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THE UNIVERSAL LACHLAN SEMILATTICE WITHOUT THE GREATEST ELEMENT

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We deal with some upper semilattices of m-degrees and of numberings of finite families. It is proved that the semilattice of all c.e. m-degrees, from which the greatest element is removed, is isomorphic to the semilattice of simple m-degrees, the semilattice of hypersimple m-degrees, and the semilattice of Σ^0_2-computable numberings of a finite family of Σ^0_2-sets, which contains more than one element and does not contain elements that are comparable w.r.t. inclusion.

INTRODUCTION

Lachlan in [1] described semilattices that are isomorphic to principal ideals of an upper semilattice of computably enumerable (c.e.) m-degrees. Currently such semilattices are conventionally referred to as *Lachlan's*. It is easy to show that the class of Lachlan semilattices coincides with a class of semilattices that are principal ideals in any fixed non-trivial semilattice of Σ^0_1-computable (i.e., computable in the classical sense) numberings of finite families of c.e. sets, and also with a class of semilattices that are segments in semilattices of Σ^0_1-computable numberings of arbitrary families of c.e. sets. In [2], it was stated that any Lachlan semilattice can be embedded as an ideal in an arbitrary non-trivial semilattice of Σ^0_n-computable numberings, for $n \geqslant 2$. In [3], it was proved that the class of Lachlan semilattices coincides with a class of distributive upper semilattices with the greatest and least elements, having Σ^0_3-presentations.

Denisov in [4] explored universal Lachlan semilattices, that is, Lachlan semilattices with the extension property for embeddings (for details, see below). He proved that the universal semilattice is unique up to isomorphism, and established that the universal property is shared by the following: the semilattice of all c.e. m-degrees, any principal upper cone of this semilattice, and the semilattice of Σ^0_1-computable numberings of a finite family of c.e. sets containing a unique element that is not maximal w.r.t. inclusion. A groundwork for this result was Denisov's extension theorem for embeddings, which subsequently was strengthened by Ershov in [5].

In the present paper we prove that the following upper semilattices are isomorphic to the universal Lachlan semilattice from which the greatest element is removed: the semilattice of simple m-degrees, the semilattice of hypersimple m-degrees, and the semilattice of Σ^0_2-computable numberings of a finite family

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consisting of Σ^0_2-sets that are pairwise non-comparable w.r.t. inclusion. Note that, for the last-mentioned case, Denisov announced this result in [4] for a two-element family; but its proof was not published.

The ideas behind our proofs are for the most part borrowed from [4, 5]. A construction in the proof of Theorem 3 for simple and hypersimple m-degrees is properly the construction from Theorem 1 in [4], in which some details have been modified and requirements added for simplicity and hypersimplicity.

1. THE UNIVERSAL SEMILATTICE

The basic notions pertaining to computability theory can be found in [6], to the lattice theory — in [7], and to the theory of numberings — in [8]. We expect the reader to be familiar with them. In the introduction to [8] is also some useful information on distributive semilattices.

For the value of a numbering ν on a natural number x, we will, following a common practice, write νx instead of $\nu(x)$, dropping the parentheses. For a partial function f, by δf we denote its domain, and by ρf its range. For a preordered set $\mathcal{A} = \langle A, \leqslant\rangle$, we denote its associated poset by $\widetilde{\mathcal{A}} = \langle \widetilde{A}, \leqslant\rangle$ (keeping the same denotation for the preorder and its associated order), and the element $\widetilde{A}$ containing $x \in A$ (an equivalence class) by $[x]_{\mathcal{A}}$ (or merely by $[x]$, if there is clarity as to which $\mathcal{A}$ is being spoken of). A preordered set $\mathcal{A}$ is called a *prelattice* (an *upper presemilattice*, a *lower presemilattice*) if $\widetilde{\mathcal{A}}$ is a lattice (an upper semilattice, a lower semilattice). A prelattice (an upper presemilattice) is said to be *distributive* if its associated lattice (upper semilattice) is distributive. In what follows, upper semilattices (upper presemilattices) will be referred to merely as semilattices (presemilattices), since lower semilattices will not be considered. For a prelattice (presemilattice) $\mathcal{A}$, writing $\mathcal{A} = \langle A, \leqslant; u, v\rangle$ ($\mathcal{A} = \langle A, \leqslant; u\rangle$) means that u and v are binary operations on $\mathcal{A}$, which on $\widetilde{\mathcal{A}}$ present operations of taking least upper and greatest lower bounds, respectively (i.e., for any $x, y \in A$, $[u(x, y)] = \sup\{[x], [y]\}$ and $[v(x, y)] = \inf\{[x], [y]\}$). If the semilattice $\mathcal{L}$ has a least or greatest element, then we denote them by $\bot_{\mathcal{L}}$ and $\top_{\mathcal{L}}$, respectively (or simply by $\bot$ and $\top$, if there is clarity as to which semilattice is being dealt with).

Let $\mathcal{L} = \langle L, \leqslant^{\mathcal{L}}; \vee^{\mathcal{L}}\rangle$ be a distributive semilattice with the greatest and least elements. Let η be a numbering for L. Following [3], we say of a pair $\langle \mathcal{L}, \eta\rangle$ that

(1) $\langle \mathcal{L}, \eta\rangle$ *is* Ω_2 if the relation $\eta x \leqslant^{\mathcal{L}} \eta y$ is Σ^0_3 in the arithmetic hierarchy and there exists a computable function u such that $\eta u(x, y) = \eta x \vee^{\mathcal{L}} \eta y$ for all $x, y \in \mathbb{N}$;

(2) $\langle \mathcal{L}, \eta\rangle$ *is* Λ_1 if there is a sequence $\{\mathcal{D}_i = \langle D_i, \leqslant_i\rangle\}_{i\in\mathbb{N}}$ of finite distributive prelattices enjoying the following properties:

(a) $D_0 \subseteq D_1 \subseteq \ldots$ is a strongly computable sequence of finite subsets in the natural series, and $\bigcup\limits_{i\in\mathbb{N}} D_i = \mathbb{N}$;

(b) for all i, $0, 1 \in D_i$, and for all $x \in D_i$, $0 \leqslant_i x \leqslant_i 1$;

(c) for $x, y \in D_i$, $x \leqslant_i y$ implies $x \leqslant_{i+1} y$, and the naturally defined mappings of $\widetilde{\mathcal{D}}_i$ into $\widetilde{\mathcal{D}}_{i+1}$ preserve least upper bounds;

(d) a ternary relation $x \leqslant_i y$ is Π^0_2 in the arithmetic hierarchy;

(e) there are sequences of functions $\{u_i : D_i^2 \to D_i\}_{i\in\mathbb{N}}$ and $\{v_i : D_i^2 \to D_i\}_{i\in\mathbb{N}}$ which are computable uniformly in i and are such that $\mathcal{D}_i = \langle D_i, \leqslant_i; u_i, v_i\rangle$;

(f) for all $x, y \in \mathbb{N}$, $\eta x \leqslant^{\mathcal{L}} \eta y \Leftrightarrow (\exists i \in \mathbb{N})(x, y \in D_i \,\&\, x \leqslant_i y)$.

If $\langle \mathcal{L}, \eta\rangle \in \Lambda_1$ then the semilattice $\mathcal{L}$ is isomorphic to a direct limit $\varinjlim_{i\in\mathbb{N}} \widetilde{\mathcal{D}}_i$. In [3], we pointed out that $\Lambda_1 \subseteq \Omega_2$ and proved that for any pair $\langle \mathcal{L}, \eta\rangle \in \Omega_2$, there is a numbering η' such that $\eta \leqslant \eta'$ and $\langle \mathcal{L}, \eta'\rangle \in \Lambda_1$.

Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be semilattices with universes L_1 and L_2, respectively. By a *morphism* from $\mathfrak{L}_1$ to $\mathfrak{L}_2$ we mean a mapping from L_1 to L_2, which is an isomorphic embedding of the semilattice $\mathfrak{L}_1$ onto an ideal of the semilattice $\mathfrak{L}_2$. If η_1 is a numbering for L_1, η_2 is one for L_2, and φ is a morphism from $\mathfrak{L}_1$ to $\mathfrak{L}_2$, then we call φ a $\mathfrak{K}$-*morphism* from $\langle \mathfrak{L}_1, \eta_1\rangle$ to $\langle \mathfrak{L}_2, \eta_2\rangle$ provided that φ is a morphism of numbered sets in the sense of [8], that is, if there exists a computable function f such that $\varphi(\eta_1 x) = \eta_2 f(x)$ for all $x \in \mathbb{N}$.

Let $\mathfrak{L} = \langle L, \leqslant^{\mathfrak{L}}; \vee^{\mathfrak{L}}\rangle$, η be a numbering for L, and $\langle \mathfrak{L}, \eta\rangle \in \Omega_2$. A pair $\langle \mathfrak{L}, \eta\rangle$ is said to be *universal* if the following conditions hold:

(1) the element $\top_{\mathfrak{L}}$ is distinct from $\perp_{\mathfrak{L}}$ and is indecomposable into the union of two strictly smaller elements;

(2) for any two pairs $\langle \mathfrak{L}_1, \eta_1\rangle, \langle \mathfrak{L}_2, \eta_2\rangle \in \Omega_2$ and for $\mathfrak{K}$-morphisms $\varphi_1 : \langle \mathfrak{L}_1, \eta_1\rangle \to \langle \mathfrak{L}_2, \eta_2\rangle$ and $\varphi_2 : \langle \mathfrak{L}_1, \eta_1\rangle \to \langle \mathfrak{L}, \eta\rangle$, $\varphi_2(\top_{\mathfrak{L}_1}) \neq \top_{\mathfrak{L}}$ implies that there exists a $\mathfrak{K}$ morphism $\varphi : \langle \mathfrak{L}_2, \eta_2\rangle \to \langle \mathfrak{L}, \eta\rangle$ such that $\varphi_2 = \varphi \circ \varphi_1$;

(3) for any $a \in L$, there is a c.e. set A for which $\{\eta x : x \in A\} = \{b \in L : b \leqslant^{\mathfrak{L}} a\}$.

We say that a semilattice $\mathfrak{L}$ is *universal* if there is a numbering η such that the pair $\langle \mathfrak{L}, \eta\rangle$ is universal.

LEMMA 1. Let $\langle \mathfrak{L}, \eta\rangle$ be a universal pair. Then the semilattice $\mathfrak{L}$ is infinite. Moreover, in the second condition specified in the definition of a universal pair, we may assume that $\varphi(\top_{\mathfrak{L}_2}) \neq \top_{\mathfrak{L}}$.

Proof. In order to state that $\mathfrak{L}$ is infinite, we take $\langle \mathfrak{L}_1, \eta_1\rangle, \langle \mathfrak{L}_2, \eta_2\rangle \in \Omega_2$ so that $\perp_{\mathfrak{L}_1} = \top_{\mathfrak{L}_1}$ and $\mathfrak{L}_2$ is infinite. Then the second condition in the definition of a universal pair ensures that there exists an ideal in $\mathfrak{L}$ isomorphic to $\mathfrak{L}_2$.

Suppose that $\varphi_2(\top_{\mathfrak{L}_1}) \neq \top_{\mathfrak{L}}$ for pairs $\langle \mathfrak{L}_1, \eta_1\rangle, \langle \mathfrak{L}_2, \eta_2\rangle \in \Omega_2$ and for $\mathfrak{K}$-morphisms $\varphi_1 : \langle \mathfrak{L}_1, \eta_1\rangle \to \langle \mathfrak{L}_2, \eta_2\rangle$ and $\varphi_2 : \langle \mathfrak{L}_1, \eta_1\rangle \to \langle \mathfrak{L}, \eta\rangle$. Let $\mathfrak{L}_2'$ be the semilattice $\mathfrak{L}_2$ with an adjoint outer greatest element and η_2' be a numbering for the universe of $\mathfrak{L}_2'$ such that $\eta_2' 0 = \top_{\mathfrak{L}_2'}$ and $\eta_2' x = \eta_2(x+1)$ for all $x \in \mathbb{N}$. By condition (2) in the definition of a universal pair, there then exists a $\mathfrak{K}$-morphism $\varphi' : \langle \mathfrak{L}_2', \eta_2'\rangle \to \langle \mathfrak{L}, \eta\rangle$ for which $\varphi_2 = \varphi' \circ \varphi_1$. As φ we may take a restriction of φ' to the universe of the semilattice $\mathfrak{L}_2$. $\square$

LEMMA 2. Let $\langle \mathfrak{L}, \eta\rangle$ be a universal pair, $\langle \mathfrak{L}_1, \eta_1\rangle \in \Omega_2$, φ be a $\mathfrak{K}$-morphism from $\langle \mathfrak{L}_1, \eta_1\rangle$ to $\langle \mathfrak{L}, \eta\rangle$, and a be an element of the universe of $\mathfrak{L}$ greater than or equal to $\varphi(\top_{\mathfrak{L}_1})$. Then there exist a pair $\langle \mathfrak{L}_2, \eta_2\rangle \in \Omega_2$ and $\mathfrak{K}$-morphisms $\varphi' : \langle \mathfrak{L}_1, \eta_1\rangle \to \langle \mathfrak{L}_2, \eta_2\rangle$ and $\psi : \langle \mathfrak{L}_2, \eta_2\rangle \to \langle \mathfrak{L}, \eta\rangle$ such that $\varphi = \psi \circ \varphi'$ and $a = \psi(\top_{\mathfrak{L}_2})$.

Proof. Let $\mathfrak{L} = \langle L, \leqslant^{\mathfrak{L}}; \vee^{\mathfrak{L}}\rangle$, $a \in L$, and $\varphi(\top_{\mathfrak{L}_1}) \leqslant^{\mathfrak{L}} a$. Let $\mathfrak{L}_2 = \langle L_2, \leqslant^{\mathfrak{L}_2}; \vee^{\mathfrak{L}_2}\rangle$ be the semilattice isomorphic to a principal ideal in $\mathfrak{L}$ generated by a and ψ be a morphism from $\mathfrak{L}_2$ to $\mathfrak{L}$ such that $\psi(\top_{\mathfrak{L}_2}) = a$. Put $\varphi' = \psi^{-1} \circ \varphi$. Clearly, φ' is a morphism for which $\varphi = \psi \circ \varphi'$.

Let f be a computable function such that $\varphi(\eta_1 x) = \eta f(x)$ for all $x \in \mathbb{N}$; A is a c.e. set with $\{\eta x : x \in A\} = \{b \in L : b \leqslant^{\mathfrak{L}} a\}$; u is a computable function with $\eta u(x, y) = \eta x \vee^{\mathfrak{L}} \eta y$ for all $x, y \in \mathbb{N}$. For every non-empty finite subset X in the natural series, we define the value $u'(X)$ by induction on the cardinality of X, setting $u'(\{x\}) = x$ and $u'(\{y_0 < \ldots < y_n < x\}) = u(u'(\{y_0, \ldots, y_n\}), x)$. Let F be the set of all non-empty finite subsets of $A \cup \rho f$ and g be a computable surjection of $\mathbb{N}$ onto F. We define a numbering η_2 for the universe of $\mathfrak{L}_2$, setting $\eta_2 x = \psi^{-1}(\eta u'(g(x)))$ for any $x \in \mathbb{N}$.

Since $\eta_2 x \leqslant^{\mathfrak{L}_2} \eta_2 y \Leftrightarrow \eta u'(g(x)) \leqslant^{\mathfrak{L}} \eta u'(g(y))$ and $\eta_2 x \vee^{\mathfrak{L}_2} \eta_2 y = \eta_2 z$, where z is defined by $g(z) = g(x) \cup g(y)$, we have $\langle \mathfrak{L}_2, \eta_2\rangle \in \Omega_2$. The equality $\psi(\eta_2 x) = \eta u'(g(x))$ shows that ψ is a $\mathfrak{K}$-morphism from $\langle \mathfrak{L}_2, \eta_2\rangle$ to $\langle \mathfrak{L}, \eta\rangle$. Let h be a computable function such that $g(h(x)) = \{f(x)\}$ for any $x \in \mathbb{N}$. Then $\varphi'(\eta_1 x) = \psi^{-1}(\varphi(\eta_1 x)) = \psi^{-1}(\eta f(x)) = \psi^{-1}(\eta u'(\{f(x)\})) = \psi^{-1}(\eta u'(g(h(x)))) = \eta_2 h(x)$; hence φ' is a $\mathfrak{K}$-morphism from $\langle \mathfrak{L}_1, \eta_1\rangle$ to $\langle \mathfrak{L}_2, \eta_2\rangle$. $\square$

An argument for the theorem below, based on the back-and-forth construction, is similar by and large to the proof of a relevant theorem in [4]. However, here, the class of numbered semilattices has been changed and definitions altered; so its proof will be provided in in full to avoid confusion.

THEOREM 1. Any two universal semilattices are isomorphic.

Proof. Let $\mathfrak{L}' = \langle L', \leqslant^{\mathfrak{L}'}; \vee^{\mathfrak{L}'}\rangle$ and $\mathfrak{L}'' = \langle L'', \leqslant^{\mathfrak{L}''}; \vee^{\mathfrak{L}''}\rangle$ be universal semilattices. We fix numberings η' and η'' such that $\langle \mathfrak{L}', \eta'\rangle$ and $\langle \mathfrak{L}'', \eta''\rangle$ are universal pairs. By Lemma 1, the sets L' and L'' are countable; we let $L' \setminus \{\top_{\mathfrak{L}'}\} = \{a_0, a_1, \ldots\}$ and $L'' \setminus \{\top_{\mathfrak{L}''}\} = \{b_0, b_1, \ldots\}$.

By induction, we define sequences of

(1) pairs $\{\langle \mathfrak{L}_i, \eta_i\rangle\}_{i\in\mathbb{N}}$ in Ω_2,

(2) $\mathfrak{K}$-morphisms $\{f_{i+1} : \langle \mathfrak{L}_i, \eta_i\rangle \to \langle \mathfrak{L}_{i+1}, \eta_{i+1}\rangle\}_{i\in\mathbb{N}}$,

(3) $\mathfrak{K}$-morphisms $\{g_i : \langle \mathfrak{L}_i, \eta_i\rangle \to \langle \mathfrak{L}', \eta'\rangle\}_{i\in\mathbb{N}}$,

(4) $\mathfrak{K}$-morphisms $\{h_i : \langle \mathfrak{L}_i, \eta_i\rangle \to \langle \mathfrak{L}'', \eta''\rangle\}_{i\in\mathbb{N}}$,

so that $g_i = g_{i+1} \circ f_{i+1}$, $h_i = h_{i+1} \circ f_{i+1}$, $a_i \in \rho g_{2i+1}$, $b_i \in \rho h_{2i+2}$, $\top_{\mathfrak{L}'} \notin \rho g_i$, and $\top_{\mathfrak{L}''} \notin \rho h_i$ for all $i \in \mathbb{N}$.

We take $\mathfrak{L}_0$ to be equal to a one-element semilattice and let η_0 be the unique possible numbering for the universe of $\mathfrak{L}_0$. Let g_0 and h_0 be the solely possible morphisms from $\mathfrak{L}_0$ to $\mathfrak{L}'$ and $\mathfrak{L}''$, respectively. Suppose that $k \in \mathbb{N}$ and that the objects $\langle \mathfrak{L}_i, \eta_i\rangle$, f_i, g_i, and h_i have been constructed, for all $i \leqslant k$. We handle two cases.

Case 1. Let $k+1 = 2s+1$. By Lemma 2, there are a pair $\langle \mathfrak{L}, \eta\rangle \in \Omega_2$ and $\mathfrak{K}$-morphisms $f : \langle \mathfrak{L}_k, \eta_k\rangle \to \langle \mathfrak{L}, \eta\rangle$ and $g : \langle \mathfrak{L}, \eta\rangle \to \langle \mathfrak{L}', \eta'\rangle$ such that $a_s \vee^{\mathfrak{L}'} g_k(\top_{\mathfrak{L}_k}) = g(\top_{\mathfrak{L}})$ and $g_k = g \circ f$. Since $\top_{\mathfrak{L}'}$ is indecomposable into the union of two smaller elements, we have $g(\top_{\mathfrak{L}}) \neq \top_{\mathfrak{L}'}$. By Lemma 1, there is a $\mathfrak{K}$-morphism $h : \langle \mathfrak{L}, \eta\rangle \to \langle \mathfrak{L}'', \eta''\rangle$ for which $h_k = h \circ f$ and $h(\top_{\mathfrak{L}}) \neq \top_{\mathfrak{L}''}$. We put $\mathfrak{L}_{k+1} = \mathfrak{L}$, $\eta_{k+1} = \eta$, $f_{k+1} = f$, $g_{k+1} = g$, and $h_{k+1} = h$.

Case 2. Let $k+1 = 2s+2$. We proceed symmetrically to the previous case. By Lemma 2, there are a pair $\langle \mathfrak{L}, \eta\rangle \in \Omega_2$ and $\mathfrak{K}$-morphisms $f : \langle \mathfrak{L}_k, \eta_k\rangle \to \langle \mathfrak{L}, \eta\rangle$ and $h : \langle \mathfrak{L}, \eta\rangle \to \langle \mathfrak{L}'', \eta''\rangle$ such that $b_s \vee^{\mathfrak{L}''} h_k(\top_{\mathfrak{L}_k}) = h(\top_{\mathfrak{L}})$ and $h_k = h \circ f$. Since $\top_{\mathfrak{L}''}$ is indecomposable into the union of smaller elements, $h(\top_{\mathfrak{L}}) \neq \top_{\mathfrak{L}''}$. By Lemma 1, there is a $\mathfrak{K}$-morphism $g : \langle \mathfrak{L}, \eta\rangle \to \langle \mathfrak{L}', \eta'\rangle$ such that $g_k = g \circ f$ and $g(\top_{\mathfrak{L}}) \neq \top_{\mathfrak{L}'}$. We put $\mathfrak{L}_{k+1} = \mathfrak{L}$, $\eta_{k+1} = \eta$, $f_{k+1} = f$, $g_{k+1} = g$, and $h_{k+1} = h$.

Now, let $v(\top_{\mathfrak{L}'}) = \top_{\mathfrak{L}''}$ and $v(a_i) = h_{2i+1}(g^{-1}_{2i+1}(a_i))$. It is easy to verify that v is an isomorphism of $\mathfrak{L}'$ onto $\mathfrak{L}''$. $\square$

Below, in treating m-reducibility and m-degrees, we will ignore sets $\varnothing$ and $\mathbb{N}$ and, consequently, assume that there exists a least m-degree consisting of computable sets. Let $i \mapsto W_i$ be the Kleene numbering of the family of all c.e. sets and $V_i = (W_i \setminus \{0\}) \cup \{1\}$. Suppose $\mathfrak{L}^e$ is a semilattice of all c.e. m-degrees and π is a numbering for the universe of $\mathfrak{L}$ such that $\pi i = \deg_m(V_i)$ for any $i \in \mathbb{N}$.

PROPOSITION 1. A pair $\langle \mathfrak{L}^e, \pi\rangle$ is universal.

Proof. Let $\mathfrak{L}^e = \langle L^e, \leqslant^{\mathfrak{L}^e}; \vee^{\mathfrak{L}^e}\rangle$. The Kuratowski–Tarski algorithm says of the relation $V_i \leqslant_m V_j$ being Σ^0_3 in the arithmetic hierarchy. Hence the relation $\pi x \leqslant^{\mathfrak{L}^e} \pi y$ likewise is Σ^0_3. Moreover, $\deg_m((V_i \oplus V_j \setminus \{0\}) \cup \{1\}) = \deg_m(V_i) \vee^{\mathfrak{L}^e} \deg_m(V_j)$. Therefore there exists a computable function u such that $\pi u(x,y) = \pi x \vee^{\mathfrak{L}^e} \pi y$ for all $x, y \in \mathbb{N}$. Thus $\langle \mathfrak{L}^e, \pi\rangle \in \Omega_2$.

Clearly, the semilattice $\mathfrak{L}^e$ is not one-element. That $\top_{\mathfrak{L}^e}$ is indecomposable is a known fact in computability theory, mentioned, in particular, in [4, 8].

Assume that $\langle \mathfrak{L}_1, \eta_1\rangle, \langle \mathfrak{L}_2, \eta_2\rangle \in \Omega_2$ and that $\varphi_2(\top_{\mathfrak{L}_1}) \neq \top_{\mathfrak{L}^e}$ for $\mathfrak{K}$-morphisms $\varphi_1 : \langle \mathfrak{L}_1, \eta_1\rangle \to \langle \mathfrak{L}_2, \eta_2\rangle$ and $\varphi_2 : \langle \mathfrak{L}_1, \eta_1\rangle \to \langle \mathfrak{L}^e, \pi\rangle$. Let μ be a numbering for the universe of $\mathfrak{L}_2$ such that $\eta_2 \leqslant \mu$ and $\langle \mathfrak{L}_2, \mu\rangle \in \Lambda_1$. The fact that $\eta_2 \leqslant \mu$ is reducible implies that φ_1 is a $\mathfrak{K}$-morphism from $\langle \mathfrak{L}_1, \eta_1\rangle$ to $\langle \mathfrak{L}_2, \mu\rangle$. By [4, Thm. 1], there is a $\mathfrak{K}$-morphism $\varphi : \langle \mathcal{L}_2, \mu\rangle \to \langle \mathcal{L}^e, \pi\rangle$ for which $\varphi_2 = \varphi \circ \varphi_1$. Clearly, φ is a $\mathfrak{K}$-morphism from $\langle \mathcal{L}_2, \eta_2\rangle$ to $\langle \mathcal{L}^e, \pi\rangle$.

Suppose $a = \deg_m(V_i) \in L^e$. Let $\{f_j\}_{j\in\mathbb{N}}$ be a computable sequence of all primitive recursive functions and g be a computable function such that $V_{g(j)} = (f_j^{-1}(V_i) \setminus \{0\}) \cup \{1\}$. We claim that $\{\pi x : x \in \rho g\} = \{b \in L^e : b \leqslant^{\mathcal{L}^e} a\}$. Since $V_{g(j)} \leqslant_m V_i$ for any $j \in \mathbb{N}$, $\{\pi x : x \in \rho g\} \subseteq \{b \in L^e : b \leqslant^{\mathcal{L}^e} a\}$. Let $b \leqslant^{\mathcal{L}^e} a$. We have $b = \deg_m(V)$ for some c.e. set V, and for some computable function h, $n \in V \Leftrightarrow h(n) \in V_i$ with all $n \in \mathbb{N}$. Let j be such that $\rho f_j = \rho h$. It is a simple matter to verify that $V_{g(j)} \equiv_m V$. Hence $b = \pi g(j)$. $\square$

Thus universal semilattices exist. Denote a unique (up to isomorphism) universal semilattice by $\mathcal{U}$. Let $\mathcal{U}'$ denote the semilattice $\mathcal{U}$ from which the greatest element $\top_{\mathcal{U}}$ is removed. Since $\top_{\mathcal{U}}$ is indecomposable in $\mathcal{U}$, $\mathcal{U}'$ is a distributive lattice lacking in the greatest element.

Along with the usual total numberings, we will also consider partial numberings. A *partial numbering* of an at most countable set S is any surjective mapping of some subset of $\mathbb{N}$ onto S. We extend the concept of a $\mathfrak{K}$-morphism. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be distributive semilattices, φ be a morphism from $\mathcal{L}_1$ to $\mathcal{L}_2$, and η_1 and η_2 be partial numberings of the universes of $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. If there is a partial computable function f such that $\delta\eta_1 \subseteq \delta f$, $f(\delta\eta_1) \subseteq \delta\eta_2$, and $\varphi(\eta_1 x) = \eta_2 f(x)$ for any $x \in \delta\eta_1$, then φ is also referred to as a $\mathfrak{K}$-*morphism* from $\langle \mathcal{L}_1, \eta_1\rangle$ to $\langle \mathcal{L}_2, \eta_2\rangle$. For the case where η_1 and η_2 are total, the concept that we have introduced coincides with the concept of a $\mathfrak{K}$-morphism as defined previously.

Let $\mathcal{L} = \langle L, \leqslant^{\mathcal{L}}; \vee^{\mathcal{L}}\rangle$ be a distributive semilattice with the least element $\perp_{\mathcal{L}}$ and η be a partial numbering for L. We say that a pair $\langle \mathcal{L}, \eta\rangle$ is *partial universal* if the following conditions hold:

(1) there are a binary relation $R \subseteq \mathbb{N}^2$, which is Σ^0_3 in the arithmetic hierarchy, and a partial computable function u such that for any $x, y \in \delta\eta$, the value $u(x, y)$ is defined and belongs to $\delta\eta$, $\eta u(x, y) = \eta x \vee^{\mathcal{L}} \eta y$, and $R(x, y) \Leftrightarrow \eta x \leqslant^{\mathcal{L}} \eta y$;

(2) for any two pairs $\langle \mathcal{L}_1, \eta_1\rangle, \langle \mathcal{L}_2, \eta_2\rangle \in \Omega_2$ and for $\mathfrak{K}$-morphisms $\varphi_1 : \langle \mathcal{L}_1, \eta_1\rangle \to \langle \mathcal{L}_2, \eta_2\rangle$ and $\varphi_2 : \langle \mathcal{L}_1, \eta_1\rangle \to \langle \mathcal{L}, \eta\rangle$ such that $\varphi_2(\top_{\mathcal{L}_1}) \neq \top_{\mathcal{L}}$ (or $\top_{\mathcal{L}}$ does not exist), there is a $\mathfrak{K}$-morphism $\varphi : \langle \mathcal{L}_2, \eta_2\rangle \to \langle \mathcal{L}, \eta\rangle$ for which $\varphi_2 = \varphi \circ \varphi_1$.

(3) for any $a \in L$, there is a c.e. set $A \subseteq \delta\eta$ with $\{\eta x : x \in A\} = \{b \in L : b \leqslant^{\mathcal{L}} a\}$.

We say that a semilattice $\mathcal{L}$ is *partial universal* if there exists a partial numbering η such that the pair $\langle \mathcal{L}, \eta\rangle$ is partial universal.

THEOREM 2. If the partial universal lattice has no greatest element then it is isomorphic to $\mathcal{U}'$.

The **proof** is similar to that for Theorem 1. We give its sketch, not going into details.

Assume that $\langle \mathcal{L}, \eta\rangle$ is a partial universal pair and that $\mathcal{L}$ has no greatest element. For $\langle \mathcal{L}, \eta\rangle$, Lemma 2 holds true. We argue in exactly the same way as we did in proving that lemma. In fact, the equality $\delta\eta = \mathbb{N}$ is properly not used in the initial proof, the only important thing being that $u'(X) \in \delta\eta$ for finite $X \subseteq \delta\eta$.

Below, as η' we take a numbering for the universe of $\mathcal{U}$ such that $\langle \mathcal{U}, \eta'\rangle$ is a universal pair. Put $\rho\eta' \setminus \{\top_{\mathcal{U}}\} = \{a_0, a_1, \ldots\}$ and $L = \{b_0, b_1, \ldots\}$, and then construct, as in the proof of Theorem 1, a sequence $\{\langle \mathcal{L}_i, \eta_i\rangle\}_{i\in\mathbb{N}}$ of pairs in Ω_2 and three sequences $\{f_{i+1} : \langle \mathcal{L}_i, \eta_i\rangle \to \langle \mathcal{L}_{i+1}, \eta_{i+1}\rangle\}_{i\in\mathbb{N}}$, $\{g_i : \langle \mathcal{L}_i, \eta_i\rangle \to \langle \mathcal{U}, \eta'\rangle\}_{i\in\mathbb{N}}$, and $\{h_i : \langle \mathcal{L}_i, \eta_i\rangle \to \langle \mathcal{L}, \eta\rangle\}_{i\in\mathbb{N}}$ of $\mathfrak{K}$-morphisms such that $g_i = g_{i+1} \circ f_{i+1}$, $h_i = h_{i+1} \circ f_{i+1}$, $a_i \in \rho g_{2i+1}$, $b_i \in \rho h_{2i+2}$, and $\top_{\mathcal{U}} \notin \rho g_i$ for all $i \in \mathbb{N}$. Since $\mathcal{L}$ has no greatest element, Lemma 1 is no longer needed in defining h at odd steps. By Lemma 2, for $\langle \mathcal{L}, \eta\rangle$, we can define f and h at even steps. With the map v defined after constructing the sequences using $v(a_i) = h_{2i+1}(g^{-1}_{2i+1}(a_i))$, we state that v is an isomorphism of $\mathcal{U}'$ onto $\mathcal{L}$. $\square$

Remark 1. We may prove that if a non one-element partial universal semilattice contains a greatest element which is indecomposable into the union of two strictly smaller ones then this semilattice is isomorphic to $\mathcal{U}$. Indeed, if in this case we remove the greatest element from the semilattice in the partial universal pair and bound the numbering then we will be faced up to a partial universal pair with a semilattice without a greatest element. (We need an analog of Lemma 1 to ground this property.)

2. EXAMPLES OF PARTIAL UNIVERSAL SEMILATTICES WITHOUT GREATEST ELEMENTS

We consider a number of distributive semilattices which lack in the greatest element and partial numberings for their universes.

Let $n \geqslant 2$ be some fixed natural number and $S_n = \{s_1, \ldots, s_n\}$ be a finite set of power n. We say that a numbering ν of a set S is $\mathbf{0}'$-*decidable* if there exists a $\mathbf{0}'$-computable function f such that $\nu x = s_{f(x)}$ for any $x \in \mathbb{N}$. Evidently, the $\mathbf{0}'$-computable numberings of S_n form an ideal in the presemilattice of all numberings of the set S_n, which is itself a presemilattice. Denote a semilattice associated with this presemilattice by $\mathcal{X} = \langle X, \leqslant^{\mathcal{X}}; \vee^{\mathcal{X}} \rangle$. Let $\{\varphi_m^{\mathbf{0}'}\}_{m \in \mathbb{N}}$ be a universal computable sequence of all unary partial $\mathbf{0}'$-computable functions and Ξ be a set of natural numbers m such that the function $\varphi_m^{\mathbf{0}'}$ is total, and is a surjective mapping of $\mathbb{N}$ onto $\{1, \ldots, n\}$. For every $m \in \Xi$, let ν_m be a numbering of S_n such that $\nu_m x = s_{\varphi_m^{\mathbf{0}'}(x)}$ for all $x \in \mathbb{N}$ and let ξm be an element of X associated with the numbering ν_m. Then ξ is a partial numbering of X with the domain Ξ.

We say that a c.e. m-degree is *simple* (*hypersimple*) if it contains a computable or simple (hypersimple) set. (As above, in treating m-degrees, we ignore the m-degrees of the sets $\varnothing$ and $\mathbb{N}$.) Sets of simple m-degrees and of hypersimple m-degrees are ideals in the semilattice $\mathcal{L}^e$. Indeed, let A be simple (hypersimple) and B be m-reduced to A by a computable function f. Then B is computable if ρf is finite. But if ρf is infinite, then we let g be a computable injection from $\mathbb{N}$ to $\mathbb{N}$ such that $\rho g = \rho f$. It is not hard to verify that the set $g^{-1}(A)$ is simple (hypersimple) and $g^{-1}(A) \equiv_m B$. Again, if A and B are simple (hypersimple) sets then the set $A \oplus B$ likewise is simple (hypersimple).

Semilattices of simple and hypersimple m-degrees are denoted by $\mathcal{L}^s = \langle L^s, \leqslant^{\mathcal{L}^s}; \vee^{\mathcal{L}^s} \rangle$ and $\mathcal{L}^{hs} = \langle L^{hs}, \leqslant^{\mathcal{L}^{hs}}; \vee^{\mathcal{L}^{hs}} \rangle$, respectively. Let $I_s = \{i \in \mathbb{N} : V_i \text{ is cofinite or simple}\}$ and $I_{hs} = \{i \in \mathbb{N} : V_i \text{ is cofinite or hypersimple}\}$. Put $\sigma = \pi \restriction I_s$ and $\chi = \pi \restriction I_{hs}$. Then σ and χ are partial numberings for the sets L^s and L^{hs}, respectively.

LEMMA 3. The semilattices $\mathcal{X}$, $\mathcal{L}^s$, and $\mathcal{L}^{hs}$ do not contain greatest elements.

Proof. We prove that $\mathcal{X}$ has no greatest element. To do this, it suffices to show that if ν is a $\mathbf{0}'$-decidable numbering of the set S_n then there exists a $\mathbf{0}'$-decidable numbering μ of S_n such that $\mu \not\leqslant \nu$. Let g be a $\mathbf{0}'$-computable function with $\nu x = s_{f(x)}$ for any $x \in \mathbb{N}$. Assume that α is a mapping of the set $\{1, \ldots, n\}$ into itself, without fixed points. Let $\{\varphi_m\}_{m \in \mathbb{N}}$ be a universal computable sequence of all unary partial computable functions. For $x < n$, put $\mu x = s_{x+1}$. For $x \geqslant n$, let

$$\mu x = \begin{cases} s_{\alpha(f(\varphi_{x-n}(x)))} & \text{if } \varphi_{x-n}(x) \text{ is defined,} \\ s_1 & \text{otherwise.} \end{cases}$$

Clearly, μ is a numbering of S_n and the function g, defined by the equality $\mu x = s_{g(x)}$, is computable with oracle $\mathbf{0}'$. If $\mu \leqslant \nu$ then the function φ_m is total for some $m \in \mathbb{N}$, and $\mu x = \nu \varphi_m(x)$ for all $x \in \mathbb{N}$. However, in this instance $\mu(m+n) = s_{\alpha(f(\varphi_m(m+n)))} \neq s_{f(\varphi_m(m+n))} = \nu \varphi_m(m+n)$, a contradiction.

Our present goal is to prove that the semilattices $\mathcal{L}^s$ and $\mathcal{L}^{hs}$ have no greatest elements. To do this, it suffices to state the following:

If A is a c.e. non-creative set then there exists a hypersimple set B such that $B \nleqslant_m A$.

Let F be a set of all finite subsets in the natural series and $\mathcal{F} = \{Y$ is a c.e. subset of $F : (\forall y_1, y_2 \in Y)(y_1 \cap y_2 = \varnothing)\}$. Evidently, there exists a computable sequence $\{Y_i\}_{i\in\mathbb{N}}$ such that $\{Y_i : i \in \mathbb{N}\} = \mathcal{F}$. We fix this sequence, assuming that for some procedure enumerating sets Y_i uniformly in i, Y_i^t stands for the set of elements of Y_i enumerated by step t. Let K be a creative set. We fix enumeration procedures for sets A and K, assuming that A^t and K^t are the parts of these sets enumerated by step t. For any $i \in \mathbb{N}$, let φ_i^t be a partial function with a finite domain such that $\varphi_i^t(x) = y$ iff the value $\varphi_i(x)$ is computed in at most t steps, and is equal to y.

We enumerate the set B in steps. For every step t, B^t denotes a finite set of numbers enumerated into B at steps not exceeding t. After every step t, values of "counters" are defined by setting

$$c'(i,t) = \max\{x \leqslant t : (\forall y < x)(y \in \delta\varphi_i^t \,\&\, (y \in B^t \leftrightarrow \varphi_i(y) \in A^t))\},$$
$$c(i,t) = \max\{c'(i,0), \dots, c'(i,t)\}.$$

It is clear that for each $i \in \mathbb{N}$, the function $\lambda t[c(i,t)]$ does not decrease, and $\lim\limits_{t\to\infty} c(i,t) = \infty$ iff B is m-reduced to A by φ_i.

We describe steps of the construction. At every step, we do the following: attach labels to natural numbers and enumerate elements into B. For every $i \in \mathbb{N}$, we use the label $[i]$. Any label can be attached only once, never to be removed, and each number is allowed to be attached at most one label. Suppose that for every $i \in \mathbb{N}$, $m(i,t)$ is equal to a total of numbers to which the label $[i]$ is attached after step t, and $x_1^i, \dots, x_{m(i,t)}^i$ are the numbers themselves, enumerated in order of attaching $[i]$. For the description of Step 1 to be correct, we put $c(i,-1) = 0$ for all $i \in \mathbb{N}$.

Step 0. No labels are attached. Put $B^0 = \varnothing$ and pass to the next step.

Step $t+1$. This step consists of three stages.

Stage I. We attach the label $[t]$ to a least number with no label. Then we seek for $i \leqslant t$ such that $c(i,t-1) < c(i,t)$. If such i are missing then we pass to the next stage at once. Otherwise, for the minimal such i, we attach $[i]$ to the least of the yet unlabelled numbers and pass to the next stage.

Stage II. For every $i \in \mathbb{N}$, if $m(i,t) > 0$ then we enumerate into B the number x_{j+1}^i for all $j < m(i,t)$ such that $j \in K^t$. Pass to the next stage.

Stage III. For every $i \leqslant t$, if each element Y_i^t contains numbers which are still not enumerated into B, and for some $y \in Y_i^t$, the set y consists of numbers to which labels $[j]$, $j > i$, are attached, then we choose (effectively) one of such y and enumerate all of its elements into B. Pass to the next step.

We have finished to describe the steps of the construction. Clearly, these steps are effective since the set $B = \bigcup\limits_{t\in\mathbb{N}} B^t$ is c.e. We claim that $B \nleqslant_m A$.

Suppose the contrary. We have $\lim\limits_{t\to\infty} c(i,t) = \infty$ for some $i \in \mathbb{N}$; let i_0 be least among all such i. Note that for each $i \in \mathbb{N}$, stage III is exercised at most once. Let t_0 be a step after which the "counters" $c(i,t)$ do not grow with all $i < i_0$ and stage III fails for i. Since stage I is realized for i_0 infinitely often, the elements $x_s^{i_0}$ are defined for all $s > 0$. Elements of the sequence $x_{m(i_0,t_0)+1}^{i_0}, x_{m(i_0,t_0)+2}^{i_0}, \dots$ may be enumerated into B only at stage II; hence, for any $s > m(i_0,t_0)$, $x_s^{i_0} \in B \Leftrightarrow s-1 \in K$. But then $K \leqslant_m B$, and $B \leqslant_m A$ implies $K \leqslant_m A$, which contradicts the initial assumption on A.

We argue to show that B is hypersimple. Assume the contrary. Then either B is cofinite, or there is $i \in \mathbb{N}$ for which the set Y_i is infinite, and for any $y \in Y_i$, $y \nsubseteq B$. The former is impossible since $B \nleqslant_m A$. Let $i_0 \in \mathbb{N}$ so that Y_{i_0} is infinite and $(\forall y \in Y_{i_0})(y \nsubseteq B)$. After some step the "counters" $c(i,t)$, $i \leqslant i_0$, will stop growing and labels $[i]$ will not be attached. Hence we can find $y \in Y_{i_0}$ such that the labels $[i]$, $i \leqslant i_0$, are never attached to the elements y. Since at least one label is always attached at stage I, it follows that at sufficiently large steps labels will be attached to all y's, stage III will be realized for i_0, and one of the elements of Y_{i_0} will be entirely enumerated into B. $\square$

Our next goal is to prove that pairs $\langle \mathcal{X}, \xi \rangle$, $\langle \mathcal{L}^s, \sigma \rangle$, and $\langle \mathcal{L}^{hs}, \chi \rangle$ are partial universal. Before we formulate a statement on being partial universal for the pairs, we give the formulation of the theorem on which our further proofs will be based.

THEOREM 3. The pairs $\langle \mathcal{X}, \xi \rangle$, $\langle \mathcal{L}^s, \sigma \rangle$, and $\langle \mathcal{L}^{hs}, \chi \rangle$ satisfy the second condition in the definition of a partial universal pair.

The proof of Theorem 3, which is technically rather complicated, will be given in Sec. 3. Preparatory to this, we state and prove a number of consequences of Theorem 3.

THEOREM 4. The pairs $\langle \mathcal{X}, \xi \rangle$, $\langle \mathcal{L}^s, \sigma \rangle$, and $\langle \mathcal{L}^{hs}, \chi \rangle$ are partial universal.

Proof. That the second condition in the definition of partial universality holds for all the three pairs follows from Theorem 3. The first condition for $\langle \mathcal{L}^s, \sigma \rangle$ and $\langle \mathcal{L}^{hs}, \chi \rangle$ is met since partial numberings σ and χ are restrictions of the numbering π, $\langle \mathcal{L}_e, \pi \rangle \in \Omega_2$, and the domains of σ and χ are closed w.r.t. a function u such that $V_{u(x,y)} = (V_x \oplus V_y \setminus \{0\}) \cup \{1\}$. We verify the third condition on these two pairs. Let $a \in L^s$ ($a \in L^{hs}$) and $a = \deg_m(V_x)$ for $x \in \delta\sigma$ ($x \in \delta\chi$). Suppose $\{f'_i\}_{i\in\mathbb{N}}$ is a computable sequence of all primitive recursive functions. For every $i, y \in \mathbb{N}$, put

$$f_i(y) = \begin{cases} y & \text{if } y = 0, 1, \\ 1 & \text{if } y \geqslant 2 \,\&\, (\exists z < y-2)(f'_i(z) = f'_i(y-2)), \\ f'_i(y-2) & \text{in all other cases.} \end{cases}$$

Then $\{f_i\}_{i\in\mathbb{N}}$ is a computable sequence of primitive recursive functions, and for every $i \in \mathbb{N}$, we have $f_i(0) = 0$, $f_i(1) = 1$, $\rho f_i = \rho f'_i \cup \{0, 1\}$, and $f_i(y_1) = f_i(y_2) \Rightarrow (y_1 = y_2) \vee (f_i(y_1) = f_i(y_2) = 1)$. Let g be a computable function such that $V_{g(i)} = W_{g(i)} = \{y \in \mathbb{N} : f_i(y) \in V_x\}$. We state that $\{\pi i : i \in \rho g\} = \{b \in L^e : b \leqslant^{\mathcal{L}^e} a\}$ in the same way as we did in proving Prop. 1. Since L^s (L^{hs}) is an ideal in $\mathcal{L}^e$ and σ (χ) is a restriction of π, we are left to show that $\rho g \subseteq \delta\sigma$ ($\rho g \subseteq \delta\chi$). This follows readily from the simplicity (hypersimplicity) of V_x and the fact that $(y_1 \neq y_2) \,\&\, (f_i(y_1) = f_i(y_2)) \Rightarrow y_1, y_2 \in V_{g(i)}$.

It remains to verify the first and second conditions on the pair $\langle \mathcal{X}, \xi \rangle$. We argue for the first. Let $R = \{\langle x, y \rangle \in \mathbb{N}^2 : (\exists i \in \mathbb{N})(\text{the function } \varphi_i \text{ is total and } (\forall z \in \delta\varphi_x^{\mathbf{0}'} \cap \varphi_i^{-1}(\delta\varphi_y^{\mathbf{0}'}))(\varphi_x^{\mathbf{0}'}(z) = \varphi_y^{\mathbf{0}'}(\varphi_i(z)))\}$ (the sequence $\{\varphi_i\}_{i\in\mathbb{N}}$ is as defined in the proof of Lemma 3). The Kuratowski–Tarski algorithm delivers $R \in \Sigma^0_3$. For $x, y \in \Xi$, we have $R(x, y) \Leftrightarrow (\exists i \in \mathbb{N})(\varphi_i \text{ is total and } (\forall z \in \mathbb{N})(\nu_x z = \nu_y \varphi_i(z))) \Leftrightarrow \nu_x \leqslant \nu_y \Leftrightarrow \xi x \leqslant^{\mathcal{X}} \xi y$. Let u be a computable function such that for any $x, y, z \in \mathbb{N}$,

$$\varphi^{\mathbf{0}'}_{u(x,y)}(z) = \begin{cases} \varphi_x^{\mathbf{0}'}(z') & \text{if } z = 2z', \\ \varphi_y^{\mathbf{0}'}(z') & \text{otherwise.} \end{cases}$$

For arbitrary $x, y \in \Xi$, we have $u(x, y) \in \Xi$, $\nu_{u(x,y)} = \nu_x \oplus \nu_y$, and hence $\xi u(x, y) = \xi x \vee^{\mathcal{X}} \xi y$.

We argue for the third condition. Let $a \in X$. Then a is associated with some $\mathbf{0}'$-decidable numbering ν. Without loss of generality, we may assume that $\nu x = s_{x+1}$ for all $x < n$. Let f be a $\mathbf{0}'$-computable function such that $\nu x = s_{f(x)}$ for all $x \in \mathbb{N}$. Suppose $\{f_i\}_{i\in\mathbb{N}}$ is a computable sequence of all primitive recursive functions, and

$$g_i(x) = \begin{cases} x + 1 & \text{if } x < n, \\ f_i(x - n) & \text{otherwise} \end{cases}$$

for all $x, i \in \mathbb{N}$. Assume h is a computable function such that $\varphi^{\mathbf{0}'}_{h(i)} = f \circ g_i$ for any $i \in \mathbb{N}$. For $i \in \mathbb{N}$ arbitrary, then, the value $h(i)$ belongs to Ξ and $\nu_{h(i)} = \nu \circ g_i$. Put $A = \rho h$. We have $A \subseteq \Xi$ and $\xi x \leqslant^{\mathcal{X}} a$ for any $x \in A$. Now, let $b \in X$ so that $b \leqslant^{\mathcal{X}} a$. Then b is associated with some numbering μ of the set S_n and there exists a computable function g for which $\mu = \nu \circ g$. We choose $i \in \mathbb{N}$ so that $\rho g = \rho f_i$. Then $\mu = \nu \circ g \equiv \nu \circ f_i \equiv \nu \circ g_i = \nu_{h(i)}$ and $b = \xi h(i) \in \xi(A)$. $\square$

COROLLARY 1. The semilattices $\mathcal{X}$, $\mathcal{L}^s$, and $\mathcal{L}^{hs}$ are isomorphic to $\mathcal{U}'$.

The **proof** follows immediately from Theorem 2, Lemma 3, and Theorem 4. $\square$

COROLLARY 2. A semilattice of m-degrees of Δ^0_2-sets is isomorphic to $\mathcal{U}'$.

Proof. There is a natural isomorphism between the semilattice of m-degrees of Δ^0_2-sets and the semilattice $\mathcal{X}$ defined by $\mathbf{0}'$-decidable numberings of the set S_n, for $n = 2$. $\square$

The corollaries below make use of the concepts of a Rogers semilattice of Σ^0_n-computable numberings and of a $\mathbf{0}'$-degree in that semilattice. All relevant definitions can be found in [9].

COROLLARY 3. Let $k \geqslant 2$, S be a finite family of Σ^0_k-sets containing at least two elements, and $\mathcal{R}^0_k(S)$ be the Rogers semilattice of Σ^0_k-computable numberings of the family S. Let $\mathbf{m}$ be the least $\mathbf{0}'$-degree in $\mathcal{R}^0_k(S)$. Then $\mathbf{m}$, treated as an upper semilattice, is isomorphic to $\mathcal{U}'$.

Proof. It is easy to show that a numbering of S presents an element of the semilattice $\mathcal{R}^0_k(S)$ lying in the least $\mathbf{0}'$-degree iff the numbering is $\mathbf{0}'$-decidable. $\square$

COROLLARY 4. Let S be a finite family of Σ^0_2-sets containing at least two elements such that different sets in S are incomparable w.r.t. inclusion. Then the Rogers semilattice $\mathcal{R}^0_2(S)$ of Σ^0_2-computable numberings of S is isomorphic to $\mathcal{U}'$.

Proof. Under the above-listed conditions, the least $\mathbf{0}'$-degree in $\mathcal{R}^0_2(S)$ coincides with the entire semilattice $\mathcal{R}^0_2(S)$. $\square$

Along with the semilattices $\mathcal{L}^s$ and $\mathcal{L}^{hs}$ dealt with in the present paper, of interest is the semilattice $\mathcal{L}^{hhs}$ of hyperhypersimple m-degrees. This semilattice was explored in [10]. It was proved that $\mathcal{L}^{hhs}$ is not a lattice, contains infinitely many minimal elements (above $\perp_{\mathcal{L}^{hhs}}$) and elements which are not underlain by the minimal, and has an undecidable elementary theory. Possibly, $\mathcal{L}^{hhs}$ is also isomorphic to $\mathcal{U}'$; but we have no idea as to how this claim can be proved or disproved (nor do we even know how to obtain a local description for $\mathcal{L}^{hhs}$).

3. THE PROOF OF THEOREM 3

First, we note that the second condition in the definition of a partial universal pair is equivalent to the following statement (hereinafter, $2\mathbb{N}$ denotes the set of even natural numbers).

Given a semilattice $\mathcal{L}' = \langle L', \leqslant^{\mathcal{L}'}; \vee^{\mathcal{L}'} \rangle$ and a numbering η' of the set L', we assume that the pair $\langle \mathcal{L}', \eta' \rangle$ is Λ_1. Let also $\eta'(2\mathbb{N})$ be a principal ideal in $\mathcal{L}'$ and f be a computable function with the domain $2\mathbb{N}$ such that $\rho f \subseteq \delta\eta$, the mapping $\eta'(2x) \mapsto \eta f(2x)$ is well defined and is an isomorphic embedding of the ideal $\eta'(2\mathbb{N})$ onto a proper ideal of the semilattice $\mathcal{L}$. Then there exists a $\mathfrak{K}$-morphism λ from $\langle \mathcal{L}', \eta' \rangle$ to $\langle \mathcal{L}, \eta \rangle$ such that $\lambda(\eta'(2x)) = \eta f(2x)$ for all $x \in \mathbb{N}$.

In fact, if the second condition in the definition of a partial universal pair holds for $\langle \mathcal{L}, \eta \rangle$ then the fact that the desired $\mathfrak{K}$-morphism λ exists may be stated by treating the numbering $\eta'' x = \eta'(2x)$ of the ideal $\eta'(2\mathbb{N})$ and keeping in mind that computable functions $f(2x)$ and $2x$ define morphisms of this ideal into $\langle \mathcal{L}, \eta \rangle$ and $\langle \mathcal{L}', \eta' \rangle$, respectively.

Conversely, let $\langle \mathfrak{L}_1, \eta_1 \rangle, \langle \mathfrak{L}_2, \eta_2 \rangle \in \Omega_2$ and $\varphi_1 : \langle \mathfrak{L}_1, \eta_1 \rangle \to \langle \mathfrak{L}_2, \eta_2 \rangle$ and $\varphi_2 : \langle \mathfrak{L}_1, \eta_1 \rangle \to \langle \mathfrak{L}, \eta \rangle$ be $\mathfrak{K}$-morphisms such that $\varphi_2(\top_{\mathfrak{L}_1}) \neq \top_{\mathfrak{L}}$ (or else $\top_{\mathfrak{L}}$ does not exist). By the result in [3] mentioned above, there is a numbering η'' of the universe of $\mathfrak{L}_2$ for which $\eta_2 \leqslant \eta''$ and $\langle \mathfrak{L}_2, \eta'' \rangle \in \Lambda_1$. Let f_1 be a computable function such that $\varphi_1(\eta_1 x) = \eta_2 f_1(x)$ for all $x \in \mathbb{N}$. We define a numbering η' for the universe of $\mathfrak{L}_2$ setting

$$\eta' x = \begin{cases} \perp_{\mathfrak{L}_2} & \text{if } x = 0, \\ \top_{\mathfrak{L}_2} & \text{if } x = 1, \\ \eta_2 f_1(y) & \text{if } x = 2y + 2, \\ \eta'' y & \text{in the remaining cases} \end{cases}$$

for all $x \in \mathbb{N}$.

We claim that $\langle \mathfrak{L}_2, \eta' \rangle \in \Lambda_1$. Let $\{\mathcal{D}'_i = \langle D'_i, \leqslant'_i \rangle\}_{i \in \mathbb{N}}$ be a sequence of finite distributive prelattices for $\langle \mathfrak{L}_2, \eta'' \rangle$ satisfying properties (a)-(f) in the definition of the class Λ_1. Assume f' is a computable function such that $\eta_2 x = \eta'' f'(x)$ for all $x \in \mathbb{N}$. Put $D_i = \{0, 1\} \cup \{2x + 3 : x \in D'_i\} \cup \{2x + 2 : x \leqslant i,\ f'(f_1(x)) \in D'_i\}$ for every $i \in \mathbb{N}$. On each of the sets D_i, we define a relation $\leqslant_i$ setting $x \leqslant_i y \Leftrightarrow f''(x) \leqslant'_i f''(y)$ for $x, y \in D_i$, where the function f'' is defined thus:

$$f''(x) = \begin{cases} x & \text{if } x = 0, 1, \\ f'(f_1(y)) & \text{if } x = 2y + 2, \\ y & \text{in the remaining cases} \end{cases}$$

for all $x \in \mathbb{N}$. Evidently, for every $i \in \mathbb{N}$, $\mathcal{D}_i = \langle D_i, \leqslant_i \rangle$ is a finite distributive prelattice, with $\widetilde{\mathcal{D}}_i \cong \widetilde{\mathcal{D}}'_i$. It is also clear that the sequence $\{\mathcal{D}_i\}_{i \in \mathbb{N}}$ enjoys all the six properties specified in the definition of Λ_1 for $\langle \mathfrak{L}_2, \eta' \rangle$.

It is easy to see that $\eta'(2\mathbb{N})$ is an ideal in $\mathfrak{L}_2$, isomorphic to $\mathfrak{L}_1$. Now, let f be a partial computable function with the domain $2\mathbb{N}$ such that $f(0) = x_0$ and $f(2x+2) = f_2(x)$, where f_2 is a computable function with $\varphi_2(\eta_1 x) = \eta f_2(x)$ for all $x \in \mathbb{N}$ and $x_0 \in \mathbb{N}$ is such that $\eta x_0 = \perp_{\mathfrak{L}}$. Then $\rho f \subseteq \delta\eta$, the mapping $\eta'(2x) \mapsto \eta f(2x)$ is well defined and is an isomorphic embedding of the ideal $\eta'(2\mathbb{N})$ onto the ideal $\varphi_2(\mathfrak{L}_1)$ of $\mathfrak{L}$. Let λ be a $\mathfrak{K}$-morphism from $\langle \mathfrak{L}_2, \eta' \rangle$ to $\langle \mathfrak{L}, \eta \rangle$ such that $\lambda(\eta'(2x)) = \eta f(2x)$ for all $x \in \mathbb{N}$. Since $\eta_2 \leqslant \eta'' \leqslant \eta'$, λ is also a $\mathfrak{K}$-morphism from $\langle \mathfrak{L}_2, \eta_2 \rangle$ to $\langle \mathfrak{L}, \eta \rangle$. We have $\varphi_2(\eta_1 x) = \eta f_2(x) = \eta f(2x + 2) = \lambda(\eta'(2x + 2)) = \lambda(\eta_2 f_1(x)) = \lambda(\varphi_1(\eta_1 x))$ for any $x \in \mathbb{N}$; so $\varphi_2 = \lambda \circ \varphi_1$.

Thus, for $\langle \mathcal{X}, \xi \rangle$, $\langle \mathfrak{L}^s, \sigma \rangle$, and $\langle \mathfrak{L}^{hs}, \chi \rangle$, we need only prove the statement formulated at the beginning of this section. Assume that the pair $\langle \mathfrak{L}', \eta' \rangle$ satisfies the conditions specified in the statement, and $\{\mathcal{D}_i = \langle D_i, \leqslant_i \rangle\}_{i \in \mathbb{N}}$ is a sequence of finite distributive prelattices for $\langle \mathfrak{L}', \eta' \rangle$ meeting the six properties in the definition of Λ_1. Let $\{u'_i, v'_i : D_i^2 \to D_i\}_{i \in \mathbb{N}}$ be computable sequences of functions such that $\mathcal{D}_i = \langle D_i, \leqslant_i; u'_i, v'_i \rangle$. For every $i \in \mathbb{N}$, we define functions u_i and v_i mapping subsets D_i into D_i by setting

$$u_i(\varnothing) = 0,\ v_i(\varnothing) = 1,$$
$$u_i(\{x_1 < \ldots < x_k\}) = u'_i(u_i(\{x_1, \ldots, x_{k-1}\}), x_k),$$
$$v_i(\{x_1 < \ldots < x_k\}) = v'_i(v_i(\{x_1, \ldots, x_{k-1}\}), x_k)$$

for all $k \geqslant 1$. We call the set $A \subseteq D_i$ an *atom of* $\mathcal{D}_i$ if $v_i(A) \nleqslant_i u_i(D_i \setminus A)$. It is easy to see that an arbitrary atom in $\mathcal{D}_i$ is a principal upper cone in $\mathcal{D}_i$, that is, if A is an atom in $\mathcal{D}_i$ then there exists $a \in D_i$ for which $A = \{x \in D_i : a \leqslant_i x\}$. In [1], the following properties of atoms are pointed out:

(1) the lattice $\widetilde{\mathcal{D}}_i$ is isomorphic to a sublattice in the lattice of all subsets of the set of atoms in $\mathcal{D}_i$; the isomorphism is given by the rule $[x]_{\mathcal{D}_i} \mapsto \{A \text{ is an atom in } D_i : x \in A\}$;

(2) for every atom A in $\mathcal{D}_{i+1}$, there is a unique set $\{A_1, \ldots, A_k\}$ of atoms in $\mathcal{D}_i$ which are pairwise incomparable w.r.t. inclusion, so that $A \cap D_i = A_1 \cup \ldots \cup A_k$.

Following [1], we bring into consideration frames and towers (see also [2, 4]). A *frame* is a pair $\mathcal{F} = (\mathcal{A}_m, \ldots, \mathcal{A}_0; c_m, \ldots, c_1)$ which consists of two finite sequences and is such that:

(F1) for $i \leqslant m$, elements of $\mathcal{A}_i$ are non-empty subsets D_i;

(F2) for $i < m$, the function $c_{i+1} : \mathcal{A}_{i+1} \to \mathcal{P}(\mathcal{A}_i)$ maps $\mathcal{A}_{i+1}$ into the set of non-empty subsets of $\mathcal{A}_i$, with $\mathcal{A}_i = \bigcup\limits_{A \in \mathcal{A}_{i+1}} c_{i+1}(A)$;

(F3) for $i < m$ and $A \in \mathcal{A}_{i+1}$, the elements $c_{i+1}(A)$ are pairwise incomparable, and $A \cap D_i = \bigcup c_{i+1}(A)$;

(F4) $\mathcal{A}_m$ is a singleton.

The number m in the definition above is called the *length* of a frame $\mathcal{F}$. We say that $\mathcal{F}$ is a frame of *kind* I if $A \cap 2\mathbb{N} = \varnothing$ for $A \in \mathcal{A}_m$; otherwise, $\mathcal{F}$ is referred to as a frame of *kind* II. A frame is *good* if A is an atom in $\mathcal{D}_i$ for any $i \leqslant m$ and any $A \in \mathcal{A}_i$. A frame that is not good is said to be *bad*. The second property of atoms maintains that for any atom A in $\mathcal{D}_m$, there exists a sole good frame of length m for which $\mathcal{A}_m = \{A\}$.

The definition of an atom implies that the property of being good for an atom is a Σ^0_2-property. With this fact in mind, we fix a computable function $\mathrm{mod}_s\mathcal{F}$ such that $\mathrm{mod}_0\mathcal{F} \leqslant \mathrm{mod}_1\mathcal{F} \leqslant \ldots$ and the frame $\mathcal{F}$ is good iff $\lim\limits_{s\to\infty} \mathrm{mod}_s\mathcal{F} < \infty$. We may assume that the function mod is injective, that is, $\mathrm{mod}_{s_1}\mathcal{F}_1 = \mathrm{mod}_{s_2}\mathcal{F}_2 \Rightarrow s_1 = s_2 \,\&\, \mathcal{F}_1 = \mathcal{F}_2$.

We denote the value of $\lim\limits_{s\to\infty} \mathrm{mod}_s\mathcal{F}$ by $\mathrm{mod}(\mathcal{F})$. If $\mathcal{F}$ is a bad frame then $\mathrm{mod}(\mathcal{F}) = \infty$; for $\mathcal{F}$ good, $\mathrm{mod}(\mathcal{F})$ is equal to some natural number, and moreover, for distinct good $\mathcal{F}$, the numbers $\mathrm{mod}(\mathcal{F})$ are distinct.

A *tower* is a pair $\mathcal{T} = (\mathcal{P}_m, \ldots, \mathcal{P}_0; \varphi_m, \ldots, \varphi_0)$ which consists of two finite sequences and is such that:

(T1) for $i \leqslant m$, elements of $\mathcal{P}_i$ are non-empty finite subsets of the natural series;

(T2) for $i \leqslant m$, the function $\varphi_i : \mathcal{P}_i \to \mathcal{P}(D_i)$ maps $\mathcal{P}_i$ into the set of subsets of D_i;

(T3) $\mathcal{P}_m$ is a singleton;

(T4) for $i \leqslant m$ and $P, Q \in \mathcal{P}_i$, $P \neq Q$ implies $P \cap Q = \varnothing$;

(T5) for $i < m$ and $P \in \mathcal{P}_{i+1}$, there is (obviously unique) $\varepsilon_{i+1}(P) \subseteq \mathcal{P}_i$ such that $P = \bigcup \varepsilon_{i+1}(P)$;

(T6) for $i < m$, let $P \in \mathcal{P}_{i+1}$, $A = \varphi_{i+1}(P)$, and $\varepsilon_{i+1}(P) = \{P_1, \ldots, P_k\}$; then $\{\varphi_i(P_1), \ldots, \varphi_i(P_k)\}$ depends only on A, and we denote this family of subsets of D_i by $c_{i+1}(A)$;

(T7) the pair $(\varphi_m(\mathcal{P}_m), \ldots, \varphi_0(\mathcal{P}_0); c_m, \ldots, c_1)$ is a frame.

A frame defined as in T7 is called the *frame of a tower* $\mathcal{T}$, and the tower itself is said to be *constructed on this frame*. The number m is referred to as the *height* of a tower $\mathcal{T}$. A set that is an element of $\mathcal{P}_m$ is called the *base* of a tower and is denoted by $\mathrm{base}(\mathcal{T})$. We say that a *tower is of kind* I (*kind* II) if its frame is of kind I (kind II). Clearly, given a frame and a sufficiently large finite set F, we can construct a tower on that frame, with the set F taken as the base.

For a tower $\mathcal{T}$, $\mathrm{mod}_s\mathcal{T}$ is conceived of as the value of the function $\mathrm{mod}_s\mathcal{F}$, and $\mathrm{mod}(\mathcal{T})$ — as the value of $\mathrm{mod}(\mathcal{F})$, where $\mathcal{F}$ is the frame of $\mathcal{T}$. A tower is *good* is its frame is good.

Let $\mathcal{F} = (\mathcal{A}_m, \ldots, \mathcal{A}_0; c_m, \ldots, c_1)$ be a frame, $k \leqslant m$, and $A \in \mathcal{A}_k$. There is a frame $(\mathcal{B}_k, \ldots, \mathcal{B}_0; d_k, \ldots, d_1)$ of length k defined by the following relations:

(1) $\mathcal{B}_k = \{A\}$;

(2) for $i < k$, $\mathcal{B}_i = \bigcup_{B \in \mathcal{B}_{i+1}} c_{i+1}(B)$;

(3) for $i < k$, $d_{i+1} = c_{i+1} \restriction \mathcal{B}_{i+1}$.

The so constructed frame is denoted by $\mathcal{F}^k_A$.

Let $\mathcal{F} = (\mathcal{A}_m, \ldots, \mathcal{A}_0; c_m, \ldots, c_1)$ and $\mathcal{G} = (\mathcal{B}_k, \ldots, \mathcal{B}_0; d_k, \ldots, d_1)$ be two frames, with $k \leqslant m$. For $i \leqslant k$ and $B \in \mathcal{B}_i$, denote by $\alpha_i(B)$ the set $\{A \in \mathcal{A}_i : B \subseteq A\}$. We say that the frame $\mathcal{G}$ *is embedded* in the frame $\mathcal{F}$, and write $\mathcal{G} \preccurlyeq \mathcal{F}$, if $\alpha_k(B) \neq \varnothing$ for $B \in \mathcal{B}_k$, and for all $i < k$, $B \in \mathcal{B}_{i+1}$, $B' \in d_{i+1}(B)$, and $A \in \alpha_{i+1}(B)$, we have $c_{i+1}(A) \cap \alpha_i(B') \neq \varnothing$.

LEMMA 4. Suppose that $\mathcal{F} = (\mathcal{A}_m, \ldots, \mathcal{A}_0; c_m, \ldots, c_1)$ and $\mathcal{G} = (\mathcal{B}_k, \ldots, \mathcal{B}_0; d_k, \ldots, d_1)$ are two good frames, $k \leqslant m$, and for (unique) $B \in \mathcal{B}_k$, there is $A \in \mathcal{A}_k$ such that $B \subseteq A$. Then $\mathcal{G} \preccurlyeq \mathcal{F}$.

Proof. Let A be an atom in $\mathcal{D}_{i+1}$. Denote by $C_i(A)$ a unique set of atoms in $\mathcal{D}_i$ which are pairwise incomparable w.r.t. inclusion and the union of elements of which is equal to $A \cap D_i$. It is not hard to see that if $B \subseteq A \subseteq D_{i+1}$ are atoms in $\mathcal{D}_{i+1}$ then for any $B' \in C_i(B)$ there is $A' \in C_i(A)$ such that $B' \subseteq A'$. Indeed, let $b \in D_i$ so that $B' = \{x \in D_i : b \leqslant_i x\}$. Since $b \in A \cap D_i$, we have $b \in A'$ for some $A' \in C_i(A)$.

Now we verify the condition on $\mathcal{G}$ being embeddable in $\mathcal{F}$. We have $\alpha_k(B) \neq \varnothing$ by the hypothesis of the lemma. For $i < k$, let $B \in \mathcal{B}_{i+1}$, $B' \in d_{i+1}(B)$, $A \in \mathcal{A}_{i+1}$, and $B \subseteq A$. Since $\mathcal{G}$ and $\mathcal{F}$ are good frames, $d_{i+1}(B) = C_i(B)$ and $c_{i+1}(A) = C_i(A)$. By the above, $A' \in \alpha_i(B')$ for some $A' \in C_i(A)$. $\square$

We say that frames $\mathcal{F} = (\mathcal{A}_m, \ldots, \mathcal{A}_0; c_m, \ldots, c_1)$ and $\mathcal{G} = (\mathcal{B}_k, \ldots, \mathcal{B}_0; d_k, \ldots, d_1)$ are *consistent at a level* i for $i \leqslant \min\{m, k\}$ if $\mathcal{G}^i_B \preccurlyeq \mathcal{F}^i_A$ for some $B \in \mathcal{B}_i$ and some $A \in \mathcal{A}_i$. We say that towers are *consistent* if so are their frames (at a level i).

Suppose that $\mathcal{T} = (\mathcal{P}_m, \ldots, \mathcal{P}_0; \varphi_m, \ldots, \varphi_0)$ and $\mathcal{S} = (\mathcal{Q}_k, \ldots, \mathcal{Q}_0; \psi_k, \ldots, \psi_0)$ are two distinct towers with disjoint bases, consistent at a level i, and that $\mathcal{F} = (\mathcal{A}_m, \ldots, \mathcal{A}_0; c_m, \ldots, c_1)$ and $\mathcal{G} = (\mathcal{B}_k, \ldots, \mathcal{B}_0; d_k, \ldots, d_1)$ are their frames. For some $B \in \mathcal{B}_i$, $A \in \mathcal{A}_i$, $P \in \mathcal{P}_i$, and $Q \in \mathcal{Q}_i$, we have $\mathcal{G}^i_B \preccurlyeq \mathcal{F}^i_A$, $B = \psi_i(Q)$, and $A = \varphi_i(P)$. We describe how to transform the towers.

For $i \leqslant j \leqslant m$ and $X \in \mathcal{P}_j$, put $X^* = X$, if $X \cap P = \varnothing$, and $X^* = X \cup Q$ if $P \subseteq X$. Letting $j < i$, we assume that the value of X^* has been defined for all $X \in \mathcal{P}_{j+1}$. Let $X \in \mathcal{P}_{j+1}$ and $X = Z_0 \cup \ldots \cup Z_l$ for $Z_0, \ldots, Z_l \in \mathcal{P}_j$. If $X \cap P = \varnothing$ then we put $Z^*_r = Z_r$ for all $r \leqslant l$. Suppose $X \subseteq P$. The construction described has the property that $X^* \setminus X = Y_1 \cup \ldots \cup Y_p$, where $Y_s \subseteq Q$, $Y_s \in \mathcal{Q}_{j+1}$, and $\psi_{j+1}(Y_s) \subseteq \varphi_{j+1}(X)$ for all $s \leqslant p$. For $s \leqslant p$, let $Y_s = Y^0_s \cup \ldots \cup Y^q_s$, where $Y^0_s, \ldots, Y^q_s \in \mathcal{Q}_j$. Since $\mathcal{G}^i_B \preccurlyeq \mathcal{F}^i_A$, it follows that for every $t \leqslant q$ we can find $r \leqslant m$ so that $\psi_j(Y^t_s) \subseteq \varphi_j(Z_r)$. We choose such r (effectively from $\mathcal{T}$ and $\mathcal{S}$) for all $s \leqslant p$ and $t \leqslant q(s)$, and put $Z^*_r = Z_r \cup \bigcup\{Y^t_s : 1 \leqslant s \leqslant p,\ t \leqslant q(s),\ r = r(s,t)\}$ for all $r \leqslant l$.

For every $j \leqslant m$, define $\mathcal{P}^*_j = \{X^* : X \in \mathcal{P}_j\}$. For $j \leqslant m$ and $X \in \mathcal{P}_j$, let $\varphi^*_j(X^*) = \varphi_j(X)$. It is a simple matter to show that the constructed pair $(\mathcal{P}^*_n, \ldots, \mathcal{P}^*_0; \varphi^*_n, \ldots, \varphi^*_0)$ is a tower with base$(\mathcal{T}) \cup Q$, which we denote by $\mathcal{T}^*$. We point out the following properties of the above construction:

(P1) frames of towers $\mathcal{T}$ and $\mathcal{T}^*$ coincide, and hence $\mathrm{mod}_s\mathcal{T} = \mathrm{mod}_s\mathcal{T}^*$ for any s;

(P2) for $j \leqslant i$ and $Y \in \mathcal{Q}_j$, either $Y \subseteq \mathrm{base}(\mathcal{T}^*)$, or $Y \cap \mathrm{base}(\mathcal{T}^*) = \varnothing$;

(P3) for $j \leqslant i$, if $Y \in \mathcal{Q}_j$ and $Y \subseteq \mathrm{base}(\mathcal{T}^*)$ then $Y \subseteq X^*$ and $\psi_j(Y) \subseteq \varphi^*_j(X^*)$ for some unique $X^* \in \mathcal{P}^*_j$;

(P4) for every $j \leqslant m$ and for $X \in \mathcal{P}_j$, there is a unique $X^* \in \mathcal{P}^*_j$ such that $X \subseteq X^*$, with $\varphi_j(X) = \varphi^*_j(X^*)$.

We say that a tower $\mathcal{T}^*$ is *obtained by transforming a tower* $\mathcal{T}$ *at a level* i *via* P *and* Q. In what follows, we will (somewhat loosely) speak of the towers $\mathcal{T}$ and $\mathcal{T}^*$ as of a single tower.

First, using a step-by-step construction in the course of which towers will be constructed and destroyed, we prove the statement formulated at the beginning of the section for pairs $\langle \mathcal{L}^s, \sigma\rangle$ and $\langle \mathcal{L}^{hs}, \chi\rangle$, and then we show, employing the elements of the construction described, that the statement is valid also for $\langle \mathcal{X}, \xi\rangle$.

Let $\langle \mathcal{L}, \eta \rangle$ be one of the pairs $\langle \mathcal{L}^s, \sigma \rangle$ or $\langle \mathcal{L}^{hs}, \chi \rangle$ and f be a partial computable function with the domain $2\mathbb{N}$ such that $\rho f \subseteq \delta \eta$ and the mapping $\varphi : \eta'(2x) \mapsto \eta f(2x)$ is well defined and is an isomorphic embedding of the ideal $\eta'(2\mathbb{N})$ onto a proper ideal of the semilattice $\mathcal{L}$. Since numberings σ and χ are restrictions of the numbering π, it follows that $\eta f(2x) = \pi f(2x) = \deg_m(V_{f(2x)})$ for all $x \in \mathbb{N}$. We claim that there is a computable sequence $\{M_i\}_{i \in \mathbb{N}}$ of c.e. sets such that $\deg_m(M_x) = \varphi(\eta' x)$ for all $x \in \mathbb{N}$ satisfying $\eta' x \in \eta'(2\mathbb{N})$, and moreover, if $\eta = \sigma$ then M_x is simple or cofinite, and if $\eta = \chi$, then M_x is hypersimple or cofinite.

The relation $\eta' x = \eta'(2y)$ is Σ^0_3, and hence there is a partial $\mathbf{0}''$-computable function g with the domain $\eta'^{-1}(\eta'(2\mathbb{N}))$ such that $\eta' x = \eta'(2g(x))$ for any $x \in \delta g$. For g, there are a $\mathbf{0}'$-computable function g_1 and a computable function g_2 with the property that for every $x \in \delta g$, $g_1(x,t) = g(x)$ with almost all t, and for any $x, t \in \mathbb{N}$, $g_1(x,t) = g_2(x,t,s)$ with almost all s. Let $M'_{x,t} = \bigcup\limits_{s \in \mathbb{N}} \{z \leqslant s : z \in V_{f(2g_2(x,t,s))}\}$. Then $M'_{x,t}$ is enumerable uniformly in x, t, and coincides with $V_{f(2g_1(x,t))}$ modulo finite sets. We have the Π^0_2-condition $(\forall t' \geqslant t)(g_1(x,t') = g_1(x,t))$ on x, t; hence there exists a computable function $g_3(x,t,s)$ which is non-decreasing in s and is such that $\lim\limits_{s \to \infty} g_3(x,t,s) = \infty$ iff the condition in question is satisfied. Let c^2 be a computable bijection of $\mathbb{N}^2$ onto $\mathbb{N}$, and

$$M'_x = \bigcup_{t \in \mathbb{N}} \{c^2(z,t) : (z \in M'_{x,t} \,\&\, \exists s(z \leqslant g_3(x,t,s))) \vee (t > 0 \,\&\, \exists s(z \leqslant g_3(x,t-1,s)))\}$$

for all $x \in \mathbb{N}$. Clearly, M'_x is enumerable uniformly in x.

Let $x \in \delta g$ and t_x be a minimal number such that $(\forall t \geqslant t_x)(g_1(x,t) = g_1(x,t_x) = g(x))$. Then for any $t < t_x$ the set $\{z : c^2(z,t) \in M'_x\}$ is finite, $\{z : c^2(z,t_x) \in M'_x\} = M'_{x,t_x}$, and $\{z : c^2(z,t) \in M'_x\} = \mathbb{N}$ with every $t > t_x$, which implies $M'_x \equiv_m M'_{x,t_x} \equiv_m V_{f(2g_1(x,t_x))} = V_{f(2g(x))}$.

Let $\{g'_x\}_{x \in \mathbb{N}}$ be a computable sequence of computable injective functions such that $\rho g'_x = \{c^2(z,t) : t \in \mathbb{N}, \exists s(z \leqslant g_3(x,t,s))\}$ for each $x \in \mathbb{N}$. Suppose $M_x = g'^{-1}_x(M'_x)$. Evidently, M_x is enumerable uniformly in x. For $x \in \delta g$, the set $\mathbb{N} \setminus \rho g'_x$ is computable and is disjoint from M'_x; therefore $M_x \equiv_m M'_x$.

Lastly, for $x \in \delta g$, the set $V_{2g(x)} = V_{2g_1(x,t_x)}$ coincides with M_{x,t_x} modulo finite sets, and the simplicity or cofiniteness (hypersimplicity or cofiniteness) of $V_{2g(x)}$ implies being simple or cofinite (hypersimple or cofinite) for the latter set; by construction, the property of being simple (hypersimple or cofinite) holds for M_{x,t_x} iff the corresponding property holds for M_x.

For a frame $\mathcal{F} = (\mathcal{A}_m, \ldots, \mathcal{A}_0; c_m, \ldots, c_1)$ of kind II, $M_{\mathcal{F}}$ denotes the set $M_{v_m(A)}$, where $A \in \mathcal{A}_m$. Let $\{M^t_{\mathcal{F}}\}_{t \in \mathbb{N}, \mathcal{F} \text{ is a frame of kind II}}$ be a strongly computable double sequence of finite sets such that $M^0_{\mathcal{F}} \subseteq M^1_{\mathcal{F}} \subseteq \ldots$ and $M_{\mathcal{F}} = \bigcup\limits_{t \in \mathbb{N}} M^t_{\mathcal{F}}$ for each frame $\mathcal{F}$ of kind II. We fix another series of the notation. Let Y_i, Y^t_i, K, and K^t be as in Lemma 3. Assume that $\{\theta_i\}_{i \in \mathbb{N}}$ is a universal computable sequence of all unary partial computable functions and $\{\theta^t_i\}_{i \in \mathbb{N}}$ is a strongly computable double sequence of unary functions with a finite domain such that $\theta_i = \bigcup\limits_{t \in \mathbb{N}} \theta^t_i$ for all $i \in \mathbb{N}$. (These are the same as φ_i and φ^t_i in Lemma 3; the designations have been changed, for φ with indices is used in the notation for towers.) Suppose that W_i denotes a c.e. set with Kleene number i (as it did in defining the numbering π) and $\{W^t_i\}_{t, i \in \mathbb{N}}$ is a strongly computable double sequence of finite sets such that $W^0_i \subseteq W^1_i \subseteq \ldots$ and $W_i = \bigcup\limits_{t \in \mathbb{N}} W^t_i$ for any $i \in \mathbb{N}$. For a tower $\mathcal{T} = (\mathcal{P}_m, \ldots, \mathcal{P}_0; \varphi_m, \ldots, \varphi_0)$ and for numbers $i \leqslant m$ and $a \in D_i$, let $p(a, i, \mathcal{T})$ denote the set $\{P \in \mathcal{P}_i : a \in \varphi_i(P)$, and if $i < m$ then $\varphi_{m-1}(P') \cap 2\mathbb{N} = \varnothing$ for $P' \in \mathcal{P}_{m-1}$ such that $P \subseteq P'\}$.

Before embarking on a direct description of the construction, we make several remarks. In the course of the construction, we build towers, destroy towers, and enumerate the set U. Numbers that will have visited the bases of the towers before some instant of the construction are referred to as *used* by this instant, and the remaining numbers — as *unused*. Every time when a new tower is constructed, its base will accept only numbers that have been unused by the instant of the construction; hence, for two distinct towers $\mathcal{T}_1$ and $\mathcal{T}_2$ ever appearing in the process, it is true that $\mathrm{base}(\mathcal{T}_1) \cap \mathrm{base}(\mathcal{T}_2) = \varnothing$. The used numbers not in the bases of the existing towers are said to be *rejected*. A set of rejected towers is denoted by D. A number may be rejected by virtue of the fact that a tower collapses in the base of which it lay — either directly (and then the entire base of the tower is rejected), or due to transformation of another tower (and then only that part of the base is rejected which does not go over to the base of the other tower).

Once some number has been rejected, it is automatically enumerated into U, and so $D \subseteq U$. However, if a non-rejected number is enumerated into U then that number does necessarily lie in the base of a tower that exists by the enumeration moment, and concurrently, also, all numbers in the base of the tower fall into U (i.e., the bases of the towers are enumerated into U "as a whole"). If, at some instant, one tower gets transformed at the expense of another tower, then the bases of both towers should not lie in U. Thus we can draw the following conclusion: at any instant of the construction, the base of any existing tower either is entirely in U or does not contain numbers from U.

For any frame $\mathcal{F}$, let $\mathcal{T}_0^{\mathcal{F}}, \mathcal{T}_1^{\mathcal{F}}, \ldots$ be a sequence of towers that will be built on this frame in the course of the construction, with the towers in the sequence arranged in order of appearance. At each moment, only finitely many elements of the sequence are defined, and the entire sequence may be either finite or infinite, depending on the frame. If some of the towers $\mathcal{T}_i^{\mathcal{F}}$ in the sequence undergoes a transformation at the expense of another tower suffering a collapse, this tower will retain the same denotation. (It is important that the frame of the tower under transformation is not altered, and all numbers in the base of the initial tower enter the base of the transformed tower; later, we will see that every tower may be transformed only finitely many times.)

In the construction, we also use extra functions $h(\mathcal{G}, t)$, where $\mathcal{G}$ is a frame. Let c' be a computable surjection of $\mathbb{N}$ onto the set $\{\langle \mathcal{F}, k, a, e\rangle : \mathcal{F} = (\mathcal{A}_m, \ldots, \mathcal{A}_0; c_m, \ldots, c_1)$ is a frame of kind I, $k \leqslant m$, $a \in \bigcup \mathcal{A}_k$, $e \leqslant m\}$ such that for any $t \in \mathbb{N}$, there are finitely many $t' \in \mathbb{N}$ for which $c'(t') = c'(t)$. Let c'' be a computable surjection of $\mathbb{N}$ onto the set of all frames sharing a similar property. If, in performing one of the actions, we need to choose among several possibilities then this choice will be made in some effective way. We pass to describe steps of the construction.

Step 0. Put $h(\mathcal{G}, 0) = 0$ for all frames $\mathcal{G}$. Pass to the next step.

Step $t+1$. This step includes four stages.

Stage I. We collapse every existing tower $\mathcal{T}$ such that $\mathrm{mod}_t\mathcal{T} < \mathrm{mod}_{t+1}\mathcal{T}$. Pass to the next stage.

Stage II. For every frame $\mathcal{F}$ of kind II such that $\mathrm{mod}_t\mathcal{F} \leqslant t$, we construct a tower on that frame, choosing as the base a sufficiently large initial interval of unused natural numbers. Further, let $c'(t) = \langle \mathcal{F}, k, a, e\rangle$, and

$$\begin{aligned} Q_1 &= \bigcup\{P \in p(a, k, \mathcal{T}_i^{\mathcal{F}}) : \text{tower } \mathcal{T}_i^{\mathcal{F}} \text{ exists}\}, \\ Q_2 &= \bigcup\{\mathrm{base}(\mathcal{T}_i^{\mathcal{F}}) : \text{tower } \mathcal{T}_i^{\mathcal{F}} \text{ exists}\}. \end{aligned}$$

If $Q_1 \subseteq \delta\theta_e^t$, $\theta_e(Q_1) \cap Q_2 = \varnothing$, and for any $q \in Q$, it is true that "q has been enumerated into $U \Leftrightarrow \theta_e(q)$ has been enumerated into U," then we construct a tower on $\mathcal{F}$, taking as the base a sufficiently large initial interval of unused natural numbers, and then collapse all the existing towers $\mathcal{T}$ such that $\mathrm{mod}_t\mathcal{T} > \mathrm{mod}_t\mathcal{F}$. Pass to the next stage.

Stage III. Let $c''(t) = \mathcal{G}$ and j be equal to the length of $\mathcal{G}$. We seek for an existing tower $\mathcal{T}_i^{\mathcal{F}} = (\mathcal{P}_m, \ldots, \mathcal{P}_0; \varphi_m, \ldots, \varphi_0)$ such that $m \geqslant j$, and for some $k \geqslant m$ and some existing (at the instant) tower

$\mathcal{S} = (\mathcal{Q}_k, \ldots, \mathcal{Q}_0; \psi_k, \ldots, \psi_0)$ constructed on some frame $\mathcal{G}'$, with $P \in \mathcal{P}_j$ and $Q \in \mathcal{Q}_j$, the following conditions hold:

(1) $\mathcal{G} = \mathcal{G}'^{j}_{\psi_j(Q)}$ and $P \in p(1, j, \mathcal{T}_i^{\mathcal{F}})$;

(2) $\mathrm{mod}_t \mathcal{T}_i^{\mathcal{F}} < \mathrm{mod}_t \mathcal{Q}$;

(3) $k \geqslant h(\mathcal{G}, t)$;

(4) $\mathcal{G} \preccurlyeq \mathcal{F}^{j}_{\varphi_j(P)}$;

(5) numbers in $\mathrm{base}(\mathcal{T}_i^{\mathcal{F}}) \cup \mathrm{base}(\mathcal{Q})$ are still not enumerated into U;

(6) $P \cap W_j^t = \varnothing$ and $Q \cap W_j^t \neq \varnothing$.

If such towers $\mathcal{T}_i^{\mathcal{F}}$ exist, among these we choose one for which the number $c^2(\mathrm{mod}_t \mathcal{T}_i^{\mathcal{F}}, i)$ is least possible, pick its corresponding tower $\mathcal{S}$ so that $P \in \mathcal{P}_j$ and $Q \in \mathcal{Q}_j$, and then subject $\mathcal{T}_i^{\mathcal{F}}$ to a level j transformation via P and Q. Put $h(\mathcal{G}, t+1) = h(\mathcal{G}, t) + 1$ and $h(\mathcal{G}', t+1) = h(\mathcal{G}', t)$ for all frames $\mathcal{G}' \neq \mathcal{G}$. If, however, $\mathcal{T}_i^{\mathcal{F}}$ with the required properties does not exist then we transform no towers and put $h(\mathcal{G}', t+1) = h(\mathcal{G}', t)$ for all frames $\mathcal{G}'$. Pass to the next stage.

Stage IV. For every frame $\mathcal{F}$ of kind I and every $x \in \mathbb{N}$, if the tower $\mathcal{T}_x^{\mathcal{F}}$ exists, its base is still not enumerated into U, and $x \in K^t$, then we enumerate all numbers in $\mathrm{base}(\mathcal{T}_x^{\mathcal{F}})$ into U. For each frame $\mathcal{F}$ of kind II and each $x \in \mathbb{N}$, if $\mathcal{T}_x^{\mathcal{F}}$ exists, its base is still not enumerated into U, and $x \in M_{\mathcal{F}}^t$, then we enumerate all numbers in $\mathrm{base}(\mathcal{T}_x^{\mathcal{F}})$ into U. Further, we seek for $i \leqslant t$ such that every element of the set Y_i^t contains numbers which are still not enumerated into U and there is $y \in Y_i^t$ such that:

(1) all elements of the set y are already used;

(2) all elements of y which are still not enumerated into U are in the bases of existing towers of height at least i.

If such i do not exist then we pass to the next step at once. Otherwise, for the least such i and for $y \in Y_i^t$ satisfying the two properties above, we enumerate into U all numbers in the bases of towers containing elements of the set y. Pass to the next step.

The description of the construction is completed. Clearly, all steps of the construction are effective, and hence the set U is c.e. The towers are constructed at stage II and are destroyed at stages I, II, and III. Every tower constructed on a bad frame is destroyed at stage I of some step to follow.

We say that a tower that exists at some moment of the construction is *final* if it does not get destroyed or transformed at a later time. Every tower is liable to just finitely many transformations. In fact, for the tower $\mathcal{T} = (\mathcal{P}_m, \ldots, \mathcal{P}_0; \varphi_m, \ldots, \varphi_0)$, consider a number $\gamma_t(\mathcal{T})$ equal to the number of elements of the set $\{P \in \mathcal{P}_i : i \leqslant m,\ P \cap W_i^t \neq \varnothing\}$; this number does not decrease with growing t and increases with $\mathcal{T}$ undergoing a transformation at step t. Thus every tower constructed on a good frame either collapses in some time or becomes final after finitely many steps.

For each good frame $\mathcal{F}$, if at least one of the towers constructed on $\mathcal{F}$ collapses at stages I and II, simultaneously, then, all towers on this frame collapse as well. At stage III, the towers constructed on $\mathcal{F}$ may be destroyed only finitely many times. Indeed, in each of such cases, for a frame $\mathcal{G}$ of length not more than that of $\mathcal{F}$, the value of $h(\mathcal{G}, t)$ increases so that at the destruction moment it does not exceed the length of $\mathcal{F}$. Thus all towers in the sequence $\mathcal{T}_0^{\mathcal{F}}, \mathcal{T}_1^{\mathcal{F}}, \ldots$ get destroyed — some time or other —, or else all but finitely many towers become final after finitely many transformations. Hence, for every good frame $\mathcal{F}$, the set

$$Q_{\mathcal{F}} = \bigcup \{\mathrm{base}(\mathcal{T}_i^{\mathcal{F}}) : \text{tower } \mathcal{T}_i^{\mathcal{F}} \text{ is defined and is final}\}$$

is computably enumerable.

If $X \subseteq \mathbb{N}$ and ε is an equivalence relation on $\mathbb{N}$ then we let $[X]_\varepsilon$ denote the set $\{y \in \mathbb{N} : (\exists x \in X)(\langle x, y\rangle \in \varepsilon)\}$. If X and ε are c.e. then $[X]_\varepsilon$ likewise are c.e. We say that the equivalence ε is *consistent with* U if either $x \in U$ and $y \in U$, or $x \notin U$ and $y \notin U$, for any pair $\langle x, y\rangle \in \varepsilon$.

Following [1], we introduce a Ψ-operator. For every c.e. set $X \subseteq \mathbb{N}$, define

(1) if $X \cap U \neq \varnothing$ and $X \not\subseteq U$ then $\Psi(U \mid X) = \deg_m(f^{-1}(U))$, where f is a total computable function such that $\rho f = X$ (clearly, $\Psi(U \mid X)$ does not depend on the choice of f);

(2) if $X \cap U = \varnothing$ or $X \subseteq U$ then $\Psi(U \mid X) = \bot_{\mathcal{L}^e}$.

We point out the following properties of a Ψ-operator:

(1) for $a \in L^e$, it is true that $a \leqslant^{\mathcal{L}^e} \deg_m(U) \Leftrightarrow$ there exists a c.e. set X such that $a = \Psi(U \mid X)$;

(2) $\Psi(U \mid X_1 \cup X_2) = \Psi(U \mid X_1) \vee^{\mathcal{L}^e} \Psi(U \mid X_2)$;

(3) if $X_1 =^* X_2$ then $\Psi(U \mid X_1) = \Psi(U \mid X_2)$ (hereinafter, $=^*$ denotes the equality modulo finite sets);

(4) if the c.e. equivalence ε is consistent with U then $\Psi(U \mid X) = \Psi(U \mid [X]_\varepsilon)$;

(5) $\Psi(U \mid X_1) \leqslant^{\mathcal{L}^e} \Psi(U \mid X_2)$ iff there is a partial computable function θ such that $X_1 \subseteq \delta\theta$, $\theta(X_1) \subseteq X_2$, and $(\forall x \in X_1)(x \in U \leftrightarrow \theta(x) \in U)$.

The third property implies that $\Psi(U \mid X) = \bot_{\mathcal{L}^e}$ for X finite, and the second property yields $X_1 \subseteq X_2 \Rightarrow \Psi(U \mid X_1) \leqslant^{\mathcal{L}^e} \Psi(U \mid X_2)$.

LEMMA 5. Let $\mathcal{F}$ be a good frame. Then only finitely many towers on $\mathcal{F}$ collapse in the course of the construction, and

(1) if $\mathcal{F}$ is a frame of kind I then the sequence $\mathcal{T}_0^{\mathcal{F}}, \mathcal{T}_1^{\mathcal{F}}, \ldots$ contains not more than finitely many members, and hence the set $Q_{\mathcal{F}}$ is finite;

(2) if $\mathcal{F}$ is a frame of kind II then $\mathcal{T}_0^{\mathcal{F}}, \mathcal{T}_1^{\mathcal{F}}, \ldots$ is infinite and $\Psi(U \mid Q_{\mathcal{F}}) = \deg_m(M_{\mathcal{F}})$.

The **proof** is by induction on $\mathrm{mod}(\mathcal{F})$. Assume that the statement of the lemma holds true for all good frames $\mathcal{F}'$ for which $\mathrm{mod}(\mathcal{F}') < \mathrm{mod}(\mathcal{F})$, and that $\mathcal{F}_1, \ldots, \mathcal{F}_k$ are all such frames. Let m be equal to the length of $\mathcal{F}$. Note that for every $i \in \mathbb{N}$, stage IV is realized at most once. Suppose $t_0 \geqslant \mathrm{mod}(\mathcal{F})$ is large enough so that:

(1) for any frame $\mathcal{F}'$, if $\mathrm{mod}(\mathcal{F}') \leqslant \mathrm{mod}(\mathcal{F})$, then $\mathrm{mod}_{t_0}\mathcal{F}' = \mathrm{mod}(\mathcal{F}')$, and if $\mathrm{mod}(\mathcal{F}') > \mathrm{mod}(\mathcal{F})$, then $\mathrm{mod}_{t_0}\mathcal{F}' > \mathrm{mod}(\mathcal{F})$;

(2) towers on $\mathcal{F}_1, \ldots, \mathcal{F}_k$ do not collapse at steps greater than or equal to t_0;

(3) if $\mathcal{F}_i$ is a frame of kind I for $i \in [1, k]$, then no towers with $\mathcal{F}_i$ are constructed at steps greater than or equal to t_0;

(4) towers with $\mathcal{F}$ do not collapse at stages III of steps greater than or equal to t_0;

(5) for every $i \leqslant m$, if stage IV is realized for i, then it is realized at a lesser step than t_0.

In view of the first condition, at steps greater than or equal to t_0, the towers with $\mathcal{F}$ cannot collapse at stages I, nor at stages II by conditions (1) and (3). Combined with (4), this means that the towers constructed on $\mathcal{F}$ fail to collapse at steps greater than or equal to t_0.

If $\mathcal{F}$ is a frame of kind II, then the sequence $\mathcal{T}_0^{\mathcal{F}}, \mathcal{T}_1^{\mathcal{F}}, \ldots$ is infinite, since a new tower with $\mathcal{F}$ is constructed at stage II of each step starting with t_0. The fifth condition for t_0 and the description of stage IV imply that $\mathrm{base}(\mathcal{T}_i^{\mathcal{F}})$ is enumerated into U at steps greater than or equal to t_0 iff $i \in M_{\mathcal{F}}$. For almost all i, therefore, the equivalence $\mathrm{base}(\mathcal{T}_i^{\mathcal{F}}) \subseteq U \Leftrightarrow i \in M_{\mathcal{F}}$ holds, whence $\Psi(U \mid Q_{\mathcal{F}}) = \deg_m(M_{\mathcal{F}})$.

Suppose that $\mathcal{F}$ is a frame of kind I and the sequence $\mathcal{T}_0^{\mathcal{F}}, \mathcal{T}_1^{\mathcal{F}}, \ldots$ is infinite. Following essentially the same argument as above, we can show that in this case $\Psi(U \mid Q_{\mathcal{F}}) = \deg_m(K) = \top_{\mathcal{L}^e}$. Furthermore, for some quadruple $\langle \mathcal{F}, k', a, e\rangle$, stage II is realized infinitely often, which means that the following two statements hold:

(1) every tower constructed on a frame other than $\mathcal{F}_1, \dots, \mathcal{F}_k, \mathcal{F}$ collapses in some time;

(2) $Q_1 \subseteq \delta\theta_e$, $\theta_e(Q_1) \cap Q_{\mathcal{F}} = \varnothing$, and $(\forall q \in Q_1)(q \in U \leftrightarrow \theta_e(q) \in U)$, where $Q_1 = \bigcup\{P \in p(a, k', \mathcal{T}_i^{\mathcal{F}}) :$ tower $\mathcal{T}_i^{\mathcal{F}}$ is final$\}$.

Evidently, Q_1 is c.e. It is also clear that the equivalence ε given by the rule $\langle x, y\rangle \in \varepsilon \Leftrightarrow x = y \vee \exists i$(tower $\mathcal{T}_i^{\mathcal{F}}$ is final and $x, y \in \mathrm{base}(\mathcal{T}_i^{\mathcal{F}})$) is c.e. and is consistent with U. We have $[Q_1]_\varepsilon = Q_{\mathcal{F}}$, and hence $\Psi(U\,|\,Q_1) = \Psi(U \mid Q_{\mathcal{F}}) = \top_{\mathcal{L}^e}$. In view of the first of the above two statements, $\mathbb{N} = Q_{\mathcal{F}_1} \cup \ldots \cup Q_{\mathcal{F}_k} \cup Q_{\mathcal{F}} \cup D$. This equality, together with statement (2) and properties of a Ψ-operator, entails $\Psi(U\,|\,Q_1) \leqslant^{\mathcal{L}^e} \Psi(U\,|\,Q_{\mathcal{F}_1}) \vee^{\mathcal{L}^e} \ldots \vee^{\mathcal{L}^e} \Psi(U \mid Q_{\mathcal{F}_k}) \vee^{\mathcal{L}^e} \Psi(U \mid D)$. However, each of the members in the right part is equal to $\perp_{\mathcal{L}^e}$ or to $\deg_m(M_{\mathcal{F}'})$ for some frame $\mathcal{F}'$ of kind II; consequently the entire right part belongs to the range of a partial numbering η. Contradiction. □

We bring into consideration a number of equivalence relations and sets. Let $i, m \in \mathbb{N}$ be arbitrary natural numbers and $d \in D_i$. Put

$$\begin{aligned}
\varepsilon &= \{\langle x, y\rangle : x, y \in D\} \cup \{\langle x, y\rangle : x, y \in \mathrm{base}(\mathcal{T}) \text{ for some final tower } \mathcal{T}\};\\
\varepsilon_i &= \{\langle x, y\rangle : x, y \in D \vee x = y\} \cup \{\langle x, y\rangle : \text{there exist } k \geqslant i, \text{ a final tower}\\
&\qquad \mathcal{T} = (\mathcal{P}_k, \ldots, \mathcal{P}_0; \varphi_k, \ldots, \varphi_0), \text{ and } P \in \mathcal{P}_i \text{ such that } x, y \in P\};\\
R_{d,i}^m &= D \cup \bigcup\{P : \text{there exist } k \geqslant i + m \text{ and a final tower}\\
&\qquad \mathcal{T} = (\mathcal{P}_k, \ldots, \mathcal{P}_0; \varphi_k, \ldots, \varphi_0) \text{ such that } P \in p(d, i, \mathcal{T})\}.
\end{aligned}$$

Clearly, $\varepsilon_i \subseteq \varepsilon$ for any $i \in \mathbb{N}$. Since the equivalence ε is consistent with U, equivalences ε_i, for all $i \in \mathbb{N}$, are also consistent with U. For any $i \in \mathbb{N}$, $d \in D_i$, and $m_1 < m_2$, we put $S_{d,i}^{m_1,m_2} = R_{d,i}^{m_1} \setminus R_{d,i}^{m_2}$ and $T_{d,i}^{m_1,m_2} = \{\mathcal{F} : \mathcal{F}$ is a good frame of length greater than or equal to $i+m_1$ and less than $i+m_2$ such that for any tower $\mathcal{T}$ on $\mathcal{F}$, the set $p(d, i, \mathcal{T})$ is not empty$\}$. Consequently the set $T_{d,i}^{m_1,m_2}$ is finite and $S_{d,i}^{m_1,m_2} = \bigcup\{P :$ there exist $\mathcal{F} \in T_{d,i}^{m_1,m_2}$, $k \in [i + m_1, i + m_2)$, and a final tower $\mathcal{T}_j^{\mathcal{F}} = (\mathcal{P}_k, \ldots, \mathcal{P}_0; \varphi_k, \ldots, \varphi_0)$ for which $P \in p(d, i, \mathcal{T}_j^{\mathcal{F}})\}$. By Lemma 5, $S_{d,i}^{m_1,m_2}$ is c.e. For any $d \in D_i$, let $\wedge(d, i)$ denote an element of D_i equal to $u_i(\{v_i'(d, 2x) : 2x \in D_i\})$. Evidently, $\wedge(d, i) \leqslant_i d$, $\wedge(d, i) \leqslant_{i+1} \wedge(d, i + 1)$, and $\eta'(\wedge(d, i)) \in \eta'(2\mathbb{N})$.

LEMMA 6. For any $m, i \in \mathbb{N}$ and any $d \in D_i$, the relation ε_i and the set $R_{d,i}^m$ are computably enumerable.

Proof. Let

$$\begin{aligned}
H_i^1 &= \{\mathcal{G} : \mathcal{G} \text{ is a frame of length} < i,\ \lim_{t\to\infty} h(\mathcal{G}, t) = \infty\};\\
H_i^2 &= \{\mathcal{G} : \mathcal{G} \text{ is a frame of length} < i,\ \lim_{t\to\infty} h(\mathcal{G}, t) < \infty\};\\
h(i, t) &= \begin{cases} \min\{h(\mathcal{G}, t) : \mathcal{G} \in H_i^1\} & \text{if } H_i^1 \neq \varnothing,\\ t & \text{otherwise.}\end{cases}
\end{aligned}$$

Let t_i be large enough so that $h(\mathcal{G}, t_i) = \lim\limits_{t\to\infty} h(\mathcal{G}, t)$ for any $\mathcal{G} \in H_i^2$. It suffices to show that

$$\begin{aligned}
\varepsilon_i &= \{\langle x, y\rangle : x, y \in D \vee x = y\} \cup \{\langle x, y\rangle : \text{at some step } t \geqslant t_i \text{ there exist } h(i, t) > k \geqslant i,\\
&\qquad \text{a tower } \mathcal{T} = (\mathcal{P}_k, \ldots, \mathcal{P}_0; \varphi_k, \ldots, \varphi_0), \text{ and } P \in \mathcal{P}_i \text{ such that } x, y \in P\};\\
R_{d,i}^m &= D \cup \bigcup\{P : \text{at some step } t \geqslant t_i \text{ there exist } h(i, t) > k \geqslant i + m \text{ and a tower}\\
&\qquad \mathcal{T} = (\mathcal{P}_k, \ldots, \mathcal{P}_0; \varphi_k, \ldots, \varphi_0) \text{ such that } P \in p(d, i, \mathcal{T})\}.
\end{aligned}$$

Indeed, it is easy to see that the right parts in both of the equalities are c.e.

The inclusions from left to right in the equalities follow readily from the fact that $\lim_{t\to\infty} h(i,t) = \infty$ and any final tower exists at all sufficiently large steps. For the inclusions from right to left, it suffices to state that if at some step $t \geqslant t_i$ there exists a tower $\mathcal{T} = (\mathcal{P}_k, \ldots, \mathcal{P}_0; \varphi_k, \ldots, \varphi_0)$ with $h(i,t) > k \geqslant i$, and $P \in \mathcal{P}_i$, then either $P \subseteq D$, or $P \subseteq Q$ and $P \in p(d,i,\mathcal{T}) \Rightarrow Q \in p(d,i,\mathcal{S})$ for some final tower $\mathcal{S} = (\mathcal{Q}_l, \ldots, \mathcal{Q}_0; \psi_l, \ldots, \psi_0)$ with $l \geqslant i$ and some $Q \in \mathcal{Q}_i$. If $\mathcal{T}$ does not collapse after step t then it becomes final after finitely many transformations, and we are done. If $\mathcal{T}$ collapses at stages I and II of some larger step then all numbers in base($\mathcal{T}$) go over to D, and $P \subseteq D$. Lastly, if our tower collapses at stage III then, by the choice of t_i, it collapses due to being transformed at a level greater than or equal to i, and all numbers in P either are rejected or go over to the base of a tower of height less than or equal to k, with a smaller module. As long as the tower exists its module remains invariant; so if we assume that for some tower $\mathcal{T}$ something is violated then we may take $\mathcal{T}$ with the least possible module and obtain a contradiction then. $\square$

LEMMA 7. For any $i, m \in \mathbb{N}$ and any $d \in D_i$, the following equalities hold:
(1) $\Psi(U \mid S^{0,m+1}_{d,i}) = \deg_m(M_{\wedge(d,i+m)})$,
(2) $R^0_{d,i+m} = [R^m_{d,i}]_{\varepsilon_{i+m}} \cup S^{0,1}_{d,i+m}$,
(3) $\Psi(U \mid R^0_{d,i}) = \Psi(U \mid R^0_{d,i+1})$.

Proof. We argue for the first equality. For every frame $\mathcal{F} \in T^{0,m+1}_{d,i}$ of length $k \in [i, i+m]$, let

$$Q'_{\mathcal{F}} = \bigcup\{P : \text{there is a final tower } \mathcal{T}^{\mathcal{F}}_j \text{ such that } P \in p(d,i,\mathcal{T}^{\mathcal{F}}_j)\}.$$

It is then easy to see that $[Q'_{\mathcal{F}}]_{\varepsilon_k} = Q_{\mathcal{F}}$. Hence $\Psi(U|Q'_{\mathcal{F}}) = \Psi(U|Q_{\mathcal{F}})$. Since $S^{0,m+1}_{d,i} = \bigcup\{Q'_{\mathcal{F}} : \mathcal{F} \in T^{0,m+1}_{d,i}\}$, by the property of a Ψ-operator, we have $\Psi(U \mid S^{0,m+1}_{d,i}) = \bigvee^{\mathcal{L}^e}\{\Psi(U \mid Q_{\mathcal{F}}) : \mathcal{F} \in T^{0,m+1}_{d,i}\}$ (assuming that $\bigvee^{\mathcal{L}^e} \varnothing = \bot_{\mathcal{L}^e}$). The set on the right-hand side of the equality may be left with just frames kind II, for the set $Q_{\mathcal{F}}$ is finite for every frame $\mathcal{F}$ of kind I. By Lemma 5, therefore, we have $\Psi(U|S^{0,m+1}_{d,i}) = \bigvee^{\mathcal{L}^e}\{\deg_m(M_{\mathcal{F}}) : \mathcal{F}$ is a frame of kind II in $T^{0,m+1}_{d,i}\}$.

For each frame $\mathcal{F} = (\mathcal{A}_k, \ldots, \mathcal{A}_0; c_k, \ldots, c_1)$, let $v(\mathcal{F}) = v_k(A)$, where $A \in \mathcal{A}_k$. By the definition of $M_{\mathcal{F}}$, we then have $M_{\mathcal{F}} = M_{v(\mathcal{F})}$. Suppose $v = u_{i+m}(\{v(\mathcal{F}) : \mathcal{F}$ is a frame of kind II in $T^{0,m+1}_{d,i}\})$. We have $\eta' v(\mathcal{F}) \in \eta'(2\mathbb{N})$ for every frame $\mathcal{F}$ of kind II and the mapping φ is an isomorphic embedding of the ideal $\eta'(2\mathbb{N})$ onto an ideal of the semilattice $\mathcal{L}^e$ for which $\varphi(x) = \deg_m(M_x)$ with any $x \in \eta'^{-1}(\eta'(2\mathbb{N}))$. It remains to show that $v \equiv_{i+m} \wedge(d, i+m)$.

Since $v(\mathcal{F}) \leqslant_k \wedge(d,k) \leqslant_{i+m} \wedge(d,i+m)$ for any frame $\mathcal{F} = (\mathcal{A}_k, \ldots, \mathcal{A}_0; c_k, \ldots, c_1) \in T^{0,m+1}_{d,i}$ of kind II, it follows that $v \leqslant_{i+m} \wedge(d,i+m)$. Suppose $\wedge(d,i+m) \not\leqslant_{i+m} v$. Then there exists an atom A in $\mathcal{D}_i$ such that $\wedge(d,i+m) \in A$ and $v \notin A$. Assume $\mathcal{F} = (\mathcal{A}_{i+m}, \ldots, \mathcal{A}_0; c_{i+m}, \ldots, c_1)$ is a good frame for which $A \in \mathcal{A}_{i+m}$. Let k be a minimal number in $[i, i+m]$ with $\wedge(d,k) \in \bigcup \mathcal{A}_k$. Suppose $B \in \mathcal{A}_k$ is such that $\wedge(d,k) \in B$. It is easy to verify that $\mathcal{F}^k_B \in T^{0,m+1}_{d,i}$ is a frame of kind II. But then $v_k(B) = v(\mathcal{F}^k_B) \leqslant_{i_m} v$ and $v_k(B) \in B \subseteq A$, which contradicts the choice of A.

We argue for the second equality. By definition, $S^{0,1}_{d,i+m} \subseteq R^0_{d,i+m}$. Let $x \in [R^m_{d,i}]_{\varepsilon_{i+m}}$. Then either $x \in D$ and $x \in R^0_{d,i+m}$, or there are $P' \in \mathcal{P}_i$ and $P'' \in \mathcal{P}_{i+m}$ such that $P' \subseteq P''$, $x \in P''$, and $P' \in p(d,i,\mathcal{T})$ for some final tower $\mathcal{T} = (\mathcal{P}_k, \ldots, \mathcal{P}_0; \varphi_k, \ldots, \varphi_0)$ of height $k \geqslant i+m$. But then $P'' \in p(d, i+m, \mathcal{T})$ and $x \in R^0_{d,i+m}$. Conversely, let $x \in R^0_{d,i+m}$. Then either $x \in D$ and $x \in [R^m_{d,i}]_{\varepsilon_{i+m}}$, or there is $P \in p(d, i+m, \mathcal{T})$ such that $x \in P$ for some final tower $\mathcal{T} = (\mathcal{P}_k, \ldots, \mathcal{P}_0; \varphi_k, \ldots, \varphi_0)$ of height $k \geqslant i+m$. If $k = i+m$ then $x \in S^{0,1}_{d,i+m}$. If, however, $k > i+m$ then $P' \in p(d,i,\mathcal{T})$ for some $P' \in \mathcal{P}_i$ with $P \supseteq P'$. It follows that, for any $y \in P'$, $y \in R^m_{d,i}$ and $\langle x, y\rangle \in \varepsilon_{i+m}$; so $x \in [R^m_{d,i}]_{\varepsilon_{i+m}}$.

We verify the third equality. In view of the first equality, $\Psi(U \mid S^{0,1}_{d,i}) = \deg_m(M_{\wedge(d,i)}) \leqslant^{\mathcal{L}^e} \deg_m(M_{\wedge(d,i+1)}) = \Psi(U|S^{0,1}_{d,i+1})$. By the properties of a Ψ-operator and the argument above, $\Psi(U|R^0_{d,i+1}) = \Psi(U|R^1_{d,i}) \vee^{\mathcal{L}^e} \Psi(U|S^{0,1}_{d,i+1}) = \Psi(U|R^1_{d,i}) \vee^{\mathcal{L}^e} \Psi(U|S^{0,1}_{d,i}) \vee^{\mathcal{L}^e} \Psi(U|S^{0,1}_{d,i+1}) = \Psi(U|R^1_{d,i} \cup S^{0,1}_{d,i}) \vee^{\mathcal{L}^e} \Psi(U|S^{0,1}_{d,i+1}) = \Psi(U \mid R^0_{d,i}) \vee^{\mathcal{L}^e} \Psi(U \mid S^{0,1}_{d,i+1})$. Since $S^{0,2}_{d,i} \subseteq R^0_{d,i}$, we have $\Psi(U \mid S^{0,1}_{d,i+1}) = \deg_m(M_{\wedge(d,i+1)}) = \Psi(U \mid S^{0,2}_{d,i}) \leqslant^{\mathcal{L}^e} \Psi(U \mid R^0_{d,i})$. Ultimately we arrive at $\Psi(U \mid R^0_{d,i+1}) = \Psi(U \mid R^0_{d,i}) \vee^{\mathcal{L}^e} \Psi(U \mid S^{0,1}_{d,i+1}) = \Psi(U \mid R^0_{d,i})$. □

To every $d \in \mathbb{N}$ we assign an m-degree $\Psi(U \mid R^0_{d,i})$, where i is such that $d \in D_i$. (By the previous lemma, such an m-degree does not depend on the choice of i.) In view of this assignment, a mapping λ from the semilattice $\mathcal{L}'$ to the semilattice $\mathcal{L}^e$ is well defined, since $c \equiv_i d \Rightarrow (c \in A \leftrightarrow d \in A)$ for any $c, d \in D_i$ and any atom A in $\mathcal{D}_i$. For any $c, d \in D_i$ and any $A \subseteq D_i$, we have $u'_i(c,d) \in A \Leftrightarrow c \in A \vee d \in A$; so λ is a semilattice homomorphism. If $d \in \mathbb{N}$ so that $\eta' d \in \eta'(2\mathbb{N})$, then $d \in D_i$, $\wedge(d,i) \equiv_i d$, $R^1_{d,i} = D$, and $\lambda(\eta' d) = \Psi(U \mid R^0_{d,i}) = \Psi(U \mid S^{0,1}_{d,i} \cup R^1_{d,i}) = \Psi(U \mid S^{0,1}_{d,i}) \vee^{\mathcal{L}^e} \Psi(U \mid R^1_{d,i}) = \deg_m(M_d) \vee \bot_{\mathcal{L}^e} = \varphi(\eta' d)$ for some sufficiently large i, and hence λ extends φ.

LEMMA 8. Let $j \in \mathbb{N}$. Then there are $m \in \mathbb{N}$ and $d \in D_j$ such that $R^0_{d,j} \subseteq [(W_j \cup U) \cap R^0_{1,j}]_{\varepsilon_j} \subseteq R^0_{d,j} \cup S^{0,m+1}_{1,j} \cup U$.

Proof. We say that an atom A in $\mathcal{D}_j$ is *j-dense* if there are infinitely many good frames $\mathcal{F}$ of length not less than j such that $A = \varphi_j(P)$, $P \in p(1, j, \mathcal{T}^{\mathcal{F}}_i)$, and $(P \cap W_j) \setminus U \neq \varnothing$ for some final tower $\mathcal{T}^{\mathcal{F}}_i = (\mathcal{P}_k, \ldots, \mathcal{P}_0; \varphi_k, \ldots, \varphi_0)$ and some $P \in \mathcal{P}_j$. Let $d = u_j(\{v_j(A) : A$ is a j-dense atom in $\mathcal{D}_j\}$. Assume m is large enough so that for any non j-dense atom A in $\mathcal{D}_j$ and any good frame $\mathcal{F}$ of length greater than m, for any final tower $\mathcal{T}^{\mathcal{F}}_i = (\mathcal{P}_k, \ldots, \mathcal{P}_0; \varphi_k, \ldots, \varphi_0)$ and any $P \in p(1, j, \mathcal{T}^{\mathcal{F}}_i)$, if $A = \varphi_j(P)$ then $(P \cap W_j) \setminus U = \varnothing$. We argue to show that d and m satisfy the desired properties.

Suppose that the first inclusion fails. Then there are a final tower $\mathcal{T}^{\mathcal{F}}_i = (\mathcal{P}_k, \ldots, \mathcal{P}_0; \varphi_k, \ldots, \varphi_0)$ and $P \in p(d, j, \mathcal{T}^{\mathcal{F}}_i)$ such that $P \nsubseteq [(W_j \cup U) \cap R^0_{1,j}]_{\varepsilon_j}$. The latter is equivalent to $P \cap (W_j \cup U) = \varnothing$. We choose $\mathcal{T}^{\mathcal{F}}_i$ with least possible $c^2(\mathrm{mod}(\mathcal{T}^{\mathcal{F}}_j), i)$. If $d \equiv_j 0$, then $R^0_{d,j} = D$ and the first inclusion holds; so we may assume that $d \not\equiv_j 0$. In view of the definition of d and the distributivity of $\mathcal{D}_j$, there exists a j-dense atom A such that $\varphi_i(P) \supseteq A$. Let $\mathfrak{F} = \{\mathcal{F}' : \mathcal{F}'$ is a frame such that $c^2(\mathrm{mod}_t \mathcal{F}', i') \leqslant c^2(\mathrm{mod}(\mathcal{F}), i)$ for some $t, i' \in \mathbb{N}\}$. Suppose t_0 is large enough so that for any $\mathcal{F}' \in \mathfrak{F}$ and any $i' \in \mathbb{N}$, such that the tower $\mathcal{T}^{\mathcal{F}'}_{i'}$ is defined by step t_0, and

(1) if $\mathcal{T}^{\mathcal{F}'}_{i'}$ ever collapses, then it has already collapsed, and if it does not, then it has become final;

(2) if $\mathrm{base}(\mathcal{T}^{\mathcal{F}'}_{i'}) \subseteq U$, then the numbers in $\mathrm{base}(\mathcal{T}^{\mathcal{F}'}_{i'})$ have already been enumerated into U;

(3) $\mathrm{base}(\mathcal{T}^{\mathcal{F}'}_{i'}) \cap W^{t_0}_j = \mathrm{base}(\mathcal{T}^{\mathcal{F}'}_{i'}) \cap W_j$.

Let $\mathcal{G}$ be a good frame of length j such that its base contains an atom A. By Lemma 4, $\mathcal{G} \preccurlyeq \mathcal{F}^j_{\varphi_j(P)}$. After step t_0, the value $h(\mathcal{G}, t)$ cannot increase since as it does, some tower must be transformed. However, the description of stage III, the minimality of $c^2(\mathrm{mod}(\mathcal{T}^{\mathcal{F}}_i), i)$, and the choice of t_0 show that exactly the tower $\mathcal{T}^{\mathcal{F}}_i$ should undergo a transformation, which it does not, since it has already become final. Hence there is $r \in \mathbb{N}$ for which $h(\mathcal{G}, t_0) = r = \lim\limits_{t \to \infty} h(\mathcal{G}, t)$. Since A is j-dense, there is a final tower $\mathcal{S} = (\mathcal{Q}_p, \ldots, \mathcal{Q}_0; \psi_p, \ldots, \psi_0)$ of height $p \geqslant \max\{k, r\}$ on some frame $\mathcal{G}'$ such that $\mathrm{mod}(\mathcal{S}) > \mathrm{mod}(\mathcal{T}^{\mathcal{F}}_i)$, and for some $Q \in \mathcal{Q}_j$, $A = \psi_j(Q)$ and $(Q \cap W_j) \setminus U \neq \varnothing$. The tower $\mathcal{S}$ is final; so the frame $\mathcal{G}'$ is good and $\mathcal{G} = \mathcal{G}'^j_{\psi_j(Q)}$. Let $t \geqslant t_0$ so that the tower $\mathcal{S}$ exists at step t, $Q \cap W^t_j \neq \varnothing$, and $c''(t) = \mathcal{G}$. The description of stage III maintains that $h(\mathcal{G}, t+1) = h(\mathcal{G}, t) + 1$, which, as noted, is an impossibility.

We argue for the second inclusion. Let $x \in [(W_j \cup U) \cap R^0_{1,j}]_{\varepsilon_j}$. If $x \in D$ then $x \in R^0_{d,j}$. But if $x \notin D$ then $x \in P$ for some final tower $\mathcal{T} = (\mathcal{P}_k, \ldots, \mathcal{P}_0; \varphi_k, \ldots, \varphi_0)$ of height $k \geqslant j$ and some $P \in p(1, j, \mathcal{T})$ such that $P \cap (W_j \cup U) \neq \varnothing$. If $x \in U$ then $x \in R^0_{d,j} \cup S^{0,m+1}_{1,j} \cup U$. If $d \in P$ then $P \subseteq R^0_{d,j}$, and again $R^0_{d,j} \cup S^{0,m+1}_{1,j} \cup U$. Lastly, let $P \nsubseteq U$ and $d \notin U$. Then $P \cap U = \varnothing$, the atom $\varphi_j(P)$ is not j-dense, and

$(P \cap W_j) \setminus U \neq \varnothing$; hence $k \leqslant m$ and $x \in S^{0,m+1}_{d,j} \subseteq R^0_{d,j} \cup S^{0,m+1}_{1,j} \cup U$. $\square$

LEMMA 9. The image of a mapping λ is an ideal of the semilattice $\mathcal{L}^e$.

Proof. Since $\lambda(\top_{\mathcal{L}'}) = \lambda(\eta' 1) = \Psi(U \mid R^0_{1,0})$ is the greatest element of the image, it suffices to show that $\Psi(U \mid W_j) \in \rho\lambda$ for any $j \in \mathbb{N}$ such that $W_j \subseteq R^0_{1,0}$. We fix arbitrary W_j with this property. Every set in the Kleene numbering has infinitely many numbers. Therefore we may assume that $j > 0$.

By Lemma 7, we have $R^0_{1,j} = [R^j_{1,0}]_{\varepsilon_j} \cup S^{0,1}_{1,j}$. Clearly, $[S^{0,j}_{1,0}]_{\varepsilon_j} = S^{0,j}_{1,0}$. We obtain $R^0_{1,j} \cup S^{0,j}_{1,0} = [R^j_{1,0}]_{\varepsilon_j} \cup [S^{0,j}_{1,0}]_{\varepsilon_j} \cup S^{0,1}_{1,j} = [R^j_{1,0} \cup S^{0,j}_{1,0}]_{\varepsilon_j} \cup S^{0,1}_{1,j} = [R^0_{1,0}]_{\varepsilon_j} \cup S^{0,1}_{1,j}$. Further, $((W_j \cup U) \cap R^0_{1,j}) \cup ((W_j \cup U) \cap S^{0,j}_{1,0}) = (W_j \cup U) \cap (R^0_{1,j} \cup S^{0,j}_{1,0}) = (W_j \cup U) \cap ([R^0_{1,0}]_{\varepsilon_j} \cup S^{0,1}_{1,j}) = (W_j \cap [R^0_{1,0}]_{\varepsilon_j}) \cup (W_j \cap S^{0,1}_{1,j}) \cup (U \cap ([R^0_{1,0}]_{\varepsilon_j} \cup S^{0,1}_{1,j})) = W_j \cup (W_j \cap S^{0,1}_{1,j}) \cup (U \cap ([R^0_{1,0}]_{\varepsilon_j} \cup S^{0,1}_{1,j})) = W_j \cup U_1$, where $U_1 = U \cap ([R^0_{1,0}]_{\varepsilon_j} \cup S^{0,1}_{1,j})$ is a c.e. subset of U. Let $X_1 = (W_j \cup U) \cap S^{0,j}_{1,0}$. Then X_1 is a c.e. subset of $S^{0,j}_{1,0}$ and $((W_j \cup U) \cap R^0_{1,j}) \cup X_1 = W_j \cup U_1$. We have $\Psi(U \mid W_j) = \Psi(U \mid W_j) \vee^{\mathcal{L}^e} \Psi(U \mid U_1) = \Psi(U \mid W_j \cup U_1) = \Psi(U \mid ((W_j \cup U) \cap R^0_{1,j}) \cup X_1) = \Psi(U \mid (W_j \cup U) \cap R^0_{1,j}) \vee^{\mathcal{L}^e} x_1$, where $x_1 = \Psi(U \mid X_1) \in \lambda(\eta'(2\mathbb{N}))$. By Lemma 8, for some $m \in \mathbb{N}$ and some $d \in D_j$, $R^0_{d,j} \subseteq [(W_j \cup U) \cap R^0_{1,j}]_{\varepsilon_j} \subseteq R^0_{d,j} \cup S^{0,m+1}_{1,j} \cup U$, and since $\Psi(U \mid [(W_j \cup U) \cap R^0_{1,j}]_{\varepsilon_j}) = \Psi(U \mid (W_j \cup U) \cap R^0_{1,j})$, we conclude that $\lambda(\eta' d) = \Psi(U \mid R^0_{d,j}) \leqslant^{\mathcal{L}^e} \Psi(U \mid (W_j \cup U) \cap R^0_{1,j}) \leqslant^{\mathcal{L}^e} \Psi(U \mid R^0_{d,j} \cup S^{0,m+1}_{1,j} \cup U) = \lambda(\eta' d) \vee^{\mathcal{L}^e} x_2$, where $x_2 = \Psi(U \mid S^{0,m+1}_{1,j} \cup U) = \lambda(\eta'(\wedge(1, j+m))) \in \lambda(\eta'(2\mathbb{N}))$. Thus $\lambda(\eta' d) \vee^{\mathcal{L}^e} x_1 \leqslant^{\mathcal{L}^e} \Psi(U \mid W_j) \leqslant^{\mathcal{L}^e} \lambda(\eta' d) \vee^{\mathcal{L}^e} x_1 \vee^{\mathcal{L}^e} x_2$. Now, $\lambda(\eta'(2\mathbb{N}))$ is an ideal in the distributive lattice $\mathcal{L}^e$; so there is $x_3 \in \lambda(\eta'(2\mathbb{N}))$ such that $\Psi(U \mid W_j) = \lambda(\eta' d) \vee^{\mathcal{L}^e} x_3$. Let $d' \in \mathbb{N}$ so that $x_3 = \lambda(\eta'(2d'))$, and for some $j' \geqslant j$, $2d' \in D_{j'}$. Eventually we arrive at $\Psi(U \mid W_j) = \lambda(\eta' u'_{j'}(d, 2d')) \in \rho\lambda$. $\square$

LEMMA 10. For $c, d \in \mathbb{N}$, if $\eta' c \not\leqslant^{\mathcal{L}'} \eta' d$ then $\lambda(\eta' c) \not\leqslant^{\mathcal{L}^e} \lambda(\eta' d)$.

Proof. We handle two cases.

Case 1. Either $\eta' c \notin \eta'(2\mathbb{N})$ and $\eta' d \in \eta'(2\mathbb{N})$, or $\eta' c \leqslant^{\mathcal{L}'} \eta' d$ if $\eta' c' \in \eta'(2\mathbb{N})$ and $\eta' c' \leqslant^{\mathcal{L}'} \eta' c$ for any $c' \in \mathbb{N}$. Let k be large enough so that $c, d \in D_k$. If $\lambda(\eta' c) \leqslant^{\mathcal{L}^e} \lambda(\eta' d)$, then $\Psi(U \mid R^0_{c,k}) \leqslant^{\mathcal{L}^e} \Psi(U \mid R^0_{d,k})$, and by the property of a Ψ-operator, there is $e \in \mathbb{N}$ such that $R^0_{c,k} \subseteq \delta\theta_e$, $\theta_e(R^0_{c,k}) \subseteq R^0_{d,k}$, and for any $x \in R^0_{c,k}$, $x \in U \Leftrightarrow \theta_e(x) \in U$. Assume $m \geqslant \max\{k, e\}$. If $c \leqslant_m u'_m(d, \wedge(c, m))$, then $\eta' c \leqslant^{\mathcal{L}'} \eta' d \vee^{\mathcal{L}'} \eta'(\wedge(c, m))$, and since $\eta' c \not\leqslant^{\mathcal{L}'} \eta' d$, we have $d \in \eta'(2\mathbb{N}) \Rightarrow c \in \eta'(2\mathbb{N})$ and $\eta'(\wedge(c, m)) \not\leqslant^{\mathcal{L}'} \eta' d$, which contradicts our choice. Suppose A is an atom in $\mathcal{D}_m$ such that $c \in A$ and $u'_i(d, \wedge(c, m)) \notin A$. Let $\mathcal{F} = (\mathcal{A}_m, \ldots, \mathcal{A}_0; c_m, \ldots, c_1)$ be a good frame of length m satisfying $A \in \mathcal{A}_m$. Since $\wedge(c, m) \notin A$, it follows that $A \cap 2\mathbb{N} = \varnothing$ and $\mathcal{F}$ is a frame of kind I. By Lemma 5, towers with $\mathcal{F}$ are constructed only at finitely many steps. Let

$$Q_1 = \bigcup \{P \in p(c, k, \mathcal{T}^{\mathcal{F}}_i) : \text{tower } \mathcal{T}^{\mathcal{F}}_i \text{ is defined and is final}\}.$$

Then $Q_1 \subseteq R^0_{c,k}$, $Q_1 \subseteq Q_{\mathcal{F}}$, and $Q_{\mathcal{F}} \cap R^0_{d,k} = \varnothing$.

Suppose t_0 is a sufficiently large step such that by this step, $\mathcal{T}^{\mathcal{F}}_i$ is defined for any i so that if the tower ever collapses, then it has already collapsed, and if it does not, then it exists and is final. Assume $t_1 \geqslant t_0$ is large enough so that $Q_1 \subseteq \delta\theta^{t_1}_e$ and all numbers in $(Q_1 \cup \theta_e(Q_1)) \cap U$ have been enumerated into U before step t_1. If we appeal to the description of stage II we see that a tower with $\mathcal{F}$ will be constructed at a step $t \geqslant t_1$ such that $c'(t) = \langle \mathcal{F}, k, c, e \rangle$, which is a contradiction with the choice of t_0.

Case 2. The previous case fails. For some $c' \in \mathbb{N}$, we then have $\eta' c' \in \eta'(2\mathbb{N})$, $\eta' c' \leqslant^{\mathcal{L}'} \eta' c$, and $\eta' c' \not\leqslant^{\mathcal{L}'} \eta' d$. For $\lambda(\eta' c) \not\leqslant^{\mathcal{L}^e} \lambda(\eta' d)$, it suffices to show that $\lambda(\eta' c') \not\leqslant^{\mathcal{L}^e} \lambda(\eta' d)$ since $\lambda(\eta' c') \leqslant^{\mathcal{L}^e} \lambda(\eta' c)$. Assume the contrary.

We have $\lambda(\eta' c') \leqslant^{\mathcal{L}^e} \lambda(\eta' d) = \Psi(U \mid R^0_{d,i})$ for some $i \in \mathbb{N}$ such that $d \in D_i$. Hence there is $e \in \mathbb{N}$ for which $W_e \subseteq R^0_{d,i}$ and $\lambda(\eta' c') = \Psi(U \mid W_e)$. In the Kleene numbering, every set has infinitely many numbers. Therefore we may assume that $e > i$. By Lemma 7, $R^0_{d,e} = [R^{e-i}_{d,i}]_{\varepsilon_e} \cup S^{0,1}_{d,e}$. Since $[S^{0,e-i}_{d,i}]_{\varepsilon_e} = S^{0,e-i}_{d,i}$, we have

$R^0_{d,e}\cup S^{0,e-i}_{d,i}=[R^{e-i}_{d,i}\cup S^{0,e-i}_{d,i}]_{\varepsilon_e}\cup S^{0,1}_{d,e}=[R^0_{d,i}]_{\varepsilon_e}\cup S^{0,1}_{d,e}$ and $W_e\cap(R^0_{d,e}\cup S^{0,e-i}_{d,i})=W_e\cap([R^0_{d,i}]_{\varepsilon_e}\cup S^{0,1}_{d,e}))=W_e$. Consequently $W_e\subseteq R^0_{d,e}\cup S^{0,e-i}_{d,i}$.

By Lemma 8, $R^0_{d',e}\subseteq[(W_e\cup U)\cap R^0_{1,e}]_{\varepsilon_e}\subseteq R^0_{d',e}\cup S^{0,m+1}_{1,e}\cup U$ for some $m\in\mathbb{N}$ and some $d'\in D_e$. We claim that $\eta'd'\in\eta'(2\mathbb{N})$. Indeed, let $\eta'd'\notin\eta'(2\mathbb{N})$. The first inclusion implies $\lambda(\eta'd')=\Psi(U\mid R^0_{d',e})\leqslant^{\mathcal{L}^e}\Psi(U\mid(W_e\cup U)\cap R^0_{1,e})\leqslant^{\mathcal{L}^e}\Psi(U\mid W_e)=\lambda(\eta'c')$, whence $\eta'd'\leqslant^{\mathcal{L}'}\eta'c'$ by the first case, and we are led to a contradiction.

Since $R^0_{d,e}\subseteq R^0_{1,e}$, $W_e\cap R^0_{d,e}\subseteq W_e\cap R^0_{1,e}\subseteq(W_e\cup U)\cap R^0_{1,e}\subseteq R^0_{d',e}\cup S^{0,m+1}_{1,e}\cup U$ in view of the second inclusion. Let k be large enough so that $d'\equiv_{e+k}\wedge(d',e+k)$. Then it is not hard to see that $R^0_{d',e}=S^{0,k+1}_{d',e}\cup D$. Thus $W_e\subseteq(W_e\cap R^0_{d,e})\cup S^{0,e-i}_{d,i}\subseteq S^{0,k+1}_{d',e}\cup S^{0,m+1}_{1,e}\cup U\cup S^{0,e-i}_{d,i}$. Hence all elements of W_e not lying in U belong to the bases of final towers of height less than or equal to $e+\max\{m,k\}$, so that $W_e\subseteq S^{0,e+\max\{m,k\}+1}_{d,i}\cup U$. By Lemma 7, we obtain $\lambda(\eta'c')=\Psi(U|W_e)\leqslant^{\mathcal{L}^e}\lambda(\eta'(\wedge(d,e+\max\{m,k\}))$. Since λ in the restriction to $\eta'(2\mathbb{N})$ is an isomorphic ideal embedding, we have $\eta'c'\leqslant^{\mathcal{L}'}\eta'(\wedge(d,e+\max\{m,k\})\leqslant^{\mathcal{L}'}\eta'd$, which contradicts the initial assumption. $\square$

Lemmas 9 and 10 show that λ is a morphism from the semilattice $\mathcal{L}'$ to the semilattice $\mathcal{L}^e$.

LEMMA 11. If $\langle\mathcal{L},\eta\rangle=\langle\mathcal{L}^s,\sigma\rangle$ then the set U is simple, and if $\langle\mathcal{L},\eta\rangle=\langle\mathcal{L}^{hs},\chi\rangle$ then U is hypersimple.

Proof. Let $\eta=\sigma$. For U to be simple, it suffices to show that for any $i\in\mathbb{N}$ such that Y_i is infinite and consists of singletons, $y\subseteq U$ for some $y\in Y_i$.

Assume the contrary, letting i be the least natural number such that Y_i is infinite, consists of singletons, and $\bigcup Y_i\cap U=\varnothing$. Going over the description of stage IV, it is easy to see that all elements of the set $\bigcup Y_i$ lie in the bases of final towers of height less than i, in which case the set $\bigcup Y_i\cap Q_{\mathcal{F}}$ will be infinite and will not contain elements of U for some good frame $\mathcal{F}$ of length less than i. Hence $Q_{\mathcal{F}}$ is infinite and $\mathcal{F}$ is a frame of kind II. Let $Y=\{j:\bigcup Y_i\cap\mathrm{base}(\mathcal{T}^{\mathcal{F}}_j)\neq\varnothing\}$. For all j such that $\mathrm{base}(\mathcal{T}^{\mathcal{F}}_j)\cap U=\varnothing$, $j\notin M_{\mathcal{F}}$; so Y is an infinite c.e. subset of $\mathbb{N}\setminus M_{\mathcal{F}}$. This, together with the simplicity of $M_{\mathcal{F}}$, leads us to a contradiction.

The second case is similar to the first, but is more complicated. Let $\eta=\chi$. For U to be hypersimple, it suffices to show that for any $i\in\mathbb{N}$ such that Y_i is infinite, $y\subseteq U$ for some $y\in Y_i$. Suppose the contrary, letting i be the least natural number such that Y_i is infinite and $y\nsubseteq U$ for any $y\in Y_i$. Let $\mathfrak{F}$ be the set of all good frames of length less than i and $Q=\bigcup\{Q_{\mathcal{F}}:\mathcal{F}\in\mathfrak{F}\}$. Addressing the description of stage IV, it is easy to see that $y\cap Q\nsubseteq U$ for any $y\in Y_i$.

Before the proof of Lemma 5, we noted that sets $Q_{\mathcal{F}}$, for any $\mathcal{F}\in\mathfrak{F}$, are c.e., and so therefore is Q. We prove that the set Q is in fact computable. We describe an algorithm which, given any $x\in\mathbb{N}$, allows the question whether $x\in Q$ to be answered within finitely many steps. Let $x\in\mathbb{N}$ be given. Then the number x will fall into the base of some tower of height m at some step t'. Let t_i and $h(i,t)$ be as in the proof of Lemma 6. We wait for a step $t\geqslant\max\{t',t_i\}$ such that $h(i,t)>m$. If x turns out to have been rejected by this step, then $x\notin Q$. If not, then x is in the base of some tower, which cannot have collapsed at stage III at the expense of transformation at a lesser level than i. If this tower is of height at least i then x may no longer fall into Q. If it is of height less than i then we wait for a step after which the towers of height less than i do not collapse, and if x fails to have fallen into D by that moment, then $x\in Q$.

Let $\mathfrak{F}_{\mathrm{I}}=\{\mathcal{F}\in\mathfrak{F}:\mathcal{F}$ is a frame of kind I$\}$ and $\mathfrak{F}_{\mathrm{II}}=\{\mathcal{F}\in\mathfrak{F}:\mathcal{F}$ is a frame of kind II$\}$. Suppose $Q_{\mathrm{I}}=\bigcup\{Q_{\mathcal{F}}:\mathcal{F}\in\mathfrak{F}_{\mathrm{I}}\}$ and $Q_{\mathrm{II}}=\bigcup\{Q_{\mathcal{F}}:\mathcal{F}\in\mathfrak{F}_{\mathrm{II}}\}$. Then $Q=Q_{\mathrm{I}}\cup Q_{\mathrm{II}}$. By Lemma 5, the set Q_{I} is finite, so that $y\cap Q_{\mathrm{II}}\nsubseteq U$ for almost all $y\in Y_i$. Put $\mathfrak{F}_{\mathrm{II}}=\{\mathcal{F}_1,\ldots,\mathcal{F}_k\}$, $Q'=\{c^2(l,j):l\in[1,k],\ j\in\mathbb{N}\}$, $Q''=\{c^2(l,j):l\in[1,k],\ j\in M_{\mathcal{F}_l}\}$, and for every finite $X\subseteq Q_{\mathrm{II}}$, let $j(X)$ be a finite subset of Q' equal to $\{c^2(l,j):(\exists x\in X)(x\in\mathrm{base}(\mathcal{T}^{\mathcal{F}_l}_j))\}$. Assume $Y=\{j(y\cap Q_{\mathrm{II}}):y\in Y_i,\ y\cap Q_{\mathrm{II}}\nsubseteq U\}$. Then Y is a c.e. set of finite subsets of Q' and $\bigcup Y$ is infinite. The infinity of $\bigcup Y$ allows us to choose an infinite c.e. set $Y'\subseteq Y$

so that $z_1 \cap z_2 = \varnothing$ for any $z_1, z_2 \in Y'$. For all $l \in [1,k]$ and $j \in \mathbb{N}$, we have $\mathrm{base}(\mathcal{T}_j^{\mathcal{F}_l}) \not\subseteq U \Rightarrow j \in M_{\mathcal{F}_l}$; so $z \setminus Q'' \neq \varnothing$ for any $z \in Y'$. However, the fact that the set $M_{\mathcal{F}_l}$ is hypersimple for every $l \in [1,k]$ implies that $Q' \setminus Q''$ is hyperimmune. Contradiction. $\square$

Thus λ is a morphism not only in $\mathfrak{L}^e$, but in $\mathfrak{L}$ as well.

LEMMA 12. The mapping λ is a $\mathfrak{K}$-morphism from $\langle \mathfrak{L}', \eta' \rangle$ to $\langle \mathfrak{L}, \eta \rangle$.

Proof. It suffices to show that there exists a computable sequence $\{Z_d\}_{d\in\mathbb{N}}$ of c.e. sets such that for any $d \in \mathbb{N}$, the following conditions hold:

(1) $\lambda(\eta' d) = \deg_m(Z_d)$;

(2) if $\eta = \sigma$ then Z_d is simple and cofinite;

(3) if $\eta = \chi$ then Z_d is hypersimple and cofinite.

We show how to enumerate the set Z_d given d. Let $d \in \mathbb{N}$ and i be the least natural number such that $d \in D_i$. Suppose $\mathfrak{F}$ is the set of all frames of length less than i. For every $H \subseteq \mathfrak{F}$, let

$$\begin{aligned} h_1(H,t) &= \begin{cases} \min\{h(\mathcal{G},t) : \mathcal{G} \in H\} & \text{if } H \neq \varnothing, \\ t & \text{otherwise;} \end{cases} \\ h_2(H,t) &= \begin{cases} \max\{h(\mathcal{G},t) : \mathcal{G} \in \mathfrak{F} \setminus H\} & \text{if } H \neq \mathfrak{F}, \\ 0 & \text{otherwise;} \end{cases} \\ R^H &= \Big(D \cup \bigcup \{P : \text{at a step } t \text{ there are } h_1(H,t) > k \geqslant i \text{ and a tower} \\ &\qquad \mathcal{T} = (\mathcal{P}_k, \ldots, \mathcal{P}_0; \varphi_k, \ldots, \varphi_0) \text{ such that } P \in p(d,i,\mathcal{T})\} \cup \{0\} \Big) \cap \\ &\qquad \{x \in \mathbb{N} : (\exists t \in \mathbb{N})(x \leqslant h_1(H,t))\}; \\ U^H &= U \cup \{x \in \mathbb{N} : (\exists t \in \mathbb{N})(x \leqslant h_2(H,t))\}. \end{aligned}$$

Note that i is computed uniformly in D, $\mathfrak{F}$ is computed uniformly in i, and h_1 and h_2 are computable functions in both arguments; hence R^H and U^H are computed uniformly in d and in H.

Let H_i^1 be as in the proof of Lemma 6. It is easy to see that if $H = H_i^1$, then $R^H =^* R_{d,i}^0$ and $U^H =^* U$; if $H \subset H_i^1$, then $U^H = \mathbb{N}$; if $H \supset H_i^1$, then the set R^H is finite. Thus $\Psi(U^H \,|\, R^H)$ is equal to $\Psi(U \,|\, R_{d,i}^0)$ for $H = H_i^1$, and to $\perp_{\mathfrak{L}^e}$ for $H \neq H_i^1$, so that $\lambda(\eta' d) = \bigvee^{\mathfrak{L}^e} \{\Psi(U^H \mid R^H) : H \subseteq \mathfrak{F}\}$. [In defining a Ψ-operator (see the text before Lemma 5), we fixed U; this factor, however, is incidental: the value $\Psi(\ldots \mid \ldots)$ may well be similarly defined for an arbitrary first component.]

Note that for any $H \subseteq \mathfrak{F}$, $0 \in U^H$ and $R^H \neq \varnothing$. Let $p^H(t)$ be a computable (in both arguments) function with $\rho p^H = R^H$. Suppose $q^H(t)$ is a function such that

$$q^H(t) = \begin{cases} 0 & \text{if } (\exists s < t)(p^H(s) = p^H(t)), \\ p^H(t) & \text{otherwise.} \end{cases}$$

Put $Z^H = \{x \in \mathbb{N} : q^H(x) \in U^H\}$. Since $U \subseteq U^H$, U^H is simple or cofinite (hypersimple or cofinite) by Lemma 11; hence the corresponding property holds also for Z^H. Furthermore, $\deg_m(Z^H) = \Psi(U^H \mid R^H)$. It remains to put $Z_d = \bigoplus\{Z^H : H \subseteq \mathfrak{F}\}$. $\square$

Thus, for pairs $\langle \mathfrak{L}^s, \sigma \rangle$ and $\langle \mathfrak{L}^{hs}, \chi \rangle$, the statement formulated at the beginning of the present section holds true, proving Theorem 3. We are left to handle the case $\langle \mathfrak{L}, \eta \rangle = \langle \mathfrak{X}, \xi \rangle$. Recall that we are dealing with $\mathbf{0}'$-decidable numberings of the set $\{s_1, \ldots, s_n\}$, for some fixed $n \geqslant 2$. Assume that the conditions of our statement hold for $\langle \mathcal{L}, \eta\rangle = \langle \mathcal{X}, \xi\rangle$, and that φ is an embedding of the ideal $\eta'(2\mathbb{N})$ into $\mathcal{X}$ given by the function f.

The idea behind the proof is as follows. First we build towers, employing an effective construction similar to the construction for the previous case (slightly modified toward simplification), and then we define a numbering that assumes equal values on the bases of the constructed towers, using an oracle $\mathbf{0}'$. We describe changes in the steps of the construction.

Step 0 remains unchanged. For $t+1$, stage I remains unchanged. At stage II, for every frame $\mathcal{F}$ with a current module less than or equal to t, a new tower is constructed iff either $\mathcal{F}$ is a frame of kind II, or $\mathcal{F}$ is a frame of kind I and no tower with this frame exists. Therefore the function c' is no longer used. Stage III is almost the same as in the previous construction, the deviation being that condition (5) is neglected in verifying the possibility for transforming towers (since U is no longer available). Stage IV is missing.

With all steps of the construction completed, we are faced, as above, with the c.e. sets D and $Q_{\mathcal{F}}$. (Moreover, as shown in Lemma 11, $Q_{\mathcal{F}}$ are even computable.) Note that towers now collapse at stages I and III only. If $\mathcal{F}$ is a good frame of kind II then the set $Q_{\mathcal{F}}$ is infinite, and if $\mathcal{F}$ is a good frame of kind I then $Q_{\mathcal{F}}$ consists of the base of one unique tower. Using the oracle $\mathbf{0}'$, we can, for $x \in \mathbb{N}$ arbitrary, effectively from x determine whether or not $x \in D$, and for $x \notin D$, find $\mathcal{F}$ and i such that x belongs to the base of a final tower $\mathcal{T}_i^{\mathcal{F}}$.

We define a sequence $\{\mu_x\}_{x\in\mathbb{N}}$ of $\mathbf{0}'$-decidable numberings so that for any $x \in {\eta'}^{-1}(\eta'(2\mathbb{N}))$, $\mu_x \equiv \varphi(\eta' x)$, and for some $\mathbf{0}'$-computable function p, $\mu_x y = s_{p(x,y)}$ with all $x, y \in \mathbb{N}$. Let the functions g and g_1 be as in the definition of sets M_x at page 175. Put $\mu_x c^2(y,z) = \nu_{f(2g_1(x,c^2(y,z)))} y$ for any $x, y, z \in \mathbb{N}$ (the numberings ν_m, for $m \in \mathbb{N}$, are as introduced in defining ξ at the beginning of Sec. 2). Verification of all the requisite properties is a simple matter.

At the moment, we define a $\mathbf{0}'$-decidable numbering τ. For $x \in D$, put $\tau x = s_1$. For x in the base of a final tower of kind II, we find a frame $\mathcal{F}$ and $i \in \mathbb{N}$ such that $x \in \mathrm{base}(\mathcal{T}_i^{\mathcal{F}})$, and then put $\tau x = \mu_{v(\mathcal{F})} i$ (for definition of $v(\mathcal{F})$, see Lemma 7). Further, assume that $\mathcal{T}_1, \mathcal{T}_2, \ldots$ are all final towers of kind I, arranged in order in which they are constructed. (This sequence may turn out to be finite or even empty in the case where the ideal $\eta'(2\mathbb{N})$ coincides with the entire semilattice $\mathcal{L}'$.) It remains to determine the value of a numbering τ on the bases of the towers in the sequence. For this, we use a step-by-step construction which is effective with oracle $\mathbf{0}'$. At step t, we define the value τx for all $x \in \mathrm{base}(\mathcal{T}_t)$. Our construction requires that we introduce a list into which natural numbers will be entered.

Step t. We seek a least $e \leqslant t$ not in the list, such that either $x \in \delta\theta_e$ and $\theta_e(x) \in \mathrm{base}(\mathcal{T}_t)$ for some $x \in \bigcup\{\mathrm{base}(\mathcal{T}_i) : i < t\}$, or $y \in \delta\theta_e$ and the value $\tau\theta_e(y)$ is already defined for some $y \in \mathrm{base}(\mathcal{T}_t)$. If e is found, then we define τ at all elements of the set $\mathrm{base}(\mathcal{T}_t)$ in a similar way, so that τz is distinct from $\tau\theta_e(z)$ for some $z \in \delta\theta_e \cap \bigcup\{\mathrm{base}(\mathcal{T}_i) : i \leqslant t\}$, and then enter e into the list and pass to the next step. If e is not found, then we pass to the next step at once.

We have finished to define τ. Clearly, τ is $\mathbf{0}'$-decidable. For any c.e. set $A \subseteq \mathbb{N}$, the value of a Ψ-operator $\Psi(\tau \,|\, A)$ is defined thus: if $A = \varnothing$ then $\Psi(\tau \,|\, A) = \bot_{\mathcal{X}}$; otherwise, $\Psi(\tau \,|\, A)$ is equal to an equivalence class of numberings containing the numbering $(\tau \circ p) \oplus \nu_\bot$, where $\nu_\bot$ is a decidable numbering for the set $\{s_1, \ldots, s_n\}$ (i.e., one presenting an element $\bot_{\mathcal{X}}$ of the semilattice $\mathcal{X}$) and p is a computable function such that $\rho p = A$. (It is easy to verify that our definition does not depend on the choice of p.) We say that an equivalence relation on $\mathbb{N}$ is *consistent with* τ if τ assumes the same value on any two equivalent numbers. It is a simple matter to check that all properties of $\Psi(U \mid \ldots)$ listed before Lemma 5 are preserved for $\Psi(\tau \mid \ldots)$, with obvious replacements of terms and necessary amendments.

Using the notation above, it is easy to formulate and prove an analog of Lemma 5, with $\deg_m(M_{\mathcal{F}})$ replaced by an equivalence class of the numbering $\mu_{v(\mathcal{F})}$. (In view of the above modification of the second stage, the proof becomes virtually obvious.) As above, we introduce equivalences ε and ε_i consistent with τ, and sets $R^m_{d,i}$ and $S^{m_1,m_2}_{d,i}$. For these, Lemma 6 (its proof being absolutely similar) and an analog of Lemma 7 (its proof being similar, with obvious amendments) hold true. The analog of Lemma 7 allows us to correctly determine a homomorphism λ of the semilattice $\mathcal{L}'$ into the semilattice $\mathcal{X}$. As above, but based on the obvious equality $\Psi(\tau \mid D) = \perp_{\mathcal{X}}$, we show that λ extends φ.

Note that U disappears from the formulation of Lemma 8, since now it is undefined. Namely, we have

LEMMA 8′. Let $j \in \mathbb{N}$. Then there exist $m \in \mathbb{N}$ and $d \in D_j$ such that $R^0_{d,j} \subseteq [W_j \cap R^0_{1,j}]_{\varepsilon_j} \subseteq R^0_{d,j} \cup S^{0,m+1}_{1,j}$.

The **proof** is similar except that all points associated with U (in the definition of a j-dense atom and so on) are obviously dropped, for the presence of numbers from U in the bases of towers is no longer a barrier to their transformation. Lemma 9 likewise is valid; the proof is analogous to the previous, relying on Lemma 8′, with all points associated with U modified in the natural way. Lemma 10 also remains valid.

LEMMA 10′. For $c, d \in \mathbb{N}$, if $\eta' c \nleqslant^{\mathcal{L}'} \eta' d$ then $\lambda(\eta' c) \nleqslant^{\mathcal{L}^e} \lambda(\eta' d)$.

Proof. We handle the same two cases as were dealt with in the proof of Lemma 10. An argument for the second case is similar to the above, with obvious modifications associated with the absence of the set U. We prove the first case.

Let k be large enough so that $c, d \in D_k$. Assume $\lambda(\eta' c) \leqslant^{\mathcal{L}^e} \lambda(\eta' d)$. We have $\Psi(\tau \mid R^0_{c,k}) \leqslant^{\mathcal{L}^e} \Psi(\tau \mid R^0_{d,k})$, and by the property of a Ψ-operator, there is $e \in \mathbb{N}$ such that $R^0_{c,k} = \delta\theta_e$, $\theta_e(R^0_{c,k}) \subseteq R^0_{d,k}$, and for any $x \in R^0_{c,k}$, $\tau x = \tau\theta_e(x)$. Since this equality holds for all $x \in \delta\theta_e$, the number e, in defining τ, will never be entered into the list. Let $t_0 \geqslant e$ be a sufficiently large step (in the construction used to define τ) such that all lesser numbers than e, ever appearing on the list, have been entered before step t_0. Suppose that m is a natural number greater than k and than the height of any of the towers $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{t_0}$. As above, let A be an atom in $\mathcal{D}_m$ containing c and not containing $u'_i(d, \wedge(c, m))$ (which exists in the conditions of the case under examination) and let $\mathcal{F} = (\mathcal{A}_m, \ldots, \mathcal{A}_0; c_m, \ldots, c_1)$ be a good frame of kind I for which $A \in \mathcal{A}_m$. Assume t_1 is such that the tower $\mathcal{T}_{t_1}$ is constructed on the frame $\mathcal{F}$.

By the choice of m, $t_1 > t_0$. Consequently $R^0_{c,k} \cap \operatorname{base}(\mathcal{T}_{t_1}) \neq \varnothing$ and $R^0_{d,k} \cap \operatorname{base}(\mathcal{T}_{t_1}) = \varnothing$. Let $x \in R^0_{c,k} \cap \operatorname{base}(\mathcal{T}_{t_1})$. We have $x \in \delta\theta_e$ and $\theta_e(x) \notin \operatorname{base}(\mathcal{T}_{t_1})$. If $\theta_e(x)$ either is in D, or belongs to the base of a final tower of kind II, or is an element of $\operatorname{base}(\mathcal{T}_i)$ for some $i < t_1$, then the number e will be entered into the list at step t_1. Otherwise, $\theta_e(x) \in \operatorname{base}(\mathcal{T}_{t_2})$ for some $t_2 > t_1$ and e will be entered at step t_2. Contradiction. □

Lemma 11 looses its significance. Lemma 12 remains valid. The proof is similar to the previous, with the following changes: instead of the set Z_d, we construct a $\mathbf{0}'$-decidable numbering ζ_d; instead of the set U^H, we treat a numbering τ^H such that for all $x \in \mathbb{N}$,

$$\tau^H x = \begin{cases} s_1 & \text{if } (\exists t \in \mathbb{N})(x \leqslant h_2(H, t)), \\ \tau x & \text{otherwise;} \end{cases}$$

the points associated with simplicity, hypersimplicity, and cofiniteness are neglected.

For $\langle \mathcal{X}, \xi \rangle$, we have thus proved the statement given at the beginning of the present section, completing Theorem 3. For each of the pairs $\langle \mathcal{L}^s, \sigma \rangle$, $\langle \mathcal{L}^{hs}, \chi \rangle$, and $\langle \mathcal{X}, \xi \rangle$, we have in fact shown something more.

Remark 1. In the proofs under this section, no use was made of the fact that $\eta'(2\mathbb{N})$ is a principal ideal. It is not hard to show that for the pair $\langle \mathcal{L}, \eta \rangle \in \{\langle \mathcal{L}^s, \sigma \rangle, \langle \mathcal{L}^{hs}, \chi \rangle, \langle \mathcal{X}, \xi \rangle\}$, the statement that we have proved is equivalent to the following:

Let $\langle \mathcal{L}_1, \eta_1 \rangle$ be any numbered semilattice, $\langle \mathcal{L}_2, \eta_2 \rangle \in \Omega_2$, and $\varphi_1 : \langle \mathcal{L}_1, \eta_1 \rangle \to \langle \mathcal{L}_2, \eta_2 \rangle$ and $\varphi_2 : \langle \mathcal{L}_1, \eta_1 \rangle \to \langle \mathcal{L}, \eta \rangle$ be $\mathfrak{K}$-morphisms. Then there exists a $\mathfrak{K}$-morphism $\varphi : \langle \mathcal{L}_2, \eta_2 \rangle \to \langle \mathcal{L}, \eta \rangle$ such that $\varphi_2 = \varphi \circ \varphi_1$.

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