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Σ-UNIFORM STRUCTURES AND Σ-FUNCTIONS. II

A. N. Khisamiev[*]

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We construct a family of Σ-uniform Abelian groups and a family of Σ-uniform rings. Conditions are specified that are necessary and sufficient for a universal Σ-function to exist in a hereditarily finite admissible set over structures in these families. It is proved that there is a set S of primes such that no universal Σ-function exists in hereditarily finite admissible sets $\mathbb{HF}(G)$ and $\mathbb{HF}(K)$, where $G = \oplus\{Z_p \mid p \in S\}$ is a group, Z_p is a cyclic group of order p, $K = \oplus\{F_p \mid p \in S\}$ is a ring, and F_p is a prime field of characteristic p.

The present paper is a continuation of [1], in which we introduced the concept of a Σ-uniform structure and derived a condition that is necessary and sufficient for a universal Σ-function to exist in a hereditarily finite admissible set over a Σ-uniform structure. Here we show how these results apply to Abelian groups and rings. We construct a family of Σ-uniform Abelian groups and a family of Σ-uniform rings. Conditions are specified that are necessary and sufficient for a universal Σ-function to exist in a hereditarily finite admissible set over structures in these families. It is proved that there is a set S of primes such that no universal Σ-function exists in hereditarily finite admissible sets $\mathbb{HF}(G)$ and $\mathbb{HF}(K)$, where $G = \oplus\{Z_p \mid p \in S\}$ is a group, Z_p is a cyclic group of order p, $K = \oplus\{F_p \mid p \in S\}$ is a ring, and F_p is a prime field of characteristic p.

We will adhere to the notation and terminology created for admissible sets in [2], for groups in [3], and for rings in [4] (see also [1]).

We start to cite the definition of a Σ-uniform structure from [1].

Definition 1. Suppose that a locally finite structure $\mathfrak{M}$ in a signature σ_0 satisfies the following conditions:

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Sobolev Institute of Mathematics, Siberian Branch, Russian Academy of Sciences, pr. Akad. Koptyuga 4, Novosibirsk, 630090 Russia; hisamiev@math.nsc.ru. Translated from *Algebra i Logika*, Vol. 51, No. 1, pp. 129-147, January-February, 2012. Original article submitted November 24, 2010; revised June 5, 2011.

(1) Let a Σ-subset Ξ_0 of natural numbers be defined without parameters. We call every element $\chi \in \Xi_0$, $0 \in \Xi_0$, an *elementary characteristic*. In addition, let a unary Σ-function $\mathfrak{b}(\chi)$, $\delta\mathfrak{b} = \Xi_0$, be defined without parameters so that for any $\chi \in \Xi_0$, the value of $\mathfrak{b}(\chi)$ is a nonempty finite set of sequences of elements of equal length in $M \setminus \Omega$, with $\mathfrak{b}(0) = \{\varnothing\}$. Every element $\bar{y} \in \mathfrak{b}(\chi)$ is called an *elementary basis* of characteristic χ and is written $\chi(\bar{y}) = \chi$. All bases $\bar{y}_i, \bar{y}_j \in \mathfrak{b}(\chi)$ generate the same subsystem, i.e., $(\bar{y}_i) = (\bar{y}_j)$. If $\bar{y} = \langle y_0, \ldots, y_{p-1}\rangle$ is an elementary basis, then $(y_i) \cap (y_0, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{p-1}) = \Omega$ for any $i < p$. If elementary bases $\bar{y}_0, \ldots, \bar{y}_{q-1}$ have pairwise distinct characteristics, then $(\bar{y}_i) \cap \big(\langle \bar{y}_0, \ldots, \bar{y}_{i-1}, \bar{y}_{i+1}, \ldots, \bar{y}_{q-1}\rangle\big) = \Omega$ for any $i < q$.

For an arbitrary sequence $\chi = \langle \chi_0, \ldots, \chi_{q-1}\rangle$ and for all bases $\bar{y}_i$ of characteristic χ_i, a sequence of the form $Y = \langle \bar{y}_0, \ldots, \bar{y}_{q-1}\rangle$ is called a *basis of characteristic* χ, where $\chi_0 \geq 0$, if $q = 1$, and $0 < \chi_i < \chi_j$ if $q > 1$ and $i < j < q$.

(2) If $\bar{y} = \langle y_0, \ldots, y_{p-1}\rangle$ is an elementary basis of characteristic $\chi \neq 0$, and $x \in (\bar{y}) \setminus \Omega$, then $\bar{y}$ is called an *elementary basis of the element* x. The elementary basis for any element $x \in \Omega$ is $\varnothing$. A sequence $f_0(x, \bar{y}) = \langle x_0, \ldots, x_{p-1}\rangle$, $x_i \in (y_i) \cup \{\varnothing\}$, such that $(x) \subseteq (x_0, \ldots, x_{p-1})$ is uniquely defined for any $x \in (\bar{y}) \setminus \Omega$. We call x_i and x, respectively, *atomwise* and *atomic* elements of characteristic χ.

For any nonatomic element $z \in M$, a sequence $f_1(z) = \langle z_0, \ldots, z_{q-1}\rangle$, $q > 1$, of atomic elements $z_i \notin \Omega$ of characteristic χ_i is uniquely defined so that $(z) = (z_0, \ldots, z_{q-1})$, $\chi_i < \chi_j$, and $i < j < q$. A basis $Y = \langle \bar{y}_0, \ldots, \bar{y}_{q-1}\rangle$, $\chi(\bar{y}_i) = \chi_i$, is called a *basis of an element* z and is denoted by $\mathfrak{B}_0(z, Y)$. If z is an atomic element, then $f_1(z) = z$. Functions f_0 and f_1 are 1-1 Σ-functions without parameters.

Let an elementary basis $\bar{y} = \langle y_0, \ldots, y_{p-1}\rangle$ of characteristic χ and $i < p$ be given. A Σ-function Cor_0 without parameters, where $\delta\mathrm{Cor}_0 = \{\langle x, y_i\rangle \mid x \in (y_i)\}$, $\rho\mathrm{Cor}_0 \subseteq \omega^+$, and $\omega^+ = \{n \in \omega \mid n > 0\}$, is defined in $\mathbb{HF}(\mathfrak{M})$ so that $\mathrm{Cor}_0(x^0, y_i) \neq \mathrm{Cor}_0(x^1, y_i)$ if $x^\varepsilon \in (y_i)$, $\varepsilon < 2$, and $x^0 \neq x^1$.

(3) Let bases Y^0 and Y^1 of the same characteristic χ and a finite substructure $\mathfrak{M}^0 \supseteq (Y^0)$ be given. Then there exists an isomorphic embedding $\varphi : \mathfrak{M}^0 \to \mathfrak{M}$ for which $\varphi Y^0 = Y^1$.

In this case we call $\mathfrak{M}$ a Σ-*uniform structure.*

Below are a number of valid results.

THEOREM 1 [1]. Let $\mathfrak{M}$ be a Σ-uniform structure and M_0 some basis. Then the family $\mathfrak{F}^{M_0}$ of all unary functions definable in $\mathbb{HF}(\mathfrak{M})$ by Σ-formulas with parameter M_0 is computable if and only if the family $\mathcal{N}^{M_0}$ of all numerical Σ-functions with parameter M_0 is computable in $\mathbb{HF}(\mathfrak{M})$.

THEOREM 2 [1]. Let $\mathfrak{M}$ be a Σ-uniform structure. The family $\mathfrak{F}$ of all unary Σ-functions in $\mathbb{HF}(\mathfrak{M})$ is computable if and only if the family $\mathcal{N}$ of all numerical Σ-functions in $\mathbb{HF}(\mathfrak{M})$ is computable.

COROLLARY 1 [1]. If $\mathfrak{M}$ is a Σ-uniform structure, then an ideal $\mathcal{I}_e(\mathfrak{M})$ of e-degrees of Σ-subsets of natural numbers in $\mathbb{HF}(\mathfrak{M})$ is principal and is generated by the e-degree of a set $\mathrm{Th}_\exists(\mathfrak{M})$ of Gödel numbers of $\exists$-sentences true in $\mathfrak{M}$.

COROLLARY 2 [1]. Let $\mathfrak{M}$ be a Σ-uniform structure. Then $\mathbb{HF}(\mathfrak{M})$ contains a universal Σ-function if and only if a principal e-ideal $\mathcal{I}_e(\mathfrak{M})$ contains a universal function for the family of all unary functions in $\mathcal{I}_e(\mathfrak{M})$.

1. GROUPS

In this section, we construct a family of Σ-uniform Abelian groups.

Let G be a periodic Abelian group, e the zero element of G, and $g \in G$. The order of an element g is denoted by $|g|$; $Z^n_{p^m}$ stands for a direct sum of n copies of a cyclic group of order p^m, where p is a prime; p_k is the kth prime. A p-component of G is denoted by G_p; i.e., $G = \oplus\{G_p \mid p \in P\}$, where P is the set of all primes.

LEMMA 1.1. Suppose G is a periodic Abelian p-group, every p-component G_p of which is finite, and $\Phi(\bar{a}, x)$ is a Σ-formula with parameter $\bar{a} = \langle a_0, \ldots, a_{m-1}\rangle$, $a_i \in G$, in a signature $\sigma = \langle U, \in, \varnothing, +, 0\rangle$. Then there exists a Σ-formula $\Phi^*(x)$ without parameters such that $\Phi(\bar{a}, x) \equiv \Phi^*(x)$ is a true formula in $\mathbb{HF}(G)$ for any $x \in \mathbb{HF}(\omega)$.

Proof. First let $(\bar{a}) \subseteq G_p$ for some p. There is no loss of generality in assuming that $G_p = (a_0) \oplus \ldots \oplus (a_{m-1})$ and $|a_i| = p^{n_i}$, with $n_i > 0$. Put

$$\Phi^* = \exists y_0 \ldots \exists y_{m-1} \left((\bar{y}) = (y_0) \oplus \ldots \oplus (y_{m-1}) \,\&\, \bigwedge_{i<m} |y_i| = p^{n_i} \,\&\, \Phi(\bar{y}, x) \right),$$

where $\Phi(\bar{y}, x)$ is obtained from $\Phi(\bar{a}, x)$ by replacing $\bar{a}$ by $\bar{y}$.

Let $\mathbb{HF}(G) \models \Phi(\bar{a}, x)$, with $x \in \mathbb{HF}(\omega)$. If a_i are taken to be values for y_i in $\Phi(\bar{y}, x)$, then $\mathbb{HF}(G) \models \Phi^*(x)$. Suppose $\mathbb{HF}(G) \models \Phi^*(x)$. Then there exists an automorphism $\varphi : \mathbb{HF}(G) \to \mathbb{HF}(G)$ for which $\varphi y_i = a_i$. Hence $\mathbb{HF}(G) \models \Phi(\bar{a}, x)$.

The general case where $(\bar{a}) \subseteq G_{p_0} \oplus \ldots \oplus G_{p_{s-1}}$ can be easily reduced to the case above. □

Let partial unary functions α, φ, and ψ be defined so that for any $k \in \delta\alpha$, the following relations hold: $\varphi(k) = [m^k_0, \ldots, m^k_{\alpha_k - 1}]$ and $\psi(k) = [n^k_0, \ldots, n^k_{\alpha_k - 1}]$, where $\alpha_k = \alpha(k)$, $m^k_i, n^k_i > 0$, and $m^k_i < m^k_j$ if $0 \leq i < j < \alpha_k$. Given these functions, we define a group of the form

$$G \rightleftharpoons G(\alpha, \varphi, \psi) = \oplus \left\{ Z^{n^k_0}_{p_k^{m^k_0}} \oplus \ldots \oplus Z^{n^k_{\alpha_k-1}}_{p_k^{m^k_{\alpha_k-1}}} \;\middle|\; k \in \delta\alpha \right\}.$$

In what follows, m_i and n_i will be used in place of m^k_i and n^k_i unless ambiguity would result.

THEOREM 1.1. A group $G = G(\alpha, \varphi, \psi)$ is Σ-uniform if and only if functions α, φ, and ψ are Σ-definable in $\mathbb{HF}(G)$.

Proof. Sufficiency. We verify whether conditions (1)-(3) in the definition of a Σ-uniform structure are valid for G. In view of Lemma 1.1, we may assume that α, φ, and ψ are Σ-definable in $\mathbb{HF}(G)$ without parameters.

(1) Define a set Ξ_0 of elementary characteristic by setting

$$\Xi_0 = \{p \mid \exists k(k \in \delta\alpha \,\&\, p = p_k)\} \cup \{0\}.$$

To show that a function $\mathfrak{b}$ is Σ-definable in $\mathbb{HF}(G)$, we introduce the following formulas. For any $k \in \delta\alpha$, put $n^k = n_0^k m_0^k + \ldots + n_{\alpha_k-1}^k m_{\alpha_k-1}^k$. Then the graph Γ_n of a function $n(k) = n^k$ is definable by a Σ-formula without parameters. Let G_k be the primary p_k-component of G. For a function $h(k) = G_k$, we have

$$h(k) = G_k \Leftrightarrow \mathbb{HF}(G) \models k \in \delta\alpha \,\&\, |G_k| = p_k^{n^k} \,\&\, \forall x \in G_k \exists s (p_k^s x = 0).$$

Consequently, h is a Σ-function without parameters in $\mathbb{HF}(G)$.

A predicate $\mathfrak{C}(k, \bar{y}) \rightleftharpoons$ '$\bar{y}$ is an elementary basis of characteristic p_k' is defined via the equivalence

$$\begin{aligned}\mathfrak{C}(k, \bar{y}) \Leftrightarrow \mathbb{HF}(G) \models{} & k \in \delta\alpha \,\&\, \exists \bar{y}^0 \in G_k \ldots \exists \bar{y}^{\alpha_k-1} \in G_k \\ & \Bigg[\bigwedge_{i<\alpha_k} \Bigg(\exists y_0^i \ldots \exists y_{n_i-1}^i \Bigg(\bar{y}^i = \langle y_0^i, \ldots, y_{n_i-1}^i \rangle \,\&\, \bigwedge_{j<n_i} |y_j^i| = p_k^{m_i} \Bigg)\Bigg) \\ & \&\, (\bar{y}^0, \ldots, \bar{y}^{\alpha_k-1}) = \oplus\{(y_j^i) \mid i < \alpha_k,\ j < n_i\} \\ & \&\, \bar{y} = \langle \bar{y}^0, \ldots, \bar{y}^{\alpha_k-1} \rangle \Bigg],\end{aligned}$$

where m_i and n_i stand for m_i^k and n_i^k. Hence $\mathfrak{C}$ is a Σ-predicate without parameters in $\mathbb{HF}(G)$.

Below we need the following:

LEMMA 1.2. There exists a computable function $\beta(k, \alpha, m, n)$ such that if $\alpha > 0$, $m = [m_0, \ldots, m_{\alpha-1}]$, $n = [n_0, \ldots, n_{\alpha-1}]$, $m_i, n_i > 0$, $m_i < m_j$, $0 \leq i < j < \alpha$, and $G_k \cong \oplus \left\{ Z_{p_k^{m_i}}^{n_i} \,\middle|\, i < \alpha \right\}$, then the value of $\beta(k, \alpha, m, n)$ is equal to the number of sequences $\bar{y} = \langle y_0^0, \ldots, y_{n_0-1}^0, \ldots, y_0^{\alpha-1}, \ldots, y_{n_{\alpha-1}-1}^{\alpha-1} \rangle$, $|y_j^i| = p^{m_i}$, $j < n_i$, $i < \alpha$, such that $G_k = \oplus\{(y_j^i) \mid j < n_i, i < \alpha\}$.

This lemma implies that a function $\mathfrak{b}$ satisfies the equivalence

$$\begin{aligned}\mathfrak{b}(p_k) = B_{p_k} \Leftrightarrow \mathbb{HF}(G) \models{} & k \in \delta\alpha \,\&\, \exists m \exists m_0 \ldots \exists m_{\alpha_k-1} \exists n \exists n_0 \ldots \exists n_{\alpha_k-1} \exists \gamma \\ & \exists \bar{y}^0 \ldots \exists \bar{y}^{\gamma-1} \Bigg(\varphi(k) = m \,\&\, m = [m_0, \ldots, m_{\alpha_k-1}] \,\&\, \psi(k) = n \\ & \&\, n = [n_0, \ldots, n_{\alpha_k-1}] \,\&\, \gamma = \beta(k, \alpha, m, n) \,\&\, \bigwedge_{i<\gamma} \mathfrak{C}(k, \bar{y}^i) \\ & \&\, \bigwedge_{s<j<\gamma} \bar{y}^s \neq \bar{y}^j \,\&\, \forall i < \gamma\ (\bar{y}^i \in B_{p_k}) \\ & \&\, \forall \bar{y} \in B_{p_k} \exists i < \gamma (\bar{y} = \bar{y}^i) \Bigg).\end{aligned}$$

Consequently, the graph $\Gamma_{\mathfrak{b}}$ of the function $\mathfrak{b}$ is a Σ-predicate without parameters in $\mathbb{HF}(G)$.

Given any sequence $\chi = \langle p_{k_0}, \ldots, p_{k_{m-1}} \rangle$, for $m = 1$, p_{k_0} is equal to 0 or to a prime numbered k_0, and for $m > 1$, we have $0 < p_{k_i} < p_{k_j}$, with $i < j < m$. For all elementary bases $\bar{y}^i \in B_{p_{k_i}}$ of characteristic p_{k_i}, a sequence of the form $Y = \langle \bar{y}^0, \ldots, \bar{y}^{m-1} \rangle$ is called a *basis of characteristic* χ.

We have thus proved the validity of condition (1).

(2) Let an arbitrary element $z \in G \backslash \{e\}$ be given and its order $|z|$ be equal to $p_{k_0}^{l_0} \dots p_{k_{q-1}}^{l_{q-1}}$, $q > 1$, $k_0 < \dots < k_{q-1}$, $0 < l_i < m_{\alpha_{k_i}-1} \rightleftharpoons s_i$, $i < q$. Put $z_i = p_{k_0}^{l_0} \dots p_{k_{i-1}}^{l_{i-1}} p_{k_{i+1}}^{l_{i+1}} \dots p_{k_{q-1}}^{l_{q-1}} z$. Then $z_i \in G_{k_i}$ and $z = z_0 + \dots + z_{q-1}$, with $z_i \neq e$ and $(z) = (z_0) \oplus \dots \oplus (z_{q-1})$. Define $f_1(z) = \langle z_0, \dots, z_{q-1} \rangle$. If $z \in G_k$ then $f_1(z) = z$. It is easy to verify that the graph Γ_{f_1} is a Σ-predicate without parameters in $\mathbb{HF}(G)$.

Let $x \in (\bar{y})$, $x \neq e$, be given; here $\bar{y} = \langle \bar{y}^0, \dots, \bar{y}^{\alpha_k - 1} \rangle$ is an elementary basis of characteristic $p_k \rightleftharpoons p$, $\bar{y}^i = \langle y_0^i, \dots, y_{n_i-1}^i \rangle$, $(\bar{y}) = (\bar{y}^0) \oplus \dots \oplus (\bar{y}^{\alpha_k - 1}) = G_k$, $(\bar{y}^i) = (y_0^i) \oplus \dots \oplus (y_{n_i-1}^i)$, $|y_j^i| = p_k^{m_i}$, and $j < n_i$. Then $x = x^0 + \dots + x^{\alpha_k - 1}$ and $x^i = x_0^i + \dots + x_{n_i-1}^i$, where $x^i \in (\bar{y}^i)$ and $x_j^i \in (y_j^i)$. For any $i < \alpha_k$ and any $j < n_i$, put

$$(x_j^i)' = \begin{cases} x_j^i & \text{if } x_j^i \neq e; \\ \varnothing & \text{otherwise.} \end{cases}$$

Therefore, in a one-to-one correspondence with every element $x \in (Y)$, $x \neq e$, is a sequence $f_0(x, Y) = \langle (x_j^i)' \mid i < \alpha_k,\, j < n_i \rangle$. For any element $x \in (y_j^i)$, there exists a unique number α for which $x = (\alpha - 1) y_j^i$. Put $\mathrm{Cor}_0(x, y_j^i) = \alpha$ and $\mathrm{Cor}(e, \varnothing) = 1$. This immediately implies that Cor_0 is a Σ-function without parameters, and if $x^0 \neq x^1$ then $\mathrm{Cor}_0(x^0, y_j^i) \neq \mathrm{Cor}_0(x^1, y_j^i)$.

(3) Is obvious. The sufficiency is proved.

Necessity. First we argue for two lemmas.

LEMMA 1.3. If $G = \oplus \{G_p \mid p \in P\}$, $|G_p| < \omega$, is a Σ-uniform group, then new elementary bases can be defined so as to be contained in primary components relative to which G is again Σ-uniform.

Proof. Every element $e \neq x \in G$ is uniquely represented as $x = x_{q_0} + \dots + x_{q_{m-1}}$, where $e \neq x_{q_j} \in G_{q_j}$ and $q_0 < \dots < q_{m-1}$ are primes. Denote the set $\{q_0, \dots, q_{m-1}\}$ by Q_x. Let an (old) elementary basis $\bar{y} = \langle y^0, \dots, y^{n-1} \rangle$ of characteristic χ be given and $Q_{\bar{y}} \rightleftharpoons \cup \{Q_{y^i} \mid i < n\} = \{q_0, \dots, q_{t-1}\}$. For every $q \in Q_{\bar{y}}$, put $I_q = \{i < n \mid q \in Q_{y^i}\}$. Suppose $I_q = \{i_0, \dots, i_{k-1}\}$. Then a sequence of the form $\bar{y}_q = \langle y_q^{i_0}, \dots, y_q^{i_{k-1}} \rangle$ is called a (*new*) *elementary basis of characteristic* $\langle \chi, q \rangle$ and is denoted $\bar{y}^*$. An empty sequence likewise is called an elementary basis of characteristic 0. Since $(\bar{y}) = (y^0) \oplus \dots \oplus (y^{n-1})$ for the basis $\bar{y}$, we have

$$(\bar{y})_q \rightleftharpoons (\bar{y}_q) = (y_q^{i_0}) \oplus \dots \oplus (y_q^{i_{k-1}}), \tag{1}$$

$$(\bar{y}) = \oplus \{(\bar{y}_q) \mid q \in Q_{\bar{y}}\}. \tag{2}$$

At the moment, we prove that conditions (1)-(3) in the definition of a Σ-uniform structure are satisfied for the new bases.

(1) Introduce the formula

$$\Phi(\chi, \bar{y}, q, \bar{y}^*) \rightleftharpoons \chi \in \Xi_0 \,\&\, \bar{y} \in B_\chi \,\&\, \exists n \exists y^0 \dots \exists y^{n-1} \big(\bar{y} = \langle y^0, \dots, y^{n-1} \rangle$$
$$\&\, (\bar{y})_q \neq \{e\}$$

$$\& \exists m \exists i_0 \ldots \exists i_{m-1} \forall k < m \forall j < m \big((k < j \to i_k < i_j) \\ \& (y_q^{i_k}) \neq \{e\} \,\&\, (\bar{y})_q = (y_q^{i_0}) \oplus \ldots \oplus (y_q^{i_{m-1}}) \\ \& \bar{y}^* = \langle y_q^{i_0}, \ldots, y_q^{i_{m-1}} \rangle \big) \big).$$

Put

$\Xi_0^* = \{\langle \chi, q\rangle \mid \mathbb{HF}(G) \models \exists \bar{y} \exists \bar{y}^* \Phi(\chi, \bar{y}, q, \bar{y}^*)\}$,

$\mathfrak{b}^*(\chi, q) = B^*_{\langle \chi, q\rangle} \Leftrightarrow \mathbb{HF}(G) \models \forall \bar{y}^* \in B^*_{\langle \chi, q\rangle} \exists \bar{y} \in B_\chi \big(\Phi(\chi, \bar{y}, q, \bar{y}^*) \,\&\, \forall \bar{y} \in B_\chi \big((\bar{y})_q \neq \{e\} \to \exists \bar{y}^* \Phi(\chi, \bar{y}, q, \bar{y}^*) \big) \,\&\, \bar{y}^* \in B^*_{\langle \chi, q\rangle} \big)$.

On the set Ξ_0^*, define a lexicographic ordering.

Let $\bar{y}_0^*$ and $\bar{y}_1^*$ be elementary bases of characteristic $\langle \chi, q\rangle$, which are obtained from old bases $\bar{y}_0$ and $\bar{y}_1$ of characteristic χ. Since $(\bar{y}_0) = (\bar{y}_1)$, we have $(\bar{y}_0^*) = (\bar{y}_1^*)$ in view of (1).

Now let bases $\bar{y}_1^*, \ldots, \bar{y}_\alpha^*$ of pairwise distinct characteristics $\langle \chi_1, q_1\rangle, \ldots, \langle \chi_\alpha, q_\alpha\rangle$ be given. We claim that $(\bar{y}_1^*) \cap (\bar{y}_2^*, \ldots, \bar{y}_\alpha^*) = \{e\}$. Indeed, suppose $x \in (\bar{y}_1^*) \cap (\bar{y}_2^*, \ldots, \bar{y}_\alpha^*)$. Then $x \in G_{q_1}$. Assume $q_2, \ldots, q_m \neq q_1$ but $q_{m+1}, \ldots, q_\alpha = q_1$. Then $x \in (\bar{y}_{m+1}^*, \ldots, \bar{y}_\alpha^*)$. Since $\langle \chi_1, q_1\rangle \neq \langle \chi_k, q_k\rangle$, $m < k \leq \alpha$, we have $\chi_1 \neq \chi_k$. Let $\bar{y}_1^*, \ldots, \bar{y}_\alpha^*$ be obtained from old bases $\bar{y}_1, \ldots, \bar{y}_\alpha$ of characteristics $\chi_1, \ldots, \chi_\alpha$. This implies $(\bar{y}_1) \cap (\bar{y}_{m+1}, \ldots, \bar{y}_\alpha) = \{e\}$. Since $\bar{y}_i^* \subseteq (\bar{y}_i)$, $1 \leq i \leq \alpha$, we have $(\bar{y}_1^*) \cap (\bar{y}_{m+1}^*, \ldots, \bar{y}_\alpha^*) = \{e\}$; i.e., $x = e$.

(2) Let $\bar{y}^*$ be an elementary basis of characteristic $\langle \chi, q\rangle$, which is obtained from a basis $\bar{y} = \langle y^0, \ldots, y^{n-1}\rangle$, and $x \in (\bar{y})^* \setminus \{e\}$. Then $(\bar{y}^*) = (\bar{y})_q = (y_q^{i_0}) \oplus \ldots \oplus (y_q^{i_{m-1}})$ for some $i_j < n$. With this in mind, we can uniquely define a sequence $\langle x_0, \ldots, x_{m-1}\rangle$ so that $x_j \in (y_q^{i_j})$, $x = x_0 + \ldots + x_{m-1}$, and $x \in (x_0, \ldots, x_{m-1})$. For every $j < m$, put

$$x_j' = \begin{cases} x_j & \text{if } x_j \neq e; \\ \varnothing & \text{otherwise.} \end{cases}$$

Set $f_0^*(x, \bar{y}^*) = \langle x_0', \ldots, x_{m-1}'\rangle$.

Define a function f_1^*. Let an element $z \in G \setminus \{e\}$ be given and $f_1(z) = \langle z^0, \ldots, z^{e-1}\rangle$ be a function for which $z^k \in (\bar{y}_k) \setminus \{e\}$, $(z) = (z^0, \ldots, z^{l-1})$, $\bar{y}_k = \langle y_{k,0}, \ldots, y_{k,n_k-1}\rangle$, $k < l$. In view of (2), we have $(\bar{y}_k) = \oplus\{(\bar{y}_k)_q \mid q \in Q_{\bar{y}_k}\}$. This implies that for any $k < l$, there exist a subset $Q_k = \{q_0^k, \ldots, q_{m_k-1}^k\} \subseteq Q_{\bar{y}_k}$ and elements $e \neq z_j^k \in (\bar{y}_k)_{q_j^k}$, $j < m_k$, such that

$$z^k = z_0^k + \ldots + z_{m_k-1}^k, \ (z^k) = (z_0^k, \ldots, z_{m_k-1}^k). \tag{3}$$

In this event we put $f_1^*(z) = \langle z_0^0, \ldots, z_{m_0-1}^0, \ldots, z_0^{e-1}, \ldots, z_{m_{e-1}-1}^{e-1}\rangle$. It follows from (3) that $(z) = (z_0^0, \ldots, z_{m_{e-1}-1}^{e-1})$. If $\bar{y}^*$ is an elementary basis and $z \in (\bar{y}^*)$, then $f_1^*(z) = z$.

The functions f_0^* and f_1^* are defined so that $\Gamma_{f_0^*}$ and $\Gamma_{f_1^*}$ are Σ-predicates without parameters in $\mathbb{HF}(G)$.

We define a function Cor_0. Let an elementary basis $\bar{y}^* = \langle y_0, \ldots, y_{n-1}\rangle$ be given. For any element $x \in (y_i)$, $i < n$, there exists a unique number α such that $x = (\alpha - 1)y_i$. In this instance we set

$\mathrm{Cor}_0(x, y_i) = \alpha$ and $\mathrm{Cor}(e, \varnothing) = 1$. This implies that Cor_0 is a Σ-function without parameters, and if $x^0 \neq x^1$ then $\mathrm{Cor}_0(x^0, y_i) \neq \mathrm{Cor}_0(x^1, y_i)$.

(3) Let $\bar{y}_0^*$ and $\bar{y}_1^*$ be two elementary bases of the same characteristic $\langle \chi, q \rangle$, which are obtained from old bases $\bar{y}_0$ and $\bar{y}_1$ of characteristic χ. In addition, suppose $G_0 \supseteq (\bar{y}_0^*)$ is a finite subgroup. We may also assume that $G_0 \supseteq (\bar{y}_0)$. Since G is Σ-uniform relative to the old bases, and $\chi(\bar{y}_0) = \chi(\bar{y}_1)$, there exists an isomorphism $\varphi : G^0 \to G$ such that $\varphi \bar{y}_0 = \bar{y}_1$. In view of the equalities $(\bar{y}_0^*) = (\bar{y}_0)_q$ and $(\bar{y}_1^*) = (\bar{y}_1)_q$, we have $\varphi \bar{y}_0^* = \bar{y}_1^*$. Now let bases $\bar{Y}^\varepsilon = \langle \bar{y}_0^\varepsilon, \ldots, \bar{y}_{n-1}^\varepsilon \rangle$ of characteristic $\chi = \langle \langle \chi_0, q_0 \rangle, \ldots, \langle \chi_{n-1}, q_{n-1} \rangle \rangle$ be given. Since $\langle \chi_i, q_i \rangle \neq \langle \chi_j, q_j \rangle$ for any $i < j < n$, it follows that $(\bar{Y}^\varepsilon) = (\bar{y}_0^\varepsilon) \oplus \ldots (\bar{y}_{n-1}^\varepsilon)$. From this point on, the proof that the required isomorphic embedding φ exists proceeds similarly to the previous. □

Below by a basis is meant a new basis and asterisks in symbols $\bar{y}^*$, Ξ_0^*, χ^*, B_χ^*, and $\mathfrak{b}^*(\chi)$ are omitted. With this in mind, we have

LEMMA 1.4. Let G be a Σ-uniform group. If $\bar{y} = \langle y_0, \ldots, y_{n-1} \rangle$ is an elementary basis, a subgroup (y_i) is pure in G for some $i < n$, and $|y_i| = p^m$, then every element $g \in G_p$ of order p^k, $k \leq m$, belongs to $(\bar{y})$.

Proof. To be specific, let $i = 0$. First we show that any element of order p^m belongs to $(\bar{y})$. Assume to the contrary that there exists some element $b_0 \in G_p \setminus (\bar{y})$ with $|b_0| = p^m$. There are two cases to consider.

(a) Let (b_0) be pure in G_p. Then G has the following decompositions: $G_p = (y_0) \oplus (a_1) \ldots \oplus (a_{l-1})$ and $G_p = (b_0) \oplus (b_1) \oplus \ldots \oplus (b_{l-1})$, where $|a_i| = |b_i|$, with $1 \leq i \leq l$. Given the basis $\bar{y}$, we define a sequence $\bar{v}$ as follows. Let $y_i = \alpha_{i,0} y_0 + \alpha_{i,1} a_1 + \ldots + \alpha_{i,l-1} a_{l-1}$, $0 < i < l$. Put $v_i = \alpha_{i,0} b_0 + \alpha_{i,1} b_1 + \ldots + \alpha_{i,l-1} b_{l-1}$. Take an isomorphism $\varphi : G_p \to G_p$, where $\varphi y_0 = b_0$ and $\varphi a_i = b_i$, such that $\varphi y_i = v_i$. Every Σ-formula $\Phi(x_0, \ldots, x_{n-1})$ without parameters true in $\mathbb{HF}(G)$ for $\langle y_0, \ldots, y_{n-1} \rangle$ will also be true for $\langle v_0, \ldots, v_{n-1} \rangle$. Let $\bar{y} \in B_\chi$. Then $\bar{v} \in B_\chi$. At the same time, $b_0 \in (\bar{v}) \setminus (\bar{y})$, a contradiction. Hence $b_0 \in (\bar{y})$.

(b) Let (b_0) not be pure in G_p. Then an element $z_0 = y_0 + b_0$ has order p^m. We prove that (z_0) is pure in G_p. Suppose on the contrary that there exists an element u_0 for which $p^{m-1} z_0 = p^m u_0$. This yields $p^{m-1} y_0 + p^{m-1} b_0 = p^m u_0$. Since (b_0) is not pure, there exists an element u_1 such that $p^{m-1} b_0 = p^m u_1$. This implies $p^{m-1} y_0 = p^m (u_0 - u_1)$, which is a contradiction with (y_0) being pure. Therefore, (z_0) is pure. In view of (a), $z_0 \in (\bar{y})$. Hence $b_0 \in (\bar{y})$.

Now let an element $x \in G_p$ have order p^k, $k \leq m$. Then $y_0 + x$ has order p^m; so $y_0 + x$ and, hence, $x \in (\bar{y})$. □

Lemma 1.4 entails the following:

COROLLARY 1.1. Every elementary basis $\bar{y} \neq \varnothing$ contained in G_p generates G_p. In other words, $(\bar{y}) = G_p$.

Proof. Let p^m be the greatest order of elements in the group G_p. We claim that there exists an elementary basis $\bar{y}$ containing some element x of order p^m. Indeed, suppose that the order of an element $g \in G$ is equal to p^m. Then $f_1(g) = \langle g_0, \ldots, g_{e-1} \rangle$, where $g_i \in (\bar{y}_i)$, $\bar{y}_i$ is an elementary basis, and $(g) = (g_0, \ldots, g_{e-1})$. Since $|g| = p^m$, there exists an element g_i of order p^m. Hence the subgroup (g_i) is pure in G_p. By Lemma 1.4, therefore, every element of order less than p^m is contained in $(\bar{y}_i)$; i.e., $(\bar{y}_i) = G_p$. Assume G_p has another elementary basis $\varnothing \neq \bar{v} \subseteq G_p$ such that $(\bar{y}_i) \neq (\bar{v})$. Then $\chi(\bar{y}_i) \neq \chi(\bar{v})$. This yields $(\bar{y}_i) \cap (\bar{v}) = \{e\}$. On the other hand, $(\bar{v}) \subseteq (\bar{y}_i)$, a contradiction. Thus $(\bar{v}) = G_p$. □

We finish to argue for the necessity. Corollary 1.1 entails the equivalence

$$\begin{aligned}
&\alpha(p) = \alpha \,\&\, \varphi(p) = \langle m_0, \ldots, m_{\alpha-1}\rangle \,\&\, \psi(p) = \langle n_0, \ldots, n_{\alpha-1}\rangle \\
\Leftrightarrow\, &\mathbb{HF}(G) \models \exists\chi\exists B_\chi\exists\bar{y}\exists y_0^0 \ldots \exists y_{n_0-1}^0 \ldots \exists y_0^{\alpha-1} \ldots \exists y_{n_{\alpha-1}-1}^{\alpha-1}\Bigg(\chi \in \Xi_0 \setminus \{0\} \\
&\& \,\mathfrak{b}(\chi) = B_\chi \,\&\, \bar{y} \in B_\chi \,\&\, (\bar{y}) \subseteq G_p \\
&\& \,\bar{y} = \langle y_0^0, \ldots, y_{n_0-1}^0 \ldots, y_0^{\alpha-1} \ldots, y_{n_{\alpha-1}-1}^{\alpha-1}\rangle \,\&\, \Bigg(\bigwedge_{i<\alpha}\Bigg(\bigwedge_{j<n_j} |y_j^i| = p^{m_i}\Bigg)\Bigg)\Bigg).
\end{aligned}$$

Hence graphs Γ_α, Γ_φ, and Γ_ψ are Σ-predicates without parameters in $\mathbb{HF}(G)$. The necessity is proved, completing the proof of Theorem 1.1. □

Theorems 1 and 1.1 can be combined to yield

COROLLARY 1.2. Let $G \rightleftharpoons G(\alpha, \varphi, \psi)$ be a group and functions α, φ, and ψ be Σ-definable in $\mathbb{HF}(G)$. Then a universal Σ-function exists in $\mathbb{HF}(G)$ if and only if the family $\mathcal{N}^G$ of all numerical Σ-functions in $\mathbb{HF}(G)$ is computable.

COROLLARY 1.3. Let functions α, φ, and ψ have partial computable extensions. Then a universal Σ-function exists in $\mathbb{HF}(G)$, where $G \rightleftharpoons G(\alpha, \varphi, \psi)$, if and only if the family $\mathcal{N}^G$ of all numerical Σ-functions in $\mathbb{HF}(G)$ is computable.

Proof. Indeed, suppose α', φ', and ψ' are extensions of α, φ, and ψ, respectively, and $P_0 = \{p \in P \mid \mathbb{HF}(G) \models \exists x(|x| = p)\}$. Since the functions α', φ', and ψ' are Σ-definable in $\mathbb{HF}(G)$, functions $\alpha = \alpha' \restriction P_0$, $\varphi = \varphi' \restriction P_0$, and $\psi = \psi' \restriction P_0$ likewise are Σ-definable, and the result now follows from Corollary 1.2. □

LEMMA 1.5. Let functions α, φ, and ψ be Σ-definable in $\mathbb{HF}(G)$, where $G \rightleftharpoons G(\alpha, \varphi, \psi)$. An arbitrary subset A of natural numbers is Σ-definable in $\mathbb{HF}(G)$ if and only if it is e-reducible to a set of the form

$$S = \{[k, m_0^k, n_0^k], \ldots, [k, m_{\alpha_k-1}^k, n_{\alpha_k-1}^k] \mid k \in \delta\alpha\},$$

where $\alpha(k) = \alpha_k$, $\varphi(k) = [m_0^k, \ldots, m_{\alpha_k-1}^k]$, $\psi(k) = [n_0^k, \ldots, n_{\alpha_k-1}^k]$, and $m_i^k < m_j^k$ if $i < j < \alpha_k$.

Proof. The sufficiency follows from [5, Thm. 1.1] and the property of being Σ-definable for S.

Necessity. Let a set $A \subseteq \omega$ be Σ-definable in $\mathbb{HF}(G)$. By virtue of Theorem 1.1, the group G is Σ-uniform. In view of Corollary 1, the set A is e-reducible to a set $\mathrm{Th}_\exists(G)$. We argue to show that $\mathrm{Th}_\exists(G)$ is e-reducible to S.

Let a set $\mathfrak{B} = \{[k, m_i^k, n_i^k] \mid k, i, m_i^k, n_i^k \in \omega,\ m_i^k, n_i^k > 0\}$ be given and $B \subseteq \mathfrak{B}$ be a finite subset such that $B = B_{k_0} \cup \ldots \cup B_{k_t-1}$, $k_i \neq k_j$, and $i < j < t$, where $B_{k_i} =$

$\{[k_i, m_0^{k_i}, n_0^{k_i}], \ldots, [k_i, m_{l_i-1}^{k_i}, n_{l_i-1}^{k_i}]\}$, $m_s^{k_i} \neq m_r^{k_i}$, and $s < r < l_i$. Given these, we define a group $G(B) = G(B_0) \oplus \ldots \oplus G(B_{k_t-1})$, where $G(B_k) = Z^{n_0^k}_{p_k^{m_0^k}} \oplus \ldots \oplus Z^{n_{l-1}^k}_{p_k^{m_{l-1}^k}}$, $k = k_0, \ldots, k_{t-1}$, and if $k = k_i$ then $l = l_i$. Let $\mathfrak{B}^* = \{B \mid B \subseteq \mathfrak{B}, B = B_{k_0} \cup \ldots \cup B_{k_t-1}\}$, write $H \hookrightarrow G(B)$ for the fact that a finite group H is embeddable in $G(B)$, and assume that Φ is a set of all $\exists$-sentences in a signature $\sigma = \langle +, 0 \rangle$. Given these, we define a set $W = \{\langle H, B, \varphi \rangle \mid H \hookrightarrow G(B), B \in \mathfrak{B}^*, \varphi \in \Phi, H \models \varphi\}$. It is easy to verify that the set W is c.e. We prove the equality

$$\mathrm{Th}_\exists(G) = \{\varphi \mid \exists H \exists B(\langle H, B, \varphi \rangle \in W \,\&\, B \subseteq S)\}. \tag{4}$$

Indeed, let $\varphi \in \mathrm{Th}_\exists(G)$. Then there exists $B \subseteq S$, $B \in \mathfrak{B}^*$, such that $G(B) \models \varphi$. If we put $H = G(B)$ we conclude that the formula in the right part of (4) is valid for φ. Assume now that φ belongs to the right part of (4). Then H is embeddable in $G(B)$ and $H \models \varphi$, whence $G(B) \models \varphi$. Since $B \subseteq S$, $G(B)$ is embeddable in G. Hence $\varphi \in \mathrm{Th}_\exists(G)$, proving (4). Thus $\mathrm{Th}_\exists(G) \leq_e S$. □

Lemma 1.5 and Corollary 2 give rise to the following:

COROLLARY 1.4. Let functions α, φ, and ψ be Σ-definable in $\mathbb{HF}(G)$, with $G \rightleftharpoons G(\alpha, \varphi, \psi)$. Then $\mathbb{HF}(G)$ contains a universal Σ-function if and only if a principal e-ideal $\mathcal{I}_e(S)$ generated by a set S contains a function that is universal for the family of all unary functions in $\mathcal{I}_e(S)$.

COROLLARY 1.5. Let α, φ, and ψ be partial computable functions. Then a universal Σ-function exists in $\mathbb{HF}(G)$, with $G \rightleftharpoons G(\alpha, \varphi, \psi)$.

Proof. In fact, the existence of a universal function is underpinned by the fact that the set S is c.e. in the e-ideal $\mathcal{I}_e(S)$ generated by S. □

COROLLARY 1.6. There exists a set S of primes such that an admissible set $\mathbb{HF}(G_S)$, $G_S = \oplus\{Z_p \mid p \in S\}$, contains no universal Σ-function.

Proof. In fact, a set S of natural numbers such that an e-ideal $\mathcal{I}_e(S)$ generated by S does not contain a universal function was constructed in [6]. This, combined with Corollary 1.4, yields the result. □

2. RINGS

In this section, we construct a family of Σ-uniform rings.

Let $F^m_{p^n}$ be a direct sum of m copies of a field of degree n over a prime field of characteristic p, treated in a ring signature $\sigma = \langle +, \cdot, 0 \rangle$. Assume partial unary functions α, φ, and ψ are defined so that for any $k \in \delta\alpha$, the following hold: $\varphi(k) = [m_0^k, \ldots, m_{\alpha_k-1}^k]$ and $\psi(k) = [n_0^k, \ldots, n_{\alpha_k-1}^k]$, where $\alpha_k = \alpha(k)$, $m_i^k, n_i^k > 0$, and $m_i^k < m_j^k$ if $0 \leq i < j < \alpha_k$. Below, unless ambiguity would result, m_i and n_i are written in place of m_i^k and n_i^k. We introduce a ring of the form

$$K \rightleftharpoons K(\alpha, \varphi, \psi) = \oplus \left\{ F^{n_0^k}_{p_k^{m_0^k}} \oplus \ldots \oplus F^{n_{\alpha_k-1}^k}_{p_k^{m_{\alpha_k-1}^k}} \,\middle|\, k \in \delta\alpha \right\},$$

where p_k is the kth prime.

LEMMA 2.1. For every Σ-formula $\Phi(\bar{a}, x)$ with parameter $\bar{a} = \langle a_0, \ldots, a_{n-1} \rangle$, $a_j \in K$, $K \rightleftharpoons K(\alpha, \varphi, \psi)$, in a signature $\sigma = \langle U, \in, \varnothing, +, \cdot, 0 \rangle$, there exists a Σ-formula $\Phi^*(x)$ without parameters such that $\Phi(\bar{a}, x) \equiv \Phi^*(x)$ is a true formula in $\mathbb{HF}(K)$ for any $x \in \mathbb{HF}(\omega)$.

Proof. Assume that for some k, it is true that $(\bar{a}) \subseteq H_k = \{x \in K \mid p_k x = 0\}$, where $(\bar{a})$ is a subring generated by a set $\{a_0, \ldots, a_{n-1}\}$. There is no loss of generality in assuming that $H_k = (a_0) \times \ldots \times (a_{n-1})$, (a_i) is a field of degree l_i over a prime field $P_i \subseteq (a_i)$, and a_i is a root of an irreducible polynomial f_i of degree l_i over P_i.

We define Σ-predicates without parameters as follows.

Let $\mathrm{Field}(F, p^n) \Leftrightarrow$ 'F is a field of cardinality p^n in K' $\Leftrightarrow \mathbb{HF}(K) \models |F| = p^n \,\&\, \forall x \in F(U(x) \,\&\,$ (field axioms whose quantifiers are bounded by a set F)).

Denote by e_F the unit of a subfield $F \subseteq K$. If the predicate $\mathrm{Field}(P, p)$ is true in $\mathbb{HF}(K)$, then P is a prime field of characteristic p and $P = \{0, e_P, 2e_P, \ldots, (p-1)e_P\}$.

Suppose $\mathrm{Pol}(f, n, P, p) \Leftrightarrow$ 'f is a polynomial of degree n over a prime field P of characteristic p' $\Leftrightarrow \mathbb{HF}(K) \models \mathrm{Field}(P, p) \,\&\, \exists s_1 < p \ldots \exists s_n < p(f = \langle e_P, s_1 e_P, \ldots, s_n e_P \rangle)$.

Thus $f(x) = x^n + (s_1 e_P) x^{n-1} + \ldots + s_n e_P$.

Assume $\mathrm{Ind}(f, n, P, p) \Leftrightarrow$ 'a polynomial f of degree n over a prime field P of characteristic p is irreducible' $\Leftrightarrow \mathbb{HF}(K) \models \mathrm{Pol}(f, n, P, p) \,\&\, \exists a_1 \in P \ldots \exists a_n \in P(f = \langle e_P, a_1, \ldots, a_n \rangle \,\&\, \forall s \forall t(s + t = n \,\&\, 1 \leq s \,\&\, 1 \leq t \to \forall b_1 \in P \ldots \forall b_s \in P \forall c_1 \in P \ldots \forall c_t \in P(a_1 \neq b_1 + c_1 \vee \ldots \vee a_n \neq b_s c_t)))$.

Put

$$\Phi^* = \exists y_0 \ldots \exists y_{n-1} \exists P_0 \ldots \exists P_{n-1} \Bigg((\bar{y}) = (y_0) \times \ldots \times (y_{n-1}) \\ \& \bigwedge_{i<n} \big(\mathrm{Field}((y_i), p^{l_i}) \,\&\, \mathrm{Field}(P_i, p) \,\&\, P_i \subseteq (y_i) \,\&\, \mathrm{Ind}(f_i, l_i, P_i, p) \\ \& f_i(y_i) = 0 \big) \,\&\, \Phi(\bar{y}, x) \Bigg),$$

where $\Phi(\bar{y}, x)$ is obtained from $\Phi(\bar{a}, x)$ by replacing $\bar{a}$ by $\bar{y}$.

Let $\mathbb{HF}(K) \models \Phi(\bar{a}, x)$, with $x \in \mathbb{HF}(\omega)$. If we take a_i to be values for y_i in $\Phi(\bar{y}, x)$, then it is obvious that $\mathbb{HF}(K) \models \Phi^*(x)$. Suppose $\mathbb{HF}(G) \models \Phi^*(x)$. Then there exists an automorphism $\varphi : \mathbb{HF}(K) \to \mathbb{HF}(K)$ for which $\varphi y_i = a_i$. Hence $\mathbb{HF}(K) \models \Phi(\bar{a}, x)$.

The general case where $(\bar{a}) \subseteq H_{k_0} \times \ldots \times K_{k_{s-1}}$ can be readily reduced to the case above. □

THEOREM 2.1. If a ring $K \rightleftharpoons K(\alpha, \varphi, \psi)$ and functions α, φ, and ψ are Σ-definable in $\mathbb{HF}(K)$, then K is Σ-uniform.

The **proof** proceeds by verifying the validity of conditions (1)-(3) in the definition of a Σ-uniform structure. In view of Lemma 2.1, we may assume that α, φ, and ψ are Σ-definable in $\mathbb{HF}(K)$ without parameters.

(1) Let $\Xi_0 = \{p \mid \exists k(k \in \delta\alpha \,\&\, p = p_k\} \cup \{0\}$ be a set of elementary characteristics. We argue to show that a function $\mathfrak{b}(p) = B_p$, where B_p is the set of all elementary bases of characteristic p, is definable in $\mathbb{HF}(K)$ by a Σ-formula without parameters.

For any $k \in \delta\alpha$, put $n^k = n_0^k m_0^k + \ldots + n_{\alpha_k-1}^k m_{\alpha_k-1}^k$. Then the graph Γ_n of a function $n(k) = n^k$ is definable by a Σ-formula without parameters. For $h(k) = H_k$, the following equivalence holds:

$$h(k) = H_k \Leftrightarrow \mathbb{HF}(K) \models \forall x \in H_k(U(x) \,\&\, k \in \delta\alpha \,\&\, |H_k| = p_k^{n^k} \,\&\, p_k x = 0).$$

Hence the graph Γ_h is a Σ-predicate without parameters in $\mathbb{HF}(K)$.

On a set of all polynomials of degree n over a given prime field P, a lexicographic ordering is defined via a predicate of the form

$$\begin{aligned}\mathrm{Ord}(f_0, f_1, n, P, p) \Leftrightarrow \mathbb{HF}(K) \models \bigwedge_{\varepsilon<2} \Bigg(& \mathrm{Pol}(f_\varepsilon, n, P, p) \\ & \&\, \exists s_1^\varepsilon < p \ldots \exists s_n^\varepsilon < p \Bigg(f_\varepsilon = \langle e_P, s_1^\varepsilon e_P, \ldots, s_n^\varepsilon e_P \rangle \\ & \&\, \exists k < n \Bigg(\bigwedge_{i<k} s_i^0 = s_i^1 \,\&\, s_k^0 < s_k^1 \Bigg) \vee f_0 = f_1 \Bigg)\Bigg).\end{aligned}$$

To define an elementary basis, Σ-predicates without parameters are introduced as follows.

Let $\mathrm{Ind}^*(f, n, P, p) \Leftrightarrow$ 'f is the least irreducible polynomial of degree n over a prime field P of characteristic p' $\Leftrightarrow \mathbb{HF}(K) \models \mathrm{Ind}(f, n, P, p) \,\&\, \forall g \in \mathrm{Ind}(g, n, P, p)(f \leq g)$.

Suppose $\mathrm{Val}(f, F, y, z, p) \Leftrightarrow$ 'z is equal to the value of a polynomial f over the prime subfield of a field F for an element $y \in F$' $\Leftrightarrow \mathbb{HF}(K) \models \exists m \exists n \exists s_1 < p \ldots \exists s_n < p(\mathrm{Field}(F, p^m) \,\&\, f = \langle e_F, s_1 e_F, \ldots, s_n e_F \rangle \,\&\, y \in F \,\&\, z = y^n + s_1 e_F y^{n-1} + \ldots + s_n e_F)$.

Define an elementary basis via the predicate

$$\begin{aligned}\mathfrak{C}(k, Y) \Leftrightarrow & \text{ '}\bar{y} \text{ is an elementary basis of characteristic } p_k\text{'} \\ & \Leftrightarrow \mathbb{HF}(K) \models k \in \delta\alpha \,\&\, \exists \bar{y}^0 \in H_k \ldots \exists \bar{y}^{\alpha_k-1} \in H_k \Bigg[\bigwedge_{i<\alpha_k} \exists y_0^i \ldots \exists y_{n_i-1}^i \\ & \Bigg(\bar{y}^i = \langle y_0^i, \ldots, y_{n_i-1}^i \rangle \,\&\, \bigwedge_{j<n_i} \exists F_j^i \exists P_j^i \exists f_j^i (\mathrm{Field}(F_j^i, p_k^{m_i}) \\ & \&\, P_j^i \subseteq F_j^i \,\&\, \mathrm{Field}(P_j^i, p_k) \,\&\, \mathrm{Ind}^*(f_j^i, m_i, P_j^i, p_k) \\ & \&\, \mathrm{Val}(f_j^i, F_j^i, y_j^i, 0, p_k)) \Bigg) \,\&\, (\bar{y}^0, \ldots, \bar{y}^{\alpha_k-1}) \\ & = \oplus\{(y_j^i) \mid i < \alpha_k,\ j < n_i\} \,\&\, \bar{y} = \langle \bar{y}^0, \ldots, \bar{y}^{\alpha_k-1} \rangle \Bigg],\end{aligned}$$

where α_k, m_i, and n_i are defined in the same way as at the beginning of Sec. 2 and $(\bar{y}^0, \ldots, \bar{y}^{\alpha_k-1})$ is a subring generated by a set $\{y_j^i \mid i < \alpha_k,\ j < n_i\}$ in K.

For our further reasoning, we need

LEMMA 2.2. There exists a computable numerical function $\beta(k,\alpha,m,n)$ satisfying the following: if $\alpha > 0$, $m = [m_0,\dots,m_{\alpha-1}]$, $n = [n_0,\dots,n_{\alpha-1}]$, $m_i, n_i > 0$, $m_i < m_j$, $0 \le i < j < \alpha$, $H_k \cong \oplus\{F_0^i \oplus \dots \oplus F_{n_i-1}^i \mid i < \alpha\}$, and F_j^i is a field of degree m_i over a prime field of characteristic p_k, then the value of $\beta(k,\alpha,m,n)$ is equal to the number of sequences $\bar{y} = \langle y_0^0,\dots,y_{n_0-1}^0,\dots,y_0^{\alpha-1},\dots,y_{n_{\alpha-1}-1}^{\alpha-1}\rangle \rightleftharpoons \langle y_j^i \mid i < \alpha, j < n_i\rangle$ such that the element y_j^i is a root of the least irreducible polynomial f_j^i of degree m_i over a prime field $P_j^i \subseteq (y_j^i)$ and $H_k = \oplus\{(y_j^i) \mid i < \alpha,\ j < n_i\}$, where (y_j^i) is a subfield of characteristic p_k generated by y_j^i.

Proof. It suffices to appeal to the fact that there exists an algorithm which, given numbers k, α, m, and n, enumerates all sequences $\bar{y}$ in the ring H_k having the properties mentioned in the lemma. □

Define a function $\mathfrak{b}$ via the equivalence $\mathfrak{b}(p_k) = B_{p_k} \Leftrightarrow \mathbb{HF}(K) \models k \in \delta\alpha \,\&\, \exists m \exists m_0 \dots \exists m_{\alpha_k-1} \exists n \exists n_0 \dots \exists n_{\alpha_k-1} \exists\gamma \exists\bar{y}^0 \dots \exists\bar{y}^{\gamma-1}\Big(\varphi(k) = m \,\&\, m = [m_0,\dots,m_{\alpha_k-1}] \,\&\, \psi(k) = n \,\&\, n = [n_0,\dots,n_{\alpha_k-1}] \,\&\, \gamma = \beta(k,\alpha,m,n) \,\&\, \bigwedge\limits_{i<\gamma} \mathfrak{C}(k,\bar{y}^i) \,\&\, \bigwedge\limits_{i<j<\gamma} \bar{y}^i \ne \bar{y}^j \,\&\, \forall i < \gamma\ (\bar{y}^i \in B_{p_k}) \,\&\, \forall \bar{y} \in B_{p_k} \exists i < \gamma (\bar{y} = \bar{y}^i)\Big)$. By Lemma 2.2, the graph $\Gamma_{\mathfrak{b}}$ of the function $\mathfrak{b}$ is a Σ-predicate without parameters in $\mathbb{HF}(K)$.

The concept of a basis is defined as follows. For any sequence $\chi = \langle p_{k_0},\dots,p_{k_{m-1}}\rangle$ (here p_{k_0} either is 0 or is a prime numbered k_0 for the case $m = 1$, and $0 < p_{k_i} < p_{k_j}$, $i < j < m$, for the case $m > 1$) and for all elementary bases $\bar{y}^i \in B_{p_{k_i}}$ of characteristic p_{k_i}, a sequence of the form $Y = \langle \bar{y}^0,\dots,\bar{y}^{m-1}\rangle$ is called a *basis of characteristic* χ.

(2) Let an arbitrary element $z \in K \setminus \{0\}$ be given and its order $|z|^+$ in the additive group K^+ of a ring K be equal to $p_{k_0}\dots p_{k_{q-1}}$, $k > 1$. Put $z_i = p_{k_0}\dots p_{k_{i-1}} p_{k_{i+1}} \dots p_{k_{q-1}} z$. Then $p_{k_i} z_i = 0$, $z_i \ne 0$, $z_i \in H_{k_i}$, $z = z_0 + \dots + z_{q-1}$, and $(z) = (z_0,\dots,z_{q-1})$. Set $f_1(z) = (z_0,\dots,z_{q-1})$. If $z \in H_k$ then $f_1(z) = z$. It is easy to verify that the graph Γ_{f_1} is a Σ-predicate without parameters in $\mathbb{HF}(K)$.

Assume $x \in (\bar{y}) \setminus \{0\}$, where $\bar{y} = \langle \bar{y}^0,\dots,\bar{y}^{\alpha_k-1}\rangle$ is an elementary basis of characteristic $p_k \rightleftharpoons p$, $\bar{y}^i = \langle y_0^i,\dots,y_{n_i-1}^i\rangle$, $(\bar{y}) = (\bar{y}^0) \oplus \dots \oplus (\bar{y}^{\alpha_k-1}) = H_k$, $(\bar{y}^i) = (y_0^i) \oplus \dots \oplus (y_{n_i-1}^i) \cong F_{p^{m_i}}^{n_i}$, and $(y_j^i) = F_j^i \cong F_{p^{m_i}}$. Let $x = x^0 + \dots + x^{\alpha_k-1}$ and $x^i = x_0^i + \dots + x_{n_i-1}^i$, where $x^i \in (\bar{y}^i)$ and $x_j^i \in (y_j^i)$. For any $i < \alpha_k$ and any $j < n_i$, put

$$(x_j^i)' = \begin{cases} x_j^i & \text{if } x_j^i \ne 0; \\ \varnothing & \text{otherwise.} \end{cases}$$

Thus in a one-to-one correspondence with every element $x \in (Y)$ is a sequence $f_0(x,Y) = \langle (x_j^i)' \mid i < \alpha_k,\ j < n_i\rangle$.

By the definition of an elementary basis, for any i and any j, we can uniquely define a least irreducible polynomial $f_j^i = \langle e_j^i, s_1^i e_j^i, \dots, s_{m_i}^i e_j^i\rangle$ of degree m_i over a prime field $P_j^i \subseteq F_j^i$, where e_j^i is the unit of the field P_j^i, whose root is y_j^i. For every element $x \in (y_j^i)$, therefore, there exists a uniquely defined polynomial $g^i_j(z) = s^i_{j0} e^i_j z^{k^i_j} + \ldots + s^i_{jk^i_j} e^i_j$, $s^i_{j0} \neq 0$, $k^i_j < m_i$, and $g^i_j(y^i_j) = x$. Put $\mathrm{Cor}_0(x, y^i_j) = [s^i_{j0}, \ldots, s^i_{jk^i_j}] + 1$ and $\mathrm{Cor}(0, \varnothing) = 1$. This immediately implies that Cor_0 is a Σ-function without parameters, and if $x^0 \neq x^1$ then $\mathrm{Cor}_0(x^0, y^i_j) \neq \mathrm{Cor}_0(x^1, y^i_j)$.

(3) First let elementary bases $\bar{y}_\varepsilon = \langle \bar{y}^0_\varepsilon, \ldots, \bar{y}^{\alpha_k - 1}_\varepsilon \rangle$ and $\bar{y}^i_\varepsilon = \langle y^{i,0}_\varepsilon, \ldots, y^{i,n_i-1}_\varepsilon \rangle$, $\varepsilon < 2$, of the same characteristic $p_k \rightleftharpoons p$ be given. By the definition of an elementary basis, $y^{i,j}_\varepsilon$ is a root of the least irreducible polynomial $f^{i,j}_\varepsilon(x)$ over a prime field $P^{i,j}_\varepsilon \subseteq (y^{i,j}_\varepsilon)$ of degree m_i. Hence the coefficients of $f^{i,j}_\varepsilon$ depend only on i and on the unit $e^{i,j}_\varepsilon$ of the field $P^{i,j}_\varepsilon$; i.e., $f^{i,j}_\varepsilon = x^{m_i} + s^{i,1}_\varepsilon e^{i,j}_\varepsilon x^{m_i - 1} + \ldots + s^{i,m_i}_\varepsilon e^{i,j}_\varepsilon$, with $s^{i,l}_\varepsilon < p$ and $l \leq m_i$. Therefore, the mapping $\varphi^i_j : y^{i,j}_0 \to y^{i,j}_1$ extends to an isomorphism $\Psi^i_j : (y^{i,j}_0) \to (y^{i,j}_1)$. Since $H_k = (\bar{y}_\varepsilon) = \oplus\{(y^{i,j}_\varepsilon) \mid i < \alpha_k, j < n_i\}$, the isomorphisms ψ^i_j extend to an isomorphism $\psi_p : (\bar{y}_0) \to (\bar{y}_1)$.

Next let bases $Y^\varepsilon = \langle \bar{y}^{p_0}_\varepsilon, \ldots, \bar{y}^{p_{q-1}}_\varepsilon \rangle$ of the same characteristic $\langle p_0, \ldots, p_{q-1} \rangle$ be given. By virtue of the fact that $(Y^\varepsilon) = \oplus\{(\bar{y}^{p_i}_\varepsilon) \mid i < q\}$, the isomorphisms Ψ_{p_i} extend to an isomorphism $\Psi : (Y^0) \to (Y^1)$. The theorem is proved. □

Theorems 2 and 2.1 can be combined to yield

COROLLARY 2.1. Let a ring $K \rightleftharpoons K(\alpha, \varphi, \psi)$ and functions α, φ, and ψ be Σ-definable in $\mathbb{HF}(K)$. Then a universal Σ-function exists in $\mathbb{HF}(K)$ if and only if the family $\mathcal{N}^K$ of all numerical Σ-functions in $\mathbb{HF}(K)$ is computable.

COROLLARY 2.2. Let functions α, φ, and ψ have partial computable extensions. Then a universal Σ-function exists in $\mathbb{HF}(K)$, $K \rightleftharpoons K(\alpha, \varphi, \psi)$, if and only if the family $\mathcal{N}^K$ of all numerical Σ-functions in $\mathbb{HF}(K)$ is computable.

LEMMA 2.3. Let functions α, φ, and ψ be Σ-definable in $\mathbb{HF}(K)$, $K \rightleftharpoons K(\alpha, \varphi, \psi)$. Then an arbitrary subset A of natural numbers is Σ-definable in $\mathbb{HF}(K)$ if and only if A is e-reducible to a set of the form

$$S = \{[k, m^k_0, n^k_0], \ldots, [k, m^k_{\alpha_k - 1}, n^k_{\alpha_k - 1}] \mid k \in \delta\alpha\},$$

where $\alpha(k) = \alpha_k$, $\varphi(k) = [m^k_0, \ldots, m^k_{\alpha_k - 1}]$, $\psi(k) = [n^k_0, \ldots, n^k_{\alpha_k - 1}]$, and $m^k_i < m^k_j$ if $i < j < \alpha_k$.

The **proof** is similar to the proof of Lemma 1.5. □

This, together with Corollary 2, entails

COROLLARY 2.3. Let functions α, φ, and ψ be Σ-definable in $\mathbb{HF}(G)$, $G \rightleftharpoons G(\alpha, \varphi, \psi)$. Then $\mathbb{HF}(G)$ contains a universal Σ-function if and only if a principal e-ideal $\mathcal{I}_e(S)$ generated by a set S contains a function that is universal for the family of all unary functions in $\mathcal{I}_e(S)$.

As in the case of groups, we have

COROLLARY 2.4. Let α, φ, and ψ be partial computable functions. Then $\mathbb{HF}(K)$, $G \rightleftharpoons K(\alpha, \varphi, \psi)$, contains a universal Σ-function.

COROLLARY 2.5. There exists a set S of primes such that (K_S), $K_S = \oplus\{F_p \mid p \in S\}$, contains no universal Σ-function.

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