Algebra and Logic, Vol. 50, No. 5, November, 2011
(Russian Original Vol. 50, No. 5, September-October, 2011)

A MAL'TSEV BASIS FOR A PARTIALLY COMMUTATIVE NILPOTENT METABELIAN GROUP

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UDC 512.5

Keywords: *partially commutative nilpotent metabelian group, variety, Mal'tsev basis.*

We find a canonical representation for elements of a partially commutative group in a variety of soluble groups of derived length two and nilpotency class at most $c \geqslant 1$.

1. PRELIMINARY INFORMATION AND THE NOTATION

The objective of the paper is to find a canonical representation for elements of a partially commutative group in a variety of soluble groups of derived length two and nilpotency class at most $c \geqslant 1$.

We start by introducing some necessary definitions and designations. As usual, for elements x and y of a group G, their commutator $x^{-1}y^{-1}xy$ is denoted by $[x, y]$. For $n \geqslant 3$, we put

$$[x_1, x_2, \ldots, x_n] = [[x_1, x_2, \ldots, x_{n-1}], x_n].$$

Denote by $\mathfrak{A}^2$ a variety of all metabelian groups, i.e., all groups satisfying an identity $[[x, y], [z, v]] = 1$. The lower central series $G = \gamma_1(G) \geq \gamma_2(G) \geq \ldots$ of a group G is given by the rule $\gamma_{i+1}(G) = [\gamma_i(G), G]$. A variety $\mathfrak{N}_c$ of nilpotent groups of nilpotency class at most c, $c \geqslant 1$, consists of all groups G for which $\gamma_{c+1}(G) = 1$.

Hereinafter, Γ is a finite undirected graph without loops, whose vertex set $\{x_1, \ldots, x_r\}$ is denoted by X. If vertices x_i and x_j are adjacent in Γ then we write $(x_i, x_j) \in \Gamma$.

For any variety $\mathfrak{M}$ of groups and for the graph Γ, a partially commutative group $F(\mathfrak{M}, \Gamma)$ is defined as follows: the generating set of $F(\mathfrak{M}, \Gamma)$ coincides with the vertex set X of Γ, and defining

*Supported by RFBR, project No. 09-01-00099.

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Translated from *Algebra i Logika*, Vol. 50, No. 5, pp. 647-658, September-October, 2011. Original article submitted December 3, 2010; revised February 25, 2011.

relations are of the form $x_i x_j = x_j x_i$ if x_i and x_j adjacent vertices of Γ. In addition, the group $F(\mathfrak{M}, \Gamma)$ belongs to the variety $\mathfrak{M}$. Thus $F(\mathfrak{M}, \Gamma)$ is represented as

$$F(\mathfrak{M}, \Gamma) = \langle X \mid x_i x_j = x_j x_i \Longleftrightarrow (x_i, x_j) \in \Gamma;\ \mathfrak{M} \rangle \tag{1}$$

in $\mathfrak{M}$. The graph Γ is said to be *defining* for the group $F(\mathfrak{M}, \Gamma)$. For convenience, we denote the group $F(\mathfrak{A}^2, \Gamma)$ by S_Γ, and the $F(\mathfrak{N}_c \wedge \mathfrak{A}^2, \Gamma)$ by $G_{\Gamma,c}$.

Having a convenient canonical representation of elements is useful in studying properties of a group. A representation of elements for a partially commutative group $F(\mathfrak{M}, \Gamma)$ defined by representation (1) in the variety of all groups was specified in [1], where it is underpinned by the idea of expressing elements of the group in terms of a product of mutually commuting blocks. A handy canonical representation for suitable degrees of elements in the commutator subgroup of a partially commutative metabelian group S_Γ can be found in [2], in which elements of the ring $\mathbb{Z}(S_\Gamma / S'_\Gamma)$ are treated as degree exponents. The representation in [2] made it possible to obtain a number of helpful properties for S_Γ and its universal theory.

For elements of a torsion-free finitely generated nilpotent group, a canonical representation derives by reason of the fact that such a group has a Mal'tsev basis. Recall the definition of a Mal'tsev basis.

Let G be a torsion-free finitely generated nilpotent group. We know that G has a central series of the form

$$G = G_1 > G_2 > \ldots > G_{s+1} = 1$$

with infinite cyclic factors (see [3]). Take elements $a_1, \ldots, a_s$ satisfying $G_i = \mathrm{gp}\langle a_i, G_{i+1} \rangle$. An ordered system $\{a_1, \ldots, a_s\}$ of elements is called a *Mal'tsev basis* for G. Every element $g \in G$ is uniquely represented as

$$g = a_1^{t_1} \ldots a_s^{t_s}, \quad t_i \in \mathbb{Z}.$$

We prove that a partially commutative group $G_{\Gamma,c}$ has a Mal'tsev basis, which can be obtained by refining the lower central series of $G_{\Gamma,c}$. This is equivalent to being torsion free for factors in the lower central series of $G_{\Gamma,c}$.

Despite the fact that partially commutative metabelian groups S_Γ, being approximated by torsion-free nilpotent groups, do not contain elements of finite order [2], it is not obvious that $G_{\Gamma,c}$ lacks elements of finite order. In fact, it is easy to point out a torsion-free metabelian group G for which the quotient $G/\gamma_2(G)$ contains elements of finite order. Such is, for instance, a metabelian group generated by two elements $\{x, y\}$ and defined by one relation $x^2[x, y] = 1$.

It is well known that for any group G and for elements $u_1, u_2 \in \gamma_n(G)$ and $v \in \gamma_m(G)$, the following congruence holds:

$$[u_1 u_2, v] \equiv [u_1,\, v][u_2, v] \ (\mathrm{mod} \gamma_{n+m+1}(G)) \tag{2}$$

(see, e.g., [4]).

Denote by G_c a free group in the variety $\mathfrak{N}_c \wedge \mathfrak{A}^2$, and by $X = \{x_1, \ldots, x_r\}$ its basis. For any $c \geqslant 1$, a set B_c of commutators is defined by induction. Put $B_1 = X$. For $c \geqslant 2$, the set B_c consists of all commutators w of weight c having the form

$$w = [x_i, x_j, x_{j_1}, \ldots, x_{j_{c-2}}], \tag{3}$$

where $1 \leqslant j < i \leqslant r$, $j \leqslant j_1 \leqslant \ldots \leqslant j_{c-2}$.

PROPOSITION 1 [5]. For $c \geqslant 1$, elements of B_c constitute a basis for a free Abelian group $\gamma_c(G_c)$.

On a set X, we introduce the following order:

$$x_1 < x_2 < \ldots < x_r.$$

Denote by $w(n)$ the nth letter in a commutator of form (3), with $1 \leqslant n \leqslant c$. The order above is extended to a set $B = \bigcup_{c=1}^{\infty} B_c$ of commutators as follows:

(1) for $u, v \in B_c$, put $u > v$ if $u(1) = v(1), \ldots, u(n-1) = v(n-1)$, with $u(n) > v(n)$;

(2) for $u \in B_p$ and $v \in B_q$, put $u > v$ whenever $p > q$.

Proposition 1 gives rise to a known result on a canonical representation of elements for the group G_c.

PROPOSITION 2. The set

$$\bigcup_{m=1}^{c} B_m$$

on which the order is defined as above is a Mal'tsev basis for G_c obtained by refining the lower central series of G_c.

2. CANONICAL REPRESENTATION OF ELEMENTS FOR A PARTIALLY COMMUTATIVE NILPOTENT METABELIAN GROUP

Let $\mathfrak{M}$ be some variety of groups. It is known that for any groups G_λ, $\lambda \in \Lambda$, $\mathfrak{M}$ contains a group G, which is called an $\mathfrak{M}$-*product*, or *verbal product*, of G_λ. An $\mathfrak{M}$-product G of groups G_λ is defined by setting

$$G = \mathfrak{M} \prod G_\lambda.$$

The group G contains subgroups isomorphic to G_λ and possesses the following property:

for an arbitrary group H in a variety $\mathfrak{M}$ and for any homomorphisms θ_λ of groups G_λ in H, there exists a homomorphism of the group $\mathfrak{M} \prod G_\lambda$ into H such that its restriction to the component G_λ coincides with θ_λ for every λ.

Obviously, an $\mathfrak{M}$-product of $\mathfrak{M}$-free cyclic groups $\mathrm{gp}\langle g_\lambda \rangle$, $\lambda \in \Lambda$, is an $\mathfrak{M}$-free group of rank $|\Lambda|$ freely generated by a set $\{g_\lambda,\ \lambda \in \Lambda\}$.

The definitions and properties of a verbal product readily imply the following:

PROPOSITION 3. Let $G_i = F(\mathfrak{M}, \Gamma_i)$, $1 \leqslant i \leqslant n$, be partially commutative groups in some variety $\mathfrak{M}$, whose defining graphs Γ_i are disjoint, i.e., $\Gamma = \bigsqcup \Gamma_i$. Then a partially commutative group $F(\mathfrak{M}, \Gamma)$ is isomorphic to an $\mathfrak{M}$-product of groups G_i.

LEMMA 1. Let $\mathfrak{M}$ be some variety of groups, Γ a graph, and $F(\mathfrak{M}, \Gamma)$ a partially commutative group in $\mathfrak{M}$. Suppose that $\Gamma_1, \ldots, \Gamma_m$ are all connected components of the graph Γ, and for $j = 1, \ldots, m$, elements x_{i_j} sit in Γ_j. Then a group generated by elements in $\{x_{i_1}, \ldots, x_{i_m}\}$ is free in the variety $\mathfrak{M}$.

Proof. Assume $v(x_{i_1}, \ldots, x_{i_m}) = 1$ is a certain relation between the elements $x_{i_1}, \ldots, x_{i_m}$. We need to prove that $v(x_{i_1}, \ldots, x_{i_m})$ is an identity in $\mathfrak{M}$.

To defining relations for a group $F(\mathfrak{M}, \Gamma)$ we add the commutativity relation $[x_p, x_q] = 1$ for all vertices x_p and x_q sitting in one connected component. Denote the resulting partially commutative group by $\overline{F}$. In view of Proposition 3, $\overline{F}$ is isomorphic to an $\mathfrak{M}$-product of $\mathfrak{M}$-free Abelian groups A_j. There exists a natural homomorphism of $F(\mathfrak{M}, \Gamma)$ onto $\overline{F}$.

Consider a retraction of a group A_j onto a cyclic group $\mathrm{gp}\langle x_{i_j} \rangle$. Extend such a retraction to a homomorphism of the group $\overline{F}$ onto the group F, which is an $\mathfrak{M}$-product of groups $\langle x_{i_j} \rangle$, $j = 1, \ldots, m$. As noted, the group F is free in $\mathfrak{M}$. We have thus obtained a homomorphism of the group $F(\mathfrak{M}, \Gamma)$ onto F under which the element $v(x_{i_1}, \ldots, x_{i_m})$, on the one hand, is mapped to itself, and on the other hand, its image is equal to the identity element. Since $x_{i_1}, \ldots, x_{i_m}$ are free generators for F, $v(x_{i_1}, \ldots, x_{i_m})$ will be an identity in the variety given. The lemma is proved.

LEMMA 2. Let

$$v = [x_{j_1}, x_{j_2}, \ldots, x_{j_c}] \tag{4}$$

be some commutator. If x_{j_1} and x_{j_m} are distinct adjacent vertices in Γ, then a commutator v' obtained by permuting x_{j_1} and x_{j_m} in v is equal to v in the group $G_{\Gamma, c}$.

Proof. Suppose $m = 2$. Then both commutators v and v' are equal to the identity element in $G_{\Gamma, c}$.

Let $m > 2$. Every metabelian group satisfies the Jacobi identity

$$[x, y, z][y, z, x][z, x, y] = 1.$$

In every metabelian group G, as is known, for any permutation a on a set $\{3, \ldots, n\}$,

$$[g_1, g_2, g_3, \ldots, g_n] = [g_1, g_2, g_{a(3)}, \ldots, g_{a(n)}].$$

Interchanging x_{j_3} and x_{j_m} in representation (4) yields a commutator of the form

$$[x_{j_1}, x_{j_2}, x_{j_m}, \ldots], \tag{5}$$

which is equal to v in the group $G_{\Gamma,c}$.

Applying the Jacobi identity to elements x_{j_1}, x_{j_2}, and x_{j_m} gives

$$[x_{j_1}, x_{j_2}, x_{j_m}][x_{j_2}, x_{j_m}, x_{j_1}][x_{j_m}, x_{j_1}, x_{j_2}] = 1.$$

Since $[x_{j_1}, x_{j_m}] = 1$, we have

$$[x_{j_1}, x_{j_2}, x_{j_m}] = [x_{j_2}, x_{j_m}, x_{j_1}]^{-1}.$$

Therefore,

$$[x_{j_m}, x_{j_2}, x_{j_1}] = [[x_{j_2}, x_{j_m}]^{-1}, x_{j_1}] = [x_{j_2}, x_{j_m}, x_{j_1}]^{-1}. \tag{6}$$

From (5) and (6), we derive

$$v \equiv [x_{j_m}, x_{j_2}, x_{j_1}, \ldots] \pmod{\gamma_{c+1}(G_{\Gamma,c})}. \tag{7}$$

We have $\gamma_{c+1}(G_{\Gamma,c}) = 1$. Therefore, if we move x_{j_1} to the mth place in (7) we obtain the statement of the lemma.

The set of vertices of Γ which do in fact occur in representation (3) for a commutator w is called the content of w and is denoted by $\sigma(w)$. By Δ_w we denote a subgraph of Γ generated by the vertex set $\sigma(w)$.

COROLLARY 1. Suppose that a commutator v has form (4) and vertices x_{j_1}, x_{j_m}, $2 \leqslant m \leqslant c$, sit in one connected component of the graph Δ_v. Then a commutator v' obtained by permuting x_{j_1} and x_{j_m} in v is equal to v in the group $G_{\Gamma,c}$.

Proof. Let $\{x_{j_1}, \ldots, x_{j_m}\}$ be some path between the vertices x_{j_1} and x_{j_m} in the graph Δ_v. That a permutation of neighboring vertices does not change the commutator v follows from Lemma 2. Therefore, $v = v'$ in $G_{\Gamma,c}$.

COROLLARY 2. A commutator v of form (4) is equal to the identity element in the group $G_{\Gamma,c}$ if and only if vertices x_{j_1} and x_{j_2} sit in one connected component of the graph Δ_v.

Proof. Suppose

$$[x_{j_1}, x_{j_2}, \ldots, x_{j_c}] = 1, \tag{8}$$

and elements x_{j_1} and x_{j_2} belong to distinct connected components Δ_1 and Δ_2 of the graph Δ_v. In each connected component, we fix one vertex, with the vertex x_{j_1} fixed in Δ_1 and the vertex x_{j_2} fixed in Δ_2. Note that x_{j_2} is the least vertex in the graph $\Delta_{[x_{j_1}, x_{j_2}, \ldots, x_{j_c}]}$. Consider an endomorphism φ of the group $G_{\Gamma,c}$ onto the subgroup H generated by the fixed vertices, under which vertices $x_l \in \Delta_s$ are mapped to fixed vertices of Δ_s. Apply this endomorphism to (8). By Proposition 4, H is a free group in the variety $\mathfrak{A}^2 \wedge \mathfrak{N}_c$. The image $[x_{j_1}, x_{j_2}, \varphi(x_{j_3}), \ldots, \varphi(x_{j_c})]$ of an element $[x_{j_1}, x_{j_2}, \ldots, x_{j_c}]$ in H is equal to some element of B_c, which is obtained via a suitable permutation of letters in the element $[x_{j_1}, x_{j_2}, \varphi(x_{j_3}), \ldots, \varphi(x_{j_c})]$, starting with the third. In view of Proposition 1, commutators in B_c are not equal to the identity element. Therefore, equality (8) is impossible.

Let vertices x_{j_1} and x_{j_2} belong to one connected component of the graph Δ_v. Consider some path $\{x_{j_1}, x_i, \ldots, x_j, x_{j_2}\}$ between the two vertices. Interchanging x_{j_1} and x_j in the commutator

$[x_{j_1}, x_{j_2}, \dots, x_{j_c}]$ yields (by Cor. 1) a commutator that is equal to $[x_{j_1}, x_{j_2}, \dots, x_{j_c}]$, and at the same time, is equal to the identity element. The corollary is proved.

For $l = 1, \dots, r$, we denote by $l(w)$ the number of elements x_l occurring in representation (3) for a commutator w and call it the *multiplicity* of the element x_l in the commutator w. An ordered tuple $\pi(w) = (1(w), \dots, r(w))$ is called a *tuple of multiplicities* of a commutator w.

LEMMA 3. Let $W = \{w_t \mid t \in T\}$ be a set of commutators of weight c with equal content $\sigma(w_t) \subseteq X$. If there exists a nontrivial dependence between elements of W in the Abelian group $\gamma_c(G_{\Gamma, c})$, then a nontrivial dependence exists also between elements having equal tuples of multiplicities in W.

Proof. Let $v = [x_{j_1}, \dots, x_{j_c}]$ be an element of W. Formula (2) implies that for any integers $l_1, \dots, l_c$ in the group $G_{\Gamma,c}$,

$$[x_{j_1}^{l_1}, \dots, x_{j_c}^{l_c}] = [x_{j_1}, \dots, x_{j_c}]^{l_1 \cdot \ldots \cdot l_c}.$$

Suppose that between elements w_t there exists a nontrivial dependence like

$$\prod_{t \in T} w_t^{\beta_t} = 1, \tag{9}$$

where $\beta_t \in \mathbb{Z}$.

For any $l_i \in \mathbb{Z}$, a mapping of the form

$$x_i \longrightarrow x_i^{l_i}, \ i = 1, \dots, r, \tag{10}$$

extends to an endomorphism of the group $G_{\Gamma,c}$. We apply this endomorphism to (9). For $l_i \neq 0$, we obtain a new nontrivial relation between commutators w_t. In this event the exponents β_t of the elements w_t having equal tuples $\pi(w_t)$ are multiplied by equal numbers. Obviously, if we choose different values for $l_1, \dots, l_r$ we face a nontrivial dependence between elements having equal tuples of multiplicities in W. The lemma is proved.

In a similar way, we can prove the following:

LEMMA 4. Let $W = \{w_t \mid t \in T\}$ be a set of commutators of weight c. If there exists a nontrivial dependence between elements of W in the Abelian group $\gamma_c(G_{\Gamma, c})$, then a nontrivial dependence exists also between elements having equal contents in W.

Now we define subsets B'_c of B_c for all $c \geqslant 1$.

Let $B'_1 = B_1 = X$. The set B'_2 is obtained from B_2 by removing commutators $[x_i, x_j]$ equal to the identity element in the group $G_{\Gamma,2}$, i.e., those commutators for which $(x_i, x_j) \in \Gamma$.

Let $c \geqslant 3$. Two commutators

$$w = [x_i, x_j, \dots], \quad w' = [x_{i'}, x_j, \dots]$$

in B_c with equal contents (written $\sigma(w) = \sigma(w')$) and equal tuples of multiplicities (written $\pi(w) = \pi(w')$) are said to be *equivalent* if vertices x_i and $x_{i'}$ sit in one connected component of the graph Δ_w. An equivalence class containing a commutator w is denoted by $[w]$.

It is worth observing that all commutators in one equivalence class $[w]$ have equal images in the group $G_{\Gamma,c}$. Furthermore, for all commutators $w_p \in [w]$, vertices $w_p(2)$ are equal and are equipped with a least number in $\sigma(w_p)$.

In an equivalence class $[w]$, we choose a greatest commutator with respect to the order $<$ defined above. Commutators in B_c that are greatest in their equivalence class are referred to as highest. We drop from B_c all nonhighest commutators, and also commutators $v = [x_l, x_j, \ldots]$ for which vertices x_l and x_j belong to one connected component of the graph Δ_v. In other words, by Corollary 2, along with nonhighest commutators, we exclude those that are equal to the identity element in $G_{\Gamma,c}$. Denote the remaining set of commutators by B'_c. Let

$$\overline{B}_c = \bigcup_{m=1}^{c} B'_m.$$

We have

THEOREM. Elements of $\overline{B}_c$ constitute a Mal'tsev basis for the group $G_{\Gamma,c}$.

Proof. The statement of the theorem is equivalent to asserting that elements of B'_c freely generate Abelian groups $\gamma_c(G_{\Gamma,c})$ for any $c \geqslant 1$, which we will prove below.

For any element $g \in \gamma_c(G_{\Gamma,c})$, we choose its preimage h in the group $\gamma_c(G_c)$ under the natural homomorphism $G_c \longrightarrow G_{\Gamma,c}$. By Proposition 1, the element h can be expressed via elements of B_c. Every element of B_c is mapped, under $G_c \longrightarrow G_{\Gamma,c}$, to a commutator equal to the identity element, or, by Corollary 1, to a commutator equal to an element of B'_c. Therefore, elements of B'_c generate a group $\gamma_c(G_{\Gamma,c})$.

Suppose that there exists a nontrivial dependence between elements of B'_c in the group $\gamma_c(G_{\Gamma,c})$. In view of Lemmas 3 and 4, therefore, a nontrivial dependence exists also between elements of B'_c with equal contents σ and equal tuples π of multiplicities.

Let w_p, $p \in P$, be the set of elements of B'_c having equal contents σ and equal tuples π of multiplicities, and

$$\prod_{p \in P} w_p^{\alpha_p} = 1, \tag{11}$$

where $\alpha_p \in \mathbb{Z}$ is a nontrivial dependence. Denote by $\triangle$ a subgraph of Γ generated by a vertex set σ. Every element w_p, $p \in P$, has the form

$$w_p = [x_{j_p}, x_j, x_{i_3}, \ldots, x_{i_c}],$$

where $x_{j_p} = w_p(1)$ is the greatest vertex in the connected component of $\triangle$ containing that vertex.

Different commutators w_p, $p \in P$, have different first elements $w_p(1)$. Furthermore, the vertex x_j is least among all vertices of the graph $\triangle$. There is no loss of generality in assuming that $j = 1$.

The definition of a set B'_c implies that vertices $Y = \{x_1, w_p(1), p \in P\}$ belong to distinct connected components of $\triangle$. Fix one vertex in each connected component of the graph $\triangle$. In the connected components containing the vertices in Y, we fix just these. Consider a subgraph $\overline{\triangle}$ of $\triangle$

generated by the fixed vertices. The graph $\overline{\triangle}$ is totally disconnected. In view of Lemma 1, a group generated by vertices of $\overline{\triangle}$ is a free group in the variety $\mathfrak{A}^2 \wedge \mathfrak{N}_c$, which we denote by G.

Consider a retraction of the group $G_{\Gamma,\,c}$ onto the group $G_{\triangle,\,c}$, under which all vertices in σ are mapped to themselves, while the remaining vertices are mapped to identity elements. Also consider an endomorphism of the group $G_{\triangle,c}$ onto G, under which all vertices in a connected component of $\triangle$ are mapped to a vertex fixed in that component.

We have thus obtained a homomorphism ψ of the group $G_{\Gamma,c}$ onto the group G under which all fixed vertices are left invariant. Images of different commutators w_p in B'_c under the homomorphism ψ are different commutators $\psi(w_p)$, since $\psi(w_p)(1) = w_p(1)$, $p \in P$, and $\psi(w_p)(2) = x_1$.

Interchanging, if necessary, letters in commutators $\psi(w_p)$ starting with the third, we arrive at different commutators constituting a basis B_c for the subgroup $\gamma_c(G)$. Since G is free in $\mathfrak{A}^2 \wedge \mathfrak{N}_c$, dependence (12) ought to be trivial. The proof is completed.

A consequence of the theorem is the following:

COROLLARY 3. Let S_Γ be a partially commutative metabelian group. Then quotient groups $S_\Gamma/\gamma_c(S_\Gamma)$ do not contain elements of finite order for any $c \geqslant 2$.

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