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DISTRIBUTIVE LATTICES OF NUMBERINGS

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We study into a semilattice of numberings generated by a given fixed numbering via operations of completion and taking least upper bounds. It is proved that, except for the trivial cases, this semilattice is an infinite distributive lattice every principal ideal in which is finite. The least upper and the greatest lower bounds in the semilattice are invariant under extensions in the semilattice of all numberings. Isomorphism types for the semilattices in question are in one-to-one correspondence with pairs of cardinals the first component of which is equal to the cardinality of a set of non-special elements, and the second — to the cardinality of a set of special elements, of the initial numbering.

INTRODUCTION

A semilattice of numberings generated by a set of complete numberings of a family under completion and taking least upper bounds was dealt with in [1-3], and completions of Σ^0_n-computable numberings — in [4]. In this paper we describe a semilattice generated by a single numbering under the above-mentioned operations.

1. SOME DEFINITIONS AND PRELIMINARY RESULTS

Basic definitions and the notation pertaining to the theory of numberings are contained in [5]. Let $z \mapsto \langle \langle z\rangle_1, \langle z\rangle_2\rangle$ be a computable bijection from $\mathbb{N}$ to $\mathbb{N}^2$. Values of the inverse function at arguments x and y are denoted by $\langle x, y\rangle$. Let $K_{n+1}(x_0, x_1, \ldots, x_n)$ be the universal Kleene function for a class of all n-ary partial computable functions, where $n \geqslant 1$, and $K(x) \leftrightharpoons K_2(\langle x\rangle_1, \langle x\rangle_2)$ be a universal unary partial computable function.

A completion of a numbering α of a family S relative to an element $a \in S$ is defined as follows:

$$\alpha_a(x) \leftrightharpoons \begin{cases} \alpha(K(x)) & \text{if } K(x)\downarrow, \\ a & \text{if } K(x)\uparrow. \end{cases}$$

In [1, 4, 5] are the following properties of completion of numberings for the family S (here, $a, b \in S$):

(1) if $\alpha \leqslant \beta$ then $\alpha_a \leqslant \beta_a$;

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(2) if α is complete relative to a, i.e., a is a special element of the numbering α, then $\alpha_a \equiv \alpha$;
(3) if α is not complete relative to a then $\alpha < \alpha_a$;
(4) if $\alpha_a \leqslant \beta \oplus \gamma$ then either $\alpha \leqslant \beta$ or $\alpha \leqslant \gamma$;
(5) if $a \neq b$ and $\alpha_a \leqslant \beta_b$ then $\alpha_a \leqslant \beta$.

2. THE LATTICE OF TREES

2.1. Finite models. Let $\mathfrak{M} = \langle M; \sigma\rangle$ be an infinite model of a signature $\sigma = \langle \sqsubseteq, P_0, P_1, \ldots\rangle$, where the relation $\sqsubseteq$ is a partial order, and $P_0, P_1, \ldots$ are unary predicates, on M. To denote finite submodels of $\mathfrak{M}$ we use the letter $\mathcal{F}$, possibly with indices. A universe of model $\mathcal{F}$ (with indices) is denoted by F (with the same indices), and the number of elements in F — by $|F|$. If $\mathcal{F}_1$ is a submodel of $\mathcal{F}_2$ then we write $\mathcal{F}_1 \leqslant \mathcal{F}_2$.

Letting $a, b \in F$, we call a a *successor* of b if $a \sqsubset b$ and there is no $c \in F$ such that $a \sqsubset c \sqsubset b$. If a is the successor of b then the elements a and b are referred to as *neighboring.* By $\max(\mathcal{F})$ we denote the set of maximal elements, and by $\min(\mathcal{F})$ the set of minimal elements, in $\mathcal{F}$. For $x \in F$, put $\widehat{x^+} \rightleftharpoons \{y \in F \mid y \sqsubseteq x\}$ and $\widehat{x^-} \rightleftharpoons \{y \in F \mid y \sqsubset x\}$. Submodels with universes $\widehat{x^+}$ and $\widehat{x^-}$ are called, respectively, a *principal downcone* and a *strict principal downcone* in $\mathcal{F}$.

2.2. p-Homomorphisms. Elements $a, b \in M$ are said to be *p-indiscernible* if $P_i(a) \leftrightarrow P_i(b)$ for any $i \geqslant 0$; otherwise, a and b are conceived of as *p-discernible.* If $\varphi : M \to M$ is a partial mapping such that a and $\varphi(a)$ are p-indiscernibles for any $a \in \mathrm{dom}(\varphi)$ then we say that φ possesses the property of being *p-indiscernible.*

Definition 1. A mapping $\varphi : F_1 \to F_2$ is called a *p-homomorphism* of $\mathcal{F}_1$ into $\mathcal{F}_2$ if φ is monotone under $\sqsubseteq$ and possesses the property of being p-indiscernible.

Let $\Phi(\mathcal{F}_1, \mathcal{F}_2) \rightleftharpoons \{\varphi \mid \varphi \text{ is a } p\text{-homomorphism of } \mathcal{F}_1 \text{ into } \mathcal{F}_2\}$. We write $\mathcal{F}_1 \preccurlyeq \mathcal{F}_2$ to express the fact that there exists a p-homomorphism of $\mathcal{F}_1$ into $\mathcal{F}_2$. If $\mathcal{F}_1 \preccurlyeq \mathcal{F}_2$ and $\mathcal{F}_2 \preccurlyeq \mathcal{F}_1$ then we write $\mathcal{F}_1 \sim \mathcal{F}_2$, and the models $\mathcal{F}_1$ and $\mathcal{F}_2$ are said to be *equivalent.* The relation $\preccurlyeq$ is a preorder, and $\sim$ is an equivalence, on the set of finite submodels of $\mathfrak{M}$. Note that if $\mathcal{F}_1 \leqslant \mathcal{F}_2$ then the identity mapping of the set F_1 into itself is a p-homomorphism of $\mathcal{F}_1$ into $\mathcal{F}_2$.

Definition 2. A model $\mathcal{F}$ is said to be *p-dense* if there is no p-homomorphism of $\mathcal{F}$ into a proper submodel of $\mathcal{F}$.

For any model $\mathcal{F}$, there exists a p-dense model $\mathcal{F}'$, equivalent to $\mathcal{F}$. As $\mathcal{F}'$, among the models equivalent to $\mathcal{F}$, we can take one that has a least number of elements.

LEMMA 1. p-Dense finite models $\mathcal{F}_1$ and $\mathcal{F}_2$ are equivalent if and only if they are isomorphic.

Proof. Clearly, isomorphic models are equivalent. Let $\mathcal{F}_1 \sim \mathcal{F}_2$ and $\varphi_i \in \Phi(\mathcal{F}_i, \mathcal{F}_{3-i})$ for $i = 1, 2$. Then $\varphi_{3-i} \circ \varphi_i$ is a p-homomorphism of $\mathcal{F}_i$ onto some submodel of $\mathcal{F}_i$ which coincides with $\mathcal{F}_i$ since $\mathcal{F}_i$ is p-dense. Hence the mappings φ_1 and φ_2 are surjective, and these are bijective because F_1 and F_2 are finite sets. For $i = 1, 2$, assume that the mapping $\bar{\varphi}_i$ acts from $F_i \times F_i$ to $F_{3-i} \times F_{3-i}$ in such a way that the pair $\langle x, y\rangle$ is translated into a pair $\langle \varphi_i(x), \varphi_i(y)\rangle$, for all $x, y \in F_i$. Also, let $\bar{F}_i \rightleftharpoons \{\langle x, y\rangle \in F_i^2 \mid x \sqsubseteq y\}$.

Since φ_i is bijective, $\bar{\varphi}_i$ is likewise, and the property of φ_i being monotone implies $\bar{\varphi}_i(\bar{F}_i) \subseteq \bar{F}_{3-i}$. Consequently $\bar{\varphi}_1 \circ \bar{\varphi}_2$ is bijective, and $\bar{\varphi}_1(\bar{\varphi}_2(\bar{F}_2)) \subseteq \bar{F}_2$; hence $\bar{\varphi}_1(\bar{\varphi}_2(\bar{F}_2)) = \bar{F}_2$. We have $\bar{\varphi}_1^{-1}(\bar{F}_2) = \bar{\varphi}_1^{-1}(\bar{\varphi}_1(\bar{\varphi}_2(\bar{F}_2))) = \bar{\varphi}_2(\bar{F}_2) \subseteq \bar{F}_1$, the mapping φ_1^{-1} is monotone, and φ_1 is an isomorphism from $\mathcal{F}_1$ onto $\mathcal{F}_2$. $\square$

2.3. Finite trees. By a *finite tree*, or merely *tree*, we mean a finite submodel of $\mathfrak{M}$ in which every two elements, larger than some third one, are mutually comparable. A finite tree is denoted by the letter $\mathcal{D}$, possibly with indices. A universe of the tree $\mathcal{D}$ (with indices) is denoted by the letter D (with the same indices). Consider the following two operations over trees.

Let $\mathcal{D}$ be a tree and a be an element of M such that $x \sqsubseteq a$ for any $x \in D$. Put $a \cdot \mathcal{D} \rightleftharpoons \langle a \cdot D; \sigma \rangle$, where $a \cdot D \rightleftharpoons D \cup \{a\}$. Then $a \cdot \mathcal{D}$ is a finite tree, which we call a *w-descent* of $\mathcal{D}$.

A tree $\mathcal{D}$ is called a *direct sum* of the trees $\mathcal{D}_1, \ldots, \mathcal{D}_m$ if there are trees $\mathcal{D}'_1, \ldots, \mathcal{D}'_m$ such that $D = \bigcup\limits_{i=1}^{m} D'_i$ and $\mathcal{D}_i \cong \mathcal{D}'_i$ for all $i \in [1, m]$, and for all $i, j \in [1, m]$, if $i \neq j$, $a \in D'_i$, and $b \in D'_j$, then a and b are incomparable in $\mathfrak{M}$. To denote the direct sum we write $\mathcal{D} = \mathcal{D}_1 \dotplus \cdots \dotplus \mathcal{D}_m = \biguplus\limits_{i=1}^{m} \mathcal{D}_i$. Any tree $\mathcal{D}'$ such that $\mathcal{D}' \cong \mathcal{D}_i$ for some $i \in [1, m]$ is called a *direct summand* of the direct sum $\mathcal{D}$.

If a tree $\mathcal{D}$ is representable as $\mathcal{D} = \biguplus\limits_{i=1}^{m} \mathcal{D}_i$, where $m > 1$, then $\mathcal{D}$ is called a *decomposable* tree; otherwise, we say that $\mathcal{D}$ is *indecomposable*. Clearly, the indecomposability of a tree is equivalent to its having a greatest element. Any indecomposable tree either is one-element or is a descent of some tree with a smaller number of elements. The decomposable tree, in turn, is representable as a direct sum of indecomposable trees, each of which has a smaller number of elements. We can thus treat trees as inductive structures using, in definitions and proofs, induction on the number of elements in a tree and its representation via the above-mentioned operations.

LEMMA 2. Let $\mathcal{D}$, $\mathcal{D}'$, and $\mathcal{D}''$ be trees and $\mathcal{D} = \biguplus\limits_{i=1}^{m} \mathcal{D}_i$. The following statements hold:

(1) $\mathcal{D}_i \preccurlyeq \mathcal{D}$ for all $i \in [1, m]$;

(2) $\mathcal{D} \preccurlyeq \mathcal{D}'$ iff $\mathcal{D}_i \preccurlyeq \mathcal{D}'$ for all $i \in [1, m]$;

(3) if $\mathcal{D}' = \biguplus\limits_{i=1}^{n} \mathcal{D}'_i$, and the trees $\mathcal{D}_1, \ldots, \mathcal{D}_m$ are indecomposable, then $\mathcal{D} \preccurlyeq \mathcal{D}'$ iff there is $j \in [1, n]$ such that $\mathcal{D}_i \preccurlyeq \mathcal{D}'_j$ for every $i \in [1, m]$;

(4) if $a \cdot \mathcal{D}' \preccurlyeq \mathcal{D}''$ for $a \in M$ then $\mathcal{D}' \preccurlyeq \mathcal{D}''$.

The **proof** is obvious. □

2.4. p-Dense trees. Using induction on the number of elements of a tree, we couch the following:

Definition 3. A tree $\mathcal{D}$ is said to be *p-dense relative to the neighborhood of the subtrees* if it is one-element or satisfies the following:

(1) either $\mathcal{D}$ is indecomposable, or all of its indecomposable direct summands are pairwise incomparable under $\preccurlyeq$;

(2) all non-empty principal strict downcones in $\mathcal{D}$ are p-dense relative to the neighborhood of the subtrees.

LEMMA 3. A tree $\mathcal{D}$ is p-dense if and only if the following hold:

(1) every two neighboring elements of $\mathcal{D}$ are p-discernible.

(2) $\mathcal{D}$ is p-dense relative to the neighborhood of the subtrees.

Proof. For one-element trees, the statement of the lemma is obvious. Assume that the lemma is valid for $|D| > 1$ and for all trees the number of elements in which is smaller than $|D|$.

Let $\mathcal{D}$ be p-dense. If $a \sqsubset b$ are the neighboring p-indiscernible elements in $\mathcal{D}$, then we define a mapping φ from D to D as follows: $\varphi(a) \rightleftharpoons b$ and $\varphi(x) \rightleftharpoons x$ for $x \neq a$. It is clear that $\varphi \in \Phi(\mathcal{D}, \mathcal{D})$ and $\varphi(D) \subset D$; the latter clashes with $\mathcal{D}$ being p-dense. Now, let $\mathcal{D}$ not be p-dense relative to the neighborhood of the subtrees. If $\mathcal{D} = \biguplus\limits_{i=1}^{m} \mathcal{D}_i$ for $m > 1$, the trees $\mathcal{D}_1, \ldots, \mathcal{D}_m$ are indecomposable, and $\mathcal{D}_i \preccurlyeq \mathcal{D}_j$ for $i \neq j$, then $\mathcal{D} \preccurlyeq \biguplus_{k \neq i} \mathcal{D}_k$ by Lemma 2(3), which is again a contradiction with $\mathcal{D}$ being p-dense. Thus item (1) in Definition 3 holds true for $\mathcal{D}$. Hence, for some $a \in D$, the strict principal cone $\widehat{a^-}$ is not empty and is not p-dense relative to the neighborhood of the subtrees.

By the inductive assumption, there is $\varphi' \in \Phi(\widehat{a^-}, \widehat{a^-})$ such that $\varphi'(\widehat{a^-}) \subset \widehat{a^-}$. We define a mapping φ from D to D as follows: $\varphi(x) \rightleftharpoons \varphi'(x)$, for $x \in \widehat{a^-}$, and $\varphi(x) \rightleftharpoons x$ for $x \notin \widehat{a^-}$. It is not hard to verify that $\varphi \in \Phi(\mathcal{D}, \mathcal{D})$ and $\varphi(D) \subset D$; the latter is again a contradiction with the fact that $\mathcal{D}$ is p-dense.

Inversely, assume that any two neighboring elements of D are p-discernible and $\mathcal{D}$ is p-dense relative to the neighborhood of the subtrees. Let $\varphi \in \Phi(\mathcal{D}, \mathcal{D})$ and $\varphi(D) \subset D$. Suppose D has the greatest element a. If $\varphi(x) = a$ for some $x \in \widehat{a^-}$, then $\varphi(b) = a$ for $b \in D$ such that $x \sqsubseteq b \sqsubset a$, and b is the successor of a, which is impossible. Hence $\varphi(\widehat{a^-}) \subseteq \widehat{a^-}$. By the inductive assumption, the tree $\widehat{a^-}$ is p-dense and, consequently, $\varphi(\widehat{a^-}) = \widehat{a^-}$. Hence $\varphi(a) \sqsubset a$. The element $\varphi(a)$ cannot be a successor of a; so $\varphi(a) \sqsubset b \sqsubset a$ for some $b \in D$. It follows that $\varphi(c) = b \not\sqsubseteq \varphi(a)$ for some $c \sqsubset a$, a contradiction.

Thus $\mathcal{D} = \biguplus_{i=1}^{m} \mathcal{D}_i$ for some $m > 1$, where the trees $\mathcal{D}_1, \ldots, \mathcal{D}_m$ are indecomposable and are subtrees of $\mathcal{D}$. Since the tree $\mathcal{D}$ is p-dense relative to the neighborhood of the subtrees, $\varphi(D_i) \subseteq D_i$ for all $i \in [1, m]$. By the inductive assumption, $\mathcal{D}_i$ is p-dense for every $i \in [1, m]$. Hence, for all $i \in [1, m]$, $\varphi(D_i) = D_i$ and $\varphi(D) = D$, a contradiction. $\square$

LEMMA 4. Let $\mathcal{F}$ be a finite model. Then the number of p-dense trees which can be p-homomorphically mapped to $\mathcal{F}$ (in other words, the number of p-dense trees-preimages of $\mathcal{F}$) is finite.

The **proof** is by induction on $|F|$. If $|F| = 1$ then every p-dense tree-preimage of $\mathcal{F}$ is one-element. Let $|F| > 1$ and $b \in \max(\mathcal{F})$. By the inductive assumption, the number of p-dense trees-preimages of $\mathcal{F}' \rightleftharpoons \langle F \setminus \{b\}; \sigma \rangle$ is finite. Let $\mathcal{D}$ be an indecomposable p-dense tree-preimage of $\mathcal{F}$, but not of $\mathcal{F}'$. Then $\mathcal{D}$ contains a as the greatest element, and for some $\varphi \in \Phi(\mathcal{D}, \mathcal{F})$, $\varphi(a) = b$. We have $\mathcal{D} = a \cdot \mathcal{D}'$ for some tree $\mathcal{D}'$ such that $a \notin D'$. By Lemma 3, the maximal elements of $\mathcal{D}'$, which are successors of a in $\mathcal{D}$, are p-discernible with a, and so $\varphi(D') \subseteq F' = F \setminus \{b\}$. By the same lemma, the tree $\mathcal{D}'$ is p-dense. In this way there exist not more than finitely many isomorphism types for $\mathcal{D}'$ and, hence, for $\mathcal{D}$.

Thus the number of indecomposable p-dense trees-preimages of $\mathcal{F}$ is finite. Let it be equal to a natural number n. Then the number of all p-dense trees-preimages of $\mathcal{F}$ is at most 2^n. $\square$

The statement proved above may turn out to be untrue if we consider all p-dense finite model-preimages rather than p-dense tree-preimages. We look at the family of models depicted in Fig. 1. All elements finished in a dark color are called *a-elements*, and b_i and c_i, $i \geqslant 1$, are referred to as, respectively, b- and *c-elements*. Assume that the a-elements are all pairwise p-indiscernible, and that the b- and c-elements are likewise. Also, suppose that elements belonging to different groups are mutually p-discernible. Then any model $\mathcal{D}_i$, $i \geqslant 2$, can be p-homomorphically mapped to $\mathcal{D}_1$, and moreover, these models are all p-dense (but are not trees).

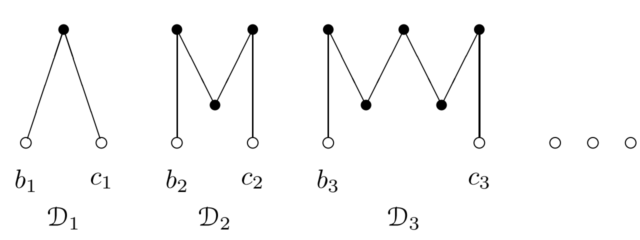

Fig. 1

2.5. A distributive lattice of trees. Let Ω be some subset of the set of all finite trees in $\mathfrak{M}$, which is closed under taking direct sums of finitely many trees, and let $\perp \notin \Omega$. Clearly, $\langle \Omega/_\sim; \preccurlyeq \rangle$ is a partially ordered set. Put $\Omega_\perp \rightleftharpoons \Omega \cup \{\perp\}$. We will assume that $\perp$ is an indecomposable element of $\Omega_\perp$, which is smaller than all elements of the set Ω. Let $m \geqslant 1$, $a \in M$, and $\mathcal{D}_i \in \Omega_\perp$ for $i \in [1, m]$. We set $a \cdot \perp = \langle \{a\}; \sigma \rangle$, and set $\biguplus\limits_{i=1}^{m} \mathcal{D}_i = \biguplus\{\mathcal{D}_i \mid i \in [1, m], \mathcal{D}_i \neq \perp\}$ if $\mathcal{D}_i \neq \perp$ for some $i \in [1, m]$, and $\biguplus\limits_{i=1}^{m} \mathcal{D}_i = \perp$ otherwise.

PROPOSITION 1. Let Ω be a set of all finite trees in $\mathfrak{M}$. Then the partially ordered set $\langle \Omega_\perp/_\sim; \preccurlyeq \rangle$ is a distributive lattice with a least element in which every principal downcone is finite.

Proof. In view of Lemma 4, any principal downcone is finite. This implies that if $\langle \Omega_\perp/_\sim; \preccurlyeq \rangle$ is an upper semilattice then it is also a lattice.

We claim that $(\mathcal{D}_1 \dotplus \mathcal{D}_2)/_\sim = \sup\{\mathcal{D}_1/_\sim, \mathcal{D}_2/_\sim\}$ for $\mathcal{D}_1, \mathcal{D}_2 \in \Omega_\perp$. If $\mathcal{D}_1 = \perp$ or $\mathcal{D}_2 = \perp$ then the result follows from the definition of a direct sum on $\Omega_\perp$. But if $\mathcal{D}_1, \mathcal{D}_2 \neq \perp$ then we appeal to Lemma 2(2).

Thus $\langle \Omega_\perp/_\sim; \preccurlyeq \rangle$ is a lattice. To prove distributivity, it suffices to state that $\mathcal{D}_1 \sqcap (\mathcal{D}_2 \sqcup \mathcal{D}_3) \preccurlyeq (\mathcal{D}_1 \sqcap \mathcal{D}_2) \sqcup (\mathcal{D}_1 \sqcap \mathcal{D}_3)$. We may assume that the left part is not equal to $\perp$. Let $\biguplus\limits_{i=1}^{m} \mathcal{D}'_i$ be the decomposition of $\mathcal{D}_1 \sqcap (\mathcal{D}_2 \sqcup \mathcal{D}_3)$ into a direct sum of indecomposable trees, for some $m \geqslant 1$. Then $\mathcal{D}'_i \preccurlyeq \mathcal{D}_1$ and $\mathcal{D}'_i \preccurlyeq \mathcal{D}_2 \dotplus \mathcal{D}_3$, for every $i \in [1, m]$. This, together with Lemma 2(3), implies that $\mathcal{D}'_i \preccurlyeq \mathcal{D}_2$ or $\mathcal{D}'_i \preccurlyeq \mathcal{D}_3$, for any $i \in [1, m]$. Let $\mathcal{D}^* = \biguplus\{\mathcal{D}'_i \mid i \in [1, m], \mathcal{D}'_i \preccurlyeq \mathcal{D}_2\}$ and $\mathcal{D}^{**} = \biguplus\{\mathcal{D}'_i \mid i \in [1, m], \mathcal{D}'_i \preccurlyeq \mathcal{D}_3\}$. By Lemma 2(2), $\mathcal{D}_1 \sqcap (\mathcal{D}_2 \sqcup \mathcal{D}_3) \preccurlyeq \mathcal{D}^* \dotplus \mathcal{D}^{**} \preccurlyeq (\mathcal{D}_1 \sqcap \mathcal{D}_2) \sqcup (\mathcal{D}_1 \sqcap \mathcal{D}_3)$. $\Box$

2.6. $\mathbb{Z}$-trees. Let $\mathbb{Z} = \langle Z; \sigma_0 \rangle$ be a model of a signature $\sigma_0 = \langle P_0, P_1, \ldots \rangle$ and $P_0(\mathbb{Z}) \neq \varnothing$. Put

$$W_Z \rightleftharpoons \{(z, i, s) \mid z \in Z,\ s \geqslant 1,\ 1 \leqslant i \leqslant s\},$$

$$K_Z \rightleftharpoons \{w_1 \ldots w_n \mid n \geqslant 1,\ w_i \in W_Z,\ i \in [1, n]\}.$$

In defining K_Z we have used concatenation of the elements of the set W_Z; in other words, K_Z is a set of all non-empty words in the alphabet W_Z. We define a model $\mathcal{K}_\mathbb{Z} = \langle K_Z; \sigma \rangle$ of signature σ. For $k_1, k_2 \in K_Z$, put $k_2 \sqsubseteq k_1$ if k_1 is a prefix of k_2. For $k \in K_Z$ terminating at (z, i, s), we set $k^* \rightleftharpoons z$ and $P_j(k) \rightleftharpoons P_j(k^*)$, where $j \geqslant 0$.

It is clear that $\langle K_Z; \sqsubseteq \rangle$ is a partially ordered set; moreover, it is a tree. We call $\mathbb{Z}$ the *urmodel* model of model $\mathcal{K}_\mathbb{Z}$.

Definition 4. A finite submodel $\mathcal{D}$ of $\mathcal{K}_\mathbb{Z}$ is called a $\mathbb{Z}$-*tree* if the following hold:

(1) D is closed w.r.t. prefixes;

(2) for $k \in K_Z \cup \{\Lambda\}$ (Λ is the empty word), if $k(z, i, s) \in D$, then exactly s elements of the form $k(z', j, s)$ belong to D, all with distinct $j \in [1, s]$;

(3) for $k \in \min(\mathcal{D})$, we have $P_0(k)$.

A set of all $\mathbb{Z}$-trees is denoted by $\Omega(\mathbb{Z})$. Suppose $\perp \notin \Omega(\mathbb{Z})$. As above, we denote by $\Omega(\mathbb{Z})_\perp$ the set $\Omega(\mathbb{Z}) \cup \{\perp\}$.

2.7. Operations over $\mathbb{Z}$-trees. For the $\mathbb{Z}$-trees, we can refine the operations of descent and taking direct sums.

Descent of a tree. Let $w = (z, 1, 1) \in W_Z$, $P_0(z)$, and $\mathcal{D} \in \Omega(\mathbb{Z})_\perp$. Put $w\perp \rightleftharpoons \langle \{w\}; \sigma \rangle$, and for $\mathcal{D} \neq \perp$, $w\mathcal{D} \rightleftharpoons \langle \{w\} \cup \{wk \mid k \in D\}; \sigma \rangle$. Clearly, $w\mathcal{D} \cong w \cdot \widetilde{\mathcal{D}}$, where $\widetilde{\mathcal{D}} \rightleftharpoons \{wk \mid k \in D\}$ and $\mathcal{D} \cong \widetilde{\mathcal{D}}$.

Direct sum. Let $m \geqslant 1$ and $\mathcal{D}_i \in \Omega(\mathbb{Z})$ for $i \in [1,m]$. Assume $\max(\mathcal{D}_i) = \{w_{i,j} \mid 1 \leqslant j \leqslant s_i\}$. For every $i \in [1,m]$ and every $j \in [1,s_i]$, put $\tilde{w}_{i,j} \rightleftharpoons \left(w^*_{i,j}, \sum\limits_{l<i} s_l + j, \sum\limits_{i=1}^{m} s_i\right)$. Define $D \leftrightharpoons \bigcup\limits_{i=1}^{m} \bigcup\limits_{j=1}^{s_i} \{\tilde{w}_{i,j}k \mid w_{i,j}k \in D_i\}$ and $\mathcal{D} = \langle D; \sigma\rangle$. Obviously, $\mathcal{D} = \biguplus\limits_{i=1}^{m} \mathcal{D}_i$.

Now, let $\mathcal{D}_1, \ldots, \mathcal{D}_m \in \Omega(\mathbb{Z})_\perp$. If there exists $i \in [1,m]$ such that $\mathcal{D}_i \neq \perp$ then we put $\biguplus\limits_{i=1}^{m} \mathcal{D}_i \rightleftharpoons \biguplus\{\mathcal{D}_i \mid \mathcal{D}_i \neq \perp,\, i \in [1,m]\}$. Otherwise, we define $\biguplus\limits_{i=1}^{m} \mathcal{D}_i \rightleftharpoons \perp$.

It is clear that the operations applied as above to our $\mathbb{Z}$-trees yield $\mathbb{Z}$-trees again. Thus the set $\Omega(\mathbb{Z})$ is closed under taking direct sums of finitely many $\mathbb{Z}$-trees.

Every $\mathcal{D}$ in $\Omega(\mathbb{Z})_\perp$ satisfies one of the following four conditions:

(1) $\mathcal{D} = \perp$;

(2) $\mathcal{D} = \langle\{w\}; \sigma\rangle$, where $w = (w^*, 1, 1)$, and $P_0(w)$ holds true;

(3) $\mathcal{D} = w\mathcal{D}'$, where $\mathcal{D}' \in \Omega(\mathbb{Z})$;

(4) for some $m > 1$, there exist indecomposable $\mathcal{D}_1, \ldots, \mathcal{D}_m \in \Omega(\mathbb{Z})$ such that $\mathcal{D} = \biguplus\limits_{i=1}^{m} \mathcal{D}_i$.

2.8. p-Dense $\mathbb{Z}$-trees. We point out an algorithm for constructing, given a $\mathbb{Z}$-tree, a p-dense $\mathbb{Z}$-tree that will be equivalent to that $\mathbb{Z}$-tree.

Glueing elements. For each $\mathcal{D} \in \Omega(\mathbb{Z})_\perp$, we define a tree $\mathcal{D}^\# \in \Omega(\mathbb{Z})_\perp$ using induction on the number of elements in D. This new operation will enjoy the following properties: if $\mathcal{D}$ is indecomposable then $\mathcal{D}^\#$ is also indecomposable, and if $\mathcal{D} \neq \perp$, then $\mathcal{D}^\# \neq \perp$.

If $\mathcal{D} = \perp$ or D is one-element then we put $\mathcal{D}^\# \rightleftharpoons \mathcal{D}$. If, for some $m > 1$, $\mathcal{D} = \biguplus\limits_{i=1}^{m} \mathcal{D}_i$, where $\mathcal{D}_1, \ldots, \mathcal{D}_m$ are indecomposable $\mathbb{Z}$-trees then we assume that $\mathcal{D}^\# \rightleftharpoons \biguplus\limits_{i=1}^{m} \mathcal{D}_i^\#$. Let $\mathcal{D} = w\mathcal{D}'$ for some $\mathcal{D}' \in \Omega(\mathbb{Z})$. There is $m \geqslant 1$ for which $\mathcal{D}' = \biguplus\limits_{i=1}^{m} \mathcal{D}_i$ with indecomposable $\mathcal{D}_1, \ldots, \mathcal{D}_m \in \Omega(\mathbb{Z})$. For all $i \in [1,m]$, $\mathcal{D}_i^\# = w_i\bar{\mathcal{D}}_i$, where $\bar{\mathcal{D}}_i \in \Omega(\mathbb{Z})_\perp$. For $i \in [1,m]$, we put $\mathcal{D}'_i \rightleftharpoons \bar{\mathcal{D}}_i$ if w and w_i are p-indiscernible, and $\mathcal{D}'_i \rightleftharpoons \mathcal{D}_i^\#$ otherwise. Define $\mathcal{D}^\# \rightleftharpoons w\biguplus\limits_{i=1}^{m} \mathcal{D}'_i$. Using induction on $|D|$ it is easy to show that every two neighboring elements in $\mathcal{D}^\#$ are p-discernible.

Removing subtrees. For $\mathcal{D} \in \Omega(\mathbb{Z})_\perp$, the $\mathbb{Z}$-tree $\mathcal{D}^\circ$ is defined by induction on the number of elements in D. If $\mathcal{D} = \perp$ then we put $\mathcal{D}^\circ = \perp$. If $\mathcal{D} = w\mathcal{D}'$ for $\mathcal{D}' \in \Omega(\mathbb{Z})_\perp$ then we define $\mathcal{D}^\circ \rightleftharpoons w(\mathcal{D}')^\circ$. Let $\mathcal{D} = \biguplus\limits_{i=1}^{m} \mathcal{D}_i$ for some $m > 1$ and $\mathcal{D}_1, \ldots, \mathcal{D}_m$ be indecomposable. Choose numbers $s \in [1,m]$ and $i_1, \ldots, i_s \in [1,m]$ so that for any two distinct $p, q \in [1,s]$, $\mathcal{D}^\circ_{i_p} \not\preccurlyeq \mathcal{D}^\circ_{i_q}$, and for every $j \in [1,m]$, there is $p \in [1,s]$ such that $\mathcal{D}^\circ_j \preccurlyeq \mathcal{D}^\circ_{i_p}$. Put $\mathcal{D}^\circ \rightleftharpoons \biguplus\limits_{p=1}^{s} \mathcal{D}^\circ_{i_p}$.

Using induction on $|D|$ it is easy to state the following properties of removing: if $\mathcal{D}$ has no p-indiscernible neighbors then $\mathcal{D}^\circ$, too, has none, and if $\mathcal{D} \in \Omega(\mathbb{Z})$, then $\mathcal{D}^\circ$ is not equal to $\perp$ and is p-dense relative to the neighborhood of the subtrees.

LEMMA 5. A tree $\mathcal{D}^{\#\circ}$ is a p-dense $\mathbb{Z}$-tree, equivalent to a $\mathbb{Z}$-tree $\mathcal{D}$.

Proof. That $\mathcal{D}^{\#\circ}$ is p-dense follows from Lemma 3 in view of the above-indicated properties behind the new operations. We show that $\mathcal{D} \sim \mathcal{D}^{\#\circ}$.

By induction on $|D|$, we prove that $\mathcal{D} \sim \mathcal{D}^\#$ for any tree $\mathcal{D} \in \Omega(\mathbb{Z})_\perp$. If $\mathcal{D} = \perp$ or $|D| = 1$ then $\mathcal{D}^\# = \mathcal{D}$. If $\mathcal{D} = \biguplus\limits_{i=1}^{m} \mathcal{D}_i$ for some $m > 1$ and for indecomposable $\mathcal{D}_1, \ldots, \mathcal{D}_m \in \Omega(\mathbb{Z})$, then $\mathcal{D}^\# \sim \mathcal{D}$ by the inductive assumption and Lemma 2(2). Assume that $\mathcal{D} = w\mathcal{D}'$ for some $\mathcal{D}' \in \Omega(\mathbb{Z})$ and that there is $m \geqslant 1$ for which $\mathcal{D}' = \biguplus_{i=1}^{m} \mathcal{D}_i$ with indecomposable $\mathcal{D}_1, \dots, \mathcal{D}_m \in \Omega(\mathbb{Z})$. For $i \in [1, m]$, the objects w_i, $\bar{\mathcal{D}}_i$, and $\mathcal{D}_i'$ are defined in the same way as in the description of glueing. By our definitions, in view of the inductive assumption, $\mathcal{D}_i' \preccurlyeq \mathcal{D}_i^{\#} \preccurlyeq \mathcal{D}_i$ for all $i \in [1, m]$. This, together with Lemma 2(2), yields $\biguplus_{i=1}^{m} \mathcal{D}_i' \preccurlyeq \mathcal{D}'$ and $\mathcal{D}^{\#} \preccurlyeq \mathcal{D}$.

For every $i \in [1, m]$, by φ_i we denote a p-homomorphism from $\mathcal{D}_i$ to $\mathcal{D}_i^{\#}$, which exists by the inductive assumption. Define a mapping $\varphi_i' : D_i \to D^{\#}$ for each $i \in [1, m]$. Let ψ_i be an isomorphism of $\mathcal{D}_i'$ onto a submodel of the $\mathbb{Z}$-tree $\mathcal{D}^{\#}$ translating maximal elements of D_i' into maximal elements of $D^{\#} \setminus \{w\}$. If w_i and w are p-discernible then $\mathcal{D}_i' = \mathcal{D}_i^{\#}$, in which case we put $\varphi_i' \rightleftharpoons \psi_i \circ \varphi_i$. Otherwise, $\mathcal{D}_i^{\#} = w_i\mathcal{D}_i'$ and there exists an isomorphism ψ_i' of the model $\langle D_i^{\#} \setminus \{w_i\}; \sigma\rangle$ onto a $\mathbb{Z}$-tree $\mathcal{D}_i'$. In this case for $k \in D_i$ we put $\varphi_i'(k) \rightleftharpoons w$, if $\varphi_i(k) = w_i$, and $\varphi_i'(k) \rightleftharpoons \psi_i(\psi_i'(\varphi_i(k)))$ if $\varphi_i(k) \neq w_i$. It is not hard to verify that $\varphi_i' \in \Phi(\mathcal{D}_i, \mathcal{D}^{\#})$ in all of the cases. By Lemma 2(2), $\mathcal{D}' \preccurlyeq \mathcal{D}^{\#}$. Keeping in mind that $\mathcal{D} = w\mathcal{D}'$ and the fact that w is the greatest element of $\mathcal{D}^{\#}$, we obtain $\mathcal{D} \preccurlyeq \mathcal{D}^{\#}$.

By induction on D we show that $\mathcal{D} \sim \mathcal{D}^{\circ}$ for any $\mathcal{D} \in \Omega(\mathbb{Z})_{\perp}$. If $\mathcal{D} = \perp$ then $\mathcal{D}^{\circ} = \mathcal{D}$. If $\mathcal{D} = w\mathcal{D}'$ for $\mathcal{D}' \in \Omega(\mathbb{Z})_{\perp}$ then $\mathcal{D}^{\circ} = w(\mathcal{D}')^{\circ}$, $\mathcal{D}' \sim (\mathcal{D}')^{\circ}$ by the inductive assumption, and $\mathcal{D} \sim \mathcal{D}^{\circ}$ again. Assume that $\mathcal{D} = \biguplus_{i=1}^{m} \mathcal{D}_i$ for $m > 1$ and for indecomposable $\mathbb{Z}$-trees $\mathcal{D}_1, \dots, \mathcal{D}_m$, and that the numbers s and $i_1, \dots, i_s$ are chosen in the same way as in the description of removing subtrees. By the inductive assumption, for every $p \in [1, s]$ we have $\mathcal{D}_{i_p}^{\circ} \preccurlyeq \mathcal{D}_{i_p}$, and by Lemma 2(2), $\mathcal{D}^{\circ} \preccurlyeq \mathcal{D}$. Furthermore, by the inductive assumption, for every $j \in [1, m]$ there is $p \in [1, s]$ such that $\mathcal{D}_j \preccurlyeq \mathcal{D}_j^{\circ} \preccurlyeq \mathcal{D}_{i_p}^{\circ}$, and $\mathcal{D} \preccurlyeq \mathcal{D}^{\circ}$ again by Lemma 2(2). □

2.9. A distributive lattice of $\mathbb{Z}$-trees. The set $\Omega(\mathbb{Z})$ is a subset of the set of all finite trees in model $\mathcal{K}_{\mathbb{Z}}$ which is closed under taking direct sums. This fact and Proposition 1 imply that the partially ordered set $\langle \Omega(\mathbb{Z})_{\perp}/_{\sim}, \preccurlyeq\rangle$ is a lattice with the least element $\perp/_{\sim}$ in which every principal downcone is finite. For $\mathcal{D}_1, \mathcal{D}_2 \in \Omega(\mathbb{Z})_{\perp}$, therefore, the expressions $\mathcal{D}_1 \sqcup \mathcal{D}_2 \rightleftharpoons \mathcal{D}_1 \dotplus \mathcal{D}_2$ and $\mathcal{D}_1 \sqcap \mathcal{D}_2$, which denote elements of the respective classes $\sup\{\mathcal{D}_1/_{\sim}, \mathcal{D}_2/_{\sim}\}$ and $\inf\{\mathcal{D}_1/_{\sim}, \mathcal{D}_2/_{\sim}\}$, are meaningful. Below we argue to state that such a lattice is distributive.

Let $\mathcal{D}_1, \mathcal{D}_2 \in \Omega(\mathbb{Z})_{\perp}$. Put $l(\mathcal{D}_1, \mathcal{D}_2) \rightleftharpoons \{\mathcal{D} \in \Omega(\mathbb{Z}) \mid \mathcal{D}$ is indecomposable, $\mathcal{D}$ is p-dense, and $\mathcal{D} \preccurlyeq \mathcal{D}_1, \mathcal{D}_2\}$. We make the convention that the direct sum of the empty set of direct summands is equal to $\perp$. Lemma 5 gives rise to the following:

LEMMA 6. Let $\mathcal{D}_1, \mathcal{D}_2 \in \Omega(\mathbb{Z})_{\perp}$. Then $\mathcal{D}_1 \sqcap \mathcal{D}_2 \sim \biguplus\{\mathcal{D} \mid \mathcal{D} \in l(\mathcal{D}_1, \mathcal{D}_2)\}$.

Proof. Assume $\mathcal{D}' \rightleftharpoons \biguplus\{\mathcal{D} \mid \mathcal{D} \in l(\mathcal{D}_1, \mathcal{D}_2)\}$. Then $\mathcal{D}' \preccurlyeq \mathcal{D}_1 \sqcap \mathcal{D}_2$. On the other hand, if $\mathcal{D}_1 \sqcap \mathcal{D}_2 \neq \perp$ then it follows by Lemma 5 that there exists a p-dense $\mathcal{D}'' \in \Omega(\mathbb{Z})$ such that $\mathcal{D}'' \sim \mathcal{D}_1 \sqcap \mathcal{D}_2$. If $\mathcal{D}'' = \biguplus_{i=1}^{m} \mathcal{D}_i''$ for some $m \geqslant 1$ and for indecomposable $\mathcal{D}_1'', \dots, \mathcal{D}_m'' \in \Omega(\mathbb{Z})$ then $\mathcal{D}_i'' \in l(\mathcal{D}_1, \mathcal{D}_2)$ for all $i \in [1, m]$, and $\mathcal{D}'' \preccurlyeq \mathcal{D}'$. □

PROPOSITION 2. Let $\mathbb{Z} = \langle Z; \sigma_0\rangle$ be any model of the signature σ_0 containing at least two p-discernibles, and let $P_0(\mathbb{Z}) \neq \varnothing$. Then the partially ordered set $\langle \Omega(\mathbb{Z})_{\perp}/_{\sim}; \preccurlyeq\rangle$ is an infinite distributive lattice with the least element $\perp/_{\sim}$ in which every principal downcone is finite.

Proof. That the above partially ordered set is a lattice with the least element $\perp/_{\sim}$ in which every principal downcone is finite was pointed out above. We argue for the distributivity.

It suffices to show that $\mathcal{D}_1 \sqcap (\mathcal{D}_2 \sqcup \mathcal{D}_3) \preccurlyeq (\mathcal{D}_1 \sqcap \mathcal{D}_2) \sqcup (\mathcal{D}_1 \sqcap \mathcal{D}_3)$. By Lemma 6, the left-hand side is equal to $\biguplus\{\mathcal{D} \mid \mathcal{D} \in l(\mathcal{D}_1, \mathcal{D}_2 \dotplus \mathcal{D}_3)\}$. By Lemma 2(3), $l(\mathcal{D}_1, \mathcal{D}_2 \dotplus \mathcal{D}_3) = l(\mathcal{D}_1, \mathcal{D}_2) \cup l(\mathcal{D}_1, \mathcal{D}_3)$. Let $D^* \rightleftharpoons \biguplus\{\mathcal{D} \mid \mathcal{D} \in l(\mathcal{D}_1, \mathcal{D}_2)\}$ and $D^{**} \rightleftharpoons \biguplus\{\mathcal{D} \mid \mathcal{D} \in l(\mathcal{D}_1, \mathcal{D}_3)\}$. We obtain $\mathcal{D}_1 \sqcap (\mathcal{D}_2 \sqcup \mathcal{D}_3) \preccurlyeq \mathcal{D}^* \dotplus \mathcal{D}^{**} \sim$

$(\mathcal{D}_1 \sqcap \mathcal{D}_2) \sqcup (\mathcal{D}_1 \sqcap \mathcal{D}_3)$.

We are left to state the infiniteness of the lattice. Let a and b be p-discernible in Z, $\mathbb{Z} \models P_0(a)$, with $w_1 = (a, 1, 1)$ and $w_2 = (b, 1, 1)$. Then the trees

$$\mathcal{D}_1 \rightleftharpoons \langle \{w_1\}; \sigma \rangle, \quad \mathcal{D}_2 = w_2\mathcal{D}_1, \quad \mathcal{D}_3 = w_1\mathcal{D}_2, \quad \mathcal{D}_4 = w_2\mathcal{D}_3, \quad \ldots$$

are pairwise non-equivalent. $\square$

LEMMA 7. Let $\mathcal{D}_1, \mathcal{D}_2 \in \Omega(\mathbb{Z})$, $\mathcal{D}_1 = w_1\mathcal{D}'_1$, and $\mathcal{D}_2 = w_2\mathcal{D}'_2$. Then $\mathcal{D}_1 \sqcap \mathcal{D}_2 \sim (\mathcal{D}_1 \sqcap \mathcal{D}'_2) \sqcup (\mathcal{D}'_1 \sqcap \mathcal{D}_2)$ if w_1 and w_2 are p-discernible, and $\mathcal{D}_1 \sqcap \mathcal{D}_2 \sim w_1(\mathcal{D}'_1 \sqcap \mathcal{D}'_2)$ otherwise.

Proof. By Lemma 6, $\mathcal{D}_1 \sqcap \mathcal{D}_2 \sim \biguplus\{\mathcal{D} \mid \mathcal{D} \in l(\mathcal{D}_1, \mathcal{D}_2)\}$.

First, suppose that w_1 and w_2 are p-discernible. In view of $\mathcal{D}'_1 \preccurlyeq \mathcal{D}_1$ and $\mathcal{D}'_2 \preccurlyeq \mathcal{D}_2$, $(\mathcal{D}_1 \sqcap \mathcal{D}'_2) \sqcup (\mathcal{D}'_1 \sqcap \mathcal{D}_2) \preccurlyeq \mathcal{D}_1 \sqcap \mathcal{D}_2$. Let $\mathcal{D} \in l(\mathcal{D}_1, \mathcal{D}_2)$. We have $\mathcal{D} = w\mathcal{D}'$ for some $\mathcal{D}' \in \Omega(\mathbb{Z})_\perp$. Let $\varphi_1 \in \Phi(\mathcal{D}, \mathcal{D}_1)$ and $\varphi_2 \in \Phi(\mathcal{D}, \mathcal{D}_2)$. Then either $\varphi_1(w) \neq w_1$ or $\varphi_2(w) \neq w_2$. In the former case $\mathcal{D} \preccurlyeq \mathcal{D}'_1$ and $\mathcal{D} \preccurlyeq \mathcal{D}'_1 \sqcap \mathcal{D}_2$. In the latter case $\mathcal{D} \preccurlyeq \mathcal{D}'_2$ and $\mathcal{D} \preccurlyeq \mathcal{D}_1 \sqcap \mathcal{D}'_2$. In either case $\mathcal{D} \preccurlyeq (\mathcal{D}_1 \sqcap \mathcal{D}'_2) \sqcup (\mathcal{D}'_1 \sqcap \mathcal{D}_2)$. Hence $\mathcal{D}_1 \sqcap \mathcal{D}_2 \preccurlyeq (\mathcal{D}_1 \sqcap \mathcal{D}'_2) \sqcup (\mathcal{D}'_1 \sqcap \mathcal{D}_2)$.

Next, assume that w_1 and w_2 are p-indiscernible. In virtue of $\mathcal{D}'_1 \preccurlyeq \mathcal{D}_1$ and $\mathcal{D}'_2 \preccurlyeq \mathcal{D}_2$, $\mathcal{D}'_1 \sqcap \mathcal{D}'_2 \preccurlyeq \mathcal{D}_1 \sqcap \mathcal{D}_2$. For any $\mathbb{Z}$-tree $\mathcal{D}$, $\mathcal{D} \preccurlyeq \mathcal{D}_1, \mathcal{D}_2$ implies $w_1\mathcal{D} \preccurlyeq \mathcal{D}_1, \mathcal{D}_2$, and so $w_1(\mathcal{D}'_1 \sqcap \mathcal{D}'_2) \preccurlyeq \mathcal{D}_1 \sqcap \mathcal{D}_2$. Consider an arbitrary $\mathcal{D} \in l(\mathcal{D}_1, \mathcal{D}_2)$. For some $\mathcal{D}' \in \Omega(\mathbb{Z})_\perp$, $\mathcal{D} = w\mathcal{D}'$. If the triples w and w_1 are p-discernible, then $\mathcal{D} \preccurlyeq \mathcal{D}'_1, \mathcal{D}'_2$ and $\mathcal{D} \preccurlyeq \mathcal{D}'_1 \sqcap \mathcal{D}'_2 \preccurlyeq w_1(\mathcal{D}'_1 \sqcap \mathcal{D}'_2)$. But if w and w_1 are p-indiscernible, then $\mathcal{D}' \preccurlyeq \mathcal{D}'_1, \mathcal{D}'_2$, since $\mathcal{D}$ is dense, and $\mathcal{D}' \preccurlyeq \mathcal{D}'_1 \sqcap \mathcal{D}'_2$ entails $\mathcal{D} \sim w_1\mathcal{D}' \preccurlyeq w_1(\mathcal{D}'_1 \sqcap \mathcal{D}'_2)$. In both of the cases $\mathcal{D} \preccurlyeq w_1(\mathcal{D}'_1 \sqcap \mathcal{D}'_2)$, and hence $\mathcal{D}_1 \sqcap \mathcal{D}_2 \preccurlyeq w_1(\mathcal{D}'_1 \sqcap \mathcal{D}'_2)$. $\square$

3. THE LATTICE OF NUMBERINGS

3.1. The model $\mathbb{S}$ and the set N_α. Let S be an at most countable set containing at least two elements. For a numbering α of the set S, N_α denotes the least subset of the set of all numberings of S which contains α and is closed under completion, direct sum, and equivalence of the numberings. We fix a numbering α of the family S with a non-empty set NS of non-special elements. Let $\mathbb{S} \rightleftharpoons \langle S; NS, P_1, P_2, \ldots \rangle$, where the predicates $P_1, P_2, \ldots$ are such that for any $a \in S$, there is a unique $i \geqslant 1$ for which $P_i(a)$ and every one of the predicates P_i, $i \geqslant 1$, is not more than a singleton. In other words, each element of the set S is distinguished by a unique predicate P_i, $i \geqslant 1$. Put $\Omega \rightleftharpoons \Omega(\mathbb{S})_\perp$.

We point out the following property of model $\mathcal{K}_\mathbb{S}$: for $k_1, k_2 \in K_S$, the elements k_1 and k_2 are p-indiscernible iff $k_1^* = k_2^*$.

Let $\mathcal{D} \in \Omega(\mathbb{S})$, $k \in D$, and $k = k'w$, where $k' \in K_S \cup \{\Lambda\}$. Put

$$\widehat{k^+} \rightleftharpoons \{(w^*, 1, 1)k_1 \mid kk_1 \in D,\ k_1 \in K_S \cup \{\Lambda\}\},$$

$$\widehat{k^-} \rightleftharpoons \{k_1 \mid kk_1 \in D,\ k_1 \in K_S\}.$$

Clearly, these cones are isomorphic to the previous.

3.2. Numberings generated by $\mathbb{S}$-trees. For $\mathcal{D} \in \Omega$, we define a numbering $\alpha_\mathcal{D} \in N_\alpha$ (up to equivalence of the numberings) as follows:

(1) if $\mathcal{D} = \perp$ then $\alpha_\mathcal{D} \rightleftharpoons \alpha$;

(2) if $\mathcal{D} = w\mathcal{D}'$ then $\alpha_D \rightleftharpoons (\alpha_{\mathcal{D}'})_{w^*}$;

(3) if for $m > 1$ there are indecomposable $\mathbb{S}$-trees $\mathcal{D}_1, \ldots, \mathcal{D}_m$ such that $\mathcal{D} = \biguplus_{i=1}^{m} \mathcal{D}_i$ then $\alpha_D \rightleftharpoons \bigoplus_{i=1}^{m} \alpha_{\mathcal{D}_i}$.

It is not hard to show that this definition is sound. By induction on the complexity of $\mathcal{D}$, we can easily state that $\alpha \leqslant \alpha_{\mathcal{D}}$ for all $\mathcal{D} \in \Omega$. By definition, $\alpha_{\mathcal{D}_1 \dotplus \mathcal{D}_2} \equiv \alpha_{\mathcal{D}_1} \oplus \alpha_{\mathcal{D}_2}$. The definition of N_α implies that for any numbering $\beta \in N_\alpha$ there is a tree $\mathcal{D} \in \Omega$ with $\beta \equiv \alpha_{\mathcal{D}}$.

3.3. An isomorphism between $\langle \Omega/_{\sim}; \preccurlyeq\rangle$ and $\langle N_\alpha/_{\equiv}; \leqslant\rangle$.

THEOREM 1. For $\mathcal{D}_1, \mathcal{D}_2 \in \Omega$, $\mathcal{D}_1 \preccurlyeq \mathcal{D}_2$ if and only if $\alpha_{\mathcal{D}_1} \leqslant \alpha_{\mathcal{D}_2}$.

The **proof** is by induction on $\mathcal{D}_1$. If $\mathcal{D}_1 = \perp$ then $\mathcal{D}_1 \preccurlyeq \mathcal{D}_2$ and $\alpha_{\mathcal{D}_1} \equiv \alpha \leqslant \alpha_{\mathcal{D}_2}$. If $\mathcal{D}_1 = \biguplus_{i=1}^{m} \mathcal{D}'_i$ for some $m > 1$ and for indecomposable $\mathbb{S}$-trees $\mathcal{D}'_1, \ldots, \mathcal{D}'_m$ then Lemma 2(2), in view of the inductive assumption and the definition of $\alpha_{\mathcal{D}_1}$, gives $\mathcal{D}_1 \preccurlyeq \mathcal{D}_2 \Leftrightarrow (\forall i \in [1,m])(\mathcal{D}'_i \preccurlyeq \mathcal{D}_2) \Leftrightarrow (\forall i \in [1,m])(\alpha_{\mathcal{D}'_i} \leqslant \alpha_{\mathcal{D}_2}) \Leftrightarrow \alpha_{\mathcal{D}_1} \leqslant \alpha_{\mathcal{D}_2}$. Lastly, let $\mathcal{D}_1 = w\mathcal{D}'$.

Assume $\mathcal{D}_1 \preccurlyeq \mathcal{D}_2$, $\varphi \in \Phi(\mathcal{D}_1, \mathcal{D}_2)$, and $\varphi(w) = k$. Then $\mathcal{D}_1 \preccurlyeq \widehat{k^+} = w\widehat{k^-}$ and $\mathcal{D}' \preccurlyeq \widehat{k^+}$. By the inductive assumption, $\alpha_{\mathcal{D}'} \leqslant \alpha_{\widehat{k^+}}$. Hence $\alpha_{\mathcal{D}_1} \equiv (\alpha_{\mathcal{D}'})_{w^*} \leqslant (\alpha_{\widehat{k^+}})_{w^*} \equiv ((\alpha_{\widehat{k^-}})_{w^*})_{w^*} \equiv (\alpha_{\widehat{k^-}})_{w^*} \equiv \alpha_{\widehat{k^+}} \leqslant \alpha_{\mathcal{D}_2}$.

Suppose, now, that $\alpha_{\mathcal{D}_1} \leqslant \alpha_{\mathcal{D}_2}$. We have $\alpha_{\mathcal{D}'} \leqslant (\alpha_{\mathcal{D}'})_{w^*} \equiv \alpha_{\mathcal{D}_1} \leqslant \alpha_{\mathcal{D}_2}$, and by the inductive assumption, $\mathcal{D}' \preccurlyeq \mathcal{D}_2$. Using induction on $\mathcal{D}_2$ we prove that $\mathcal{D}_1 \preccurlyeq \mathcal{D}_2$. There are three cases to consider:

(1) Let $\mathcal{D}_2 = \perp$. Then $\mathcal{D}' = \perp$, $D_1 = \{w\}$, $w^* \in NS$, and $\alpha_{\mathcal{D}_1} \equiv \alpha_{w^*} \nleqslant \alpha \equiv \alpha_{\mathcal{D}_2}$, which contradicts the initial assumption. This case is, therefore, impossible.

(2) Let $\mathcal{D}_2 = v\mathcal{D}''$ for some $v = (v^*, 1, 1) \in W_S$. If $w^* = v^*$ then $\mathcal{D}_1 \preccurlyeq \mathcal{D}_2$. Let $w^* \neq v^*$. By the completion property, $(\alpha_{\mathcal{D}'})_{w^*} \equiv \alpha_{\mathcal{D}_1} \leqslant \alpha_{\mathcal{D}_2} \equiv (\alpha_{\mathcal{D}''})_{v^*}$ yields $\alpha_{\mathcal{D}_2} \equiv (\alpha_{\mathcal{D}'})_{w^*} \leqslant \alpha_{\mathcal{D}''}$. Hence $\mathcal{D}_1 \preccurlyeq \mathcal{D}'' \preccurlyeq \mathcal{D}_2$ by the inductive assumption.

(3) For some $m > 1$ and for indecomposable $\mathbb{S}$-trees $\mathcal{D}''_1, \ldots, \mathcal{D}''_m$, $\mathcal{D}_2 = \biguplus_{i=1}^{m} \mathcal{D}''_i$. By the complete numbering property, $(\alpha_{\mathcal{D}'})_{w^*} \equiv \alpha_{\mathcal{D}_1} \leqslant \alpha_{\mathcal{D}_2} \equiv \bigoplus_{i=1}^{m} \alpha_{\mathcal{D}''_i}$ gives $\alpha_{\mathcal{D}_1} \equiv (\alpha_{\mathcal{D}'})_{w^*} \leqslant \alpha_{\mathcal{D}''_{i_0}}$ for some $i_0 \in [1,m]$. By the inductive assumption, $\mathcal{D}_1 \preccurlyeq \mathcal{D}''_{i_0} \preccurlyeq \mathcal{D}_2$. $\square$

COROLLARY 1. The correspondence $\mathcal{D} \mapsto \alpha_{\mathcal{D}}$ induces an isomorphism of the distributive lattice $\langle \Omega/_{\sim}; \preccurlyeq\rangle$ onto the semilattice $\langle N_\alpha/_{\equiv}; \leqslant\rangle$.*

3.4. Invariance of greatest lower bounds.

THEOREM 2. Let $\mathcal{D}_1, \mathcal{D}_2 \in \Omega$ and γ be a numbering of some subset of S such that $\gamma \leqslant \alpha_{\mathcal{D}_1}, \alpha_{\mathcal{D}_2}$. Then $\gamma \leqslant \alpha_{\mathcal{D}_1 \sqcap \mathcal{D}_2}$.

Proof. If $\mathcal{D}_1 \preccurlyeq \mathcal{D}_2$ or $\mathcal{D}_2 \preccurlyeq \mathcal{D}_1$ then the statement of the theorem is obvious. We may so assume that $\mathcal{D}_1 \not\preccurlyeq \mathcal{D}_2$ and $\mathcal{D}_2 \not\preccurlyeq \mathcal{D}_1$. In particular, $\mathcal{D}_1 \neq \perp \neq \mathcal{D}_2$. We prove the theorem by induction on $|D_1| + |D_2|$.

Assume that for some $i \in \{1, 2\}$ there exists $m > 1$ such that $\mathcal{D}_i = \biguplus_{j=1}^{m} \mathcal{D}'_j$. In view of the lattice of $\mathbb{S}$-trees being distributive, we have $\mathcal{D}_1 \sqcap \mathcal{D}_2 \sim \biguplus_{j=1}^{m} (\mathcal{D}'_j \sqcap \mathcal{D}_{3-i})$. Since $\gamma \leqslant \alpha_{\mathcal{D}_i} \equiv \bigoplus_{j=1}^{m} \alpha_{\mathcal{D}'_j}$, by virtue of a known result in the theory of numberings, there are numberings $\gamma_1, \ldots, \gamma_m$ of the subsets of S such that $\gamma \equiv \bigoplus_{j=1}^{m} \gamma_j$ and $\gamma_j \leqslant \alpha_{\mathcal{D}'_j}$ for all $j \in [1,m]$. By the inductive assumption, $\gamma_j \leqslant \alpha_{\mathcal{D}'_j \sqcap \mathcal{D}_{3-i}}$ for every $j \in [1,m]$. Consequently $\gamma \equiv \bigoplus_{j=1}^{m} \gamma_j \leqslant \bigoplus_{j=1}^{m} \alpha_{\mathcal{D}'_j \sqcap \mathcal{D}_{3-i}} \equiv \alpha_{\biguplus_{j=1}^{m} (\mathcal{D}'_j \sqcap \mathcal{D}_{3-i})} \equiv \alpha_{\mathcal{D}_1 \sqcap \mathcal{D}_2}$.

It remains to consider the case where $\mathcal{D}_1 = w_1\mathcal{D}'_1$ and $\mathcal{D}_2 = w_2\mathcal{D}'_2$ for some $\mathcal{D}'_1, \mathcal{D}'_2 \in \Omega$. We partition this case into two. Let $\beta_1 \equiv \alpha_{\mathcal{D}'_1}$ and $\beta_2 \equiv \alpha_{\mathcal{D}'_2}$. Recall that

*After the article had been prepared for publication, V. L. Selivanov draw my attention to the fact that, for the case where α is a minimal numbering of a finite set, the above-indicated isomorphism had been established in [2].

$$\alpha_{\mathcal{D}_1} \equiv (\beta_1)_{w_1^*} = \begin{cases} \beta_1(K(x)) & \text{if } K(x)\!\downarrow, \\ w_1^* & \text{if } K(x)\!\uparrow, \end{cases}$$

$$\alpha_{\mathcal{D}_2} \equiv (\beta_2)_{w_2^*} = \begin{cases} \beta_2(K(x)) & \text{if } K(x)\!\downarrow, \\ w_2^* & \text{if } K(x)\!\uparrow. \end{cases}$$

Let f_1 and f_2 be computable functions such that $\gamma(x) = (\beta_1)_{w_1^*}(f_1(x))$ and $\gamma(x) = (\beta_2)_{w_2^*}(f_2(x))$ for all $x \in \mathbb{N}$. Put $A_1 \rightleftharpoons \{x \in \mathbb{N} \mid K(f_1(x))\!\downarrow\}$ and $A_2 \rightleftharpoons \{x \in \mathbb{N} \mid K(f_2(x))\!\downarrow\}$.

Case 1. Let $w_1^* = w_2^*$. It is easy to see that $\gamma(x) = w_1^*$ for any $x \notin A_1 \cap A_2$. If $A_1 \cap A_2 = \varnothing$ then $\gamma(x) = w_1^*$ for any $x \in \mathbb{N}$ and any $\gamma \leqslant \alpha_{\mathcal{D}_1 \sqcap \mathcal{D}_2}$. Let $A_1 \cap A_2 \neq \varnothing$ and f be a computable surjective mapping of $\mathbb{N}$ onto $A_1 \cap A_2$. Then $K(f_1(f(x)))$ and $K(f_2(f(x)))$ are defined for any $x \in \mathbb{N}$, and the numbering $\gamma_1 \rightleftharpoons \gamma \circ f$ is reduced to β_1 and to β_2 by functions $K \circ f_1 \circ f$ and $K \circ f_2 \circ f$, respectively. By the inductive assumption, $\gamma_1 \leqslant \alpha_{\mathcal{D}_1' \sqcap \mathcal{D}_2'}$. Let $\beta_3 \equiv \alpha_{\mathcal{D}_1' \sqcap \mathcal{D}_2'}$. In view of Lemma 7, $\mathcal{D}_1 \sqcap \mathcal{D}_2 \sim w_1(\mathcal{D}_1' \sqcap \mathcal{D}_2')$, and

$$\alpha_{\mathcal{D}_1 \sqcap \mathcal{D}_2} \equiv (\beta_3)_{w_1^*} = \begin{cases} \beta_3(K(x)) & \text{if } K(x)\!\downarrow, \\ w_1^* & \text{if } K(x)\!\uparrow. \end{cases}$$

We now need to show that $\gamma \leqslant (\beta_3)_{w_1^*}$. Let g' be a computable function such that $\gamma_1(x) = \beta_3(g'(x))$ for all $x \in \mathbb{N}$, and $g = g' \circ f^{-1}$. Then the domain of g is equal to $A_1 \cap A_2$, and for all $x \in \mathrm{dom}(g)$, we have $\gamma(x) = \beta_3(g(x))$. Let $c \in \mathbb{N}$ be such that $K(\langle c, x\rangle) = g(x)$ with all x. We show that $\gamma \leqslant (\beta_3)_{w_1^*}$ (by a function $\langle c, x\rangle$). Indeed, if $x \in A_1 \cap A_2$ then $K(\langle c, x\rangle)$ is defined and is equal to $g(x)$. Therefore $(\beta_3)_{w_1^*}(\langle c, x\rangle) = \beta_3(K(\langle c, x\rangle)) = \beta_3(g(x)) = \gamma(x)$. If, however, $x \notin A_1 \cap A_2$ then $K(\langle c, x\rangle)$ is undefined and $\gamma(x) = w_1^* = (\beta_3)_{w_1^*}(\langle c, x\rangle)$.

Case 2. Let $w_1^* \neq w_2^*$. Then $A_1 \cup A_2 = \mathbb{N}$. If $A_i = \varnothing$ for some $i \in \{1, 2\}$ then $\gamma(x) = w_i^*$ for all $x \in \mathbb{N}$ and $\gamma \leqslant \alpha_{\mathcal{D}_1 \sqcap \mathcal{D}_2}$. Let $A_1, A_2 \neq \varnothing$ and g_1 and g_2 be computable surjective mappings of $\mathbb{N}$ onto A_1 and A_2, respectively. Put $\gamma_1 \rightleftharpoons \gamma \circ g_1$ and $\gamma_2 \rightleftharpoons \gamma \circ g_2$. The numbering γ_1 is reduced to β_1 by the function $K \circ f_1 \circ g_1$, and to $(\beta_2)_{w_2^*}$ by the function $f_2 \circ g_1$, that is, $\gamma_1 \leqslant \alpha_{\mathcal{D}_1'}, \alpha_{\mathcal{D}_2}$. By the inductive assumption, $\gamma_1 \leqslant \alpha_{\mathcal{D}_1' \sqcap \mathcal{D}_2}$. Similarly, $\gamma_2 \leqslant \alpha_{\mathcal{D}_1 \sqcap \mathcal{D}_2'}$. By Lemma 7, $\mathcal{D}_1 \sqcap \mathcal{D}_2 \sim (\mathcal{D}_1' \sqcap \mathcal{D}_2) \sqcup (\mathcal{D}_1 \sqcap \mathcal{D}_2')$ and $\gamma_1 \oplus \gamma_2 \leqslant \alpha_{\mathcal{D}_1' \sqcap \mathcal{D}_2} \oplus \alpha_{\mathcal{D}_1 \sqcap \mathcal{D}_2'} \equiv \alpha_{(\mathcal{D}_1' \sqcap \mathcal{D}_2) \sqcup (\mathcal{D}_1 \sqcap \mathcal{D}_2')} \equiv \alpha_{\mathcal{D}_1 \sqcap \mathcal{D}_2}$.

We claim that $\gamma \equiv \gamma_1 \oplus \gamma_2$. It is clear that $\gamma_1, \gamma_2 \leqslant \gamma$; hence $\gamma_1 \oplus \gamma_2 \leqslant \gamma$. For all $x \in \mathbb{N}$,

$$(\gamma_1 \oplus \gamma_2)(x) = \begin{cases} \gamma_1(y) & \text{if } x = 2y, \\ \gamma_2(y) & \text{otherwise.} \end{cases}$$

Let R be a computable set such that $R \subseteq A_1$ and $\mathbb{N} \setminus R \subseteq A_2$. For any $x \in \mathbb{N}$, put

$$g(x) \rightleftharpoons \begin{cases} 2g_1^{-1}(x) & \text{if } x \in R, \\ 2g_2^{-1}(x) + 1 & \text{otherwise.} \end{cases}$$

The function g is computable and reduces γ to $\gamma_1 \oplus \gamma_2$. $\square$

COROLLARY 2. For any $\mathcal{D}_1, \mathcal{D}_2 \in \Omega$, the numberings $\alpha_{\mathcal{D}_1}$ and $\alpha_{\mathcal{D}_2}$ have a greatest lower bound in the semilattice of all numberings of the set S equal to $\alpha_{\mathcal{D}_1 \sqcap \mathcal{D}_2}$.

3.5. The lattice of numberings.

THEOREM 3. Let α be a numbering of a set S containing at least two elements for which there exists at least one non-special element. Then the semilattice $\langle N_\alpha/_\equiv; \leqslant\rangle$ is an infinite constructivizable distributive lattice with a least element in which every principal downcone is finite. Moreover, the least upper and the greatest lower bounds in $\langle N_\alpha/_\equiv; \leqslant\rangle$ are invariant under extensions of the latter in the semilattice of all numberings of the set S.

Proof. The theorem follows immediately from Proposition 2 and Corollaries 1, 2. □

We give a characterization of isomorphism types for distributive lattices of numberings generated by some fixed numbering using the operations of completion and taking least upper bounds. For any numbering α of any at most countable non-empty family S, $NS(\alpha, S)$ denotes the set of non-special elements of α, with $\lambda_1(\alpha, S) \rightleftharpoons |NS(\alpha, S)|$ and $\lambda_2(\alpha, S) \rightleftharpoons |S \setminus NS(\alpha, S)|$. If $\lambda_1(\alpha, S) = 0$ then the lattice $\langle N_\alpha/_\equiv; \leqslant\rangle$ is one-element for any value $\lambda_2(\alpha, S) > 0$, and so below we assume that the value of λ_1 at any pair ⟨a numbering, a numbered set⟩ is other than zero.

THEOREM 4. For arbitrary numberings α_1 and α_2 of arbitrary at most countable non-empty sets S_1 and S_2, respectively, the distributive lattices $\langle N_{\alpha_1}/_\equiv; \leqslant\rangle$ and $\langle N_{\alpha_2}/_\equiv; \leqslant\rangle$ are isomorphic if and only if the pairs $\langle \lambda_1(\alpha_1, S_1), \lambda_2(\alpha_1, S_1)\rangle$ and $\langle \lambda_1(\alpha_2, S_2), \lambda_2(\alpha_2, S_2)\rangle$ coincide.

Proof. Let $\Omega_1 \rightleftharpoons \Omega(\mathbb{S}_1)_\perp$ and $\Omega_2 \rightleftharpoons \Omega(\mathbb{S}_2)_\perp$.

Sufficiency. Let f be a bijection from S_1 onto S_2 such that $f(NS(\alpha_1, S_2)) = NS(\alpha_2, S_2)$. We extend f to W_{S_1}, setting $f((s, i, m)) \rightleftharpoons (f(s), i, m)$ for $s \in S_1$, and then to K_S setting $f(w_1 \dots w_n) \rightleftharpoons f(w_1) \dots f(w_n)$. On the model $\mathbb{S}_2$, we define values of the predicates so that the elements s and $f(s)$ are p-indiscernible, for any $s \in S_1$. Consequently f is an isomorphism of model $\mathcal{K}_{\mathbb{S}_1}$ onto model $\mathcal{K}_{\mathbb{S}_2}$. Hence f can be extended to an isomorphism of $\langle \Omega_1; \preccurlyeq\rangle$ onto $\langle \Omega_2; \preccurlyeq\rangle$. Appealing to Corollary 1, we see that $\langle N_{\alpha_1}/_\equiv; \leqslant\rangle$ is isomorphic to $\langle N_{\alpha_2}/_\equiv; \leqslant\rangle$.

Necessity. Let $\langle N_{\alpha_1}/_\equiv; \leqslant\rangle$ be isomorphic to $\langle N_{\alpha_2}/_\equiv; \leqslant\rangle$. Then $\langle \Omega_1; \preccurlyeq\rangle \cong \langle \Omega_2; \preccurlyeq\rangle$ by Corollary 1. Obviously, the minimal elements of $\Omega_1 \setminus \{\perp\}$ are the $\mathbb{S}_1$-trees p-equivalent to one-element $\mathbb{S}_1$-trees. The number of the last-mentioned trees, in turn, is equal to the number of elements in $NS(\alpha_1, S_1)$; hence the cardinal $\lambda_1(\alpha_1, S_1)$ is defined by an isomorphism type of model $\langle \Omega_1/_\sim; \preccurlyeq\rangle$. Thus $\lambda_1(\alpha_1, S_1) = \lambda_1(\alpha_2, S_2)$.

Let $\mathcal{D}_1$ be a one-element $\mathbb{S}_1$-tree. It is easy to see that a p-dense $\mathbb{S}_1$-tree $\mathcal{D}_2$ satisfies the following: (1) $\mathcal{D}_1 \preccurlyeq \mathcal{D}_2$ and $\mathcal{D}_2 \not\preccurlyeq \mathcal{D}_1$; (2) $\mathcal{D}_1$ is a unique (up to p-equivalence) $\mathbb{S}_1$-tree meeting the first condition iff $D_2 = \{w_1, w_1 w_2\}$, where $\{w_2\} = D_1$ and $w_1^* \notin NS(\alpha_1, S_1)$. For any fixed $\mathcal{D}_1$, therefore, there exist exactly $\lambda_2(\alpha_1, S_1)$ possibilities for $\mathcal{D}_2$. In this way the cardinal $\lambda_2(\alpha_1, S_1)$ is also defined by an isomorphism type of model $\langle \Omega_1/_\sim; \preccurlyeq\rangle$, and $\lambda_2(\alpha_1, S_1) = \lambda_2(\alpha_2, S_2)$. □

Thus the pair $\langle \lambda_1(\alpha, S), \lambda_2(\alpha, S)\rangle$ is a characteristic of an isomorphism type for the lattice $\langle N_\alpha/_\equiv; \leqslant\rangle$. Since the trivial cases $NS(\alpha, S) = \varnothing$ and $|S| = 1$ are left out of consideration, we have $\lambda_1(\alpha, S) > 0$ and $\lambda_1(\alpha, S) + \lambda_2(\alpha, S) > 1$. We claim that any pair with these restrictions may well be realized.

THEOREM 5. Let S be an at most countable set containing at least two elements, and let the cardinal numbers λ_1 and λ_2 be such that $\lambda_1 + \lambda_2 = |S|$ and $\lambda_1 > 0$. Then there exists a numbering α of the family S for which $\lambda_1(\alpha, S) = \lambda_1$ and $\lambda_2(\alpha, S) = \lambda_2$.

Proof. Let NS be a subset of S such that $|NS| = \lambda_1$. By [6, Cor. 2], there is a numbering α of the set S for which $NS(\alpha, S) = NS$. Consequently $\lambda_1(\alpha, S) = \lambda_1$ and $\lambda_2(\alpha, S) = \lambda_2$. □

Remark 1. If we treat Σ^0_{n+2}-computable numberings as in [4] then, using [6, Cor. 1], for every pair $\langle \lambda_1, \lambda_2\rangle$ of cardinal numbers such that $\lambda_1 > 0$ and $1 < \lambda_1 + \lambda_2 \leqslant \omega$, we can construct a family $S \subseteq \Sigma^0_{n+2}$

and a Σ^0_{n+2}-computable numbering α of S so that $\lambda_1(\alpha, S) = \lambda_1$ and $\lambda_2(\alpha, S) = \lambda_2$. Note that in the former case all numberings in N_α will be Σ^0_{n+2}-computable.

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