Algebra and Logic, Vol. 49, No. 4, 2010

S-EMBEDDED SUBGROUPS OF FINITE GROUPS

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UDC 512.542

Keywords: *finite group, permutable group, S-embedded subgroup, supersoluble group, p-nilpotent group.*

A subgroup H of G is said to be S-embedded in G if G has a normal subgroup N such that HN is s-permutable in G and $H \cap N \leqslant H_{sG}$, where H_{sG} is the largest s-permutable subgroup of G contained in H. S-embedded subgroups are used to give novel characterizations for some classes of groups. New results are obtained and a number of previously known ones are generalized.

INTRODUCTION

Throughout the paper, all groups considered are finite and G denotes a group. Terminology and notation are standard, as in [1, 2].

Recall that a subgroup H of G is said to be *permutable* with a subgroup T of G if $HT = TH$. A subgroup H of G is said to be *s-permutable* [3] or *s-quasinormal* [4] in G if H is permutable with every Sylow subgroup P of G. A subgroup H of G is said to be *c-normal* in G if there exists a normal subgroup K of G such that $HK = G$ and $H \cap K \leq H_G$, where H_G is the maximal normal subgroup of G contained in H [5]. A subgroup H of G is said to be *nearly s-normal* in G if there exists $N \trianglelefteq G$ such that $HN \trianglelefteq G$ and $H \cap N \leq H_{sG}$, where H_{sG} is the largest s-permutable subgroup of G contained in H [6]. By using s-permutability, c-normality, and nearly s-normality of some subgroups, many interesting results have been derived (see, e.g., Sec. 4 below and [7]). As a development, the following new concept was introduced in [8].

Definition 1.1. Let H be a subgroup of G. We say that H is *S-embedded* in G if there exists a normal subgroup N such that HN is s-permutable in G and $H \cap N \leq H_{sG}$, where H_{sG} is the largest s-permutable subgroup of G contained in H.

It is easy to see that all subgroups, independently of whether they are normal, permutable, s-permutable, c-normal, or nearly s-normal, are S-embedded subgroups. However, the converse is not true (see, e.g., [8, Ex. 1.4]).

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In this paper, we continue to study the influence of S-embedded subgroups on the structure of groups. New results are obtained and a number of previously known ones are generalized.

1. PRELIMINARIES

We cite some basic results which are useful in the sequel.

LEMMA 1.1 [8, Lemma 2.1]. Let G be a group and $H \leq G$. Then:

(1) If H is S-embedded in G and $H \leq K \leq G$, then H is S-embedded in K.

(2) Suppose $N \trianglelefteq G$ and $N \leq H$. Then H is S-embedded in G if and only if H/N is S-embedded in G/N.

(3) Let N be a normal π'-subgroup of G and H a π-subgroup of G. If H is S-embedded in G, then HN/N is S-embedded in G/N.

It is easy to verify the following:

LEMMA 1.2. Let $N \trianglelefteq G$ and $H \leq G$. If H is s-permutable in G, then $H \cap N$ is s-permutable in G.

LEMMA 1.3 [9, Lemmas 2.6, 2.7]. Let $H \leq G$.

(1) If H is s-permutable in G, then H is subnormal in G.

(2) If H is s-permutable in G, and H is a p-group for some prime p, then $O^p(G) \leq N_G(H)$.

LEMMA 1.4 [10; 9, Lemma 2.5(6)]. If H is a subnormal π-subgroup of G, then $H \leq O_\pi(G)$.

Let $\mathfrak{F}$ be a class of groups. We say that $\mathfrak{F}$ is *S-closed* if every subgroup of G belongs to $\mathfrak{F}$ whenever $G \in \mathfrak{F}$. A subgroup H of G is $\mathfrak{F}$-*supplemented* in G if G has a subgroup $T \in \mathfrak{F}$ such that $G = HT$. In this case we call T an $\mathfrak{F}$-*supplement* of H in G. In particular, if $\mathfrak{F}$ is the class of all supersoluble groups (of all p-nilpotent groups), then an $\mathfrak{F}$-supplement is referred to as a *supersoluble supplement* (a *p-nilpotent supplement*).

The following lemma is obvious.

LEMMA 1.5. Let $\mathfrak{F}$ be a formation of groups. Suppose that a subgroup H of G has an $\mathfrak{F}$-supplement in G.

(1) If $N \trianglelefteq G$, then HN/N has an $\mathfrak{F}$-supplement in G/N.

(2) If $H \leq K \leq G$ and $\mathfrak{F}$ is S-closed, then H has an $\mathfrak{F}$-supplement in K.

LEMMA 1.6 [6, Lemma 2.8]. Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime divisor of $|G|$. If either P is cyclic or P is not cyclic but every maximal subgroup of P has a supersoluble supplement in G, then G is soluble.

LEMMA 1.7 [11, Lemma 3.10]. Take two distinct prime divisors p and q of $|G|$ and a noncyclic Sylow p-subgroup P of G. If every maximal subgroup of P has a q-closed supplement in G, then G is q-closed.

LEMMA 1.8 [1, Lemma II.7.9]. Let P be a nilpotent normal subgroup of G. If $P \cap \Phi(G) = 1$, then P is a direct product of some minimal normal subgroups of G.

LEMMA 1.9 [12, Lemma 2.3]. Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and E a normal subgroup of G such that $G/E \in \mathfrak{F}$. If E is cyclic, then $G \in \mathfrak{F}$.

2. MAIN RESULTS

THEOREM 2.1. A group G is supersoluble if and only if there exists a normal subgroup H of G such that G/H is supersoluble and all maximal subgroups of every noncyclic Sylow subgroup of H not having a supersoluble supplement in G are S-embedded in G.

Proof. The necessity being obvious, we need only prove the sufficiency. Suppose the contrary, letting G be a counterexample with $|G||H|$ minimal.

(1) G/E is supersoluble for every nontrivial normal p-subgroup E of G contained in H, where p is a prime. Obviously, $(G/E)/(H/E) \cong G/H$ is supersoluble. Assume that T/E is a noncyclic Sylow q-subgroup of H/E and T_1/E is a maximal subgroup of T/E, where q is a prime divisor of $|H/E|$.

If $q = p$, then T is a noncyclic Sylow p-subgroup of H and T_1 is a maximal subgroup of T. By hypothesis, either T_1 has a supersoluble supplement in G or T_1 is S-embedded in G. By Lemmas 1.5(1) and 1.1(2), either T_1/E has a supersoluble supplement in G/E or T_1/E is S-embedded in G/E.

Now suppose that $q \neq p$. In this case there exists a Sylow q-subgroup Q of H such that $T = QE$. Let $Q_1 = Q \cap T_1$. It is easy to see that Q_1 is a maximal subgroup of Q and $T_1 = Q_1E$. By hypothesis, either Q_1 has a supersoluble supplement in G or Q_1 is S-embedded in G. By Lemmas 1.5(1) and 1.1(3), either T_1/E has a supersoluble supplement in G/E or T_1/E is S-embedded in G/E. This shows that $(G/E, H/E)$ satisfies the hypothesis. Since G is chosen minimal, G/E is supersoluble.

(2) G is soluble.

Lemmas 1.1(1) and 1.5(2) imply that the hypothesis is still true for (H, H). If $H < G$, then H is supersoluble by the choice of G. It follows that G is soluble.

Now assume that $H = G$. Let p be the smallest prime divisor of $|G|$. Then $p = 2$ by the Feit–Thompson theorem. If $O_2(G) \neq 1$, then $G/O_2(G)$ is supersoluble by (1), and so G is soluble. Let $O_2(G) = 1$ and P be a Sylow 2-subgroup of G. If P is cyclic, then G is 2-nilpotent by [13, (10.1.9)]. It follows that G is soluble. Suppose that P is noncyclic. By Lemma 1.6, there exists a maximal subgroup P_1 of P such that P_1 has no supersoluble supplement in G. By hypothesis, therefore, P_1 is S-embedded in G. Hence there exists $K \trianglelefteq G$ such that P_1K is s-permutable in G and $P_1 \cap K \leq (P_1)_{sG}$. By Lemmas 1.3(1) and 1.4, $(P_1)_{sG} \leq O_2(G) = 1$, and consequently $P_1 \cap K = 1$.

Let $C = [K]P_1$ and K_2 be a Sylow 2-subgroup of K. Then $|K_2| \leqslant 2$. Hence K is soluble by [13, (10.1.9)], which implies that C is soluble. Since $C = P_1K$ is s-permutable in G, C is subnormal in G by Lemma 1.3(1), and so C is contained in some soluble normal subgroup D of G (see [10]).

Let Q/D be a Sylow 2-subgroup of G/D. Since $P_1 \leq C \leq D$, $|Q/D| \leqslant 2$, and so G/D is soluble. This means that G is soluble.

(3) G has a unique minimal normal subgroup N contained in H, $G = [N]M$, where M is a maximal subgroup of G, and $N = O_p(H) = F(H) = C_H(N)$, for some prime $p \in \pi(G)$.

Let N be a minimal normal subgroup of G contained in H. In view of (2), N is an elementary Abelian p-group, for some prime p dividing $|G|$. The class of all supersoluble groups is a saturated formation. By (1), N is the unique minimal normal subgroup of G contained in H, and $N \nsubseteq \Phi(G)$. Hence there exists a maximal subgroup M of G such that $G = [N]M$. Since $C = C_H(N) = C_G(N) \cap H \trianglelefteq G$, $(C \cap M)^G = (C \cap M)^{NM} = (C \cap M)^M = C \cap M$. Hence $C \cap M$ is normal in G. It follows that $C \cap M = 1$. Thus $C = C \cap NM = N(C \cap M) = N$. Since $N \leq O_p(H) \leq F(H) \leq F(G) \leq C_G(N)$, $F(H) \leq C_G(N) \cap H = C = N$.

(4) N is a Sylow p-subgroup of H and N is not cyclic.

In view of (1), G/N is supersoluble. If N is cyclic, then G is supersoluble by Lemma 1.9, a contradiction. Hence N is not cyclic. Let q be the largest prime divisor of $|H|$ and Q a Sylow q-subgroup of H. Then QN/N is a Sylow q-subgroup of H/N. Since G/N is supersoluble, H/N is supersoluble, and hence $QN/N \trianglelefteq H/N$. Therefore, $QN \trianglelefteq H$.

Let P be a Sylow p-subgroup of H. If $q = p$, then $P = Q = QN \trianglelefteq H$. In view of (3), $N = O_p(H) = P$ is a Sylow p-subgroup of H. Assume that $q > p$. Then $QP = QNP$ is obviously a subgroup of H. If $QP < G$, then it follows from Lemmas 1.1(1) and 1.5(2) that (QP, QP) satisfies the hypothesis. By the minimal choice of (G, H), QP is supersoluble. It follows that $Q \trianglelefteq QP$, and so $QN = Q \times N$. Hence $Q \leq C_H(N) = N$ by (3), a contradiction.

Now we assume that $G = PQ = H$. Obviously, $q \neq p$ and Q is not a normal subgroup of G by (3). Suppose that $N < P$. Since N is not cyclic, P is not cyclic. By Lemma 1.7, P has a maximal subgroup P_1 which has no q-closed supplement in G. Consequently, P_1 has no supersoluble supplement in G. By hypothesis, P_1 is S-embedded in G, that is, there exists a normal subgroup K of G such that P_1K is s-permutable in G and $P_1 \cap K \leq (P_1)_{sG}$. By Lemmas 1.3(1) and 1.4, $(P_1)_{sG} \leq O_p(G) = O_p(H) = N$.

Let $P_1 \cap K = 1$. Then p^2 does not divide $|K|$. If $K \neq 1$, then $N \leq K$, and so p^2 divides $|K|$ since N is not cyclic, a contradiction.

If $K = 1$, then P_1 is s-permutable in G. Consequently, P_1 is subnormal in G by Lemma 1.3(1). It follows that $P_1 \trianglelefteq P_1Q$ and $Q \leq N_G(P_1)$. Thus $P_1 \triangleleft G$, whence $P_1 \leq O_p(G) = N$. In view of $QN \trianglelefteq H = G$ and the Frattini argument, $G = QNN_G(Q) = NN_G(Q) = P_1N_G(Q)$. This means that P_1 has a q-closed supplement in G, a contradiction. Hence $P_1 \cap K \neq 1$.

By Lemmas 1.3 and 1.4, $(P_1)_{sG} \leq P_1 \cap O_p(G) = P_1 \cap N$. On the other hand, $Q \leq N_G((P_1)_{sG})$ by Lemma 1.3(2). Hence $1 \neq (P_1)_{sG} \leq ((P_1)_{sG})^G = ((P_1)_{sG})^{PQ} = ((P_1)_{sG})^P \leq (P_1 \cap N)^P = P_1 \cap N \leq N$. It follows that $((P_1)_{sG})^G = N = P_1 \cap N$. Thus $N \leq P_1$. By using the Frattini argument again, we obtain $G = QNN_G(Q) = NN_G(Q) = P_1N_G(Q)$. This implies that P_1 has a q-closed supplement in G, a contradiction. Therefore, $N = P$ is a Sylow p-subgroup of H.

(5) The final contradiction.

Let M_p be a Sylow p-subgroup of M and $P = NM_p$. Since $G = [N]M$, P is a Sylow p-subgroup of G. Let P_1 be a maximal subgroup of P containing M_p and $N_1 = N \cap P_1$. Since $|N : N_1| = |N : N \cap P_1| = |NP_1 : P_1| = |P : P_1| = p$, N_1 is a maximal subgroup of N, and so $N_1 \trianglelefteq N$. Let T be an arbitrary supplement of N_1 in G. Then $G = N_1T = NT$ and $N = N \cap N_1T = N_1(N \cap T)$. This implies that $N \cap T \neq 1$. Since $N \cap T \trianglelefteq NT = G$ and N is a unique minimal normal subgroup of G, we have $N \cap T = N$. Hence $G = T$ is not supersoluble. This shows that N_1 does not have a supersoluble supplement in G. By hypothesis, therefore, there exists $K \trianglelefteq G$ such that N_1K is s-permutable in G and $N_1 \cap K \leq (N_1)_{sG}$. In view of (3), we see that $N \cap K = 1$ or $N \leq K$.

If $N \cap K = 1$, then $N_1 = N_1(N \cap K) = N \cap N_1K$. Since $N \trianglelefteq G$ and N_1K is s-permutable in G, $N_1 = N \cap N_1K$ is s-permutable in G by Lemma 1.2. Thus, for any Sylow q-subgroup Q of G, where $q \neq p$, N_1Q is a subgroup of G. By Lemma 1.3, N_1 is subnormal in N_1Q. It follows that $N_1 \trianglelefteq N_1Q$. Therefore, $Q \leq N_G(N_1)$. On the other hand, $N_1 = N \cap P_1 \trianglelefteq P$. This implies that $N_1 \trianglelefteq G$. Thus $N_1 = 1$ and $|N| = p$, which contradicts (4).

Now assume that $N \leq K$. Then $N_1 = N_1 \cap N \leq N_1 \cap K \leq (N_1)_{sG} \leq N_1$. This means that $N_1 = (N_1)_{sG}$ is s-permutable in G. Consequently, $O^p(G) \leq N_G(N_1)$ by Lemma 1.3(2). It follows that $N_1 = N \cap P_1 \trianglelefteq PO^p(G) = G$. Hence $N_1 = 1$ and $|N| = p$. The final contradiction completes the proof.

COROLLARY 2.1.1. A group G is supersoluble if and only if all maximal subgroups of every noncyclic Sylow subgroup of G having no supersoluble supplement in G are S-embedded in G.

COROLLARY 2.1.2. Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and G a group. Then $G \in \mathfrak{F}$ if and only if there exists a normal subgroup H of G such that $G/H \in \mathfrak{F}$ and all maximal subgroups of every noncyclic Sylow subgroup of H having no supersoluble supplement in G are S-embedded in G.

Proof. The necessity being obvious, we need only prove the sufficiency. Assume the contrary, letting G be a counterexample with $|G||H|$ minimal. Since $H/H = 1$ is supersoluble, H is supersoluble by Lemmas 1.1(1) and 1.5(2) and Theorem 2.1. Let p be the largest prime divisor of $|H|$ and P a Sylow p-subgroup of H. Then P is the characteristic subgroup of $H \trianglelefteq G$, and hence $P \trianglelefteq G$.

Let N be a minimal normal subgroup of G contained in P. Obviously, $(G/N)/(H/N) \cong G/H \in \mathfrak{F}$. In view of Lemmas 1.5(1) and 1.1, the hypothesis is still true for G/N (with respect to H/N). The minimal choice of G implies that $G/N \in \mathfrak{F}$. Since $\mathfrak{F}$ is a saturated formation, N is a unique minimal normal subgroup of G contained in P and $N \nsubseteq \Phi(G)$. It is easy to see that $N = O_p(H) = P$ (see proofs of Thm. 2.1(3), (4)). If N is cyclic, then $G \in \mathfrak{F}$ by Lemma 1.9, which contradicts the choice of G. Thus we may assume that N is not cyclic.

Let N_1 be a maximal subgroup of N. Following the same line of argument as was used in proving Theorem 2.1(5), we see that N is cyclic, and consequently $G \in \mathfrak{F}$. The corollary is proved.

THEOREM 2.2. A group G is nilpotent if and only if, for every $p \in \pi(G)$ and every Sylow p-subgroup P of G, the following conditions hold:

(i) $N_G(P)/C_G(P)$ is a p-group;

(ii) all maximal subgroups of P are S-embedded in G.

Proof. The necessity is obvious. We argue for the sufficiency. First, in view of Theorem 2.1, G is supersoluble. Let q be the largest prime divisor of $|G|$ and Q a Sylow q-subgroup of G. Then $Q \trianglelefteq G$.

Let N be a minimal normal subgroup of G contained in Q. Consider a factor group $\overline{G} = G/N$. Let $\overline{P}$ be a Sylow p-subgroup of $\overline{G}$. Then there exists a Sylow p-subgroup P of G such that $\overline{P} = PN/N$. Obviously, $N_{\overline{G}}(\overline{P}) = N_G(P)N/N$ and $C_{\overline{G}}(\overline{P}) \geqslant C_G(P)N/N$. Hence $N_{\overline{G}}(\overline{P})/C_{\overline{G}}(\overline{P})$ is a p-subgroup. Suppose P_1/N is a maximal subgroup of PN/N. If $p = q$, then $N \leq P$. Hence P_1 is a maximal subgroup of P. By (ii), P_1 is S-embedded in G. It follows from Lemma 1.1(2) that P_1/N is S-embedded in G/N. If $p \neq q$, then $P_1 = P_1 \cap PN = (P_1 \cap P)N$. It is easy to see that $P_1 \cap P$ is a maximal subgroup of P. By hypothesis, $P_1 \cap P$ is S-embedded in G, and consequently $P_1/N = (P_1 \cap P)N/N$ is S-embedded in G/N by Lemma 1.1(3). This shows that G/N satisfies the hypothesis. By induction, G/N is nilpotent.

The class of all nilpotent groups is a saturated formation; therefore, N is a unique minimal normal subgroup of G contained in Q, and $\Phi(G) = 1$. Hence there exists a maximal subgroup M such that $G = NM$. Since G is soluble, N is an elementary Abelian group. Consequently, $N \cap M \trianglelefteq G$ and $N \cap M = 1$. Now $Q = Q \cap NM = N(Q \cap M)$ and $Q \cap M \subseteq Q \subseteq F(G) \subseteq C_G(N)$. Hence $Q \cap M \trianglelefteq G$. It follows that $Q \cap M = 1$. Hence $N = Q$ and $Q \leq C_G(Q)$. In view of (i), $N_G(Q)/C_G(Q)$ is a q-group. This implies that $N_G(Q) = C_G(Q) = G$. Consequently, $Q \leq Z(G)$. Since G/Q is nilpotent, G is as well. The theorem is proved.

THEOREM 2.3. Let P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P not having a p-nilpotent supplement in G is S-embedded in G, then G is p-nilpotent.

Proof. Suppose the contrary, letting G be a counterexample of minimal order. Then:

(1) Every maximal subgroup of P is S-embedded in G.

If not, then P_1 has a maximal subgroup P which has a p-nilpotent supplement T in G. Let H be a non-p-nilpotent subgroup of G which contains P and is such that every proper subgroup of H is p-nilpotent. Then H is a minimal nonnilpotent group by [14, Thm. IV.5.4]. In view of [2, Thm. 3.4.11], H has the following properties:

(i) $|H| = p^\alpha q^\beta$, where p and q are different primes;

(ii) $H = [H_p]H_q$, where H_p is a normal Sylow p-subgroup of H and H_q is a cyclic Sylow q-subgroup of H;

(iii) $H_p/\Phi(H_p)$ is a chief factor of H.

Since $G = P_1T$, $H = H \cap P_1T = P_1(H \cap T) = P_1L$, where $L = H \cap T$. We claim that L is a proper subgroup of H. Otherwise, H is contained in T, and so H is p-nilpotent, a contradiction. Thus $L < H$, and hence L is nilpotent. Let $L = L_q \times L_p$. Obviously, L_q is a Sylow q-subgroup of H. Since $P \subseteq H$ and $H = P_1L$, $L_p \neq 1$ and L_p is not contained in $\Phi = \Phi(H_p)$. Now we consider a factor group H/Φ. The fact that $L_q \leq N_H(L_p)$ implies that $L_q\Phi/\Phi \leq N_{H/\Phi}(L_p\Phi/\Phi)$. On the other hand, $L_p\Phi/\Phi \trianglelefteq H_p/\Phi$ since H_p/Φ is an elementary Abelian group. Hence $L_p\Phi/\Phi \trianglelefteq H/\Phi$. Since $L_p\Phi/\Phi \neq 1$ and H_p/Φ is a chief factor of H, we have $L_p\Phi/\Phi = H_p/\Phi$. It follows that $L_p = H_p$. Consequently, $L = H$. This contradiction proves (1).

(2) $O_{p'}(G) = 1$. If $O_{p'}(G) \neq 1$, then we may choose a minimal normal subgroup N of G such that $N \leq O_{p'}(G)$. It is clear that $(|G/N|, p-1) = 1$ and PN/N is a Sylow p-subgroup of G/N. Let P_1/N be a maximal subgroup of PN/N. Then there exists a maximal subgroup P_2 of P such that $P_1 = P_2N$. By hypothesis and Lemma 1.1(3), $P_1/N = P_2N/N$ is S-embedded in G/N. Since G is chosen minimal, G/N is p-nilpotent. It follows that G is p-nilpotent, a contradiction. Thus $O_{p'}(G) = 1$.

(3) G is soluble. Suppose the contrary. By the Feit–Thompson theorem, $p = 2$. Assume $O_2(G) \neq 1$. If $O_2(G)$ is a Sylow 2-subgroup of G, then G is obviously soluble, a contradiction. Thus $O_2(G)$ is not a Sylow 2-subgroup of G. By (1) and Lemma 1.1(2), $G/O_2(G)$ satisfies the hypothesis. Hence $G/O_2(G)$ is 2-nilpotent by the choice of G. It follows that G is soluble, a contradiction. We have $O_2(G) = 1$.

Let P_1 be a maximal subgroup of P. By (1), P_1 is S-embedded in G. Hence there exists $K \trianglelefteq G$ such that $C = P_1K$ is s-permutable in G, and $P_1 \cap K \leq (P_1)_{sG}$. By Lemmas 1.3 and 1.4, $(P_1)_{sG} \leq O_2(G) = 1$. Therefore, $C = [K]P_1$.

Let K_2 be a Sylow 2-subgroup of K. Then, clearly, $|K_2| \leqslant 2$. Hence, in view of [13, (10.1.9)] and the Feit–Thompson theorem, K is soluble, and so therefore is C. Since C is subnormal in G, C is contained in some soluble normal subgroup M of G, as follows by Lemma 1.3(1) and [10]. Obviously, 2^2 does not divide $|G/M|$. Hence, by [13, (10.1.9)] and the Feit–Thompson theorem, G/M is soluble. This implies that G is soluble, a contradiction.

(4) $O_p(G) \neq 1$. This follows directly from (2) and (3).

(5) $O_p(G)$ is a unique minimal normal subgroup of G, and $\Phi(G) = 1$. Let N be an arbitrary minimal normal subgroup of G. By virtue of (2) and (3), N is an elementary Abelian p-group and $N \leq O_p(G)$. By Lemma 1.1(2), the hypothesis holds for G/N. The minimal choice of G implies that G/N is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, N is a unique minimal normal subgroup of G and $\Phi(G) = 1$. By Lemma 1.8, $O_p(G) = N$.

(6) $|O_p(G)| \geq p^2$. Assume that $|O_p(G)| = p$. In view of (5), $F(G) = O_p(G) = C_G(O_p(G))$, and so $G/O_p(G) \cong G/C_G(O_p(G))$ is isomorphic to some subgroup of $\mathrm{Aut}(O_p(G))$. Since $|\mathrm{Aut}(O_p(G))| = p - 1$ and $(|G|, p-1) = 1$, $G/O_p(G) = 1$. It follows that $G = O_p(G)$ is p-nilpotent, a contradiction.

(7) The final contradiction. By virtue of (5), there exists a maximal subgroup M of G such that $G = [O_p(G)]M$. Let $P = O_p(G)M_p$ be a Sylow p-subgroup of G, where M_p is a Sylow p-

subgroup of M, and P_1 be a maximal subgroup of P such that $M_p \leq P_1$. Obviously, $O_p(G) \nleq P_1$. In view of (1), there exists a normal subgroup K of G such that P_1K is s-permutable in G, and $P_1 \cap K \leq (P_1)_{sG}$.

If $K = 1$, then P_1 is s-permutable in G, and hence P_1Q is a subgroup of G, for every Sylow q-subgroup Q of G with $q \neq p$. By Lemma 1.3(1), P_1 is subnormal in G. This implies that $P_1 \trianglelefteq P_1Q$ and $Q \leq N_G(P_1)$. It follows that $P_1 \trianglelefteq G$. Since $O_p(G)$ is the unique minimal normal subgroup of G, $O_p(G) \leq P_1$, a contradiction. We have $K \neq 1$.

In view of Lemmas 1.3(1) and 1.4, $P_1 \cap K \leq (P_1)_{sG} \leq O_p(G) \leq K$. If $P_1 \cap K \neq 1$, then $1 \neq (P_1)_{sG} \leq P_1 \cap O_p(G)$. By Lemma 1.3(2), $O^p(G) \leq N_G((P_1)_{sG})$. Thus $(P_1)_{sG} \leq ((P_1)_{sG})^G = ((P_1)_{sG})^{PO^p(G)} = ((P_1)_{sG})^P \leq (P_1 \cap O_p(G))^P = P_1 \cap O_p(G) \leq O_p(G)$. Hence, by (5), we have $((P_1)_{sG})^G = O_p(G) = P_1 \cap O_p(G)$. It follows that $O_p(G) \leq P_1$, a contradiction. Hence $P_1 \cap K = 1$, and therefore p^2 does not divide $|K|$. Since $K \neq 1$, $O_p(G) \leq K$ and p^2 divides $|K|$ by (6). The finial contradiction completes the proof.

COROLLARY 2.3.1. Let p be the smallest prime divisor of $|G|$ and P a Sylow p-subgroup of G. If every maximal subgroup of P is S-embedded in G, then G is p-nilpotent.

COROLLARY 2.3.2. If every maximal subgroup of every Sylow subgroup of a group G is S-embedded in G, then G is a Sylow tower group of supersoluble type.

Proof. Let p be the smallest prime divisor of $|G|$ and P a Sylow p-subgroup of G. By Corollary 2.3.1, G is p-nilpotent. Let N be a normal p-supplement of G. Clearly, N satisfies the hypothesis by Lemma 1.1(1). By induction, therefore, N is a Sylow tower group of supersoluble type. This shows that G is a Sylow tower group of supersoluble type.

THEOREM 2.4. Let p be a prime dividing $|G|$ and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent, and every maximal subgroup of P not having a p-nilpotent supplement in G is S-embedded in G, then G is p-nilpotent.

Proof. If $p = 2$, then the assertion follows from Theorem 2.3. Hence we need only prove the theorem for the case where p is an odd prime. Assume the contrary, letting G be a counterexample of minimal order. Then:

(1) Every maximal subgroup of P is S-embedded in G (see proof of Theorem 2.3(1)).

(2) $O_{p'}(G) = 1$. Suppose $O_{p'}(G) \neq 1$. Consider a factor group $G/O_{p'}(G)$. By Lemma 1.1(3) and [2, Lemma 3.6.10], $G/O_{p'}(G)$ satisfies the hypothesis. Thus $G/O_{p'}(G)$ is p-nilpotent by the choice of G. It follows that G is p-nilpotent, a contradiction.

(3) If $P \leq H < G$, then H is p-nilpotent. Since $N_H(P) \leq N_G(P)$, $N_H(P)$ is p-nilpotent. Hence H satisfies the hypothesis in view of Lemma 1.1(1). The minimal choice of G implies that H is p-nilpotent.

(4) G is p-soluble. Since G is not p-nilpotent, by the Thompson theorem (see [13, (10.4.1)]), there exists a characteristic subgroup T of P such that $N_G(T)$ is not p-nilpotent. Since $N_G(P)$ is p-nilpotent, we may choose a characteristic subgroup L of P such that $N_G(L)$ is not p-nilpotent, but $N_G(K)$ is p-nilpotent for every characteristic subgroup K of P with $L < K \leq P$. Since

$L \operatorname{char} P \trianglelefteq N_G(P)$, $L \trianglelefteq N_G(P)$ and $N_G(P) \leq N_G(L)$. Obviously, $N_G(P) < N_G(L)$. Hence, by (3), we obtain $N_G(L) = G$. This means that $O_p(G) \neq 1$ and $N_G(K)$ is p-nilpotent for every characteristic subgroup K of P satisfying $O_p(G) < K \leq P$. Now, by applying the Thompson theorem again, we see that $G/O_p(G)$ is p-nilpotent, and consequently G is p-soluble.

(5) The final contradiction. Let N be a minimal normal subgroup of G. Then N is an elementary Abelian p-group by virtue of (2) and (4). It is easy to see that G/N satisfies the hypothesis. Hence G/N is p-nilpotent by the choice of G. Since the class of all p-nilpotent groups is a saturated formation, N is the unique minimal normal subgroup of G, and $\Phi(G) = 1$. Thus $G = [N]M$ for some maximal subgroup M of G, and $N = C_G(N) = F(G) = O_p(G)$.

Let M_p be a Sylow p-subgroup of M such that $P = NM_p$ and M_1 be a maximal subgroup of P containing M_p. Since $N_G(P)$ is p-nilpotent, $P \neq N$ and $M_p \neq 1$. Clearly, $N \nleq M_1$. By (1), M_1 is S-embedded in G, and so there exists some normal subgroup K of G such that M_1K is s-permutable in G, and $M_1 \cap K \leq (M_1)_{sG}$. If $K = 1$, then M_1 is s-permutable in G. By Lemma 1.3(2), $O^p(G) \leq N_G(M_1)$. Thus $M_1 \trianglelefteq PO^p(G) = G$. This implies that $N \leq M_1$, a contradiction. Thus $K \neq 1$, and hence $N \leq K$.

If $M_1 \cap K \neq 1$, then the same argument as was used at step (7) in the proof of Theorem 2.3 shows that $N \leq M_1$, a contradiction. Therefore, $M_1 \cap K = 1$. In view of $N \leq K$, $M_1 \cap N \leq M_1 \cap K = 1$. Since $P = NM_p = NM_1$ and M_1 is a maximal subgroup of P, we have $|N| = p$. Therefore, $M \cong G/N = G/C_G(N)$ is isomorphic to some subgroup of $\operatorname{Aut}(N)$ with order dividing $p-1$. It follows that N is a Sylow p-subgroup of G. Hence $G = N_G(N) = N_G(P)$ is p-nilpotent. The final contradiction completes the proof.

Analogously, we can prove the following:

THEOREM 2.5. Let $\mathfrak{F}$ be a saturated formation containing all p-nilpotent groups and H a normal subgroup of G such that $G/H \in \mathfrak{F}$. Let p be a prime dividing the order of H and P a Sylow p-subgroup of H. If $N_G(P)$ is p-nilpotent, and every maximal subgroup of P not having a p-nilpotent supplement in G is S-embedded in G, then $G \in \mathfrak{F}$.

3. SOME APPLICATIONS

The results obtained in Sec. 2 have many corollaries. Here we state only those special cases that can be found in the literature.

COROLLARY 3.1 [14, Thm. VI.10.3]. A group G is supersoluble if every Sylow subgroup of G is cyclic.

COROLLARY 3.2 [15]. If every maximal subgroup of every Sylow subgroup of a group G is normal in G, then G is supersoluble.

COROLLARY 3.3 [16]. If every maximal subgroup of every Sylow subgroup of a group G not having a supersoluble supplement in G is normal in G, then G is supersoluble.

COROLLARY 3.4 [15]. Let G be a group with a normal subgroup H such that G/H is supersoluble. If every maximal subgroup of every Sylow subgroup of H is normal in G, then G is supersoluble.

COROLLARY 3.5 [6]. A group G is supersoluble if and only if there exists a normal subgroup H of G such that G/H is supersoluble and every maximal subgroup of every noncyclic Sylow subgroup of H not having a supersoluble supplement in G is nearly s-normal in G.

COROLLARY 3.6 [16]. Let G be a group with a normal subgroup H such that G/H is supersoluble. If every maximal subgroup of every Sylow subgroup of H not having a supersoluble supplement in G is normal in G, then G is supersoluble.

COROLLARY 3.7 [5]. If every maximal subgroup of every Sylow subgroup of a group G is c-normal in G, then G is supersoluble.

COROLLARY 3.8 [17]. If every maximal subgroup of every Sylow subgroup of G not having a supersoluble supplement in G is c-normal in G, then G is supersoluble.

COROLLARY 3.9 [5]. Let H be a normal subgroup of G such that G/H is supersoluble. If every maximal subgroup of every Sylow subgroup of H is c-normal in G, then G is supersoluble.

COROLLARY 3.10 [17]. Let H be a normal subgroup of G such that G/H is supersoluble. If all maximal subgroups of every Sylow subgroup of H not having a supersoluble supplement in G are c-normal in G, then G is supersoluble.

COROLLARY 3.11 [18]. Let $\mathfrak{F}$ be a S-closed saturated formation containing all supersoluble groups. Suppose that G has a normal subgroup H such that $G/H \in \mathfrak{F}$. If every maximal subgroup of every Sylow subgroup of H is c-normal in G, then $G \in \mathfrak{F}$.

COROLLARY 3.12 [15]. If every maximal subgroup of every Sylow subgroup of a group G is s-permutable in G, then G is supersoluble.

COROLLARY 3.13 [19]. Let H be a soluble normal subgroup of G such that G/H is supersoluble. If all maximal subgroups of every Sylow subgroup of H are s-permutable in G, then G is supersoluble.

COROLLARY 3.14 [20]. Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and G a group with a normal subgroup H such that $G/H \in \mathfrak{F}$. If every maximal subgroup of every Sylow subgroup of H is s-permutable in G, then $G \in \mathfrak{F}$.

COROLLARY 3.15 [21]. If every maximal subgroup of every Sylow subgroup of a group G either is s-quasinormal in G or is c-normal in G, then G is supersoluble.

COROLLARY 3.16 [21]. Let E be a normal subgroup of G such that G/E is supersoluble. If every maximal subgroup of every Sylow subgroup of E either is s-quasinormal in G or is c-normal in G, then G is supersoluble.

COROLLARY 3.17 [22]. Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. If every maximal subgroup of P is c-normal in G, then G is p-nilpotent.

COROLLARY 3.18 [22]. Let p be an odd prime dividing $|G|$ and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent, and every maximal subgroup of P is c-normal in G, then G is p-nilpotent.

COROLLARY 3.19. [22]. Let N be a normal subgroup of G, p an odd prime dividing the order of N, and P a Sylow p-subgroup of N. Also let $\mathfrak{F}$ be a saturated formation containing the class of all p-nilpotent groups and $G/N \in \mathfrak{F}$. If $N_G(P)$ is p-nilpotent, and every maximal subgroup of P is c-normal in G, then $G \in \mathfrak{F}$.

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