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ABELIAN GROUPS WITH NORMAL ENDOMORPHISM RINGS

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A ring is said to be normal if all of its idempotents are central. It is proved that a mixed group A with a normal endomorphism ring contains a pure fully invariant subgroup $G \oplus B$, the endomorphism ring of a group G is commutative, and a subgroup B is not always distinguished by a direct summand in A. We describe separable, coperiodic, and other groups with normal endomorphism rings. Also we consider Abelian groups in which the square of the Lie bracket of any two endomorphisms is the zero endomorphism. It is proved that every central invariant subgroup of a group is fully invariant iff the endomorphism ring of the group is commutative.

Let A be an Abelian group. Denote by R the endomorphism ring $E(A)$ of A, by $C = Z(R)$ the center of R, with $p^\omega A = \bigcap\limits_{n=1}^{\infty} p^n A$ and $A^1 = \bigcap\limits_{n=1}^{\infty} nA$, and by $r(A)$ the rank of A; $\mathbb{Z}$ is a ring or group of integers, $\mathbb{Q}$ is a field or group of all rational numbers, $\widehat{\mathbb{Z}}_p$ is a ring or group of p-adic integers, and P is the set of all primes. By writing $H \le A$ we mean that H is a subgroup of A; $H \le fiA$ signifies that H is a fully invariant subgroup of A i.e., $\varphi H \subseteq H$ for every $\varphi \in R$; $H \le ciA$ means that H is a central invariant subgroup of A, i.e., $\alpha H \subseteq H$ for every $\alpha \in C$; unless otherwise stated, A_p is a p-component and $t(A)$ is the periodic part of A. If $H \subseteq A$ and $\varphi \in R$, then $\varphi|H$ is a restriction of φ to H. A subgroup $G \le A$ is said to be *pure* in A if $G \cap nA = nG$ for every natural number n. If $G \cap p^n A = p^n G$ for a given prime p and for every natural n, then G is called a *p-pure* subgroup. If $G \cap pA = pG$ for every prime p, then G is called a *weakly pure* subgroup. An Abelian p-group is *cocyclic* if it is cyclic or isomorphic to a group Z_{p^∞}. Note that the center C of the endomorphism ring R of a group A can be identified with its biendomorphism ring $\mathrm{End}_R A$.

Recall that a ring whose idempotents are all central is said to be *normal* [1, item 0.6]. The endomorphism ring of an Abelian group is normal if and only if all of its direct summands are fully invariant (see, e.g., [1, Assert. 3.28]).

If $\varphi \in R$ and $H \le ciA$, then $\varphi H \le ciA$ and $\varphi^{-1} H = \{a \in A \mid \varphi a \in H\} \le ciA$. In particular, all endomorphic images of A as well as kernels of all of its endomorphisms are ci-subgroups of A. It is not hard to verify that if $H \le fiB$ and $B \le ciA$, then $H \le ciA$, and if $B \le A$, $H \le B$, $H \le ciA$, and $B/H \le fiA/H$, then $B \le ciA$.

The following example says of a possible situation where $H \le ciB$ and $B \le fiA$, but $H \not\le ciA$.

Example 1. Let $S = \mathbb{Z}[\sqrt{-5}]$ and A be a reduced torsion-free group with $E(A) \cong S$. (That such a group exists follows from Corner's known result on countable torsion-free rings [2, Thm. 29.2].) Furthermore, let $0 \ne a \in A$ and $B = E(A)a$. Then $B \le fiA$. Elements of the ring S are algebraic integers and A is a

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torsion-free group; so all nonzero endomorphisms of A are monomorphisms, and $B \cong S^+$. The fact that $S^+ \cong \mathbb{Z} \oplus \mathbb{Z}$ implies that $C(E(B)) \cong \mathbb{Z}$. Hence $\langle a \rangle \leq ciB$. However, $\langle a \rangle$ is not a *ci*-subgroup of A.

We also point out the following properties.

(1) Let $A = \bigoplus\limits_{j\in J} A_j$ ($A = \prod\limits_{j\in J} A_j$) and $e_j\colon A \to A_j$ be respective projections. If $\alpha \in C$, then $\alpha' = e_j\alpha e_j \in Z(E(A_j))$.

Indeed, $e_j = e_j^2$ implies $\alpha e_j = \alpha e_j^2 = e_j\alpha e_j$. If now $\varphi \in E(A_j)$, then we extend φ to an endomorphism $\overline{\varphi}$ of A (assuming that $\overline{\varphi} = 0$ on a direct summand complementary to A_j). Then $\varphi = (e_j\overline{\varphi}e_j)|A_j$. By assumption, $\overline{\varphi}\alpha = \alpha\overline{\varphi}$, whence $(e_j\overline{\varphi}e_j)(e_j\alpha e_j) = (e_j\alpha e_j)(e_j\overline{\varphi}e_j)$. Treating this equality on A_j, we see that $\varphi\alpha' = \alpha'\varphi$, i.e., $\alpha' \in Z(E(A_j))$.

Recall that if $A = B \oplus G$ then we may conceive of $E(A)$ as a ring of matrices of the form $r = \begin{pmatrix} \alpha & \gamma \\ \delta & \beta \end{pmatrix}$, where $\alpha \in E(B)$ and $\beta \in E(G)$ while $\gamma \in \mathrm{Hom}(G, B)$ and $\delta \in \mathrm{Hom}(B, G)$. Obviously, if $r \in C$ then $\gamma = 0$ and $\delta = 0$.

(2) Let $A = B \oplus G$. The inclusion $r = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in C$ holds iff $\alpha \in Z(E(B))$, $\beta \in Z(E(G))$, and $\varphi\alpha = \beta\varphi$, $\psi\beta = \alpha\psi$ for any $\varphi \in \mathrm{Hom}(B, G)$ and any $\psi \in \mathrm{Hom}(G, B)$.

Verification is straightforward.

(3) If $B = \varphi A$ for some $\varphi \in R$, and $H \leq ciB$, then $H \leq ciA$.

Let $b = \varphi a \in B$, $\alpha \in C$, and $f \in E(B)$. Then $\alpha f(b) = \alpha(f\varphi)a = (f\varphi)\alpha a = f(\varphi\alpha a) = f\alpha(\varphi a) = f\alpha b$, i.e., $(\alpha|B)f = f(\alpha|B)$. Hence $\alpha H \subseteq H$.

(4) Let $A = \bigoplus\limits_{j\in J} A_j$ and $H_j \leq ciA_j$ for every $j \in J$. Then $H = \bigoplus\limits_{j\in J} H_j \leq ciA$.

This follows from property (2).

(5) Let $A = \bigoplus\limits_{j\in J} A_j$, $e_j\colon A \to A_j$ be respective projections, and $H \leq ciA$. Then $\bigoplus\limits_{j\in J} e_jH \leq ciA$.

(6) Let $A = \bigoplus\limits_{j\in J} A_j$ and $A_j \leq fiA$ for every $j \in J$. A subgroup H is a *ci*-subgroup of A iff $H = \bigoplus\limits_{j\in J} H_j$, where $H_j \leq ciA_j$ for every $j \in J$.

We verify the necessity. If $e_j\colon A \to A_j$ are respective projections, then $e_j \in C$ for every $j \in J$. This implies $H_j = e_jH = H \cap A_j$. The condition $H_j \leq ciA_j$ follows from property (2). Property (1) entails the sufficiency.

In a p-group, the center of its endomorphism ring is isomorphic either to a ring $\mathbb{Z}_{p^k}$ (if the group is bounded and p^k serves as the least upper bound of the order of its elements), or to $\widehat{\mathbb{Z}}_p$ (otherwise) [2, Thm. 19.7]. Therefore, every subgroup of a periodic group is a *ci*-subgroup.

Let A be a separable torsion-free group and denote by $\Omega(A)$ the set of types of all direct summands of rank 1 in A. Types $s, t \in \Omega(A)$ are assumed to be *equivalent* if there are $r_1, \dots, r_n \in \Omega(A)$ such that types r_i, r_{i+1} are comparable for all $i = 0, \dots, n$, where $r_0 = s$, $r_{n+1} = t$. If $\Omega(A) = \bigcup\limits_{k\in K} \Omega_k$ is a partition of the set $\Omega(A)$ into disjoint equivalence classes, then $A = \bigoplus\limits_{k\in K} A_k$, where A_k are separable groups, $\Omega(A_k) = \Omega_k$, the summands A_k are fully invariant in A, and the center $Z(E(A_k))$ is isomorphic to a subring of the field $\mathbb{Q}$ [2, Sec. 19, Exercise 7]. It is easy to see that $pZ(E(A_k)) = Z(E(A_k))$ iff $pA_k = A_k$.

PROPOSITION 1. Let $A = \bigoplus\limits_{k\in K} A_k$ be a separable torsion-free group, where $\Omega(A) = \bigcup\limits_{k\in K} \Omega_k$ is a partition of the set $\Omega(A)$ into disjoint equivalence classes. A subgroup H of A is a *ci*-subgroup of A if and only if $H = \bigoplus\limits_{k\in K} H_k$, where $H_k = A_k \cap H$ and $pH_k = H_k$ for every prime p with $pA_k = A_k$.

The **proof** follows from property (6).

Proposition 1, in particular, implies that every *ci*-subgroup of a divisible torsion-free group is a direct summand.

PROPOSITION 2. If all *ci*-subgroups of A are fi-subgroups of A, then R is a commutative ring.

Proof. Let $0 \neq a \in A$ and $\varphi \in R$. Since $Ca \leq fiA$, $\varphi a = \alpha a$ for some $\alpha \in C$. We have $(\varphi - \alpha)a = 0$. Therefore, $Ca \subseteq \ker(\varphi - \alpha)$. Hence $\varphi|Ca = \alpha|Ca$. If now $\psi \in R$, and $\beta \in C$ is such that $\psi|Ca = \beta|Ca$, then $(\varphi\psi - \psi\varphi)|Ca = (\alpha\beta - \beta\alpha)|Ca = 0$. This implies $\varphi\psi = \psi\varphi$ since a is arbitrary.

Note that Abelian groups with commutative endomorphism rings were studied in a series of papers. In particular, periodic and splitting groups with commutative endomorphism rings were described in [3]. (These results are reflected in [2, Sec. 19] and [4, Sec. 111, Exercise 12].) Groups all endomorphic images of which are fully invariant were dealt with in [5, 6]. Denote the class of Abelian groups with normal endomorphism rings by $\mathbf{N}$. Now, if $G \oplus B$ is a direct summand of a group in $\mathbf{N}$ then $\mathrm{Hom}(B, G) = \mathrm{Hom}(G, B) = 0$. In the same way as for groups with commutative endomorphism rings such as in [3, Lemma 5], therefore, it is not hard to show that every p-component of a mixed group in $\mathbf{N}$ is cocyclic, and moreover, the complementary direct summand is p-divisible. Specifically, the endomorphism ring of every periodic group in $\mathbf{N}$ is commutative. In treating groups in the class $\mathbf{N}$, we can confine ourselves to the reduced case. In fact, if $A \in \mathbf{N}$ is a nonreduced mixed group, then $A = T \oplus Q$, where T is a reduced periodic group and Q is isomorphic to the additive group of rational numbers. A nonzero torsion-free group in $\mathbf{N}$ should be either reduced or divisible of rank 1. The endomorphism ring of every nonreduced group in $\mathbf{N}$ is commutative. Clearly, direct summands of a group in $\mathbf{N}$ also belong to $\mathbf{N}$.

Recall that a ring S is *right invariant* [2, Sec. 19] if there is $c \in S$ with $ab = bc$ for any $a, b \in S$. If $ab = ca$ for any $a, b \in S$ and for some $c \in S$, then S is *left invariant*.

A ring without nonzero nilpotent elements is said to be *reduced* [1, item 0.6]. If e is an idempotent of a right or left invariant ring S, or, if S is reduced, then the equalities $eS(1 - e) = 0$ and $(1 - e)Se = 0$ imply that e is central. In other words, left or right invariant rings as well as reduced rings are normal.

It is easy to construct indecomposable groups with nonreduced endomorphism rings. Indeed, if such a group is properly embedded as a fully invariant direct summand, we arrive at decomposable groups having a normal but nonreduced endomorphism ring.

If a ring $E(A)$ is right (left) invariant, then all endomorphic images (kernels of endomorphisms) of A are fully invariant. Groups with right invariant (*subcommutative*) endomorphism rings were taken up in [7, 8]. For a separable torsion-free group, the property of an endomorphism ring being right invariant implies its being commutative [2, Sec. 19, Exercise 6]. Analogously, we can prove that a separable torsion-free group with left invariant endomorphism ring enjoys a similar property. It is not hard to see that the endomorphism ring of a separable torsion-free group in the class $\mathbf{N}$ is also commutative, and so is the endomorphism ring of a vector torsion-free group in $\mathbf{N}$.

LEMMA 1. If $A = \bigoplus_{j \in J} A_j$, where $A_j \leq fiA$ and $A_j \in \mathbf{N}$, then $A \in \mathbf{N}$.

Proof. Let B be a direct summand in A, with $A = B \oplus G$. Since $A_j \leq fiA$, we have $A_j = (A_j \cap B) \oplus (A_j \cap G)$. The required result now follows from the decomposition $A = \bigoplus_{j \in J} ((A_j \cap B) \oplus (A_j \cap G))$, where $\left(\bigoplus_{j \in J} (A_j \cap B) \right) = B$.

Let A be a nonzero reduced group satisfying the descending chain condition for direct summands (in particular, A is a group of finite rank). A ring $E(A)$ is normal iff $A = \bigoplus_{j=1}^{n} A_j$, where $A_j \leq fiA$ and $A_j \in \mathbf{N}$. The endomorphism ring of a torsion-free group of rank 1 is commutative. For a decomposable torsion-free group of rank 2, the property of being normal for its endomorphism ring implies being commutative for that ring. We know that every indecomposable torsion-free group of rank 2 has a commutative endomorphism ring [9, Thms. 3.2, 3.3]. An indecomposable torsion-free group of rank 3 may have a noncommutative endomorphism ring.

Example 2. We construct a group A using a vector space V over a field $\mathbb{Q}$. If Π is some set of primes, then $\mathbb{Q}^{(\Pi)}$ denotes a subgroup of $\mathbb{Q}$ generated by all rational numbers whose denominators are powers of the primes in Π. Let p_1, p_2, p_3, q, p be distinct primes and e_1, e_2, e_3 be elements of V independent over $\mathbb{Q}$. Further, let $E_1 = \mathbb{Q}^{(p_1,p_3,q)}e_1$, $E_2 = \mathbb{Q}^{(p_2)}e_2$, $E_3 = \mathbb{Q}^{(p_3)}e_3$, $G = \langle E_2, E_3, \mathbb{Q}^{(q)}(e_2+e_3)\rangle$, and $A = \langle E_1, G, p^{-1}(e_1+e_2)\rangle \subset V$. Since there exists a nonzero homomorphism $G/E_2 \to E_1$, and $pA \subseteq E_1 \oplus G$, the ring $E(A)$ is not commutative. We claim that A is indecomposable. Assume $A = C \oplus D$. Note that $E_1, E_2 \le fiA$. Therefore, $E_i = (E_i \cap C) \oplus (E_i \cap D)$ $(i = 1, 2)$. Since E_1 and E_2 are indecomposable groups, either $E_i \subseteq C$ or $E_i \subseteq D$. There are two cases to consider.

(1) Let $E_1 \subseteq C$ and $E_2 \subseteq D$. Then $p^{-1}(e_1+e_2) = c + d$ for some $c \in C$ and some $d \in D$. This yields $e_1 + e_2 = pc + pd$; hence $e_1 = pc$ and $e_2 = pd$, which is impossible since no one of the elements e_1 and e_2 is divisible by p in the group A.

(2) Let $E_1, E_2 \subseteq D$. Since $q^\omega A = q^\omega C \oplus q^\omega D$, we have $e_2 + e_3 \in D$, provided that $q^\omega C = 0$; hence $C = 0$. In view of the equality $q^\omega A = E_1 \oplus \mathbb{Q}^{(q)}(e_2+e_3)$ and the inclusion $E_1 \subseteq D$, we obtain $q^\omega D = E_1$.

We have $e_3 = c + d$ for some $c \in C$ and some $d \in D$. Consequently, $e_2 + e_3 = c + (d + e_2) \in q^\omega A$. The fact that $d + e_2 \in q^\omega D = E_1$ implies that $d + e_2 = e_1'$ for some $e_1' \in E_1$. Now let $\pi \colon A \to D$ be a projection. Then $e_2 = e_1' - \pi e_3$. Here $e_1' \in p_3^\omega A$ and $\pi e_3 \in p_3^\omega A$, yielding $e_2 \in p_3^\omega A$, a contradiction.

We pass to mixed groups in the class $\mathbf{N}$.

THEOREM 1. Let A be a reduced mixed group with a normal endomorphism ring $R = E(A)$, $\Pi = \{p \in P \mid A_p \neq 0\}$, $B = \bigcap_{p\in\Pi} p^\omega A$, and $G = t(A)^-$ be the closure in the Z-adic topology of A of its subgroup $t(A)$. Then $G \cap B = 0$ and $G \oplus B$ is a pure fully invariant subgroup of A, the ring $E(G)$ is commutative, and the group G is, up to isomorphism, representable as

$$\bigoplus_{p\in\Pi} G_p \subseteq G \subseteq \prod_{p\in\Pi} G_p = S, \qquad (*)$$

where $G_p = A_p$ are cyclic p-groups, and the subgroup G is pure in S.

Proof. A subgroup $G/t(A)$ coincides with the divisible part of a group $A/t(A)$, and since $t(A)$ is a pure subgroup of A, the subgroup G is pure in A.

Since A_p is a cyclic p-group $(p \in \Pi)$, it follows that $A = A_p \oplus E_{(p)}$ for some $E_{(p)} \le A$, with $pE_{(p)} = E_{(p)}$ (cf. paragraph after Prop. 2). For every natural n, we have $A = A_{p_1} \oplus \ldots \oplus A_{p_n} \oplus B_n = A_{p_1} \oplus \ldots \oplus A_{p_{n+1}} \oplus B_{n+1}$, where $B_{n+1} \subseteq B_n$. If $B = \bigcap_n B_n$, then $t(A) \cap B = 0$. Specifically, B is a torsion-free group. And since $(p_1 \ldots p_n)B_n = B_n$, $pB = B$ for every $p \in \Pi$ (which follows from the divisibility of B_n in respective cases and the fact that $A_{p_n} \cap B_m = 0$ for $m \geqslant n$). Notice that these properties of B were also proved in [6, Lemma 2.3] for groups whose homomorphic images are all fully invariant, and in [10], for groups with commutative endomorphism rings. Now let $E = \bigcap_{p\in\Pi} E_{(p)}$. We have $E_{(p_n)} \subseteq B_n$, whence $B \subseteq E$. On the other hand, E (being an intersection of fi-subgroups $E_{(p)}$) is an fi-subgroup, and so $E = (A_{p_1} \cap E) \oplus \ldots \oplus (A_{p_n} \cap E) \oplus (B_n \cap E)$. Consequently, $E \subseteq B_n \cap E$, and hence $E \subseteq B$. Thus $B = E$. The subgroup B is pure in A. Indeed, $pB = B$ for $p \in \Pi$, and if $p \in P \setminus \Pi$, then B is p-pure as an intersection of p-pure subgroups (since $A_p = 0$ for $p \in P \setminus \Pi$).

We claim that $G \cap B = 0$. Let $x \in B$ and $x + t(A) \in (A/t(A))^1 = G/t(A)$. Then $x = a_{p_1} + \ldots + a_{p_n} + p^n a_n$ for every prime p and natural n, where $a_n \in B_n$, $a_{p_i} \in A_{p_i}$ $(i = 1, \ldots, n)$. Since $x \in B$, we have $a_{p_1} + \ldots + a_{p_n} = 0$ and $x \in p^n B_n$, i.e., $x \in A^1 \cap B = B^1$ (the last equality follows from the fact that B is pure in A). Since B is a torsion-free group, B^1 is its divisible subgroup. By virtue of A being reduced, $B^1 = 0$, and hence $G \cap B = 0$. Clearly, $B \subseteq \bigcap_{p \in \Pi} p^\omega A$, and since $p^\omega A \subseteq B_n$ for every n, we have $B = \bigcap_{p \in \Pi} p^\omega A$. The condition $pG \neq G$ entails $pB = B$, and so the subgroup $G \oplus B$ is pure in A. The fact that $t(A) = t(G) \leq fiA$ implies $G = t(A)^- \leq fiA$. Consequently, $G \oplus B \leq fiA$ (as a sum of fi-subgroups G and B).

For all $p \in \Pi$, we have projections $\pi_p \colon G \to G_p$, which generate a homomorphism $f \colon G \to \prod G_p = S$ with kernel $\cap E_{(p)} = E = B$. Since $G \cap B = 0$, f is a monomorphism, and so G can be identified with a subgroup of S containing $\oplus G_p = t(G)$. The subgroup $t(G)$ is pure in S and the quotient group $G/t(G)$ is divisible. Therefore, G is a pure subgroup of S. Let $E(G) \to E(t(G))$ be a ring homomorphism assigning every $\varphi \in E(G)$ its restriction to $t(G)$. If $\varphi \neq 0$, but $\varphi(t(G)) = 0$, then $\varphi(G)$ is a divisible subgroup, which contradicts the property of being reduced. Consequently, $E(G)$ is embedded in the commutative ring $\prod E(G_p)$.

Below is an example showing that a subgroup B of a mixed group A with commutative endomorphism ring is not necessarily distinguished by a direct summand.

Example 3. Let G be a group of the form $(*)$ in Theorem 1, $G \neq \prod G_p$, $|\Pi| = \aleph_0$, $p_1, p_2 \in P \setminus \Pi$, $\overline{A} = \overline{G} \oplus \widehat{\mathbb{Z}}_{p_1} \oplus \widehat{\mathbb{Z}}_{p_2}$, where $\overline{G}$ is an algebraically compact closure of G ($\overline{G}$ coincides with $\prod G_p$), and $0 \neq B$ be a proper pure subgroup of $\widehat{\mathbb{Z}}_{p_1}$. Furthermore, let $a = g + b + v$, where $g \in \overline{G}$, $\langle g \rangle \cap G = 0$ (g is an element of infinite order), $b \in \widehat{\mathbb{Z}}_{p_1} \setminus G$, $0 \neq v \in \widehat{\mathbb{Z}}_{p_2}$, and $A/(G \oplus B)$ be a pure hull of the subgroup $\langle a + (G \oplus B) \rangle$ in $\overline{A}/(G \oplus B)$. Then A is a pure subgroup of $\overline{A}$. Notice that $\bigcap_{p \in \Pi} p^\omega A = B$. In fact, for any $x \in A$, there are $n, m \in \mathbb{Z}$ such that $nx = (mg + g_0) + (mb + b_0) + mv$ for some $g_0 \in G$ and some $b_0 \in B$. Here, if $m \neq 0$ then $mg + g_0, mb + b_0 \neq 0$. Since $\widehat{\mathbb{Z}}_{p_i}$ are p-divisible for $p \in \Pi$, the condition that $x \in \bigcap_{p \in \Pi} p^\omega A$ implies that $mg + g_0 \in \bigcap_{p \in \Pi} p^\omega \overline{A}$. This is the case only if $mg + g_0 = 0$, which clashes with the choice of g. Assume now that $A = B \oplus C$ for some $C \leq A$. We have $\overline{A} = \overline{B} \oplus \overline{C}$. Since $\overline{G}, \widehat{\mathbb{Z}}_{p_2} \leq fi\overline{A}$, it follows that $\overline{C} = \overline{G} \oplus \widehat{\mathbb{Z}}_{p_2}$, whence $a = b_0 + c = b + (g + v)$ ($b_0 \in B$, $c \in C$). Here $b = b_0$, a contradiction. Since $\overline{A}$ is an algebraically compact closure of the group A and the group $\overline{A}$ has a commutative endomorphism ring, the endomorphism ring of A, too, is commutative.

In the same way as for groups with commutative endomorphism rings, we can show [2, Prop. 19.6; 3, Thm. 4] that the quotient group A/B of A having a normal ring $E(A)$ is, up to isomorphism, representable as $\bigoplus_{p \in \Pi} G_p \subseteq A/B \subseteq \prod_{p \in \Pi} G_p = S$, where G_p are cyclic p-groups, $G_p \cong A_p$, $\Pi = \{p \in P \mid A_p \neq 0\}$, and the subgroup A/B is p-pure in S for every $p \in \Pi$ and has a commutative endomorphism ring. Moreover, if $\Pi = P$, then $B = 0$ (since B is a divisible subgroup in this case), and the ring $E(A)$ is commutative.

In view of $B \leq fiA$, the map $\Psi \colon E(A) \to E(B)$, $\Psi(f) = f|B$ is a ring homomorphism.

In Corollaries 1-6 below, we assume that A is a reduced mixed group, and if $R = E(A)$ is a normal ring, then B and G are the subgroups of A as defined in Theorem 1, and $\Pi = \{p \in P \mid A_p \neq 0\}$.

COROLLARY 1. If R is a normal ring and $|A| > 2^{\aleph_0}$, then $B \neq 0$ and $|A| \leqslant |B|^{\aleph_0}$.

Proof. Let $\overline{A}$ be an algebraically compact closure of the group A and $\overline{G}$ and $\overline{B}$ be algebraically compact closures (in $\overline{A}$) of the subgroups G and B. Then $\overline{G} \oplus \overline{B}$ is direct summand in $\overline{A}$, with $\overline{A} = \overline{G} \oplus \overline{B} \oplus C$, where C is isomorphic to the algebraically compact closure of $A/(G \oplus B)$. Therefore, $A \cap C \subseteq B \cap C = 0$. Hence, if π is a projection of the group $\overline{A}$ onto $\overline{G} \oplus \overline{B}$, then $\pi|A$ is a monomorphism. Since $\overline{G} = \prod G_p$, we have $|\overline{G}| = 2^{\aleph_0}$ for $|\Pi| = \aleph_0$; if, however, $|\Pi| < \aleph_0$, then $\overline{G} = G = \oplus G_p$ is a finite group. Therefore, if θ is a projection of $\overline{A}$ onto $\overline{B}$, then $|\theta A| = |A/G| = |A| \leqslant |\overline{B}|$. Notice that $|\overline{B}| \leqslant |B|^{\aleph_0}$ [4, Sec. 34, Exercise 9].

COROLLARY 2. Let a group A be splitting and $A = t(A) \oplus B$. A ring $R = E(A)$ is normal if and only if a ring $E(t(A))$ is commutative, $pB = B$ for every p with $A_p \neq 0$, and $E(B)$ is a normal ring.

COROLLARY 3. Let R be a normal ring and all endomorphisms of a group A be defined on its subgroup $B \oplus G$. A ring R is commutative if and only if $\Psi(R)$ is a commutative subring of $E(B)$.

Proof. The necessity is obvious.

Sufficiency. Let $\varphi, \psi \in R$ and $\alpha = \varphi\psi - \psi\varphi$. Then $\alpha(B \oplus G) = 0$ by assumption, hence $\alpha A \subseteq B$ (in view of the fact that $p(A/G) = A/G$ for $p \in P \setminus \Pi$), and so $\alpha^2 = 0$. If, however, endomorphisms are defined on the subgroup $B \oplus G$, then $\alpha(B \oplus G) = 0$ immediately implies $\alpha = 0$.

COROLLARY 4. If R is a normal ring and E is a pure torsion-free subgroup of A, then $E \subseteq B$.

Proof. For $p \in \Pi$, we have $A = A_p \oplus E_{(p)}$. If $x \in E$, then $x = a + e$, where $a \in A_p$ and $e \in E_{(p)}$. If now $p^n a = 0$, then $p^n x = p^n e \in p^\omega E$. This implies that the p-height of an element x is infinite, for E is a torsion-free group. Hence, $pE = E$ for every $p \in \Pi$.

Recall that a mixed group A is *separable* if its elements each is contained in a direct summand of A, which is a direct sum of rank 1 groups, i.e., torsion-free groups of rank 1 and cocyclic primary groups.

COROLLARY 5. If A is a separable group with a normal endomorphism ring R, then R is commutative, A is splitting, $A = t(A) \oplus B$, where B is a completely decomposable torsion-free group with commutative endomorphism ring, and $pB = B$ for every $p \in \Pi$.

Proof. According to Corollary 4, a torsion-free direct summand of A is contained in B. This, in view of the property of being separable, immediately implies that A is splitting.

It is easy to see that every nonzero p-component of a reduced group A having a reduced endomorphism ring is a cyclic group of prime order [11, Thm. 2]. An example with $\prod_{p \in \Pi} \mathbb{Z}_p$, where $\mathbb{Z}_p$ is a cyclic group of prime order p and $|\Pi| = \aleph_0$, shows that groups with reduced endomorphism rings are generally not splitting.

COROLLARY 6. If A is a group without nonzero nilpotent endomorphisms, then a subgroup G has a commutative endomorphism ring, G_p are cyclic groups of prime order, and the quotient A/B is, up to isomorphism, representable as $\bigoplus_{p \in \Pi} G_p \subseteq A/B \subseteq \prod_{p \in \Pi} G_p = S$, where $G_p \cong A_p$, and moreover, the subgroup A/B is p-pure in S for every $p \in \Pi$.

Note that if A is a reduced algebraically compact group, then A_p, as a rule, denotes its p-adic component, in which case the p-primary component of A is contained in the p-adic component. Which p-component is spoken of is usually clear from the context.

COROLLARY 7. Let A be a reduced coperiodic group and $R = E(A)$. Then the following conditions are equivalent:

(1) R is a normal ring;

(2) R is a commutative ring;

(3) A is an algebraically compact group representable as $A = G \oplus B$, where $G \cong \prod_{p \in \Pi} A_p$ and $B \cong \prod_{p \in \Pi_1} B_p$; here A_p are p-primary components of the group A, all A_p are cyclic groups, and B_p are p-adic components of the group B; also $B_p \cong \widehat{\mathbb{Z}}_p$ for every $p \in \Pi_1$, and $\Pi \cap \Pi_1 = \varnothing$.

Proof. (1) ⇒ (3). We have $A^1 \subseteq B$. This implies that $A^1 = A^1 \cap B = B^1 = 0$ (since B^1 is a divisible subgroup of a torsion-free group B), and so A is algebraically compact according to [4, Prop. 54.2]. The other statements follow from the fact that an algebraically compact closure of a subgroup G coincides with $\prod G_p$, and this subgroup is distinguished by a direct summand in A. Implications (3) ⇒ (2) and (2) ⇒ (1) are obvious.

Let $[\varphi, \psi] = \varphi\psi - \psi\varphi$ (the *Lie bracket* or *commutator*). P. A. Krylov posed the problem of studying Abelian groups A in which $[\varphi, \psi]^2 = 0$ for any $\varphi, \psi \in R = E(A)$ (cf. Cor. 3 above). Denote the class of such groups by BL_2. Clearly, a direct summand of a group in BL_2 also belongs to BL_2. If multiplication in R is replaced by *commutation*, i.e, $\varphi \circ \psi = \varphi\psi - \psi\varphi$, we face the *Lie endomorphism ring* $R^{(-)}$ of A [12, Chap. 5, Sec. 10]. A ring R is commutative iff $R^{(-)}$ is a nilpotent ring of index 2. The problem of studying properties of $R^{(-)}$ is also due to Krylov.

The class BL_2 is close to the class $\mathbf{N}$. It might be interesting to compare structures of groups in the two classes. Therefore, we lay out some facts on the groups in BL_2.

If B and G are Abelian groups, then by $\mathrm{Hom}(B, G)B$ we denote the *trace* of B in G, i.e., a subgroup generated by all homomorphic images of B in G. We write 1_A for the identity automorphism of A.

LEMMA 2. (1) If $A = \bigoplus_{j \in J} A_j$, where $A_j \le fiA$, then $A \in \mathrm{BL}_2$ if and only if $A_j \in \mathrm{BL}_2$ for every $j \in J$.

(2) If $A \in \mathrm{BL}_2$ and $A = B \oplus G$, then $\alpha(\mathrm{Hom}(B, G)B) = 0$ and $\beta(\mathrm{Hom}(G, B)G) = 0$ for any $\alpha \in \mathrm{Hom}(G, B)$ and any $\beta \in \mathrm{Hom}(B, G)$.

Proof. (1) Is obvious.

(2) Let θ be the projection $A \to G$ and $g = \gamma b$ for some $\gamma \in \mathrm{Hom}(B, G)$ and some $b \in B$. Now let $f \in E(A)$ be such that $f|B = \gamma$ and $f|G = \alpha$. We have $[\theta, f]g = -\alpha g$ and $[\theta, f]b = \gamma b$. Consequently, $[\theta, f]^2 b = -\alpha\gamma b = 0$. This entails $\alpha(\mathrm{Hom}(B, G)B) = 0$, for γ and b are arbitrary.

Denote by A' a subgroup of A generated by all of its subgroups of the form $[\xi, \eta]A$, i.e., $A' = \langle [\xi, \eta]A \mid \xi, \eta \in E(A) \rangle$ (the *E-derived subgroup* of A). Clearly, the ring $E(A)$ is commutative iff $A' = 0$. If $A = B \oplus G$, then it may so happen that $B', G' = 0$, but $A' \neq 0$, as follows from the next lemma.

LEMMA 3. If $A = B \oplus G$, then $A' = \langle \mathrm{Hom}(B, G)B, \mathrm{Hom}(G, B)G, B', G' \rangle$.

Proof. Let $\pi \colon A \to B$ and $\theta \colon A \to G$ be projections, $\gamma \in \mathrm{Hom}(B, G)$, and $0 \neq b \in B$. If $f \in E(A)$ is such that $f|B = \gamma$ and $f|G = 1_G$, then $[\theta, f]b = \gamma b$. This proves that $\mathrm{Hom}(B, G)B \subseteq A'$. If now $\xi, \eta \in E(A)$, then $[\xi, \eta]b = [(\pi + \theta)\xi, (\pi + \theta)\eta]b = [\pi\xi, \pi\eta]b + (\pi\xi\theta\eta b - \pi\eta\theta\xi b) + (\theta\xi\pi\eta b + \theta\xi\theta\eta b - \theta\eta\pi\xi b - \theta\eta\theta\xi b)$. Here $[\pi\xi, \pi\eta]b \in B'$, the second summand belongs to $\mathrm{Hom}(G, B)G$, and the third to $\mathrm{Hom}(B, G)B$. A similar argument being true for elements of G implies that A' coincides with the subgroup considered.

LEMMA 4. Let $A = B \oplus G$, where $G \le fiA$. Then the condition that $A \in \mathrm{BL}_2$ is equivalent to the fact that $B, G \in \mathrm{BL}_2$, $[\varphi, \psi](\mathrm{Hom}(B, G)B) = 0$, for any $\varphi, \psi \in E(G)$, and $\beta(B') = 0$ for any $\beta \in \mathrm{Hom}(B, G)$. In particular, if $E(B)$ and $E(G)$ are commutative rings, then $A \in \mathrm{BL}_2$.

Proof. Necessity. We extend $\varphi, \psi \in E(G)$ to endomorphisms of the group A, setting $\varphi|B = \beta \in \mathrm{Hom}(B, G)$ and $\psi|B = 0$. For $b \in B$, we have $[\varphi, \psi]b = -\psi\beta b$, whence $[\varphi, \psi]^2 b = -[\varphi, \psi]\psi\beta b$. Hence, $[\varphi, \psi]\psi(\mathrm{Hom}(B, G)B) = 0$. Symmetrically, $[\varphi, \psi]\varphi(\mathrm{Hom}(B, G)B) = 0$.

If we extend φ and ψ by setting $\varphi|B = 1_B + \beta$ and $\psi|B = 1_B - \beta$, then $[\varphi, \psi]b = 2\beta b - \varphi\beta b - \psi\beta b$. This yields $2[\varphi, \psi]\beta b = 0$. If, however, we put $\varphi|B = 1_B + 2\beta$ and $\psi|B = 1_B - \beta$, then $3[\varphi, \psi]\beta b = 0$ similarly to the above; hence $[\varphi, \psi]\beta b = 0$. Thus, $[\varphi, \psi](\mathrm{Hom}(B, G)B) = 0$.

The second null equality can be proved in a similar manner. Fix some $\beta \in \mathrm{Hom}(B, G)$. Extend $\xi, \eta \in E(B)$ to $\overline{\xi}, \overline{\eta} \in E(A)$, setting $\overline{\xi}|G = 1_G$, $\overline{\eta}|G = 0$, $\overline{\xi}|B = \xi + \beta$, and $\overline{\eta}|B = \eta$. For $b \in B$, we have

$[\overline{\xi}, \overline{\eta}]b = [\xi, \eta]b + \beta\eta b$. Since $[\xi, \eta]^2 b = 0$ and $[\overline{\xi}, \overline{\eta}]\beta\eta b = 0$, it follows that $[\overline{\xi}, \overline{\eta}]^2 b = \beta\eta[\xi, \eta]b = 0$. In view of b being arbitrary, we obtain $\beta(\eta[\xi, \eta]B) = 0$, and symmetrically, $\beta(\xi[\xi, \eta]B) = 0$.

If we extend ξ and η by setting $\overline{\xi}|G = 1_G$, $\overline{\eta}|G = 1_G$, $\overline{\xi}|B = \xi + \beta$, and $\overline{\eta}|B = \eta - \beta$, then $[\overline{\xi}, \overline{\eta}]b = [\xi, \eta]b + \beta\eta b + \beta\xi b - 2\beta b$. Hence $[\overline{\xi}, \overline{\eta}]^2 b = -2\beta[\xi, \eta]b = 0$. If, however, we put $\overline{\xi}|G = 1_G$, $\overline{\eta}|G = 1_G$, $\overline{\xi}|B = \xi + 2\beta$, and $\overline{\eta}|B = \eta - \beta$, then $[\overline{\xi}, \overline{\eta}]b = [\xi, \eta]b + 2\beta\eta b + \beta\xi b - 3\beta b$, yielding $[\overline{\xi}, \overline{\eta}]^2 b = -3\beta[\xi, \eta]b = 0$. Hence, $\beta([\xi, \eta]B) = 0$, and $\beta(B') = 0$ since ξ and η are arbitrary.

Sufficiency. Let $\pi\colon A \to B$ and $\theta\colon A \to G$ be projections and $\gamma, \delta \in E(A)$. We have $[\gamma, \delta] = (\pi + \theta)[\gamma, \delta](\pi + \theta) = \pi[\gamma, \delta]\pi + \theta[\gamma, \delta]\pi + \theta[\gamma, \delta]\theta$ (it should be taken into account that $\pi[\gamma, \delta]\theta = 0$). Here we may assume that $\theta[\gamma, \delta]\theta \in E(G)$. It remains to verify how $[\gamma, \delta]$ acts on B. If $b \in B$ then $[\gamma, \delta]b = [\pi\gamma, \pi\delta]b + \theta\gamma(\pi\delta b) - \theta\delta(\pi\gamma b) + [\theta\gamma, \theta\delta]b$. The last three summands belong to the trace of B in G, and so they are annulled under the action of $[\gamma, \delta]$. Consequently, $[\gamma, \delta]^2 b = \theta\gamma(\pi\delta[\pi\gamma, \pi\delta]b) - \theta\delta(\pi\gamma[\pi\gamma, \pi\delta]b) + [\theta\gamma, \theta\delta][\pi\gamma, \pi\delta]b = 0$. Since $\pi\gamma, \pi\delta \in E(B)$, these summands each belongs to a corresponding homomorphic image in G of the subgroup $[\pi\gamma, \pi\delta]B$.

LEMMA 5. If $A = \bigoplus_{i \in I} A_i$, $|I| > 1$, then $A \in \mathrm{BL}_2$ if and only if $A_i \in \mathrm{BL}_2$, $\alpha_i(\mathrm{Hom}(A_j, A_i)A_j) = 0$, $[\varphi_i, \psi_i](\mathrm{Hom}(A_j, A_i)A_j) = 0$, and $\alpha_i(A_i') = 0$ for any $\alpha_i \in \mathrm{Hom}(A_i, A_k)$ and any $\varphi_i, \psi_i \in E(A_i)$, where $j, k \in I \setminus \{i\}$.

Proof. The necessity follows from Lemmas 2 and 4.

Sufficiency. Let $B_j = \bigoplus_{i \in I \setminus \{j\}} A_i$, $\pi\colon A \to A_j$ and $\theta\colon A \to B_j$ be projections, and $\gamma, \delta \in E(A)$. If $a \in A_j$ then $[\gamma, \delta]a = [(\pi + \theta)\gamma, (\pi + \theta)\delta]a = [\pi\gamma, \pi\delta]a + [\pi\gamma, \theta\delta]a + [\theta\gamma, \pi\delta]a + [\theta\gamma, \theta\delta]a = [\pi\gamma, \pi\delta]a + \pi\gamma\theta\delta a - \theta\delta\pi\gamma a + \theta\gamma\theta\delta a - \theta\delta\theta\gamma a + \theta\gamma\pi\delta a - \pi\delta\theta\gamma a$. Here $\theta\delta a \in \mathrm{Hom}(A_j, B_j)A_j$ and $(\pi\gamma)|B_j \in \mathrm{Hom}(B_j, A_j)$, and so $\pi\gamma\theta\delta a = 0$. Analogously, $\pi\delta\theta\gamma a = 0$. Furthermore, $\theta\delta\pi\gamma a, \theta\gamma\theta\delta a, \theta\delta\theta\gamma a, \theta\gamma\pi\delta a \in \mathrm{Hom}(A_j, B_j)A_j$, and since homomorphisms from $\mathrm{Hom}(B_j, A_j)$ occur as factors in the brackets $[\pi\gamma, \theta\delta]$ and $[\theta\gamma, \pi\delta]$, the elements given are annulled under the action of these brackets. Keeping in mind that $[\pi\gamma, \pi\delta]^2 a = 0$ and $[\theta\gamma, \theta\delta](\mathrm{Hom}(A_j, B_j)A_j) = 0$, eventually we arrive at $[\gamma, \delta]^2 a = 0$.

THEOREM 2. If $A \in \mathrm{BL}_2$, then every nonzero p-component of A either is a cyclic group or is a direct sum of a cyclic group B_p and a group Z_{p^∞}; moreover, if $B_p \neq 0$ in the latter case, then the complementary direct summand is p-divisible.

Proof. Every cyclic direct summand B_p of a p-component A_p of A is a direct summand in A, and $A = B_p \oplus G$. If a p-component of G contained E as a cyclic direct summand, then there would exist nonzero homomorphisms $B_p \to E$ and $E \to B_p$, and moreover, the trace of one of the groups, say, of B_p in E, would coincide with E, which is a contradiction with Lemma 2(2). Therefore, if A_p is a reduced group, then $A_p = B_p$; otherwise $A_p = B_p \oplus D_p$. A decomposable divisible p-group is isomorphic to a direct sum of groups Z_{p^∞}, and Lemma 5 implies that D_p is indecomposable, i.e., $D_p \cong Z_{p^\infty}$. We have $A = (B_p \oplus D_p) \oplus C$. Lastly, the condition that $pC \neq C$ implies that there exists a nonzero composition of homomorphisms $C \to B_p \to D_p$, which clashes with Lemma 5.

COROLLARY 8. Let A be a periodic group. The inclusion $A \in \mathrm{BL}_2$ holds if and only if each one of its nonzero p-components either is a cyclic group or is a direct sum of a cyclic group and a group Z_{p^∞}. In particular, if A is a reduced group, then its endomorphism ring is commutative.

Lemmas 2 and 4 imply that a divisible group belongs to BL_2 iff its nonzero p-components are all of rank 1, and so is the torsion-free part (if it is nonzero).

THEOREM 3. Let $0 \neq D$ be the divisible part of a group A, with $A = B \oplus D$. The inclusion $A \in \mathrm{BL}_2$ holds if and only if $B, D \in \mathrm{BL}_2$, the E-derived subgroup B' of B is periodic, and if $D_p, B_p \neq 0$, then

$B/B_p = p(B/B_p)$; moreover, the condition that $0 \neq t(D) \neq D$ implies that B is periodic, in which case $A = \left(\bigoplus_{p\in\Pi} A_p\right) \oplus Q$, where Π is a set of primes, every A_p either is a cyclic p-group or is a direct sum of a cyclic p-group and a group Z_{p^∞}, and $Q \cong \mathbb{Q}$.

Proof. Necessity. If $0 \neq b \in B$ is an element of infinite order, then there exists a homomorphism $\alpha\colon B \to D$ with $\alpha b \neq 0$, since D is injective; moreover, if the torsion-free part D_0 of D is other than zero, then α can be chosen so that $\alpha b \in D_0$ and $\gamma\alpha b \neq 0$ for some $\gamma \in \mathrm{Hom}(D_0, Z_{p^\infty})$. By virtue of Lemma 5, the condition that $0 \neq t(D) \neq D$ implies that B is periodic. Lastly, if $B_p \neq 0$, then B_p is a cyclic group by Theorem 2, and so $B = B_p \oplus E_{(p)}$ for some subgroup $E_{(p)} \subseteq A$. If now $pE_{(p)} \neq E_{(p)}$, then (provided that $D_p \neq 0$) there exists a nonzero composition of homomorphisms $E_{(p)} \to B_p \to D_p$, which contradicts Lemma 5.

Sufficiency. Let $0 \neq t(D) \neq D$. We have $A = B \oplus t(D) \oplus D_0$, where $B \oplus t(D) \leq fiA$, and $E(t(D))$ is a commutative ring. According to Corollary 8, $B \oplus t(D) \in \mathrm{BL}_2$. The trace of the group D_0 in $B \oplus t(D)$ is contained in the subgroup $t(D)$, $E(t(D))$ and $E(D_0)$ are commutative rings, and Lemma 4 implies that $A \in \mathrm{BL}_2$. If $D_0 = 0$, then $E(D)$ is a commutative ring and $D \leq fiA$. Furthermore, if $B = B_p \oplus E_{(p)}$ then $pE_{(p)} = E_{(p)}$ for $B_p, D_p \neq 0$ by assumption. In this case $B' = (E_{(p)})'$ by Lemma 3, and hence $(B')_p = 0$. This yields $\beta(B') = 0$ for every $\beta \in \mathrm{Hom}(B, D)$. By Lemma 4, $A \in \mathrm{BL}_2$. Finally, let D be a torsion-free group. Then $r(D) = 1$, $D \leq fiA$, $E(D)$ is a commutative ring, and since B' is a periodic group, $\beta(B') = 0$ for every $\beta \in \mathrm{Hom}(B, D)$; hence again $A \in \mathrm{BL}_2$ by Lemma 4.

COROLLARY 9. If $0 \neq D$ is the divisible part of a group A, $A = B \oplus D$, and $0 \neq B$ is a torsion-free group, then $A \in \mathrm{BL}_2$ if and only if $E(B)$ and $E(D)$ are commutative rings.

Proof. The necessity follows from Theorem 3, and the sufficiency from Lemma 4.

COROLLARY 10. Let $A = t(A) \oplus R$ be a splitting group ($t(A)$, $R \neq 0$). We represent A as $A = T \oplus B \oplus t(D) \oplus Q$, where $T \oplus B$ and $D = t(D) \oplus Q$ are, respectively, the reduced and divisible parts of A. The group A belongs to BL_2 if and only if $t(D) \cong \bigoplus_{p\in\Pi} \mathbb{Z}_{p^\infty}$, $r(Q) \leqslant 1$, $T = \bigoplus_{p\in\Pi_1} T_p$, every T_p is a cyclic p-group, $pB = B$ for $p \in \Pi' = \Pi \cap \Pi_1$, $E(B)$ is a commutative ring, and if $Q \neq 0$, then $t(D) = 0$.

Proof. The necessity follows from Theorem 3.

Sufficiency. If $Q = 0$ then $t(A) = T \oplus t(D) \leq fiA$. Denote by G the trace of B in $t(A)$. We may write G in the form $G = G_1 \oplus G_2$, where $G_1 = \bigoplus_{p\in\Pi_1\setminus\Pi} G_p \subseteq T$ and $G_2 = \bigoplus_{p\in\Pi} G_p \subseteq t(D)$ ($pG = G$ for $p \in \Pi'$, and so $G \cap T_p = 0$ for such p). Since the subgroup $\bigoplus_{p\in\Pi_1\setminus\Pi} T_p$ is fully invariant in $t(A)$ and the rings $E(T)$ and $E(t(D))$ are commutative, it follows that $[\varphi, \psi]G = 0$ for any $\varphi, \psi \in E(t(A))$. Therefore, $A \in \mathrm{BL}_2$ by Lemma 4. If $Q \cong \mathbb{Q}$, then $A = B \oplus T \oplus Q$, where $T \oplus Q \leq fiA$ and $E(B)$ and $E(T \oplus Q)$ are commutative rings. By Lemma 4, $A \in \mathrm{BL}_2$ again.

THEOREM 4. (1) Let A be a completely decomposable torsion-free group and $A = B \oplus D$, where D is the divisible part of A. The inclusion $A \in \mathrm{BL}_2$ holds if and only if the following are satisfied:

(a) if $D \neq 0$, then $r(D) = 1$, and B is a direct sum of rank 1 groups of mutually incomparable types;

(b) if $D = 0$, then $A = \bigoplus_{i\in I} A_i$, where either $r(A_i) = 1$ or $A_i = B_i \oplus C_i$, $r(B_i) = 1$, C_i is a direct sum of rank 1 groups of mutually incomparable types greater than $t(B_i)$, and moreover, types of direct summands of rank 1 in groups A_i and A_j are incomparable, for $i \neq j$.

(2) Let A be a separable (vector) torsion-free group and $A = B \oplus D$, where D is the divisible part of A. The inclusion $A \in \mathrm{BL}_2$ holds if and only if the following are satisfied:

(a) if $D \neq 0$, then $r(D) = 1$ and B is a direct sum (direct product) of rank 1 groups of mutually incomparable types;

(b) if $D = 0$, then $A = \bigoplus_{i \in I} A_i$ $(A = \prod_{i \in I} A_i)$, where either $r(A_i) = 1$ or $A_i = B_i \oplus C_i$, $r(B_i) = 1$, and C_i is a separable (vector) group in which types of direct summands of rank 1 are incomparable and are greater than $t(B_i)$; moreover, types of direct summands of rank 1 in A_i and in A_j are incomparable, for $i \neq j$.

Proof. (1) The necessity follows from Lemma 5. Indeed, for a direct summand $N_1 \oplus N_2 \oplus N_3$ in A, where $r(N_i) = 1$, the following relations for types are impossible: $t(N_1) = t(N_2)$ and $t(N_1) \leqslant t(N_2) \leqslant t(N_3)$.

The sufficiency in item (a) follows from Lemma 4 since $D \leq fiA$ and $E(B)$ and $E(D)$ are commutative rings, and in (b) from the fact that $A_i \leq fiA$, where $A_i \in \mathrm{BL}_2$ in accordance with Lemma 4.

(2) Direct summands of separable groups are themselves separable groups. Furthermore, the paragraph before Proposition 1 implies that $A = \bigoplus_{i \in I} A_i$, where $\Omega(A_i) = \Omega_i$ and $A_i \leq fiA$, i.e., types in Ω_i and in Ω_j are incomparable, for $i \neq j$ (see [2, Sec. 19, Exercise 7]). With these facts in mind, we can prove the remaining statements similarly to how we did in item (1).

THEOREM 5. Let A be a coperiodic group, D its divisible part, $A = B \oplus D$, and $D = t(D) \oplus D_0$. Then $A \in \mathrm{BL}_2$ if and only if A is algebraically compact, $t(D) \cong \bigoplus_{p \in \Pi} Z_{p^\infty}$, where Π is a set of primes, $r(D_0) \leqslant 1$, and moreover, the following statements hold:

(a) if $0 \neq t(D) \neq D$, then $B = \bigoplus_{p \in \Pi_1} B_p$, where each B_p is a cyclic p-group, and Π_1 is a finite set of primes;

(b) if $D_0 = 0$, then $B = G \oplus C$, $G = \prod_{p \in \Pi_1} B_p$, where each B_p is a cyclic p-group, $C \cong \prod_{p \in \Pi_2} \widehat{\mathbb{Z}}_p$, Π_1 and Π_2 are sets of primes such that $\Pi \cap \Pi_1 \cap \Pi_2 = \varnothing$, and if $D \neq 0$, then the set $\Pi_1 \cap \Pi_2$ is finite.

Proof. Necessity. We have $A^1 = D \oplus B^1$. If $B_p \neq 0$ and $B = B_p \oplus E_{(p)}$, then $B^1 = E^1_{(p)}$. This implies that B^1 is a divisible torsion-free subgroup of B; hence $B^1 = 0$ and $A^1 = D$. Therefore, the group A is algebraically compact [4, Prop. 54.2]. If $0 \neq t(D) \neq D$ then B is a periodic group by Theorem 3. Every periodic algebraically compact group is bounded [4, Cor. 40.3], which proves item (a). If $D_0 = 0$, then the closure $G = (t(B))^-$ in the $\mathbb{Z}$-adic topology of the periodic part $t(B)$ is distinguished in B by a direct summand, and $B = G \oplus C$ $(\Pi_1 = \{p \in P \mid B_p \neq 0\})$. If $D_p, B_p \neq 0$, then $pC = C$, and so $\Pi \cap \Pi_1 \cap \Pi_2 = \varnothing$. If the set $\Pi_1 \cap \Pi_2$ is infinite, then the trace of the group C in G is a mixed group. This, on a condition that $D \neq 0$, clashes with Theorem 3 in view of Lemma 3.

Sufficiency. In (a), $A \in \mathrm{BL}_2$ by Theorem 3. If (b) holds, then $D \leq fiA$, $G \leq fi(G \oplus C)$, and $E(G)$ and $E(C)$ are commutative rings. By Lemma 4, therefore, $G \oplus C \in \mathrm{BL}_2$. By Lemma 3, $(G \oplus C)' = \mathrm{Hom}(C, G)C$. The set $\Pi_1 \cap \Pi_2$ is finite; so $(G \oplus C)'$ is a periodic group, and $\beta(G \oplus C)' = 0$ for every $\beta \in \mathrm{Hom}(G \oplus C, D)$ by virtue of $\Pi \cap \Pi_1 \cap \Pi_2 = \varnothing$. Hence, $A \in \mathrm{BL}_2$ by Lemma 4.

LEMMA 6. Let S be an associative ring and $a, b \in S$. In S, the commutation operation $a \circ b = ab - ba$ is associative if and only if every commutator $[a, b] = ab - ba$ of S lies in its center $Z(S)$.

Proof. Necessity. We have $[[b, c], a] = bca - cba - abc + acb$, $[b, [c, a]] = bca - bac - cab + acb$. Equating the right parts, we obtain $0 = abc - bac - cab + cba = [[a, b], c]$. This yields $[a, b] \in Z(S)$ since c is arbitrary.

The sufficiency is obvious.

Let A be an Abelian group, $R = E(A)$, and $R^{(-)}$ be the Lie endomorphism ring of A (cf. paragraph after Cor. 7). Of interest is the problem of studying Abelian groups A whose endomorphism rings R satisfy the identity $[x_1, \ldots, x_n] = 0$, i.e., Abelian groups with Lie endomorphism rings $R^{(-)}$ nilpotent of index n. According to Lemma 6, being nilpotent of index 3 for a ring $S^{(-)}$ is equivalent to its being associative.

Since every central endomorphism of A acts invariantly on its direct summands, it follows from the proof of Lemma 2 that the property of being associative for the ring $R^{(-)}$ implies being normal for the ring $R = E(A)$.

REFERENCES

1. G. E. Puninskii and A. A. Tuganbaev, *Rings and Modules* [in Russian], Soyuz, Moscow (1998).
2. P. A. Krylov, A. A. Mikhalev, and A. A. Tuganbaev, *Abelian Groups and Their Endomorphism Rings* [in Russian], Faktorial Press, Moscow (2006).
3. T. Szele and J. Szendrei, "On Abelian groups with commutative endomorphism ring," *Acta Math. Acad. Sci. Hung.*, **2**, 309-324 (1951).
4. L. Fuchs, *Infinite Abelian Groups*, Vols. 1 and 2, Academic Press, New York (1970), (1973).
5. D. A. Lawver, "Abelian groups in which endomorphic images are fully invariant," *J. Alg.*, **29**, 232-245 (1974).
6. D. A. Lawver, "On the commutativity and generalized regularity of $\mathcal{E}(G)$," *Acta Math. Acad. Sci. Hung.*, **24**, 107-112 (1973).
7. J. D. Reid, "On subcommutative rings," *Acta Math. Acad. Sci. Hung.*, **16**, 23-26 (1965).
8. A. Orsatti, "Su di un problema di T. Szele, e J. Szendrei," *Rend. Sem. Mat. Univ. Padova*, **35**, 171-175 (1965).
9. D. Arnold, *Finite Rank Torsion Free Abelian Groups and Rings, Lect. Notes Math.*, **931**, Springer, New York (1982).
10. P. Schultz, "On a paper of Szele and Szendrei on groups with commutative endomorphism rings," *Acta Math. Acad. Sci. Hung.*, **24**, 59-63 (1973).
11. S. F. Kozhukhov, "Abelian groups without nilpotent endomorphisms," in *Abelian Groups and Modules* [in Russian], Tomsk Univ., Tomsk (1979), pp. 87-94.
12. A. G. Kurosh, *General Algebra* [in Russian], Nauka, Moscow (1973).