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GROUPS WITH ELEMENTARY ABELIAN CENTRALIZERS OF INVOLUTIONS

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An involution i of a group G is said to be almost perfect in G if any two involutions of i^G the order of a product of which is infinite are conjugated via a suitable involution in i^G. We generalize a known result by Brauer, Suzuki, and Wall concerning the structure of finite groups with elementary Abelian centralizers of involutions to groups with almost perfect involutions.

In the paper we prove a theorem that generalizes the known result in [1] concerning the structure of finite groups with elementary Abelian centralizers of involutions to groups with almost perfect involutions.

An involution i is *perfect in* G if every two non-commuting involutions in i^G are conjugated by an involution in the same class. An involution i is *almost perfect* if the condition mentioned is satisfied for pairs of involutions in i^G the order of a product of which is infinite. A group is 2-*complete* if the equation $x^2 = b$ is soluble in it, for every element b; a field with a 2-complete multiplicative group is *quadratically closed*. A proper subgroup H of G is said to be *isolated*, and (G, H) is called a *Frobenius pair* if $H \cap H^g = 1$, for every element $g \in G \setminus H$; if, in addition, $G = F\lambda H$ and $G \setminus F = \bigcup_{x\in G} x^{-1}H^{\#}x$, then G is a *Frobenius group* with kernel F and complement H.

THEOREM. Suppose G contains an almost perfect involution i and the centralizer of each involution in G is an elementary Abelian group. Then one of the following statements holds:

(1) G is an extension of an elementary Abelian 2-group $U = C_G(i)$ by an involution-free group, and every element of $G \setminus U$ acts by conjugation on U regularly;

(2) $G = F\lambda\langle i\rangle$ is a Frobenius group with a 2-complete Abelian kernel F and complement $\langle i\rangle$;

(3) G is isomorphic to a group $PGL_2(Q)$, where Q is a suitable quadratically closed field of characteristic 2.

Groups with finite involutions the centralizer of every involution in which is an Abelian 2-group were explored by Mazurov in [2, Thm. 2] except for the case of a quasicyclic Sylow 2-subgroup. Clearly, every involution of a periodic group is finite. For periodic groups with Abelian centralizers of involutions, Suchkov in [3] came up with an analog of the known Suzuki theorem (see [4]). There naturally arises the problem of generalizing the results obtained by Mazurov and Suchkov to groups with (almost) perfect involutions. The condition of being almost perfect for an involution considerably weakens the requirement for an involution to be periodic and finite, because G, in all of the three statements of our theorem, does not need to be periodic. An example illustrating the necessity of this condition is $G = A * B$, a free product of an arbitrary group A with Abelian centralizers of involutions and an arbitrary involution-free group B. For such a group, clearly, the theorem is no longer true.

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A subgroup U of a group G is *strongly isolated* if $C_G(u) \leq U$ for any non-identity element $u \in U$. A proper subgroup B of G is *strongly embedded* if B contains an involution, and for any element $g \in G \setminus B$, the subgroup $B \cap B^g$ contains none. A doubly transitive permutation group in which only unity keeps three points fixed is a *Z-group* (a *Zassenhaus group*).

The following properties of dihedral groups are well known and so given without proofs.

LEMMA 1. Let $D = \langle i, j \rangle$, where i and j are involutions. Then:

(1) $D = \langle ij \rangle \lambda \langle i \rangle = \langle ij \rangle \lambda \langle j \rangle$, $(ij)^i = (ij)^j = ji = (ij)^{-1}$, and the set $D \setminus \langle ij \rangle$ consists of involutions;

(2) if $\langle ij \rangle$ has an involution z then $z \in Z(D)$, D has no isolated and strongly embedded subgroups, i and j are not conjugate in D, and D contains three conjugacy classes of involutions: $\{z\}$, i^D, and j^D;

(3) D is a Frobenius group iff the order of an element ij is finite and odd, in which case i^D is a sole conjugacy class of involutions in the group D, $i = j^c$, where $c \in \langle ij \rangle$, $c^2 = ij$, and for an involution $k = ic$, $j = i^k$ and $i^k = j$.

(4) if the order of ij is infinite then D has two conjugacy classes of involutions: i^D and j^D.

Let G satisfy the hypotheses of the theorem, j be an involution in G, $U = C_G(j)$, and $B = N_G(U)$. We also make the convention that the notation appearing in the formulations of the lemmas below will be kept throughout.

LEMMA 2. The following statements hold:

(1) U is a strongly isolated elementary Abelian 2-subgroup of G. In particular, $U \cap U^g = 1$ for any element $g \in G \setminus B$.

(2) For $U = G$ or $B = G$, item (1) of the theorem holds true.

(3) For $U = B \neq G$, the subgroup U is equal to $\langle j \rangle$ and item (2) of the theorem holds true.

Proof. (1) By the hypotheses of the theorem, U is an elementary Abelian 2-group; moreover, U is maximal among the elementary Abelian 2-subgroups, and for any involution $u \in U$, $U = C_G(u)$. Hence the subgroup U is strongly isolated in G, and since it is Abelian, $U \cap U^g = 1$ for any element $g \in G \setminus B$.

(2) If $U = G$, there is nothing to prove. Let $U \neq B = G$. In view of the hypotheses of the theorem, all 2-elements of G are involutions. If k is an involution in $G \setminus U$ then (by Lemma 1 and the fact that U is normal in G), the dihedral group $D = \langle k, j \rangle$ is Klein's four-group and $k \in C_G(j)$, which clashes with the definition of U. Consequently the set $G \setminus U$ and the factor group G/U are involution free. By virtue of statement (1), every element of $G \setminus U$ acts on the subgroup U regularly.

(3) In view of (1), (G, U) is a Frobenius pair. We prove that j is perfect in G. Let $k \in i^G \setminus U$ and $D = \langle k, j \rangle$. If the subgroup D is finite, then j and k are conjugate in D (cf. Lemma 1(3)) and $j \in i^G$, since U is isolated in G. For the case where $D = \langle k, j \rangle$ is infinite, by the definition of an almost perfect involution it follows that there is an involution $v \in i^G$ such that $k^v = k^j$. Consequently $vj \in C_G(k)$, and either $v = j$ or $|vj| = 2$. If $|vj| = 2$ then $v \in U$ (cf. Lemma 1(2)) and $1 \neq vj \in U \cap U^k$, which is impossible. Hence $v = j$ and j is perfect. The result now follows from [5, Lemma 3]. $\square$

Based on Lemma 2, below we assume that $U \neq B \neq G$. Denote by J the set of all involutions in G.

LEMMA 3. The following statements hold:

(1) a subgroup B is strongly embedded in G;

(2) $J = j^G$, $J \cap B = j^B$, and j is an involution, perfect in G.

(3) there exists a one-one correspondence between the set of involutions of B and the set of involutions of any right coset Bg, $g \in G$; moreover, if j is a fixed involution in U then every involution $k \in U$ (including the j) is associated with a unique involution $v_k \in Bg$ such that $g^{-1}kg = v_k j v_k$;

(4) each element $g \in G$ is represented as $g = bv$, where $b \in B$ and v is some involution;

(5) for each involution $v \in G \setminus B$, the subgroup B contains a set H_v of elements, strictly real w.r.t. v, which has the same cardinality as the set of involutions in U.

Proof. (1) Clearly, all hypotheses of the theorem hold true for B. In view of Lemma 2(2), all 2-elements of B are contained in U, and by Lemma 2(1), $U \cap U^g = 1$ for any element $g \in G \setminus B$. Hence the subgroup B is strongly embedded in G.

(2) Let t be any involution in B and $k \in i^G \setminus B$. If the subgroup $D = \langle k, t \rangle$ is finite, then j and k are conjugate in D in view of Lemma 1(2),(3), since B is strongly embedded in G. If $D = \langle k, t \rangle$ is infinite then there is an involution $v \in i^G$ such that $k^v = k^t$. It follows that $vt \in C_G(k)$, and by the hypotheses of the theorem, either $v = t$ or $|vt| = 2$. In the latter case $v \in C_G(t) = U$ and $1 \neq vt \in B \cap B^k$, which is impossible by (1). Thus it is always true that $t \in i^G$.

Since t is any involution of U and j, in defining U, was chosen arbitrarily in G, $J = i^G$ is the set of all involutions in G. Keeping in mind that B is strongly embedded in G, we have $J \cap B = j^B$. If $k \in J$ and $kj \neq jk$, then $k \in J \setminus U$, and by the above, there exists an involution $v \in J \setminus U$ with $k^v = j$ and $j^v = k$. Hence J is the class of perfect involutions in G.

(3) Let $g \in G \setminus H$; the involutions k and t are not necessarily distinct in B, and $D = \langle g^{-1}kg, j \rangle$. By (2), J contains an involution v_k such that $v_k j v_k = g^{-1}kg$. In view of (1), $v_k \in Bg$. Similarly, we define an involution $v_t \in Bg$. If $k \neq t$ then $g^{-1}kg = v_k j v_k \neq g^{-1}tg = v_t j v_t$ and $v_k \neq v_t$.

Now, let v be any involution of Bg and let $g = bv$, where $b \in B$. Put $k = bjb^{-1}$. Then $g^{-1}kg = vjv$ and $v = v_k$. Thus the mapping $k \leftrightarrow v_k$ is bijective.

(4) By (3), the coset Bg contains an involution v, and $g = bv$, where $b \in B$.

(5) Let v be an involution of $G \setminus B$. By (3), the coset Bv contains a set J_v of involutions of the same cardinality as the set of involutions in U. Every involution $k \in J_v$ is represented as $k = b_k v$, where $b_k \in B$. Obviously, b_k is an element of B which is strictly real w.r.t. v, and all elements b_k are distinct. Inversely, if b is an element of B that is strictly real w.r.t. v then $k = bv$ is an involution in Bv and $b = b_k$ by Lemma 1(1). □

LEMMA 4. Let v be any involution in $G \setminus B$ and $H = B \cap B^v$. Then $H = H_v$ is an Abelian group, invertible by v, and the subgroup $B = U \lambda H$ is a Frobenius group with kernel U and complement H, isomorphic to an affine transformation group of a suitable field Q of characteristic 2.

Proof. Obviously, H is a subgroup containing the set H_v and intersecting the subgroup U at unity. In view of Lemma 3, for every involution $k \in U$, H_v contains an element b_k with $j^{b_k} = k$. Since $C_G(j) \cap H = 1$, $j^h = k$, and $h \in H$, we have $h = b_k$. Thus $H = H_v$ and $h^v = h^{-1}$ for any element $h \in H$, and H is Abelian.

Suppose $b \in B$ and $b \notin K = U \lambda H$. Then $j^b = k$ for an appropriate involution $k \in U$, and $b_k b^{-1} \in C_G(j) \setminus K$, which is a contradiction with $C_G(j) = U$. Hence $B = U \lambda H$, in which case H acts regularly on the set of involutions in U. As stated in [6], B is isomorphic to an affine transformation group of a suitable field Q of characteristic 2, and it is a Frobenius group with kernel U and complement H. Moreover, H is isomorphic to the multiplicative group of Q and U is isomorphic to its additive group. □

LEMMA 5. A subgroup H is strongly isolated in G, $N_G(H) = H \lambda \langle v \rangle = K$, and $J \cap K = J \cap Bv = v^K$. In particular, H is a 2-complete group and the field Q is quadratically closed.

Proof. That $J \cap Bv \subset J \cap K$ is obvious. On the other hand, $J \cap K = Hv \subset Bv$ and $Bv \cap J = J \cap K$. Assume $Hv \neq v^K$ and $t \in Hv \setminus v^K$. In view of Lemma 3, $v^k = t$ for some involution $k \in J$. Clearly, $vk \notin H$; on the other hand, $(vk)^2 = vt \in H$. Hence $vt \in B \cap B^k = H_k$, and in virtue of Lemma 4, H_k are non-invariant factors of the Frobenius group B which intersect pairwise at unity. Since $vt \neq 1$, we have $H_k = H$, and by Lemma 4, k inverts every element of H. Consequently $kv \in C_G(H)$. Using Lemma 4 we conclude that $vk \notin B$, and by Lemma 3, $vk = bs$ for some $s \in J$ and $b \in B$. As above, $H_s = B \cap B^s = H$ and the involution s inverts every element of H. Consider a group $C_G(H)\lambda\langle v\rangle$; it contains involutions v, k, and s. Therefore $b, s \in C_G(H)$, a contradiction. We have $J \cap K = v^K$.

Suppose that the inclusion $C_G(h) \leq H$ fails for some non-identity element $h \in H$. By Lemma 4, $C_B(h) = H$, and so there is an element $t \in C_G(h) \setminus B$. By Lemma 3, $t = bs$ for some $s \in J$ and $b \in B$. As stated above, $H_s = B \cap B^s = H$ and s inverts every element of H. The subgroup $C_G(h)\lambda\langle s\rangle$ contains elements t and b. It follows that the element $b \in B$ inverts h, which is a contradiction with Lemma 4. Consequently the subgroup H is strongly isolated in G.

Let $h \in H$. Since v and hv are conjugate in K, we obtain $hv = tvt^{-1} = t^2v$ and $h = t^2$, for an appropriate element $t \in H$. Hence H is a 2-complete group and Q is a quadratically closed field. □

Consider a permutation representation of G on a set $\Omega = \{U^x \mid x \in G\}$. The subgroup U, being an element of Ω, is denoted by α, and U^v — by β.

LEMMA 6. G is a permutation Z-group on the set Ω, with $G_\alpha = B$ and $G_{\alpha\beta} = H$.

Proof. The group G is transitive on Ω by definition. It is clear that $B = N_G(U) = G_\alpha$. In view of Lemma 5 and the fact that v was chosen arbitrarily in Lemma 4, the subgroup $B = G_\alpha$ is transitive on a set of all involutions not in B, and moreover, G_α is transitive on $\Omega \setminus \{\alpha\}$. Hence G is doubly transitive on Ω. Lemmas 4 and 5 entail $H = G_{\alpha\beta}$. If $1 \neq h \in G_{\alpha\beta\gamma}$ for some $\gamma \in \Omega$, then Lemmas 3 and 4 imply that there exists an involution $s \in J \setminus K$, inverting $h \in H$. Consequently $sv \in C_G(h) \setminus H$, which is a contradiction with Lemma 5. Since involutions j and v (or, equivalently, their centralizers — points α and β) and a point γ are chosen arbitrarily, the stabilizer of any three points in Ω is trivial. □

Proof of the theorem. We claim that G is triply transitive on Ω. In view of Lemma 5, this claim is equivalent to the subgroup H being transitive on $\Omega \setminus \{\alpha, \beta\}$. We choose arbitrary $\gamma, \delta \in \Omega \setminus \{\alpha, \beta\}$. By Lemmas 4 and 5, $G_{\alpha\gamma} = H^b$ and $G_{\alpha\gamma} = H^c$ for suitable $b, c \in B$. The group B, being an affine transformation group of a field Q (cf. Lemma 4), is a sharply doubly transitive group with stabilizer H, which is equivalent to H being transitive on a set $\{H^x \mid x \in B \setminus H\}$ of subgroups. Hence there exists $h \in H$ mapping H^b to H^c, and $\gamma^h = \delta$. Thus G is triply transitive on Ω. Therefore G satisfies all the hypotheses of Theorem 1 in [2], which says that G is isomorphic to $PGL_2(Q)$. □

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