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GENERALIZED STABILITY OF TORSION-FREE ABELIAN GROUPS

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It is shown that there exists an Abelian group that is not (P, a)*-stable.*

INTRODUCTION

Stable theories were introduced in [1] for constructing classification theory and are a generalization of the concept of a totally transcendental theory as defined in [2]. In [3], it was proved that the property of being stable for a theory is equivalent to being definable for every complete type of the theory. This property plays a fundamental part in research on stable theories.

In [4], the notion of E^*-stability (generalized stability) was introduced, and it was proved that types for E^*-stable theories are definable. A consequence of that result was stating, along with definability of types for stable theories (see [3]), that types over any P-sets in P-stable theories likewise are definable (which had been established in [5] for types over P-models). The notion of E^*-stability is a new stability scale, whose basic parameter is a mapping of types of a complete theory into types of another theory.

An interesting example of E^*-stability is (P, a)-stability, defined by adding to a language a unary predicate symbol and adding to types the condition of being algebraically closed for that predicate. In this paper, we work to prove a (P, a)-stability theorem for theories of torsion-free Abelian groups. In so doing, use is made of quantifier elimination down to positive primitive formulas in Abelian groups with a predicate distinguishing a subgroup. By virtue of this fact, the question of being (P, a)-stable for a theory reduces to asking if a system of linear equations with integer coefficients has a solution in the algebraic closure of constants involved in the system. To

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find such a solution, we have developed a technique which is based on the Gauss–Jordan method and takes into account the specific character of the situation in question.

In the light of the (P, a)-stability theorem, it is natural to ask if there exists an Abelian group that is not (P, a)-stable. This question will be answered in the affirmative at the end of the paper.

1. DEFINITIONS AND THE NOTATION

A good many definitions and the bulk of the notation are borrowed from [4]. We lay them out to make our discussion self-contained.

Finite sequences are called *tuples*. Denote the set of all tuples in A by $A^{<\omega}$, and the length of a tuple $\mathbf{s}$ by $l(\mathbf{s})$. Tuples of length n are called n*-tuples*. For simplicity, instead of $\mathbf{a} \in A^{<\omega}$ and $\mathbf{D} \subseteq A^{<\omega}$, we will often write $\mathbf{a} \in A$ and $\mathbf{D} \subseteq A$, respectively. If $\mathbf{a}$ is a tuple, then $\bigcup \mathbf{a}$ denotes a set consisting of elements of the tuple $\mathbf{a}$. If $\mathbf{D}$ is a set of tuples, then $\bigcup \mathbf{D}$ denotes a set consisting of all elements of the tuples in $\mathbf{D}$. For D a set and $\mathbf{a}$ a tuple, instead of $D \cup \bigcup \mathbf{a}$, we write merely $D \cup \mathbf{a}$.

Variables are denoted by lowercase letters, e.g., x, y, z, and sets of variables by the corresponding uppercase letters X, Y, Z. Tuples of variables are denoted by lowercase boldface letters, e.g., $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, and sets of tuples of variables by the corresponding uppercase boldface letters $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$. Unless specified otherwise, we will assume that variables that occur in different places of a same tuple are distinct.

Let L be a language and T a theory in L. A set of all L-formulas with free variables in a tuple $\mathbf{x}$ (in a set X) is denoted $F_{\mathbf{x}}(L)$ ($F_X(L)$). A set of all types in T over $\mathbf{x}$ (over X), i.e., subsets of $F_{\mathbf{x}}(L)$ ($F_X(L)$) consistent with T with respect to deducibility in T, is denoted $S^{\subseteq}_{\mathbf{x}}(T)$ ($S^{\subseteq}_X(T)$). The subsets $S^{\subseteq}_{\mathbf{x}}(T)$ and $S^{\subseteq}_X(T)$, consisting of maximal T-consistent sets of L-formulas (complete types of T) whose free variables are in $\mathbf{x}$ and in X, are denoted $S_{\mathbf{x}}(T)$ and $S_X(T)$, respectively.

If a theory T is trivial, i.e., consists of identically true sentences, then we write $S^{\subseteq}_{\mathbf{x}}(L)$, $S^{\subseteq}_X(L)$, $S_{\mathbf{x}}(L)$, and $S_X(L)$ in place of $S^{\subseteq}_{\mathbf{x}}(T)$, $S^{\subseteq}_X(T)$, $S_{\mathbf{x}}(T)$, and $S_X(T)$, respectively. In what follows, $t(X)$ ($t(\mathbf{X})$) denotes a type $t \in S_X(T)$ ($t \in S_{\bigcup \mathbf{X}}(T)$). For a formula Φ, by writing $\Phi(X)$ ($\Phi(\mathbf{X})$) we mean that free variables in Φ belong to a set X (tuples in $\mathbf{X}$). For $\mathbf{X}$ and $\mathbf{Y}$ sets of tuples of variables, t a type, and Φ a formula, we use the expressions $t(\mathbf{X}; \mathbf{Y})$ and $\Phi(\mathbf{X}; \mathbf{Y})$ instead of $t(\mathbf{X} \cup \mathbf{Y})$ and $\Phi(\mathbf{X} \cup \mathbf{Y})$. If $t \in S^{\subseteq}_X(T)$ and Y is a set of variables (of tuples of variables) in X, then $t \restriction Y$ denotes a type consisting of formulas in t, whose free variables are in Y (in tuples of Y).

For our purposes, it will be more convenient if the cardinality of a set of (object) variables is not bounded in advance. We assume that all bound variables in formulas are taken from a fixed countable set $U = \{u_i \mid i \in \omega\}$, free variables are not in that set, and unless otherwise stated, variables have no occurrences in U. Deducibility is treated in predicate calculus, with the above conditions of separation on bound and free variables.

Fix a countable set of variables $V = \{v_i \mid i \in \omega\}$. Denote by $S_n(T)$ a set $S_{\mathbf{v}}(T)$, where $\mathbf{v} = \langle v_0, \ldots, v_{n-1} \rangle$. Let $S_\omega(T) = \bigcup \{S_{\mathbf{v}}(T) \mid \mathbf{v} \in V\}$. A set of L-formulas with variables in the tuple $\mathbf{v} = \langle v_0, \ldots, v_{n-1}\rangle$ is denoted by $F_n(L)$. If w is a mapping from a set X to a set Y then, for a formula $\Phi \in F_X(L)$, $w(\Phi)$ denotes a formula obtained by replacing every free variable x in Φ by a variable $w(x)$. For a set of formulas F, put $w(F) = \{w(\Phi) \mid \Phi \in F\}$. Along with $w(F)$, we will also write $(F)^X_{w(X)}$. If Δ is a formula or set of formulas, $\mathbf{x}$ is a tuple of variables, and $\mathbf{s}$ is a tuple of terms of the same length, then $(\Delta)^{\mathbf{x}}_{\mathbf{s}}$ denotes the result of replacing in each formula $\Phi \in \Delta$ all free occurrences of variables in $\mathbf{x}$ by respective terms in $\mathbf{s}$. For a formula $\Phi(\mathbf{x})$, we write $\Phi(\mathbf{s})$ in place of $(\Phi)^{\mathbf{x}}_{\mathbf{s}}$.

Definition. Let languages L and L^* and a complete theory T in L be given. A mapping $E : S_\omega(T) \to S^{\subseteq}_\omega(L^*)$ is a *representation of types for T in L^** if the following conditions hold:

(1) *abstractness*, i.e., if w is a permutation on a set V of variables, then $E(w(t)) = w(E(t))$ for any $t \in S_\omega(T)$;

(2) *preservation of equality*, i.e., if $t \in S_\omega(T)$, $x, y \in V$, and $x = y \in t$, then $x = y \in E(t)$;

(3) *conservatism*, i.e., if $t \in S_n(T)$ and $t \subseteq t' \in S_\omega(T)$, then $E(t) = (E(t') \cap F_n(L^*))$.

(4) *continuity*, i.e., if $t \in S_\omega(T)$ and $\varphi \in E(t)$, then there exists a formula $\Phi \in t$ such that $\varphi \in E(t')$ for any $t' \in S_\omega(T)$ with $\Phi \in t'$.

Below a representation of types for T in L^* is denoted by E^*.

Definition. The mapping E^* is extended to types in any set of variables as follows.

(a) If $\mathbf{x}$ is an arbitrary n-tuple of variables, and $t \in S_{\mathbf{x}}(T)$, then we put $E^*(t) = (E^*((t)^{\mathbf{x}}_{\mathbf{v}})^{\mathbf{v}}_{\mathbf{x}}$, where $\mathbf{v} = \langle v_0, \ldots, v_{n-1}\rangle$.

(b) If X is an arbitrary set of variables, and $t \in S_X(T)$, then we put $E^*(t) = \bigcup\{E^*(t \restriction \mathbf{x}) \mid \mathbf{x} \in X\}$.

That the last definition is sound (i.e., the mapping in question satisfies the conditions of being abstract, conservative, continuous, and equality preserving and the image of a type $t \in S_X(T)$ under such a mapping is a type in $S^{\subseteq}_X(L^*)$) was shown in [4].

Definition. Let E^* be a representation of types for a complete theory T. We say that T is *E^*-stable in cardinality* λ if, for any set X with $|X| \leqslant \lambda$ and for every $t \in S_X(T)$, a type $E^*(t)$ has at most λ completions of a set $S_X(L^*)$. A theory T is said to be *E^*-stable* if it is E^*-stable in some infinite cardinality λ.

Definition. Let L be some language, X a set of variables, and $t \in S^{\subseteq}_X(L)$. Suppose that $\mathbf{X}$ and $\mathbf{Y}$ are some sets of tuples of variables of length n in X. We say that a pair $\langle \mathbf{X}, \mathbf{Y}\rangle$ is *separable* in a type t over X if there exists an L-formula $\Phi(\mathbf{z}; \mathbf{x}^0)$ such that $l(\mathbf{z}) = n$, $\mathbf{x}^0 \in X$, and a set of formulas like

$$\{\Phi(\mathbf{x}; \mathbf{x}^0) \mid \mathbf{x} \in \mathbf{X}\} \cup \{\neg\Phi(\mathbf{x}; \mathbf{x}^0) \mid \mathbf{x} \in \mathbf{Y}\}$$

is consistent with the type t. In this event the formula $\Phi(\mathbf{z}; \mathbf{x}^0)$ *separates* $\mathbf{X}$ from $\mathbf{Y}$ in $t(X)$.

Below we need the following theorem, which generalizes the definability theorem for types to the case of E^*-stable theories.

THEOREM 1.1 [4]. Let T be a complete theory in a language L and E^* a representation of types for T in L^*. Then the following conditions hold:

(1) T is E^*-stable;

(2) for any set X and for an arbitrary complete type $t \in S_X(T)$, every pair $\langle \mathbf{X}, \mathbf{Y} \rangle$ of sets of tuples of variables of equal length in X which is separable in a type $E^*(t)$ over X is separable in t over X.

Let a language L be given. Then we denote an extension of L by a unary predicate by $L^P = L \cup \{P(x)\}$, an extension of L by a set $C = \{c_\alpha \mid \alpha \in \lambda\}$ of constants in some cardinality λ by $L_c = L \cup C$, and an extension of L by a unary predicate and a set of constants by $L_c^P = L_c \cup L^P$.

For a complete theory T in a language L, we define a representation of types in a language L^P. Let $E^{(P,a)}(t(X))$ be the closure with respect to deducibility of a set of formulas which includes all formulas of t, formulas of the form $P(x)$ for $x \in X$, and a set of formulas saying that a set of solutions for a predicate P is algebraically closed relative to L. Clearly, $E^{(P,a)}$ is a representation of types.

Definition. A theory T is (P,a)-*stable* if it is $E^{(P,a)}$-stable.

In what follows, as a language L we take a group language containing one binary function symbol $+$, a unary function symbol $-$, and a constant 0. In dealing with Abelian groups (in L), we will not parenthesize terms in representations, which is possible due to the associative property of addition. Terms of the form $\underbrace{t + \ldots + t}_{n \text{ times}}$ are denoted $n \cdot t$, where n is a natural number and t is a term in L. Terms like $\underbrace{(-t) + \ldots + (-t)}_{n \text{ times}}$ are denoted $-n \cdot t$, where n is a natural number and t is a term in L. Combining the two designations, we will employ coefficients from a set of integers in term representations. It is worth observing that every formula encountered in the paper in which the designations mentioned are involved can also be written conventionally, by using only symbols of the language L. We use these designations merely for simplicity.

2. (P,a)-STABILITY OF TORSION-FREE ABELIAN GROUPS

LEMMA 2.1. Let $H_0, \ldots, H_n$ be subgroups of some Abelian group A, $a_0, \ldots, a_n \in A$, $H_0 + a_0 \subseteq \bigcup\limits_{1 \leqslant i \leqslant n} (H_i + a_i)$, and $[H_0 \colon H_n] > n!$. Then $H_0 + a_0 \subseteq \bigcup\limits_{1 \leqslant i \leqslant n-1} (H_i + a_i)$.

Proof. See [6, Chap. 7, Sec. 39, Lemma 2]. □

LEMMA 2.2. Let $A_0, \ldots, A_n$ be finite sets. Then

$$A_0 \subseteq \bigcup_{1 \leqslant i \leqslant n} A_i \iff \sum_{r \subseteq \{1,\ldots,n\}} (-1)^{|r|} \left| A_0 \cap \bigcap_{i \in r} A_i \right| = 0.$$

Proof. See [6, Chap. 7, Sec. 39, Lemma 3]. □

Since a predicate P distinguishes a subgroup, proofs for Proposition 2.1 and Lemma 2.3 (see below) repeat word for word the proofs of Proposition 4 and Lemma 4 in [6, Chap. 7, Sec. 39]. Here we cite them to make our discussion self-contained.

PROPOSITION 2.1. Let $\Phi(x_1, \ldots, x_n)$ be a positive primitive formula in a language L^P and G a structure in L^P such that $G \upharpoonright L$ is an Abelian group and a predicate P distinguishes a subgroup. Then:

(a) $\Phi(x_1, \ldots, x_n)$ defines a subgroup in the Cartesian degree G^n of a group G;

(b) for any $a_l, \ldots, a_n \in G$ and for $l \geqslant 1$, a formula $\Phi(x_1, \ldots, x_{l-1}, a_l, \ldots, a_n)$ either fails in G or specifies in A^{l-1} a coset with respect to a subgroup defined in G^{l-1} by a positive primitive formula $\Phi(x_1, \ldots, x_{l-1}, 0, \ldots, 0)$.

Proof. Note that $G \models t(0, \ldots, 0) = 0$ and

$$t(a_1 + b_1, \ldots, a_n + b_n) = t(a_1, \ldots, a_n) + t(b_1, \ldots, b_n)$$

for any term $t(x_1, \ldots, x_n)$ in L and any $a_1, \ldots, a_n, b_1, \ldots, b_n \in G$. This, together with the fact that an intersection of two groups is again a group, yields the result required. □

LEMMA 2.3. Let G be a structure in a language L^P such that $G \upharpoonright L$ is an Abelian group and a predicate P distinguishes an algebraically closed subset. Then every formula $\Phi(x_1, \ldots, x_n)$ in L^P is equivalent in $\mathrm{Th}(G)$ to a Boolean combination $\Phi^*(x_1, \ldots, x_n)$ of positive primitive formulas.

The **proof** is by induction on the number of quantifiers. It suffices to consider the case $\Phi = \forall x(\Theta_0 \vee \ldots \vee \Theta_m)$, where Θ_i, $i \leqslant m$, are positive primitive formulas or their negations. A disjunction of negations of positive primitive formulas is equivalent to a negation of one positive primitive formula. Therefore, adding a formula $\neg\exists x = x$ if necessary, we may assume that $\Phi = \forall x(\neg\Phi_0 \vee \Phi_1 \ldots \vee \Phi_m)$, where Φ_i, $i \leqslant m$, are positive primitive formulas. A formula $\forall x \neg\Phi_0$ is equivalent to a negation of a positive primitive formula. Hence we may assume that $m > 0$. Thus we need only handle the case where

$$\Phi(x_0, \ldots, x_{n-1}) = \forall y \left(\Phi_0(x_0, \ldots, x_{n-1}, y) \to \bigvee_{0 < i \leqslant m} \Phi_i(x_0, \ldots, x_{n-1}, y) \right).$$

Let B_i, $i \leqslant m$, be subgroups of G defined by respective formulas $\Phi_i(0, \ldots, 0, y)$, $i \leqslant m$. In view of Lemma 2.1, we may assume that $[B_0 \colon B_i] \leqslant m!$, $0 < i \leqslant m$. Consequently, for $b_0, \ldots, b_{n-1} \in G$ and $\alpha \subseteq \{1, \ldots, m-1\}$, the positive primitive formula

$$\Phi_0(b_0, \ldots, b_{n-1}, y) \wedge \bigwedge_{i \in \alpha} \Phi_i(b_1, \ldots, b_{n-1}, y)$$

defines in G either the empty set or a set containing a finite number $n(\alpha) = \left[B_0 \cap \bigcap_{i \in \alpha} B_i \colon (B_0 \cap \ldots \cap B_m)\right]$ of cosets with respect to a subgroup $B_0 \cap \ldots \cap B_m$. Consider a set of the form

$$V = \left\{ S \;\middle|\; S \subseteq \mathcal{P}(\{1, \ldots, m\}), \sum_{\alpha \in S} (-1)^{|\alpha|} \cdot n(\alpha) = 0 \right\},$$

where $\mathcal{P}(Z)$ denotes the power set of Z. For any $S \subseteq \mathcal{P}(\{1,\dots,m\})$, we define a formula like

$$\Phi^S(x_1,\dots,x_n) = \left(\bigwedge_{\alpha\in S} \exists y \bigwedge_{i\in S\cup\{0\}} \Phi_i(x_1,\dots,x_n,y)\right)$$
$$\wedge \left(\bigwedge_{\alpha\in\mathcal{P}(\{1,\dots,m\})\setminus S} \neg\exists y \bigwedge_{i\in S\cup\{0\}} \Phi_i(x_1,\dots,x_n,y)\right).$$

By virtue of Lemma 2.2, $\Phi(x_0,\dots,x_{n-1})$ is equivalent in $\mathrm{Th}(G)$ to a formula $\bigvee_{S\in V} \Phi^S(x_0,\dots,x_{n-1})$. If in the formula $\bigvee_{S\in V} \Phi^S(x_0,\dots,x_{n-1})$ we replace formulas $\exists x \bigwedge_{i\in\alpha\cup\{0\}} \Phi_i$ by their positive primitive counterparts (by changing bound variables in Φ_i so that existential quantifiers are beyond the scope of conjunction) we obtain the desired formula Φ^*. $\Box$

PROPOSITION 2.2. Let G be a structure in a language L_c^P such that $G \upharpoonright L$ is a torsion-free Abelian group, a predicate P distinguishes an algebraically closed subset, and an algebraically closed set C of constants is realized by elements of P. Assume that $\Phi(x_0,\dots,x_{n-1})$ is a conjunction of atomic L_c-formulas, i.e., Φ is of the form $\bigwedge_{i<m}\left(\sum_{j<n} \alpha_{i,j}\cdot x_j = a_i\right)$, where $\alpha_{i,j}$ are integer coefficients and a_i are constants in C, with $i<m$ and $j<n$.

(a) An element $\mathbf{b}$ is a solution in G for a formula $\Phi(\mathbf{x})$ if and only if $\mathbf{b}$ is a solution in G for a formula $\Phi_1(\mathbf{x})$ obtained by permutations of conjunctive terms in Φ.

(b) If d is the greatest common divisor of coefficients $\alpha_{i,0},\dots,\alpha_{i,n-1}$, then d divides a_i, and moreover, $\mathbf{b}$ is a solution in G for a formula $\Phi(\mathbf{x})$ if and only if $\mathbf{b}$ is a solution in G for a formula $\Phi_2(\mathbf{x})$ obtained by replacing the ith conjunctive term in Φ by $\sum_{j<n} \alpha'_{i,j}\cdot x_j = a'_i$, where $\alpha_{i,j} = d\cdot\alpha'_{i,j}$, $j<n$, and $a_i = d\cdot a'_i$.

(c) An element $\mathbf{b}$ is a solution in G for a formula $\Phi(\mathbf{x})$ if and only if $\mathbf{b}$ is a solution in G for a formula $\Phi_3(\mathbf{x})$ obtained by replacing the i_1th conjunctive term in Φ by $\sum_{j<n}(\alpha_{i_1,j} + c\cdot\alpha_{i_2,j})\cdot x_j = a_{i_1} + c\cdot a_{i_2}$ for $i_1, i_2 < m$, where c is an integer.

Proof. (a) Is obvious.

(b) If we put $a'_i = \sum_{j<n} \alpha'_{i,j}\cdot b_j$ and substitute this expression into the ith conjunctive term of Φ we obtain $G \models d\cdot a'_i = a_i$; i.e., d divides a_i. Since G is a torsion-free group, the result of dividing a'_i is unique and does not depend on the choice of $\mathbf{b}$. Hence $a' \in \{c_\alpha \mid \alpha \in 2^\omega\}$, and for any $\mathbf{b}'$, which is a solution for Φ, it is true that $a'_i = \sum_{j<n} \alpha'_{i,j}\cdot b'_j$; i.e., $\mathbf{b}'$ is a solution for Φ_2. That any solution of Φ_2 is a solution for Φ is obvious.

(c) Let $\mathbf{b}$ be a solution of Φ; then $\sum_{j<n}(\alpha_{i_1,j})\cdot b_j = a_{i_1}$ and $\sum_{j<n}(\alpha_{i_2,j})\cdot b_j = a_{i_2}$. Substituting these values into the i_1th conjunctive term of Φ_3 yields an identity.

Let $\mathbf{b}$ be a solution for Φ_3; then $\sum_{j<n}(\alpha_{i_2,j})\cdot b_j = a_{i_2}$ and $\sum_{j<n}(\alpha_{i_1,j} + c\cdot\alpha_{i_2,j})\cdot b_j = a_{i_1} + c\cdot a_{i_2}$. Therefore, $\sum_{j<n}(\alpha_{i_1,j})\cdot b_j + c\cdot a_{i_2} = a_{i_1} + c\cdot a_{i_2}$, and hence $\mathbf{b}$ is a solution of Φ. $\Box$

PROPOSITION 2.3. Suppose that the conditions of Proposition 2.2 hold.

(a) If $\Phi_4(x_0, \ldots, x_{n-1}) = \Phi(x_0, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_{n-1})$, then $\mathbf{b}$ is a solution for Φ if and only if $\langle b_0, \ldots, b_{i-1}, b_j, b_{i+1}, \ldots, b_{j-1}, b_i, b_{j+1}, \ldots, b_{n-1}\rangle$ is one for Φ_4.

(b) If

$$\Phi_5(\mathbf{x}) = \bigwedge_{i<m} \left(\sum_{\substack{j<n \\ j \neq j_0}} (\alpha_{i,j} - \alpha_{i,j_0} \cdot \gamma_j) \cdot y_j + \alpha_{i,j_0} \cdot y_{j_0} = a_i \right),$$

where $j_0 < n$ and $\gamma_0, \ldots, \gamma_{n-1}$ are integers, then $\langle b_0, \ldots, b_{n-1}\rangle$ is a solution for a formula Φ if and only if

$$\left\langle b_0, \ldots, b_{j_0-1}, b_{j_0} + \sum_{\substack{j<n \\ j \neq j_0}} \gamma_j \cdot b_j, b_{j_0+1}, \ldots, b_{n-1} \right\rangle$$

is one for Φ_5, and $\langle b'_0, \ldots, b'_{n-1}\rangle$ is a solution for Φ_5 if and only if

$$\left\langle b'_0, \ldots, b'_{j_0-1}, b'_{j_0} - \sum_{\substack{j<n \\ j \neq j_0}} \gamma_j \cdot b'_j, b'_{j_0+1}, \ldots, b'_{n-1} \right\rangle$$

is one for Φ.

Proof. (a) Is obvious.

(b) Substituting, we verify that the formulas $\Phi(x_0, \ldots, x_{n-1})$ and

$$\Phi_5 \left(x_0, \ldots, x_{j_0-1}, x_{j_0} + \sum_{\substack{j<n \\ j \neq j_0}} \gamma_j \cdot x_j, x_{j_0+1}, \ldots, x_{n-1} \right)$$

coincide up to grouping terms. Similarly, we argue for $\Phi_5(y_0, \ldots, y_{n-1})$ and

$$\Phi \left(y_0, \ldots, y_{j_0-1}, y_{j_0} - \sum_{\substack{j<n \\ j \neq j_0}} \gamma_j \cdot y_j, y_{j_0+1}, \ldots, y_{n-1} \right). \ \square$$

LEMMA 2.4. Assume that the conditions of Proposition 2.2 hold. Then the formula $\left(\Phi(x_0, \ldots, x_{n-1}) \wedge \bigwedge_{i<n} P(x_n) \right)$ has a solution in G.

Proof. Appealing to axioms for Abelian groups and using the notation above, we reduce Φ to the form

$$\bigwedge_{i<m} \left(\sum_{j<n} \alpha_{i,j} \cdot x_j = a_i \right), \tag{1}$$

where $\alpha_{i,j}$ are coefficients in a set of integers and a_i are constants in the set C, with $i < m$ and $j < n$.

Stage 1. Using Proposition 2.2(b), we cancel the first equality by the greatest common divisor of coefficients $\alpha_{0,0}, \ldots, \alpha_{0,n-1}$. In this case the first equality assumes the form $\sum\limits_{j<n} \alpha'_{0,j} \cdot x_j = a'_0$.

Stage 2. In a set $\{\alpha'_{0,j} \mid j < n, \alpha'_{0,j} \neq 0\}$, we choose a coefficient α'_{0,j_0} that is least in modulus. If $\alpha_{0,j_0} = 1$ or $\alpha_{0,j_0} = -1$, then we go Stage 3. Otherwise, for $j < n$, $j \neq j_0$, we divide $\alpha'_{0,j}$ by α'_{0,j_0}, i.e., set

$$\alpha'_{0,j} = \gamma_j \cdot \alpha'_{0,j_0} + \beta_{0,j},$$

where $\gamma_0, \ldots, \gamma_{n-1}$ are integers and $\beta_{0,j}$ are positive integers, which are strictly less than α'_{0,j_0} in modulus, of which at least one is distinct from zero, since $\alpha'_{0,0}, \ldots, \alpha'_{0,n-1}$ are coprime, $j < n$, and $j \neq j_0$.

Put $\beta_{0,j_0} = \alpha'_{0,j_0}$, $b_0 = a'_0$, $\beta_{i,j} = \alpha_{i,j} - \alpha_{i,j_0} \cdot \gamma_j$, $\beta_{i,j_0} = \alpha_{i,j_0}$, and $b_i = a_i$ for $0 < i < m$, $j < n$, and $j \neq j_0$. In view of Proposition 2.3(b), substituting variables

$$y_j = x_j \text{ for } j \neq j_0,\ j < n,$$
$$y_{j_0} = x_{j_0} + \sum_{\substack{j<n \\ j \neq j_0}} (\gamma_j \cdot x_j),$$

yields a formula like

$$\bigwedge_{i<m} \left(\sum_{j<n} \beta_{i,j} \cdot y_j = b_i \right), \tag{2}$$

in which all coefficients in the first equality are not greater than α'_{0,j_0} in modulus.

Further, we repeat for (2) all operations at Stage 2 until one of the coefficients $\beta_{0,0}, \ldots, \beta_{0,n-1}$ becomes equal to one, and then go to the next stage.

Stage 3. By virtue of Proposition 2.3(a), we may assume that $\beta_{0,0} = 1$. In view of Proposition 2.2(c), we subtract the first equality from each $(i+1)$th equality $\beta_{i,0}$ times, respectively, with $0 < i < m$. The coefficients at y_0 vanish in so doing, and eventually, we arrive at the following system of equalitites:

$$\left(y_0 + \sum_{0<j<n} \beta_{0,j} \cdot y_j = b_i \right) \wedge \bigwedge_{0<i<m} \left(\sum_{0<j<n} (\beta_{i,j} - \beta_{0,j} \cdot \beta_{i,0}) \cdot y_j = b_i - \beta_{0,j} \cdot b_0 \right). \tag{3}$$

If all coefficients in the second to mth equations are equal to zero, then we go to Stage 4. Otherwise, with Proposition 2.2(a) in mind, we may assume that the second equation contains nonzero coefficients.

Further, we proceed by repeating Stages 1-3 for a system like

$$\bigwedge_{0<i<m} \left(\sum_{0<j<n} (\beta_{i,j} - \beta_{0,j} \cdot \beta_{i,0}) \cdot y_j = b_i - \beta_{0,j} \cdot b_0 \right),$$

with the stipulation that variable substitutions at Stage 2 are effected also in $y_0 + \sum\limits_{0<j<n} \beta_{0,j} \cdot y_j = b_i$ while not altering the coefficient at y_0 in so doing. Ultimately we obtain a system in which the

coefficient at y_1 is equal to one in the second equation, and to zero in subsequent ones. Thereafter, we repeat Stages 1-3 for the third to mth equations while preserving coefficients at y_0 and y_1 in the first two equations. Proceeding further with the process, we arrive at a system in upper triangular form

$$\bigwedge_{i<m_0}\left(z_i+\sum_{i<j<n}\delta_{i,j}\cdot z_j=c_i\right), \tag{4}$$

where $0<m_0\leqslant m$.

Note that the resulting system has no rows like $0=c_i$, where $c_i\neq 0$, $i\geqslant m_0$, since the initial system has a solution by hypothesis. In view of Propositions 2.2 and 2.3, therefore, system (4) must necessarily have a solution in G.

Stage 4. By virtue of Proposition 2.2(c), we subtract the last equation from each of the previous equations δ_{i,m_0-1} times, respectively, so that coefficients at z_{m_0-1} vanish.

Now we repeat Stage 4 for the first m_0-1 equations in the resulting system, removing in so doing coefficients at z_{m_0-2} and so on. Ultimately we obtain a formula of the form

$$\bigwedge_{i<m_0}\left(z_i+\sum_{m_0\leqslant j<n}\delta'_{i,j}\cdot z_j=c'_i\right). \tag{5}$$

It remains to observe that $z_i=c'_i$, for $i<m_0$, and $z_j=0$ for $m_0\leqslant j<n$, will be a solution for system (5), each element of which is a constant and, hence, lies in P. Formula (1) is derived from (5) by using a sequence of transformations such as in Props. 2.2 and 2.3. By applying inverse substitutions, therefore, we obtain a solution for Φ, lying in P. □

LEMMA 2.5. Let G be a structure in a language L_c^P such that $\langle G,+\rangle$ is a torsion-free Abelian group, a predicate P distinguishes an algebraically closed subset, and an algebraically closed set C of constants is realized by elements of P. Suppose $\Phi(x_0,\ldots,x_{n-1},y_0,\ldots,y_{m-1})$ is a conjunction of atomic L_c-formulas. Then a set of solutions for the formula

$$\Phi_1(\mathbf{x})\leftrightharpoons\exists y_0\ldots\exists y_{m-1}\left(\Phi(x_0,\ldots,x_{n-1},y_0,\ldots,y_{m-1})\wedge\bigwedge_{i<n}P(x_i)\wedge\bigwedge_{i<m}P(y_i)\right)$$

coincides in G with a set of solutions for the formula

$$\Phi_2(\mathbf{x})\leftrightharpoons\exists y_0\ldots\exists y_{m-1}\left(\Phi(x_0,\ldots,x_{n-1},y_0,\ldots,y_{m-1})\wedge\bigwedge_{i<n}P(x_i)\right).$$

Proof. Let $G\models\Phi_1(\mathbf{a})$. Then $G\models\Phi_2(\mathbf{a})$.

Suppose $G\models\Phi_2(\mathbf{a})$, i.e., $G\models P(a_i)$, $i<n$, and the formula $\Phi(a_0,\ldots,a_{n-1},y_0,\ldots,y_{m-1})$ has a solution in G. By Lemma 2.4, $\Phi(a_0,\ldots,a_{n-1},y_0,\ldots,y_{m-1})\wedge\bigwedge_{i<m}P(y_i)$ has a solution in G, which implies $G\models\Phi_1(\mathbf{a})$. □

THEOREM 2.1. If T is a complete theory for a torsion-free Abelian group, then T is (P,a)-stable.

Proof. Let $\Phi(\mathbf{x}, \mathbf{x}^0)$ separate $\mathbf{X}$ from $\mathbf{Y}$ in a type $E^{(P,a)}(t)$, i.e., the set of formulas $\{\Phi(\mathbf{x}; \mathbf{x}^0) \mid \mathbf{x} \in \mathbf{X}\} \cup \{\neg\Phi(\mathbf{x}; \mathbf{x}^0) \mid \mathbf{x} \in \mathbf{Y}\}$ is consistent with the type $E^{(P,a)}(t)$. Assume t_1 is a completion of $E^{(P,a)}(t)$ in which $\{\Phi(\mathbf{x}; \mathbf{x}^0) \mid \mathbf{x} \in \mathbf{X}\} \cup \{\neg\Phi(\mathbf{x}; \mathbf{x}^0) \mid \mathbf{x} \in \mathbf{Y}\}$ is consistent. By virtue of Lemma 2.3, $\Phi(\mathbf{x}, \mathbf{x}^0)$ is equivalent in t' to a Boolean combination $\Phi_1(\mathbf{x}, \mathbf{x}^0)$ of positive primitive formulas. Suppose $P(s)$ occurs in the subformula $\exists z_1 \ldots \exists z_k \varphi$ of Φ_1, where s is a term and φ is a conjunction of atomic formulas. Replace the occurrence of φ in Φ_1 by an equivalent formula $\exists y \exists z_1 \ldots \exists z_k (y = s) \wedge \varphi'$, where φ' is obtained from φ by replacing all occurrences of the subformula $P(s)$ by a subformula $P(y)$ and y does not occur in Φ_1. This procedure applies with all occurrences of the predicate P. Therefore, we will assume that P occurs only with variables in Φ_1.

Let $\Psi(\mathbf{x}, \mathbf{x}^0)$ be a formula derived from $\Phi_1(\mathbf{x}, \mathbf{x}^0)$ by replacing all occurrences of subformulas of the form $P(y)$ by subformulas $y = y$, where y is an arbitrary variable. Lemma 2.5 implies that sets of solutions for $\Psi(\mathbf{x}, \mathbf{x}^0) \wedge \bigwedge\limits_{i<n} P(x_i)$ and $\Phi_1(\mathbf{x}, \mathbf{x}^0) \wedge \bigwedge\limits_{i<n} P(x_i)$ are equal. Hence the formula Ψ separates $\mathbf{X}$ from $\mathbf{Y}$ in a type t. By Theorem 1.1, T is a (P, a)-stable theory. $\square$

That (P, a)-unstable Abelian groups exist is shown in the following:

Example. Let $G = \bigoplus\limits_{i \in \omega} G_i$, where $G_i \simeq \mathbb{Z}_4$ are cyclic groups of order four, for $i \in \omega$. Then $T = \mathrm{Th}(G)$ is not a (P, a)-stable theory.

Indeed, take a type realized by a set D of all elements of order two to be $t(X)$. These elements each has infinitely many divisors by two, and so elements of order four do not enter the algebraic closure of D. We partition D into two disjoint subsets D_1 and D_2 so that $\mathrm{acl}(D_1) \cap \mathrm{acl}(D_2) = \varnothing$. For each $d \in D_2$, choose any of its divisors a_d by two. Clearly, $d + a_d$ is also a divisor of d. Consequently, the set $\mathrm{acl}(D \cup \{a_d \mid d \in D_2\})$ contains no divisors of elements of D_1, and we take it to be a realization of the predicate P.

By the above, a formula like $\exists y P(y) \wedge (y + y = x)$ separates subsets X_1 and X_2 of X, which are realized by sets D_1 and D_2 over a type t. By Theorem 1.1, T is not (P, a)-stable.

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