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SYMMETRY OF CUTS IN FIELDS OF FORMAL POWER SERIES

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INTRODUCTION

The theory of cuts is an effective tool for studying ordered fields. Specifically, order characteristics of cuts [1] go back to Hausdorff [2]; concepts of algebraic and transcendent cuts in an ordered field were introduced in [3, 4]; definitions of symmetric and asymmetric cuts were couched in [3]. Properties of such cuts were taken up in [5-8], and symmetric cuts in fields of restricted formal power series were treated in [5, 8-10]. The present paper continues research into the relationship between the structure of cuts in a field of formal power series and algebraic properties of the field. Our main results are Theorems 2.1 and 3.2.

1. FIELDS OF FORMAL POWER SERIES

Let G be a linearly ordered multiplicative Abelian group. Following [11], by $\mathbb{R}[[G]]$ we denote a set of formal power series of the form

$$x = \sum_{g\in G} r_g g, \tag{1}$$

where r_g are real numbers, and $\operatorname{supp} x = \{g \in G \mid r_g \neq 0\}$ is a well-antiordered subset of G; in other words, every subset $A \subset \operatorname{supp} x$ contains a maximal element. For x specified by a series such as in (1), we put $x(g) = r_g$, $g \in G$. On $\mathbb{R}[[G]]$, addition and multiplication are defined thus: if

$$y, z \in \mathbb{R}[[G]],\ y = \sum_{g\in G} {r_g}' g,\ z = \sum_{g\in G} {r_g}'' g,$$

then

$$y + z = \sum_{g\in G} ({r_g}' + {r_g}'')g,\ yz = \sum_{g\in G} r_g g,$$

where

$$r_g = \sum_{g_1 g_2 = g | g_1,\, g_2 \in G} {r_{g_1}}' {r_{g_2}}''. \tag{2}$$

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The sum in (2) turns out to be finite, and hence the coefficients r_g are well defined. Furthermore, the supports supp $(y+z)$ and supp (yz) are well antiordered. The set $\mathbb{R}[[G]]$, on which the above-mentioned operations are defined, is a field, which we call a *field of formal power series w.r.t. a group* G (details and proof can be found in [11]; in some other notation, in [1]).

Let $x \in \mathbb{R}[[G]]$. Denote by $\widehat{x}$ an Archimedean equivalence class in $\mathbb{R}[[G]]$ containing x (see [11]). We have

LEMMA 1.1. Let $x, y \in \mathbb{R}[[G]]$, $g_1, g_2 \in G$, and $g_1 < g_2$. Then $x(g_2) = y(g_2)$ if $|x - y| \in g_1$.

2. CUTS IN A FIELD OF FORMAL POWER SERIES

Let K be an ordered field and K^+ its positive cone. We call the set

$$D(A, B) = \{y - x \mid x \in A,\ y \in B\}$$

the *difference of a cut* (A, B). Denote by $V(A, B)$ a set of Archimedean classes of all elements in $D(A, B)$. The bank A in (A, B) in a linearly ordered field is said to be *short* if there is $a \in A$ such that $(a_1+(a_1-a)) \in A$ for every $a_1 \in A$. Such an element a_1 is referred to as being *close* to the bank B. A bank of the cut that is not short is said to be *long*. At least one bank of each cut is long. If both of the banks in (A, B) are long then we call (A, B) a *symmetric* cut. If, however, one of these is short then we call (A, B) an *asymmetric* cut (see [5, 6]).

THEOREM 2.1. Let G be a multiplicative linearly ordered Abelian group. Then all cuts in the field $\mathbb{R}[[G]]$ of formal power series are asymmetric.

Proof. 1. For brevity, let $K = \mathbb{R}[[G]]$. Assume that (A, B) is a cut in K. We need to find a point that belongs to one of the banks in this cut and is close to the other. In view of [6], this will imply that the given cut is asymmetric. We seek for the desired point in a series of the form

$$x^* = \sum_{g \in G} r^*_g g.$$

(a) For all $g \in G$, $g < V(A, B)$, put $r^*_g = 0$.

(b) Let $h_1 \in V(A, B)$. Define r^*_g on the set $G_{h_1} = \{g \in G \mid g > h_1\}$ as follows. By the definition of $V(A, B)$, there are $a_1 \in A$ and $b_1 \in B$ such that $h_1 = \widehat{b_1 - a_1}$. Set

$$[a_1, b_1] = \{x \in K \mid a \leq x \leq b\}.$$

Let $g > h_1$, $g \in G$. By Lemma 1.1,

$$\forall x (x \in [a_1, b_1] \Rightarrow a_1(g) = b_1(g) = x(g)). \tag{3}$$

Set $r^*_g = a_1(g)$. The value r^*_g does not depend on the choice of a_1, b_1, h_1. Indeed, let

$$h_2 \in V(A, B),\ h_2 < g,\ a_2 \in A,\ b_2 \in B,\ \widehat{b_2 - a_2} = h_2.$$

By Lemma 1.1,

$$\forall x (x \in [a_2, b_2] \Rightarrow a_2(g) = b_2(g) = x(g)). \tag{4}$$

Set $a = \max\{a_1, a_2\}$ and $b = \min\{b_1, b_2\}$. Clearly, $[a, b] = [a_1, b_1] \cap [a_2, b_2]$. Let $x \in [a, b]$. Then (3) and (4) imply

$$x(g) = a(g) = b(g) = a_1(g) = a_2(g) = r^*_g.$$

Thus r_g^* is defined for all $g \neq \min V(A,B)$.

2. (a) First, consider the case where $\min V(A,B)$ does not exist. Then r_g^* is defined for all $g \in G$. Put

$$x^* = \sum_{g \in G} r_g^* g.$$

We verify that the set $\operatorname{supp} x^*$ is inversely well ordered. Indeed, let $g_0 \in V(A,B)$, $a_0 \in A$, $b_0 \in B$, and $\widehat{b_0 - a_0} = g_0$. Then $x^*(g) = a_0(g)$ with $g > g_0$. Therefore,

$$(\operatorname{supp} x^*) \cap \{g > g_0 \mid g \in G\} = (\operatorname{supp} a_0) \cap \{g > g_0 \mid g \in G\}. \tag{5}$$

Thus (5) holds for all $g_0 \in V(A,B)$, and hence every interval of the set $(\operatorname{supp} x^*) \cap V(A,B)$ is well antiordered. Hence $(\operatorname{supp} x^*) \cap V(A,B)$ is inversely well ordered. Since $r_g^* = 0$ whenever $g < V(A,B)$, $\operatorname{supp} x^*$ is well antiordered. Thus $x^* \in \mathbb{R}[[G]]$.

(b) To be specific, assume that $x^* \in A$. Let $x_1 \in A$, $x^* < x_1$. We claim that $\widehat{x_1 - x^*} < V(A,B)$. Suppose the contrary, letting

$$\widehat{x_1 - x^*} = g_1,\ g_1 \in V(A,B). \tag{6}$$

Making use of the fact that $V(A,B)$ has no least element, we choose $a \in A$ and $b \in B$ so that $\widehat{b-a} < g_1$. By the construction of x^*, $x^*(g) = b(g) = a(g)$ hold for all $g \geq g_1$. Since $x^* < x_1 < b(g)$, it follows that $x^*(g) = x_1(g)$ for any $g \geq g_1$. Hence $\widehat{x_1 - x^*} < g_1$, which is a contradiction with (6). Thus, for all $x_1 \in A$, $x_1 > x^*$, we have

$$\widehat{x_1 - x^*} < V(A,B). \tag{7}$$

(c) We verify that $x_1 + (x_1 - x^*) \in A$. Assume to the contrary that $x_1 + (x_1 - x^*) \in B$. Since $x^* \in A$, the definition of $V(A,B)$ implies $(x_1 + (\widehat{x_1 - x^*})) - x^* \in V(A,B)$, i.e., $2(\widehat{x_1 - x^*}) \in V(A,B)$. Consequently, $(\widehat{x_1 - x^*}) \in V(A,B)$, which is a contradiction with (7). In this instance

$$\forall x((x \in A, x > x^*) \Rightarrow x + (x - x^*) \in A).$$

Hence x^* is close to B and (A,B) is asymmetric (see [6]).

3. Consider the case where $g_0 = \min V(A,B)$ exists. Now r_g^* is defined for all $g \in G$, $g > g_0$. By the definition of $V(A,B)$, there are $a \in A$ and $b \in B$ such that $\widehat{b-a} = g_0$. Let

$$a = \sum_{g \in G} r'_g g, \tag{8}$$

$$b = \sum_{g \in G} r''_g g. \tag{9}$$

At the moment, we represent the sum in (8) and in (9) as three summands, setting

$$a = \sum_{g > g_0} r'_g g + r'_{g_0} g_0 + \sum_{g < g_0} r'_g g,$$

$$b = \sum_{g > g_0} r''_g g + r''_{g_0} g_0 + \sum_{g < g_0} r''_g g.$$

By the definition of r_g^*, $r'_g = r''_g = r_g^*$ for all $g > g_0$. Therefore,

$$b - a = (r''_{g_0} - r'_{g_0}) g_0 + \sum_{g < g_0} (r''_g - r'_g) g.$$

Since $b - a > 0$, we have $(r''_{g_0} - r'_{g_0}) \geq 0$.

Let $x \in A$, $x \geq a$, $y \in B$, and $y \leq b$. It is easy to see that

$$r'_{g_0} \leq x(g_0) \leq y(g_0) \leq r''_{g_0}.$$

Therefore, there exist $r' = \sup\limits_{x \in A,\, x > a} x(g_0)$ and $r'' = \inf\limits_{y \in B,\, y < b} y(g_0)$. We verify that $r' = r''$. Set $a_0 = a$, $b_0 = b$, and $b_0(g_0) - a_0(g_0) = d_0$. The definition of g_0 implies $d_0 > 0$. Consider an element $c_0 = \frac{a_0 + b_0}{2}$. If $c_0 \in A$ then we put $a_1 = c_0$ and $b_1 = b_0$. If $c_0 \in B$ then we put $a_1 = a_0$ and $b_1 = c_0$. Obviously, $b_1(g_0) - a_1(g_0) = \frac{d_0}{2}$ in either case. If we continue this process we obtain an (nonstrictly) ascending sequence (a_n), $a_n \in A$, and a descending sequence (b_n), $b_n \in B$, for which $b_n(g_0) - a_n(g_0) = \frac{d_0}{2^n}$. Consequently,

$$\sup_{n \in \mathbb{N}} a_n(g_0) = \inf_{n \in \mathbb{N}} b_n(g_0).$$

This in turn implies that $\sup\limits_{x \in A,\, x > a} x(g_0) = \inf\limits_{y \in B,\, y < b} y(g_0)$, i.e., $r' = r''$.

4. Let $r^*_{g_0} = r'$. Set

$$x_0 = \sum_{g \geq g_0} r^*_g g.$$

To be specific, assume that

$$x_0 \in A. \tag{10}$$

The sequence (a_n) is monotonically (nonstrictly) ascending; so the number sequence $(a_n(g_0))$, too, is ascending. There are two cases to consider: (a) starting with some n_0, $(a_n(g_0))$ stabilizes; (b) such n_0 does not exist.

(a) Clearly, in this case

$$\forall n \ ((n \in \mathbb{N},\, n \geqslant n_0) \Rightarrow \forall g \leq g_0 (a_n(g) = a_{n_0}(g)).$$

This implies

$$\forall x \ ((x \in A,\ x \geq a_{n_0}) \Rightarrow \forall g \leq g_0 (x(g) = a_{n_0}(g)).$$

Set $x^* = a_{n_0}$. We verify that x^* is close to B. Let $x_1 \in A$, $x^* < x_1$. Then $\forall g \leq g_0 (x_1(g) = a_{n_0}(g))$. Hence,

$$\widehat{x_1 - x^*} < V(A, B). \tag{11}$$

We verify that $(x_1 + (x_1 - x^*)) \in A$. For brevity, let $(x_1 + (x_1 - x^*)) = q$. Now $q \in B$ would imply $\widehat{q - x^*} \in V(A, B)$. On the other hand,

$$\widehat{q - x^*} = 2(\widehat{x_1 - x^*}) = (\widehat{x_1 - x^*}),$$

and $\widehat{q - x^*} < V(A, B)$, as follows from (11). Thus, for every $x_1 \in A$, $x_1 > x^*$, we have $(x_1 + (x_1 - x^*)) \in A$. Consequently, x^* is close to B.

(b) For each $n_0 \in \mathbb{N}$, there is $n_1 \in \mathbb{N}$, $n_1 > n_0$, such that $a_{n_0}(g_0) < a_{n_1}(g_0)$. Note that $\forall n \, (n \in \mathbb{N} \Rightarrow a_n(g_0) < r')$. In this instance we set

$$x^* = x_0 = \sum_{g \geq g_0} r^*_g g. \tag{12}$$

According to (10), $x^* \in A$. For all natural n, $a_n < x^*$. In fact, if for some n we had $x^* \leq a_n$, then it would be true that

$$x^*(g_0) \leq a_n(g_0) < r',\ x^*(g_0) < r'.$$

However, $x^*(g_0) = r'$ in view of (12). Hence $x^* \in [a_n, b_n]$ for all natural n.

We verify that x^* is close to B. Let $x_1 \in A$, $x_1 > x^*$. Then $a_n < x^* < x_1 < b_n$ for all natural n. This implies $\widehat{x_1 - x^*} < g_0$, and so $\widehat{x_1 - x^*} < V(A, B)$. Further reasoning is as in (a). The theorem is proved.

A linearly ordered field K is said to be *Archimedean closed* if every linearly ordered extension of K contains at least one element which is not Archimedean equivalent to any element in the field K.

COROLLARY 2.2. Let G be a multiplicative linearly ordered Abelian group. Then the field $\mathbb{R}[[G]]$ of formal power series is Archimedean closed.

Indeed, all cuts in $\mathbb{R}[[G]]$ are asymmetric, and $\mathbb{R}[[G]]$ is Archimedean closed in view of [5, Thm. 1].

3. FIELDS OF BOUNDED FORMAL POWER SERIES

Let G be an ordered commutative group and β a (infinite) cardinal. Put

$$\mathbb{R}[[G, \beta]] = \{x \in \mathbb{R}[[G]] \mid \operatorname{card} \operatorname{supp} x < \beta\}.$$

The set $\mathbb{R}[[G, \beta]]$ is a subfield of the field $\mathbb{R}[[G]]$ of formal power series (see [1]). We call it a *field of bounded power series.*

For the regular cardinal β, in [8] it was proved that if (A, B) is a symmetric cut in $\mathbb{R}[[G, \beta]]$ then $\operatorname{cf}(\beta) \leqslant \operatorname{cf}(A, B) \leqslant \operatorname{card}(G)$. Modifying that proof relative to an arbitrary cardinal β, we obtain the following:

LEMMA 3.1. If (A, B) is a symmetric cut in $\mathbb{R}[[G, \beta]]$, then $\operatorname{cf}(\beta) \leqslant \operatorname{cf}(A, B) \leqslant \operatorname{card}(G)$.

Every element of $\mathbb{R}[[G]]$ induces a symmetric cut in $\mathbb{R}[[G, \beta]]$. Inversely, every symmetric cut in $\mathbb{R}[[G, \beta]]$ is induced by some element of $\mathbb{R}[[G]]$ (see [12]).

We say that a linearly ordered set X is *inversely similar* to an ordered set Y if there exists the bijection

$$f : X \to Y$$

such that the fact that $x_1, x_2 \in X$, $x_1 \le x_2$, implies $f(x_1) \ge f(x_2)$. A cut (A, B) in a field F is said to be *fundamental* if there are $a \in A$ and $b \in B$ such that $b - a \le \varepsilon$ for every positive $\varepsilon \in F$.

A symmetric cut (A, B) in $\mathbb{R}[[G, \beta]]$ is fundamental iff there is $x_0 \in \mathbb{R}[[G]] \setminus \mathbb{R}[[G, \beta]]$ such that $A < x_0 < B$, and $\operatorname{supp} x_0$ is inversely similar to β and is coinitial in G (see [12]).

THEOREM 3.2. Let β be a cardinal, G a commutative group, and $\aleph_0 < \beta \leqslant \operatorname{card} G$. Then the cofinality of every symmetric cut in a field $K = \mathbb{R}[[G, \beta]]$ is equal to $\operatorname{cf}(\beta)$. In particular, if β is a regular cardinal, then every symmetric cut in K has cofinality β.

Proof. Let (A, B) be a symmetric cut in $K = \mathbb{R}[[G, \beta]]$. Suppose that there exists $a \in \mathbb{R}[[G]]$ such that $A < a < B$ (see [12]). Then

$$a = \sum_{g \in G} r_g g,$$

where r_g are real numbers and $\operatorname{supp} a = \{g \in G \mid r_g \neq 0\}$ is a well-antiordered subset of G. Since $a \notin \mathbb{R}[[G, \beta]]$, $\operatorname{card} \operatorname{supp} a \geqslant \beta$.

We number all elements of the set $\operatorname{supp} a$ in increasing order so that

$$\operatorname{supp} a = \{g_\tau \mid \tau < \gamma\},$$

where γ is an ordinal, and $\beta < \gamma$. Put $r_\tau = r_{g_\tau}$. Now,

$$a = \sum_{\tau<\gamma} r_\tau g_\tau. \tag{13}$$

We proceed to construct the following two transfinite sequences of elements in the field K:

$$(x_\delta | \delta < \beta), \; (y_\delta | \delta < \beta),$$

where

$$\begin{aligned} x_\delta &= \sum_{\tau\leq\delta} r_\tau g_\tau - 1 g_\tau, \\ y_\delta &= \sum_{\tau\leq\delta} r_\tau g_\tau + 1 g_\tau. \end{aligned} \tag{14}$$

First, we mention that each x_δ, $x_\delta \in K$, is smaller than a. Hence $x_\delta \in A$. Similarly, $y_\delta \in B$. It is easy to see that the sequence (x_δ) is strictly monotonically ascending. Analogously, (y_δ) is strictly monotonically descending.

We claim that (y_δ) is coinitial in B. Let $y_0 \in B$, $y_0 = \sum_{g\in G} r'_g g$. Then

$$y_0 - a = \sum_{g\in G} (r'_g - r_g) g.$$

Since $y_0 - a > 0$, the first coefficient other than zero is $r'_{g^*} - r_{g^*} > 0$. We have

$$(y_0 - a) \sim 1 g^*. \tag{15}$$

There is an ordinal $\tau_0 < \beta$ such that $g^* > g_{\tau_0}$. Assume to the contrary that $g^* \leq g_\tau$ for all $\tau < \beta$. This implies that $r'_\tau - r_\tau = 0$ for any $\tau < \beta$. Hence $\operatorname{supp} y_0 \supset \operatorname{supp} a$. Consequently,

$$\operatorname{card} \operatorname{supp} y_0 \geqslant \operatorname{card} \operatorname{supp} a \geqslant \beta.$$

Therefore, $\operatorname{card} \operatorname{supp} y_0 \geqslant \beta$, and hence $y_0 \notin \mathbb{R}[[G, \beta]]$, a contradiction with the choice of $y_0 \in \mathbb{R}[[G, \beta]]$. Thus, there is an ordinal $\tau_0 < \beta$ for which $g^* > g_{\tau_0}$.

By (14),

$$y_{\tau_0} = \sum_{\tau\leq\tau_0 1} r_\tau g_\tau - 1 g_\tau. \tag{16}$$

In view of (13) and (16),

$$(y_{\tau_0} - a) \sim 1 g_{\tau_0}.$$

At the same time, $(y_0 - a) \sim 1 g^*$ by (15). On the other hand, $g_{\tau_0} \ll g^*$. Hence $(y_{\tau_0} - a) < (y_0 - a)$, whence $x_{\tau_0} < y_0$. Thus, there is $y_{\tau_0} < y_0$ for every $y_0 \in B$. Consequently, the transfinite sequence (y_δ) is coinitial in B.

The sequence (y_δ) is strictly descending; so $\operatorname{coi}(y_\delta) = \operatorname{cf}(\beta)$. Hence $\operatorname{coi} B = \operatorname{cf}(\beta)$. Similarly, we can show that $\operatorname{cf}(A) = \operatorname{cf}(\beta)$. The theorem is proved.

We know from [7] that every fundamental cut in F, which is not produced by any element of F (a proper fundamental cut) is symmetric. An ordered field is said to be *Dedekind complete* if every fundamental cut of the field is produced by some of its elements: that is, the field is freed of fundamental symmetric cuts.

THEOREM 3.3. If (A, B) is a proper fundamental cut of a field F, then $\operatorname{cf}(A, B) = \operatorname{cf}(F)$.

Proof. Put $\delta = \mathrm{cf}(A)$. Let $(a_\gamma)_{\gamma<\delta}$ be a strictly monotonically ascending sequence, cofinal in A. Analogously, $(b_\gamma)_{\gamma<\delta}$ is a strictly monotonically descending sequence, coinitial in B.

Put $c_\gamma = b_\gamma - a_\gamma$. The sequence $(c_\gamma)_{\gamma<\delta}$ is coinitial in the set $F^+ \setminus \{0\}$. Since this sequence is anti-isotonically isomorphic to a regular cardinal δ, we have $\mathrm{coi}(F^+ \setminus \{0\}) = \delta = \mathrm{cf}(A, B)$. Lastly, $({c_\gamma}^{-1})_{\gamma<\delta}$ is cofinal in F and is isotonically isomorphic to the regular cardinal δ. Consequently, F has cofinality $\delta = \mathrm{cf}(A, B)$, as required.

COROLLARY 3.4. If $\mathrm{cf}(\beta) \neq \mathrm{cf}(\mathbb{R}[[G, \beta]])$, then $\mathbb{R}[[G, \beta]]$ has no proper fundamental cuts, that is, this field is Dedekind complete.

By Theorem 3.2, all symmetric cuts in a field of bounded formal power series have the same cofinality. The question remains open as to whether there exist linearly ordered fields having symmetric cuts of nonequal cofinality. Furthermore, let A be some set of infinite cardinals. Does, then, there exist an ordered field K such that the set of cofinalities of all symmetric cuts in K is equal to A?

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