg}$ is a negative logic with CIP and L_2 is a superintuitionistic logic with CIP;

(2) $L_1 \cap (L_3 \Uparrow L_2)$, where $L_1 = L_{neg}$ is a negative logic with CIP, L_2 is a consistent superintuitionistic logic with CIP, and $L_3 \in \{\mathrm{Neg}, \mathrm{NC}, \mathrm{NE}\}$;

(3) $L_1 \cap (L_3 \uparrow L_2)$, where L_1, L_2, and L_3 are as in item (2).

Proof. ($\Leftarrow$) Let L_1 be a negative logic with CIP. Then $L_1 \cap L_2$ has CIP for any J-logic with CIP, as follows by Prop. 5.6. In addition, if L_2 is a consistent superintuitionistic logic with CIP, and $L_3 \in \{\mathrm{Neg}, \mathrm{NC}, \mathrm{NE}\}$, then $(L_3 \Uparrow L_2)$ and $(L_3 \uparrow L_2)$ possess CIP in view of Prop. 5.1. Consequently, logics $L_1 \cap (L_3 \Uparrow L_2)$ and $L_1 \cap (L_3 \uparrow L_2)$ also have CIP by virtue of Prop. 5.6.

($\Rightarrow$) Let L have CIP. Then both logics $L_1 = L_{neg}$ and $L_2 = L_{int}$ have CIP by Lemma 5.3. In addition, L has WIP. By Theorem 4.5, therefore, the class $\Lambda(L)$ is empty, or $\Lambda(L) = \Lambda(\mathrm{Cl})$, or $\Lambda(L) = \Lambda(L_3 \uparrow \mathrm{Cl})$, or $\Lambda(L) = \Lambda(L_3 \Uparrow \mathrm{Cl})$, where $L_3 \in \{\mathrm{Neg}, \mathrm{NC}, \mathrm{NE}\}$.

If $\Lambda(L)$ is an empty class, then L is a negative logic and, hence, is representable as $L \cap \mathrm{For}$, i.e., item (1) holds.

Let $\Lambda(L) = \Lambda(\mathrm{Cl})$. We claim that $L = L_{neg} \cap L_{int}$. Obviously, $L \subseteq L_{neg} \cap L_{int}$. We argue for the inverse inclusion. Let $\mathbf{A}$ be a finitely indecomposable algebra in $V(L)$. By Lemma 5.4, $\mathbf{A}$ has the form $(\mathbf{C} \uparrow \mathbf{B})$, where $\mathbf{C} \in V(L_{neg})$ and $\mathbf{B} \in V(L_{int})$. If both algebras $\mathbf{C}$ and $\mathbf{B}$ are nondegenerate, then an algebra G_1 embeds in $\mathbf{A}$ and, hence, belongs to $\Lambda(L)$, contrary to Lemma 5.5. Therefore, one of the algebras $\mathbf{C}$ or $\mathbf{B}$ is degenerate, i.e., $\mathbf{A}$ coincides with $\mathbf{B}$ or $\mathbf{C}$ and belongs to $V(L_{neg} \cap L_{int})$, as required.

Suppose $\Lambda(L) = \Lambda(L_0)$, where $L_0 = (L_3 \Uparrow \mathrm{Cl})$ and $L_3 \in \{\mathrm{Neg}, \mathrm{NC}, \mathrm{NE}\}$. By Theorem 3.1, $V(L)$ is an amalgamable variety. In view of Lemma 3.7, if $\mathbf{A}$ is a negative algebra, $\mathbf{B}$ is a Heyting algebra, and $\mathbf{A}^{\Lambda}, \mathbf{B} \in V(L)$, then $(\mathbf{A} \uparrow \mathbf{B}) \in V(L)$. This implies that the variety $V(L)$ contains all algebras $(\mathbf{A} \uparrow \mathbf{B})$ such that $\mathbf{A}$ is a finitely indecomposable algebra in $V(L_3)$ and $\mathbf{B}$ is a Heyting algebra in $V(L)$. Furthermore, the condition $\mathbf{B} \in V(L)$ is equivalent to $\mathbf{B} \in V(L_{int}) = V(L_2)$. Since the logic $L_3 \Uparrow L_2$ is generated by a class of algebras $(\mathbf{A} \uparrow \mathbf{B})$ such that $\mathbf{A}$ is a finitely indecomposable algebra in $V(L_3)$ and $\mathbf{B} \in V(L_2)$, we see that $L \subseteq L_3 \Uparrow L_2$ and $L \subseteq L_1 \cap (L_3 \Uparrow L_2)$.

We argue for the equality. By virtue of Lemma 2.5, it suffices to prove that every finitely indecomposable algebra of $V(L)$ is contained in $V(L_1) \cup V(L_3 \Uparrow L_2)$. Let $\mathbf{C}$ be a finitely indecomposable algebra in $V(L)$. In view of Lemma 5.4, $\mathbf{C}$ has the form $(\mathbf{A} \uparrow \mathbf{B})$, where $\mathbf{A}$ is a negative algebra in $V(L_1)$ and $\mathbf{B} \in V(L_2)$. If $\mathbf{B}$ is a degenerate algebra, then $\mathbf{C}$ is a negative algebra and coincides with $\mathbf{A}$; hence $\mathbf{C}$ is contained in $V(L_1)$.

Let $\mathbf{B}$ be a nondegenerate algebra. Then $\mathbf{C}$ contains a subalgebra $(\mathbf{A} \uparrow B_0)$, which is contained in $\Lambda(L) = \Lambda(L_0)$. Hence $\mathbf{A}$ is a finitely indecomposable algebra in $V(L_3)$. Since $\mathbf{B} \in V(L_2)$, we obtain $\mathbf{C} = (\mathbf{A} \uparrow \mathbf{B}) \in V(L_3 \Uparrow L_2)$.

The case $\Lambda(L) = \Lambda(L_0)$, where $L_0 = (L_3 \uparrow \mathrm{Cl})$ and $L_3 \in \{\mathrm{Neg}, \mathrm{NC}, \mathrm{NE}\}$, can be treated similarly.

We need only drop the condition of **A** being finitely indecomposable. □

Since there exist only finitely many negative and superintuitionistic logics with CIP [4, 11], Theorem 5.8 immediately implies

COROLLARY 5. There are only finitely many logics with CIP over JX.

6. WELL-COMPOSED J-LOGICS WITH IPR

In this section we work to obtain a description of logics with the restricted interpolation property over JX.

THEOREM 6.1. Assume that a logic L contains a logic JX, the logic L_{neg} has IPR, and

$$L = L_{neg} \cap L_0 \cap L_1,$$

where $L_0 \in \mathrm{SL}$, $\Lambda(L_0) \supseteq \Lambda(L_1)$, $L_1 \in \{\mathrm{For}, (L_2 \uparrow L_3), (L_2 \Uparrow L_3)\}$, L_2 is a negative logic with CIP, and L_3 is a superintuitionistic logic with IPR. Then L has IPR. Moreover, L has PBP.

Proof. For the case where $L_1 = \mathrm{For}$, the logic $L = L_{neg} \cap L_0$ contains Gl and, hence, has IPR (by Thm. 4.6) and PBP (by Cor. 4.7)

Now let $L_1 \neq \mathrm{For}$. First we prove that L has IPR. In view of Theorem 3.4, it suffices to show that $V(L)$ possesses RAP*. Let subdirectly irreducible algebras $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V(L)$ be given, suppose $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and $\mathbf{C}$, and assume that the three algebras all have a common opremum. There are three cases to consider.

(1) Let $\mathbf{A}$ be a negative algebra. Then $\mathbf{B}$ and $\mathbf{C}$ are also negative algebras, and so the three algebras are all in $V(L_{neg})$. By Theorem 3.4, the variety $V(L_{neg})$ has RAP* and, hence, contains an amalgam $\mathbf{D}$ for $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. Consequently, $\mathbf{D} \in V(L)$.

(2) Let $\mathbf{A}$ not be a negative algebra and let it have $\perp$ as an opremum. Then $\mathbf{B}$ and $\mathbf{C}$ satisfy the same conditions. In this instance $\mathbf{A} = \mathbf{A}_1^{\Lambda}$, $\mathbf{B} = \mathbf{B}_1^{\Lambda}$, and $\mathbf{C} = \mathbf{C}_1^{\Lambda}$, and these algebras all belong to the class $\Lambda(L)$. By Lemma 2.5, $\Lambda(L) = \Lambda(L_{neg}) \cup \Lambda(L_0) \cup \Lambda(L_1)$. Since $\Lambda(L_{neg}) = \varnothing$ and $\Lambda(L_0) \supseteq \Lambda(L_1)$, we obtain $\Lambda(L) = \Lambda(L_0)$. In view of Proposition 4.3, the class $\Lambda(L_0)$ is amalgamable, and hence $\Lambda(L)$ and $V(L)$ contain an amalgam for $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$.

(3) Let $\mathbf{A}$ not be a negative algebra and $\perp$ not be its opremum. Then $\mathbf{B}$ and $\mathbf{C}$ satisfy the same conditions. All the three algebras do not belong to $V(L_{neg}) \cup V(L_0)$ and, consequently, are contained in $V(L_1)$ by Lemma 2.5. In view of Proposition 5.1, the logic L_1 has IPR. By virtue of Theorem 3.2, $V(L_1)$ has RAP*. Therefore, there exists an amalgam for a triple $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $V(L_1)$ and hence in $V(L)$.

Thus in all cases (1)-(3), $V(L)$ has RAP* and L has IPR.

Now we prove that L has PBP. With Theorem 3.2 in mind, it suffices to show that a class of finitely indecomposable algebras in $V(L)$ has SES. By Proposition 1.3, logics L_{neg} and L_3 have PBP. By virtue of Proposition 5.1, L_1 likewise has PBP. Since L_0 has CIP by Proposition 4.3, L_0 has PBP. In view of Theorem 3.2, varieties $V(L_{neg})$, $V(L_1)$, and $V(L_0)$ possess SES.

Let $\mathbf{A}$ and $\mathbf{B}$ be finitely indecomposable algebras in $V(L)$, $\mathbf{A}$ a subalgebra of $\mathbf{B}$, and $a \in (\mathbf{B}-\mathbf{A})$. By Lemma 2.5, $\mathbf{B}$ is contained in $V(L_{neg}) \cup V(L_0) \cup V(L_1)$. Consequently $\mathbf{B}$, together with $\mathbf{A}$, enters one of the three varieties mentioned. In view of SES, a same variety will contain an algebra $\mathbf{D}$ and two homomorphisms $g, h: \mathbf{B} \to \mathbf{D}$ such that $g(x) = h(x)$ for all $x \in \mathbf{A}$ and $g(b) \neq h(b)$. Clearly, $\mathbf{D} \in V(L)$. Thus $V(L)$ has SES. □

Our present goal is to prove a representation theorem for well-composed J-logics with IPR.

THEOREM 6.2. Let a well-composed J-logic L have IPR. Then L_{neg} has IPR and L is representable as

$$L = L_{neg} \cap L_0 \cap L_1,$$

where $L_0 \in$ SL, $\Lambda(L_0) \supseteq \Lambda(L_1)$, $L_1 \in \{\mathrm{For}, (L_2 \uparrow L_3), (L_2 \Uparrow L_3)\}$, L_2 is a negative logic with CIP, and L_3 is a superintuitionistic logic with IPR.

Proof. Assume that a well-composed J-logic L has IPR. Then L_{neg} and L_{int} have IPR by Lemma 5.3.

Since L has WIP, it follows by Theorem 4.5 that the class $\Lambda(L)$ coincides with $\Lambda(L_0)$ for one of the logics L_0 in the list SL. Consequently, $\Lambda(L_0) = \Lambda(L)$, and by Proposition 2.8, L_0 is generated by the class $\Lambda(L_0)$.

Denote by K a class of subdirectly irreducible algebras in $V(L)$, which are not contained in $V(L_{neg}) \cup \Lambda(L)$. Let L_1 be a logic generated by K. We show that

$$L = L_{neg} \cap L_0 \cap L_1. \tag{1}$$

Obviously, $L \subseteq L_{neg} \cap L_0 \cap L_1$. We argue for the inverse inclusion. Let $\mathbf{A}$ be any subdirectly irreducible algebra in $V(L)$. If $\mathbf{A}$ is a negative algebra, then $\mathbf{A} \in V(L_{neg})$. If $\perp$ is an opremum of $\mathbf{A}$, then $\mathbf{A} \in \Lambda(L) = \Lambda(L_0) \subseteq V(L_0)$. In all other cases $\mathbf{A} \in V(L_1)$. Therefore, $\mathbf{A} \in V(L_{neg}) \cup V(L_0) \cup V(L_1) \subseteq V(L_{neg} \cap L_0 \cap L_1)$. This yields $V(L) \subseteq V(L_{neg} \cap L_0 \cap L_1)$ and $L \supseteq L_{neg} \cap L_0 \cap L_1$.

We claim that L_1 meets all requirements of the theorem.

Equality (1) and Lemma 2.5 imply that $\Lambda(L_0) = \Lambda(L) = \Lambda(L_{neg}) \cup \Lambda(L_0) \cup \Lambda(L_1)$, and so $\Lambda(L_0) \supseteq \Lambda(L_1)$.

Consider a class $\Lambda(L_1)$. First we show that the following relation holds:

$$\Lambda(L_1) = \{\mathbf{A}^{\Lambda} \mid (\mathbf{A} \uparrow C_1) \in K\}, \tag{2}$$

where C_1 is a three-element Heyting algebra. Since $\mathbf{A}^{\Lambda}$ is a subalgebra of $(\mathbf{A} \uparrow C_1)$, we conclude that if $(\mathbf{A} \uparrow C_1) \in K$, then $\mathbf{A}^{\Lambda} \in V(L_1)$, and hence $\mathbf{A}^{\Lambda} \in \Lambda(L_1)$.

We argue for the inverse inclusion. Let $\mathbf{A}^{\Lambda} \in \Lambda(L_1)$. Then $\mathbf{A}^{\Lambda}$ is a subdirectly irreducible algebra in the variety $V(L_1)$ generated by the class K. By a well-known theorem of Jonsson [26], the algebra $\mathbf{A}^{\Lambda}$ is a homomorphic image of a subalgebra of some ultraproduct of algebras in K. By definition, K is a class of algebras in $V(L)$ having an opremum other than $\perp$ and satisfying the inequality $\perp \neq \top$; i.e., K is definable by first-order formulas. Therefore, K is closed under ultraproducts.

Consequently, there exist an algebra $\mathbf{B}_1$, which is a subalgebra of some algebra $\mathbf{B} \in K$, and a homomorphism h of $\mathbf{B}_1$ onto $\mathbf{A}^\Lambda$. Appealing to Lemma 2.4(2), we see that the restriction h^l of h to $\mathbf{A}$ is a monomorphism of $\mathbf{A}$ onto an algebra $(\mathbf{B}_1)^l$, which is in turn a subalgebra of $\mathbf{B}^l$. Consequently, the mapping h^l can be extended to a monomorphism of the algebra $(\mathbf{A}{\uparrow}C_1)$ into $\mathbf{B}$, which translates an opremum of the algebra $(\mathbf{A}{\uparrow}C_1)$ into an opremum of the algebra $\mathbf{B}$. This yields $(\mathbf{A}{\uparrow}C_1) \in V(L)$ and $(\mathbf{A}{\uparrow}C_1) \in K$.

We prove that $\Lambda(L_1)$ is an amalgamable class. Let $\mathbf{A}^\Lambda, \mathbf{B}^\Lambda, \mathbf{C}^\Lambda \in \Lambda(L_1)$ and $\mathbf{A}^\Lambda$ be a common subalgebra of $\mathbf{B}^\Lambda$ and $\mathbf{C}^\Lambda$. By relation (2), $(\mathbf{A}{\uparrow}C_1), (\mathbf{B}{\uparrow}C_1), (\mathbf{C}{\uparrow}C_1) \in K$. Note that algebras $(\mathbf{A}{\uparrow}C_1)$, $(\mathbf{B}{\uparrow}C_1)$, and $(\mathbf{C}{\uparrow}C_1)$ are obtained from $\mathbf{A}^\Lambda$, $\mathbf{B}^\Lambda$, and $\mathbf{C}^\Lambda$ by adding a new opremum Ω; so $(\mathbf{A}{\uparrow}C_1)$ is a common subalgebra of $(\mathbf{B}{\uparrow}C_1)$ and $(\mathbf{C}{\uparrow}C_1)$, and moreover, all the three algebras have a common opremum. By Theorem 3.4, there exist a subdirectly irreducible algebra $\mathbf{D} \in V(L)$ and monomorphisms $\delta : (\mathbf{B}{\uparrow}C_1) \to \mathbf{D}$ and $\varepsilon : (\mathbf{C}{\uparrow}C_1) \to \mathbf{D}$ such that $\delta(x) = \varepsilon(x)$ for all $x \in (\mathbf{A}{\uparrow}C_1)$ and $\delta(\Omega)$ is an opremum in $\mathbf{D}$. Since $\mathbf{D}$ is not a negative algebra and its opremum is not $\perp$, we see that $\mathbf{D} \in K \subseteq V(L_1)$. An algebra $(\mathbf{D}^l)^\Lambda$ is a subalgebra of $\mathbf{D}$ and, hence, is contained in $V(L_1)$ and in $\Lambda(L_1)$. Furthermore, the restrictions of mappings δ and ε to $\mathbf{B}^\Lambda$ and $\mathbf{C}^\Lambda$, respectively, are monomorphisms in $(\mathbf{D}^l)^\Lambda$. Therefore, $(\mathbf{D}^l)^\Lambda$ is an amalgam for $\mathbf{A}^\Lambda$, $\mathbf{B}^\Lambda$, and $\mathbf{C}^\Lambda$.

Since the class $\Lambda(L_1)$ is amalgamable, by Theorem 4.5,

$$\Lambda(L_1) = \Lambda(L_{1,0}) \tag{3}$$

for a suitable logic $L_{1,0}$ in the list SL. The logic $L_{1,0}$ is generated by the class $\Lambda(L_{1,0})$, so $L_1 \subseteq L_{1,0}$.

Letting $L_2 = (L_1)_{neg}$, we show that

$$L_2 = (L_1)_{neg} = (L_{1,0})_{neg}. \tag{4}$$

Obviously, $(L_1)_{neg} \subseteq (L_{1,0})_{neg}$. We argue for the inverse inclusion.

Let $\mathbf{A}$ be a subdirectly irreducible algebra in $V(L_1)_{neg}$. Then $\mathbf{A}$ is a negative algebra in $V(L_1)$. By Jonsson's theorem, $\mathbf{A}$ is a homomorphic image of a subalgebra of some ultraproduct of algebras in the class K. As we have seen above, K is closed under ultraproducts. Furthermore, every subalgebra $\mathbf{B}$ of an algebra in K contains a subalgebra $(\mathbf{B}^l)^\Lambda \in \Lambda(L_1)$, and moreover, by Lemma 2.4, the algebras $\mathbf{B}$ and $(\mathbf{B}^l)^\Lambda$ have the same negative algebras as homomorphic images. Therefore, $\mathbf{A} \in V(L_{1,0})$. Hence $(L_1)_{neg} \supseteq (L_{1,0})_{neg}$.

From equality (4), in view of Proposition 4.3, we conclude that the fragment $L_2 = (L_1)_{neg}$ has CIP.

Put $L_3 = L_{int}$; then L_3 has IPR by Lemma 5.3. We prove that $L_1 \in \{\mathrm{For}, (L_2{\uparrow}L_3), (L_2{\Uparrow}L_3)\}$. The following two cases are possible: (1) $L_{1,0} = \mathrm{For}$; (2) $L_{1,0} \neq \mathrm{For}$.

(1) In this event $\Lambda(L_1) = \Lambda(\mathrm{For}) = \varnothing$. The class K is also empty since any algebra in K should contain a subalgebra B_0, in which case B_0 should be contained in $V(L_1)$ and hence in $\Lambda(L_1)$. Now $L_1 = \mathrm{For}$.

(2) In this instance the class $\Lambda(L_1)$ contains a two-element Boolean algebra B_0, while the class K contains a Heyting algebra C_1 (in view of relation (2)).

Note that in this case

$$(L_1)_{int} = L_{int}. \quad (5)$$

Indeed, relation (1) implies

$$L_{int} = (L_{neg})_{int} \cap (L_0)_{int} \cap (L_1)_{int} \supseteq \mathrm{For} \cap \mathrm{Cl} \cap (L_1)_{int} = (L_1)_{int} \supseteq L_{int}$$

since $C_1 \in V(L_1)$, and hence $(L_1)_{int} \subset \mathrm{Cl}$.

It remains to prove that

$$L_1 \in \{(L_1)_{neg} \uparrow L_{int},\ (L_1)_{neg} \Uparrow L_{int}\}. \quad (6)$$

With equality (5) in mind, it suffices to verify that $L_1 = (L_1)_{neg} \uparrow (L_1)_{int}$ or $L_1 = (L_1)_{neg} \Uparrow (L_1)_{int}$. We handle three cases.

(2.1) Let $L_{1,0} = \mathrm{Cl}$. Then $(L_1)_{neg} = \mathrm{For}$ by virtue of (4). In other words, L_1 is a superintuitionistic logic and $L_1 = (L_1)_{int} = (L_1)_{neg} \uparrow (L_1)_{int}$, as required.

(2.2) Let $L_{1,0} = L_4 \Uparrow \mathrm{Cl}$ for some $L_4 \in \{\mathrm{Neg}, \mathrm{NC}, \mathrm{NE}\}$. Then $L_4 = (L_{1,0})_{neg} = (L_1)_{neg} = L_2$ in view of (4). We prove that

$$L_1 = L_2 \Uparrow (L_1)_{int}. \quad (7)$$

Recall that L_1 is generated by the class K. We claim that all algebras in K belong to the variety $V(L_2 \Uparrow (L_1)_{int})$. Let $\mathbf{A}$ be any algebra in K. By Lemma 5.4, $\mathbf{A} = \mathbf{A}^l \uparrow \mathbf{A}^u$, and moreover, by the definition of K, the algebra $\mathbf{A}$ contains a subalgebra $\mathbf{A} = (\mathbf{A}^l)^\Lambda$, which belongs to the class $\Lambda(L_1) = \Lambda(L_{1,0})$. In particular, $\mathbf{A}^l$ is finitely indecomposable and is contained in $V((L_1)_{neg})$. In addition, $\mathbf{A}^u \in V((L_1)_{int})$. Consequently $\mathbf{A} \in V(L_2 \Uparrow (L_1)_{int})$. This implies $L_2 \Uparrow (L_1)_{int} \subseteq L_1$.

We argue for the inverse inclusion. We show that all subdirectly irreducible algebras in the variety $V(L_2 \Uparrow (L_1)_{int})$ belong to $V(L_1)$. Let $\mathbf{A}$ be a subdirectly irreducible algebra in $V(L_2 \Uparrow ((L_1)_{int})$. If $\mathbf{A}$ is a negative algebra, then $\mathbf{A} \in V(L_2) \subseteq V(L_1)$.

Let $\mathbf{A}$ not be a negative algebra. In view of Lemma 5.4, $\mathbf{A} = \mathbf{A}^l \uparrow \mathbf{A}^u$, where $\mathbf{A}^u$ is a subdirectly irreducible algebra in $V((L_1)_{int})$ and $\mathbf{A}^l$ is a finitely indecomposable negative algebra in $V(L_2)$. Furthermore, $(\mathbf{A}^l)^\Lambda \in \Lambda(L_{1,0}) = \Lambda(L_1)$. By virtue of (2), we obtain $(\mathbf{A}^l \uparrow C_1) \in K \subseteq V(L)$. Since $\mathbf{A}^u$ is a subdirectly irreducible Heyting algebra of $V(L)$, it follows by Lemma 3.8 that $\mathbf{A} \in V(L)$. If $\mathbf{A}^u$ is a two-element algebra, then $\mathbf{A}$ coincides with $(\mathbf{A}^l)^\Lambda$, and so $\mathbf{A} \in \Lambda(L_1) \subseteq V(L_1)$. If $\mathbf{A}^u$ contains at least three elements, then $\mathbf{A} \in K \subseteq V(L_1)$. Thus $\mathbf{A} \in V(L_1)$.

Every variety is generated by its subdirectly irreducible algebras; therefore, $V(L_2 \Uparrow (L_1)_{int}) \subseteq V(L_1)$ and $L_1 \subseteq L_2 \Uparrow (L_1)_{int}$. This implies $L_1 = L_2 \Uparrow (L_1)_{int}$, and so (7) is proved.

(2.3) Let $L_{1,0} = L_4 \uparrow \mathrm{Cl}$ for some $L_4 \in \{\mathrm{Neg}, \mathrm{NC}, \mathrm{NE}\}$. That $L_1 = L_2 \uparrow (L_1)_{int}$ is proved analogously to (2.2). We need only drop the condition of $\mathbf{A}^l$ being finitely indecomposable. □

COROLLARY 6.3. (1) There exist only finitely many JX-logics with IPR.

(2) IPR and PBP are equivalent over JX.

Proof. (1) The required statement follows from Theorem 6.2 in view of finiteness of the class SL and of the number of negative and superintuitionistic logics with IPR (see Props. 1.1-1.3).

(2) Appealing to Theorems 6.1 and 6.2, we conclude that IPR implies PBP over JX. The converse is true for all J-logics. □

The question whether IPR and PBP are equivalent on the class of all J-logics remains open.

COROLLARY 6.4. All well-composed J-logics possessing CIP, PBP, or IPR are finitely axiomatizable.

Proof. By Theorem 6.2, every well-composed J-logic with IPR is representable as an intersection of three logics of a special kind. All superintuitionistic and negative logics with IPR are finitely axiomatizable in view of Props. 1.1-1.3. Consequently, logics L_0, $L_2{\uparrow}L_3$, and $L_2{\Uparrow}L_3$ such as in Theorem 6.2 are also finitely axiomatizable by Prop. 2.7. By virtue of Proposition 2.5, this implies that all well-composed J-logics with IPR are finitely axiomatizable. □

CONCLUSION

In the representation theorems for well-composed J-logics with CIP and IPR proved above, essential use was made of the description of J-logics with WIP given in [14]. There, too, it was stated that WIP is decidable over J. A class of well-composed logics contains all superintuitionistic logics and all extensions of the logic Gl, for which CIP, IPR, and PBP were also proved decidable. It is likely that all of these properties likewise are decidable on a class of well-composed J-logics.

For the family of all J-logics, the situation is essentially more complicated, and the following problems remain open.

Problem 1. Describe all J-logics with CIP, IPR, and PBP. Are sets of such logics finite? As we have seen above, there exist only finitely many well-composed J-logics with CIP, IPR, or PBP.

Problem 2. Is it true that all J-logics with CIP, IPR, or PBP are finitely axiomatizable, or residually finite, or decidable?

Problem 3. As already noted, WIP is decidable over J. Are the other properties decidable over J?

Problem 4. For every J-logic, IPR follows form PBP. We have proved above that the converse is true for every well-composed J-logic. Are IPR and PBP equivalent on the class of all J-logics?

By comparison, on the class of all modal logic, IPR follows from PBP, but the converse is not true [21].

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