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ISOMORPHISM TYPES OF ROGERS SEMILATTICES FOR FAMILIES FROM DIFFERENT LEVELS OF THE ARITHMETICAL HIERARCHY

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We investigate differences in isomorphism types for Rogers semilattices of computable numberings of families of sets lying in different levels of the arithmetical hierarchy.

Among the many possible applications of the theory of generalized computable numberings propounded in [1], a particularly interesting and popular one is to *arithmetical numberings*, that is, numberings of families of arithmetical sets. When considering a family $\mathcal{A}$ of Σ^0_n-sets, generalized computable numberings can be characterized as follows: a numbering α of $\mathcal{A}$ is generalized computable if and only if the set $\{\langle x, i\rangle \mid x \in \alpha(i)\}$ is Σ^0_n. In what follows, such a numbering α will merely be referred to as Σ^0_n-*computable*.

We recall that if α and β are numberings of the same family of objects, then α is said to be *reducible* to β (written $\alpha \leqslant \beta$) if there exists a computable function f such that $\alpha = \beta \circ f$. We write $\alpha \equiv \beta$ if $\alpha \leqslant \beta$ and $\beta \leqslant \alpha$. An equivalence class of a numbering α (depending of course on the collection of numberings under study) will be denoted by the symbol $\deg(\alpha)$. Since $\leqslant$ is a preordering relation, $\equiv$ is an equivalence relation. If $\mathcal{A}$ is a family of Σ^0_n-sets, then $\equiv$ partitions the set $\mathrm{Com}^0_\mathrm{n}(\mathcal{A})$ of all Σ^0_n-computable numberings into equivalence classes, thus generating a degree structure, denoted by $\mathcal{R}^0_n(\mathcal{A})$ and called the *Rogers semilattice* of $\mathcal{A}$.

In this paper we continue research on isomorphism types of Rogers semilattices, started in [2]. We are interested in differences between elementary theories and isomorphism types at different arithmetical levels. In [3, 4] it was shown that for every fixed level of the arithmetical hierarchy, there exist infinitely many families with pairwise different elementary theories for their Rogers semilattices. In [2] we established that, for every n, the isomorphism type of the Rogers semilattice of some Σ^0_{n+5}-computable family $\mathcal{B}$ is different from the isomorphism type of the Rogers semilattice $\mathcal{R}^0_{n+1}(\mathcal{A})$ of an arbitrary Σ^0_{n+1}-computable family $\mathcal{A}$. In this paper we improve on this result by showing that, for every n, the isomorphism type of the Rogers semilattice of any non-trivial (i.e., non one-element) Σ^0_{n+4}-computable family $\mathcal{B}$ is different from the isomorphism type of the Rogers semilattice $\mathcal{R}^0_{n+1}(\mathcal{A})$ of any Σ^0_{n+1}-computable family $\mathcal{A}$.

For the unexplained terminology and notation relative to computability theory, our main references are in [5-7]. We will use the term *computable function* to connote completely defined computable functions. For the main concepts and notions of the theory of numberings and computable Boolean algebras, we ask the

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reader to consult [8, 9]. The basic notions, notation, and methods bearing on arithmetical numberings and their Rogers semilattices can be found in [10, 11]. For the reader's convenience, and to make our discussion more self-contained, here, we recall the definition of the Lachlan operator for numberings, and summarize some of its properties in Lemma 1 below.

Definition 1. If β is a numbering of a family $\mathcal{A}$, and C is a non-empty c.e. set, with f a computable function such that $\mathrm{range}(f) = C$, then we define $\beta_C \leftrightharpoons \beta \circ f$.

The definition does not depend on f: if we define β_C starting from any other computable function g such that $\mathrm{range}(g) = C$ then we obtain a numbering which is equivalent to one given by f. The assignment $C \mapsto \beta_C$ from c.e. sets to numberings (up to equivalence of the numberings) is called the *Lachlan operator*.

LEMMA 1 [11, Lemma 2.2]. For every pair A, B of non-empty c.e. sets and for every pair of numberings α, β,

(1) the following are equivalent:

(a) $\beta_A \leqslant \beta_B$;

(b) there is a partial computable function φ satisfying $\mathrm{dom}(\varphi) \supseteq A$, $\varphi[A] \subseteq B$, and $\beta(x) = \beta(\varphi(x))$ for all $x \in A$;

(2) if $A \subseteq B$ then $\beta_A \leqslant \beta_B$;

(3) if $\beta_A \leqslant \beta_B$ then $\beta_B \equiv \beta_{A \cup B}$;

(4) if $\alpha \leqslant \beta$ then $\alpha = \beta_C$ for some c.e. set C;

(5) if $\alpha \leqslant \beta$ and $\alpha \equiv \beta_C$ for some c.e. set C then for every γ such that $\alpha \leqslant \gamma \leqslant \beta$ there exists a c.e. set D with $C \subseteq D$ and $\gamma \equiv \beta_D$;

(6) $\beta_{A \cup B} \equiv \beta_A \oplus \beta_B$.

The three lemmas given below and the notion of an $\mathbf{X}$-computable Boolean algebra play a key role in establishing our claim. Recall that a Boolean algebra $\mathfrak{A}$ is said to be $\mathbf{X}$-*computable* if its universe, operations, and relations are $\mathbf{X}$-computable (see [9]).

In the next lemma, the symbol $[\gamma, \delta]$ denotes the following interval of degrees in $\mathcal{R}^0_{n+1}(\mathcal{A})$:

$$[\gamma, \delta] \leftrightharpoons \{\deg(\beta) \mid \gamma \leqslant \beta \leqslant \delta\}.$$

LEMMA 2. Let γ and δ be Σ^0_{n+1}-computable numberings of a family $\mathcal{A}$. If $[\gamma, \delta]$ is a Boolean algebra, then it is $\mathbf{0}^{(n+3)}$-computable.

Proof. First, we observe that given n, $\mathcal{A}$, γ, and δ as in the hypothesis of the lemma, it follows by Lemma 1(4),(5) that there exists a c.e. set C such that $\gamma \equiv \delta_C$ and

$$[\gamma, \delta] = \{\deg(\delta_X) \mid X \text{ is c.e. and } X \supseteq C\}.$$

For every i, let $U_i \leftrightharpoons C \cup W_i$. This gives an effective listing of all c.e. supersets of C. By Lemma 1(1b), for every i, j, we have $\delta_{U_i} \leqslant \delta_{U_j}$ if and only if

$$\exists p[\forall x(x \in U_i \Rightarrow \exists y(\varphi_p(x) = y \ \&\ y \in U_j)) \ \&\ \forall x \forall y(x \in U_i \ \&\ \varphi_p(x) = y \Rightarrow \delta(x) = \delta(y))].$$

Since $\delta \in \mathrm{Com}^0_{n+1}(\mathcal{A})$, $\delta_{U_i} \leqslant \delta_{U_j}$ is a Σ^0_{n+3}-relation in i, j.

Consider the equivalence relation η on ω defined by setting

$$(i, j) \in \eta \Leftrightarrow \delta_{U_i} \leqslant \delta_{U_j} \ \&\ \delta_{U_j} \leqslant \delta_{U_i}.$$

Let $B \leftrightharpoons \{x \mid \forall y(y < x \Rightarrow (x, y) \notin \eta)\}$. Define a bijection $\psi_1 : B \to [\gamma, \delta]$, letting $\psi_1(i) = \deg(\delta_{U_i})$ for all $i \in B$. It is evident that ψ_1 induces in $\mathcal{R}^0_{n+1}(\mathcal{A})$ a partially ordered set $\mathfrak{B}$, which is a Boolean algebra isomorphic to $[\gamma, \delta]$. The interval $\mathfrak{B}$ is a $\mathbf{0}^{(n+3)}$-computable partially ordered set. It follows from [9, Thm. 3.3.4; 12] that the Boolean algebra $\mathfrak{B}$ under Boolean operations is also $\mathbf{0}^{(n+3)}$-computable. □

LEMMA 3 [13]. Let $\mathfrak{F}$ be a computable atomless Boolean algebra. Then for every $\mathbf{X}$ there is an ideal J such that J is $\mathbf{X}$-c.e. and the quotient $\mathfrak{F}/J$ is not isomorphic to any $\mathbf{X}$-computable Boolean algebra.

Below, we will use the following notation. For a given c.e. set H, $\{V_i \mid i \in \omega\}$ denotes an effective listing of all c.e. supersets of the set H defined, for instance, by $V_i \coloneqq H \cup W_i$, for all i. We will assume for convenience that $V_0 = H$. Let ε_H stand for the distributive lattice of all c.e. supersets of H. For a given c.e. set $V \supseteq H$, by V^* we denote the image of V under the canonical homomorphism of ε_H onto ε_H^* (i.e., ε_H modulo the finite sets), and write $\subseteq^*$ to denote the partial ordering relation on ε_H^*. Obviously, $J^* \coloneqq \{V^* \mid V \in J\}$ is an ideal in ε_H^* if J is one in ε_H.

As is known, if $\mathfrak{A}$ is a Boolean algebra and J is an ideal of $\mathfrak{A}$, then the universe of the quotient algebra $\mathfrak{A}/J$ is given by the set of equivalence classes $[a]_J$, $a \in \mathfrak{A}$, under the equivalence relation $\equiv_J$ defined by setting

$$a \equiv_J b \Leftrightarrow \exists c_1, c_2 \in J(a \vee c_1 = b \vee c_2),$$

and the partial ordering relation is given by the rule

$$[a]_J \leqslant_J [b]_J \Leftrightarrow a - b \in J,$$

where $a - b$ stands for $a \wedge \neg b$ (see, e.g., [9]).

LEMMA 4. Let $\mathcal{B}$ be a Σ^0_{m+1}-computable family, $\beta \in \mathrm{Com}^0_{m+1}(\mathcal{B})$, and H be any c.e. set such that $\beta[H] = \mathcal{B}$ and ε_H^* is a Boolean algebra. Let $\psi_2 : \varepsilon_H \to [\beta_H, \beta]$ be the mapping given by $\psi_2(V_i) = \deg(\beta_{V_i})$ for all i, and let I be any ideal of ε_H containing all finite sets. Then ψ_2 induces an isomorphism of ε_H^*/I^* onto $[\beta_H, \beta]$ if and only if for all i, j we have the following:

(1) $V_i \in I \Rightarrow \beta_{V_i} \leqslant \beta_H$;

(2) $V_i - V_j \notin I \Rightarrow \beta_{V_i} \nleqslant \beta_{V_j}$ (where $V_i - V_j \coloneqq (V_i \setminus V_j) \cup H$).

Proof. Let H, $\mathcal{B}$, β, and ψ_2 be given. The "only if" direction is immediate. To show that the conditions stated in the lemma are also sufficient, we can argue as follows. By Lemma 1(4),(5), it follows that every γ with $\beta_H \leqslant \gamma \leqslant \beta$ is of the form $\gamma \equiv \beta_C$ for some c.e. set $C \supseteq H$. Therefore the mapping induced by ψ_2 is clearly onto.

Suppose now that $[V_i^*]_{I^*} \subseteq^*_{I^*} [V_j^*]_{I^*}$. Then $V_i^* - V_j^* \in I^*$. But $V_i^* - V_j^* = (V_i - V_j)^*$, with $V_i - V_j$ a c.e. superset of H, since ε_H^* is a Boolean algebra. Hence $V_i - V_j \in I$. On the other hand,

$$V_i = (V_i - V_j) \cup (V_i \cap V_j).$$

Now, by (1), $\beta_{V_i - V_j} \leqslant \beta_H$, so $\beta_{V_i} \equiv \beta_{V_i \cap V_j}$ by Lemma 1(3), and hence $\beta_{V_i} \leqslant \beta_{V_j}$ by Lemma 1(2), since $V_i \cap V_j \subseteq V_j$.

Finally, if $[V_i^*]_{I^*} \nsubseteq^*_{I^*} [V_j^*]_{I^*}$ then $V_i - V_j \notin I$, and therefore $\beta_{V_i} \nleqslant \beta_{V_j}$ by (2). □

THEOREM 1. For every n, every non-trivial Σ^0_{n+5}-computable family $\mathcal{B}$, and every Σ^0_{n+1}-computable family $\mathcal{A}$, the Rogers semilattices $\mathcal{R}^0_{n+5}(\mathcal{B})$ and $\mathcal{R}^0_{n+1}(\mathcal{A})$ are not isomorphic.

Proof. Let n be given, suppose that $\mathcal{B}$ is an arbitrary non-trivial Σ^0_{n+5}-computable family, and assume that $\mathcal{A}$ is any Σ^0_{n+1}-computable family. By Lemma 2, all Boolean intervals in $\mathcal{R}^0_{n+1}(\mathcal{A})$ are $\mathbf{0}^{(n+3)}$-computable Boolean algebras. Therefore, to prove the theorem, it is sufficient to do the following:

(i) find a computable atomless Boolean algebra $\mathfrak{F}$ and an ideal J of $\mathfrak{F}$ (as in Lemma 3) such that J is c.e. in $\mathbf{0}^{(n+3)}$ and $\mathfrak{F}/J$ is not isomorphic to any $\mathbf{0}^{(n+3)}$-computable Boolean algebra;

(ii) find Σ^0_{n+5}-computable numberings α and β of $\mathcal{B}$ such that the interval $[\alpha,\beta]$ of $\mathcal{R}^0_{n+5}(\mathcal{B})$ is a Boolean algebra isomorphic to $\mathfrak{F}/J$.

First, we show how to satisfy the requirement (i). Let $\mathfrak{F}$ be a computable atomless Boolean algebra. By a famous result of Lachlan [14], there exists a hyperhypersimple set H for which ε^*_H is isomorphic to $\mathfrak{F}$. We fix such a set H.

We refer the reader to [7] for details of the construction of a suitable isomorphism χ of the Boolean algebra ε^*_H onto $\mathfrak{F}$. Here, we need only notice that starting from a computable listing $\{b_0, b_1, \ldots\}$ of the elements of $\mathfrak{F}$, we can find a Σ^0_3-computable Friedberg numbering $\{B_0, B_1, \ldots\}$ of some subfamily of ε_H such that $\varepsilon^*_H = \{B^*_0, B^*_1, \ldots\}$ and $\chi(B^*_i) = b_i$.

We will use the technique for embedding posets into intervals of Rogers semilattices developed in [11]. Let J be any $\mathbf{0}^{(n+3)}$-c.e. ideal of $\mathfrak{F}$ satisfying Lemma 3, and let $\hat{J} = \{j \in \omega \mid b_j \in J\}$. Then $\hat{J}$ is a $\mathbf{0}^{(n+3)}$-c.e. set, $I^* \leftrightharpoons \{B^*_j \mid j \in \hat{J}\}$ is an ideal of ε^*_H, and $\mathfrak{F}/J$ is isomorphic to ε^*_H/I^*. Thus, as the Boolean algebra $\mathfrak{F}/J$ in the requirement (ii) above we can take ε^*_H/I^*.

Let $I \leftrightharpoons \{V \mid V \in \varepsilon_H \ \&\ V^* \in I^*\}$ and $\hat{I} = \{i \in \omega \mid V^*_i \in I^*\}$. Obviously, I is an ideal of ε_H containing all finite subsets.

LEMMA 5. The relations $V_i \in I$ (equivalently, $i \in \hat{I}$) in i and $V_i - V_j \in I$ in i, j are both $\mathbf{0}^{(n+3)}$-c.e.

Proof. First we note that for any sets A and B, the relation $A =^* B$ is c.e. relative to $A = B$. Indeed, if $D_0, D_1, \ldots$ is the canonical numbering of the family of all finite sets (see [5, 6]), then

$$A =^* B \Leftrightarrow \exists p\, \exists q (A \cup D_p = B \cup D_q).$$

Since

$$\begin{array}{lcl} i \in \hat{I} & \Leftrightarrow & \exists j (V_i =^* B_j \ \&\ j \in \hat{J}), \\ V_i - V_j \in I & \Leftrightarrow & \exists k ((V_i \cap \overline{V_j}) \cup H = V_k,\ k \in \hat{I}), \end{array}$$

routine calculations show that $\hat{I} \in \Sigma^0_{n+4}$ and that the relation $V_i - V_j \in I$ is also c.e. in $\mathbf{0}^{(n+3)}$. $\square$

Due to Lemma 4, we can now construct a suitable numbering β of $\mathcal{B}$ and consider the corresponding mapping ψ_2, which will yield an isomorphism of ε^*_H/I^* onto the interval $[\beta_H, \beta]$.

Requirements. First, it is necessary that the numbering β satisfies the requirement

$$\mathbf{B} :\ \beta[H] = \mathcal{B},$$

which guarantees that β_H is a numbering of the whole family $\mathcal{B}$. Next, in view of Lemma 4, we must make it sure that all i, j, and p meet the following requirements:

$$\begin{array}{lcl} \mathbf{P_i} :\ V_i \in I & \Rightarrow & \beta_{V_i} \leqslant \beta_H, \\ \mathbf{R_{i,j,p}} :\ V_i - V_j \notin I & \Rightarrow & \beta_{V_i} \nleqslant \beta_{V_j} \text{ via } \varphi_p, \end{array}$$

where by "$\beta_{V_i} \nleqslant \beta_{V_j}$ via φ_p" we mean that φ_p does not reduce β_{V_i} to β_{V_j} in the sense of Lemma 1(1b).

In the construction we use an oracle $\mathbf{0}^{(n+4)}$ to answer questions such as "$V_i \in I$?" or "$V_i - V_j \in I$?" and to verify some properties of c.e. sets and computable functions. Computations with $\mathbf{0}^{(n+4)}$ will ensure that $\beta \in \mathrm{Com}^0_{n+5}(\mathcal{B})$.

The strategy for $\mathbf{B}$. We fix a numbering $\alpha \in \mathrm{Com}^0_{n+5}(\mathcal{B})$ and build by stages a $\mathbf{0}^{(n+4)}$-computable function $a(x)$ with range$(a) \subseteq H$. We will "insert" the numbering α into the numbering β by setting $\beta(a(x)) = \alpha(x)$ for all x. Since we never change the values $\beta(a(x))$ we ultimately arrive at $\beta(H) = \mathcal{B}$.

The strategy for $\mathbf{R_{i,j,p}}$. Since $\mathcal{B}$ is non-trivial, we can fix two different sets $A, B \in \mathcal{B}$. To meet $\mathbf{R_{i,j,p}}$, we will destroy any possible reducibility of the numbering β_{V_i} to the numbering β_{V_j} via a partial computable function φ_p, for which $V_i \subseteq \mathrm{dom}(\varphi_p)$ and $\varphi_p[V_i] \subseteq V_j$. We choose some β-index $x \in V_i \setminus V_j$ and let $\beta(x) = A$ and $\beta(\varphi_p(x)) = B$, or, conversely, $\beta(x) = B$ and $\beta(\varphi_p(x)) = A$. Note that $x \neq \varphi_p(x)$ since $x \notin V_j$ and $\varphi_p[V_i] \subseteq V_j$.

The strategy for $\mathbf{P_i}$. Fix an infinite computable set $R \subseteq H$ and a computable partition of R into disjoint infinite computable sets R_i, $i \in \omega$. Lastly, fix a computable sequence $\{r_i\}_{i\in\omega}$ of injective unary partial computable functions such that $\mathrm{dom}(r_i) = V_i \setminus R$ and $\mathrm{range}(r_i) = R_i$. For every $i \in \hat{I}$, it is sufficient to guarantee that $\beta(x) = \beta(r_i(x))$, for all $x \in V_i \setminus R$, to meet the requirement $\mathbf{P_i}$. Indeed, by Lemma 1, if so then we have $\beta_{V_i\setminus R} \leqslant \beta_{R_i}$. Since $\beta_{V_i} \equiv \beta_{V_i\setminus R} \oplus \beta_R \leqslant \beta_{R_i} \oplus \beta_R \equiv \beta_R$, it follows that $\beta_{V_i} \leqslant \beta_H$.

Unfortunately, the **P**- and **R**-strategies described above can give rise to conflicts of two types. More precisely, the strategy for $\mathbf{R_{i,j,p}}$ may want $\beta(x) \neq \beta(\varphi_p(x))$ for some $x \in V_m$ with $m \in \hat{I}$, whereas the strategy for $\mathbf{P_m}$ forces us to have $\beta(x) = \beta(r_m(x))$. Since the functions φ and r_m are fixed *a priori*, the equality $\varphi_p(x) = r_m(x)$ is quite possible and the two strategies clash in this instance.

Moreover, for any i, j, and m, there exist infinitely many numbers p such that $\varphi_p \restriction (V_i \setminus V_j) = r_m \restriction (V_i \setminus V_j)$, and it may seem impossible to prevent conflicts whatsoever. Fortunately, this is not true: the next lemma and its corollary provide us with tools to avoid almost all conflicts between the $\mathbf{P_m}$-strategy for any fixed $m \in \hat{I}$ and all **R**-strategies. For $t \in \omega$, we denote by U_t the set $\bigcup\{V_s \mid s \leqslant t \ \&\ s \in \hat{I}\}$.

LEMMA 6. Let H and I be as above, and let V' and V be arbitrary sets of the lattice ε_H such that $V' \notin I$ and $V \in I$. Then the following statements hold:

(a) $V' - V \in \varepsilon_H$;

(b) $V' - V \notin I$, and in particular, $V' \setminus V$ is an infinite set.

Proof. (a) Follows from the trivial equality

$$(V' - V)^* = (V')^* - V^*.$$

(b) Is easily verified by contradiction using the following equality:

$$V' = (V' - V) \cup (V' \cap V). \ \square$$

COROLLARY 1. For any $i, j, t \in \omega$, if $V_i - V_j \notin I$ then $V_i \setminus (V_j \cup U_t)$ is infinite.

We give absolute priority to **R**-strategies over **P**-strategies. For any i, j, and p, we exclude all conflicts between a strategy for $\mathbf{R_{i,j,p}}$ and all strategies for $\mathbf{P_m}$, with $m \in \hat{I}$ and $m \leqslant \langle i, j, p\rangle$, in the following way. We choose a β-index x to satisfy the requirement $\mathbf{R_{i,j,p}}$, not in the set $V_i \setminus V_j$ (as in the strategy for $\mathbf{R_{i,j,p}}$ above), but in the set $V_i \setminus (V_j \cup U_{\langle i,j,p\rangle})$. And we do not pay attention to the conflicts between a strategy for fixed $\mathbf{R_{i,j,p}}$ and $\mathbf{P_m}$-strategies with $m \in \hat{I}$ and $m > \langle i, j, p\rangle$. Thus, for every fixed $m \in \hat{I}$, there will be not more than finitely many conflicts with **R**-strategies.

Conflicts of the second type between $\mathbf{R_{i,j,p}}$- and $\mathbf{P_m}$-strategies may arise even in the case $\varphi_p(x) \neq r_m(x)$, due to the function a built by the **B**-strategy. Assume that the number $y \in H \setminus R$ has become a value of the function a and that $\beta(y) = \alpha(z)$ is defined for some z. If x is chosen to meet the requirement $\mathbf{R_{i,j,p}}$ then we have to set $\beta(x)$ and $\beta(\varphi_p(x))$ equal to A or to B (with $\beta(x) \neq \beta(\varphi_p(x))$). Suppose also that $\varphi_p(x) \in R$. It may so happen that $\varphi_p(x) = r_m(y)$ for some $m \in \hat{I}$. The strategy for $\mathbf{P_m}$ forces us to have $\beta(\varphi_p(x)) = \beta(r_m(y)) = \alpha(z)$. A conflict arises if either $\alpha(z) \notin \{A, B\}$ or $\alpha(z) \in \{A, B\}$ and $\beta(\varphi_p(x)) \neq \alpha(z)$. Unfortunately, we cannot determine whether $\alpha(z) = A$ or $\alpha(z) = B$ relative to the oracle $\mathbf{0}^{(n+4)}$. Again, for every $m \in \hat{I}$, we would have not more than finitely many conflicts of this type, if we restricted the choice of x by an extra condition — $m > \langle i, j, p\rangle$.

In order to describe the construction, we need some optional notions and notation. For every $x \in \overline{R}$, we consider the set $\{x\} \cup \{r_m(x) \mid m \in \omega\}$. This set is called a *star with center* x. Obviously, stars with different centers are disjoint, and the collection of all stars form a partition of ω. For every $x \in \omega$, it is easy to compute the center x^+ of the star which contains x: namely, $x^+ = x$, if $x \notin R$, and $x^+ = r_m^{-1}(x)$ if $x \in R$ and R_m is the element of the partition $\{R_i\}_{i\in\omega}$ which contains x.

In terms of stars, our plan of preventing conflicts between **P**- and **R**-strategies, mentioned above, can be described as follows. First note that, given x, conflicts of the first type do not arise whatsoever, provided that $x \neq (\varphi_p(x))^+$, and also $x = (\varphi_p(x))^+$ whenever $\varphi_p(x) \in R_m$, $m \notin \hat{I}$. For every fixed m, we allow only finitely many strategies $\mathbf{R_{i,j,p}}$ to injure $\mathbf{P_m}$; that is, $\beta(\varphi_p(x)) \neq \beta(x)$ while $\beta(r_m(x)) \neq \beta(x)$. The stars with centers in range(a) are the sole source of conflicts of the second type. And, for every fixed m, we allow only finitely many such stars to injure $\mathbf{P_m}$.

Now we proceed to a stage-by-stage construction of a numbering β of the family $\mathcal{B}$, and an auxiliary function a. If a value $\beta(x)$, or $a(x)$, has not been explicitly modified by the end of stage $t+1$ then by default $\beta^{t+1}(x) = \beta^t(x)$ or $a^{t+1}(x) = a^t(x)$, respectively. Note that in the construction below, the functions β and a will never become undefined if will their values have been defined before.

Construction

Stage 0. Let $\beta(x)$ and $a(x)$ be undefined for all x. Go to the next stage.

Stage $t+1$. Let $t = \langle i, j, p\rangle$. We carry out three procedures, starting from $\mathbf{R_{i,j,p}}$.

Procedure $\mathbf{R_{i,j,p}}$. We verify the following conditions (which we can do relative to the oracle $\mathbf{0}^{(n+4)}$):

(i) $V_i - V_j \notin I$;

(ii) $V_i \subseteq \mathrm{dom}(\varphi_p)$ and $\varphi_p[V_i] \subseteq V_j$.

If one of (i), (ii) fails then go to Procedure $\mathbf{P_i}$. Otherwise choose the least element x of the set

$$X^t \risingdotseq \{x \mid x \in V_i \setminus (V_j \cup U_t) \ \&\ \beta^t(x)\uparrow \ \&\ \beta^t(\varphi_p(x))\uparrow\}$$

such that at least one of the following holds:

(iii) $\exists y(y \neq x \ \&\ y \in X^t \ \&\ \varphi_p(y) = \varphi_p(x))$;

(iv) $(\varphi_p(x))^+ \in \mathrm{range}(a^t) \Rightarrow \forall m(\varphi_p(x) \in R_m \Rightarrow m > t)$.

(That such x exists will be shown in item (7) below.)

If (iii) holds then we pick the least y satisfying (iii) and put $\beta(x) = A$, $\beta(y) = B$. Go to Procedure P_i.

If (iii) does not hold (but (iv) does) then we denote by z the center $(\varphi_p(x))^+$ and carry out the instructions specified in the two cases below. Go to Procedure P_i.

Case 1. Let $\varphi_p(x) \in R$, $z \neq x$, and $z \notin \mathrm{range(a^t)}$. If $\beta^t(z)\uparrow$ then put $\beta(z) = A$. Denote by C the set $\beta(z)$, and by D an element of the set $\{A, B\} \setminus \{C\}$. Put $\beta(x) = D$ and $\beta(\varphi_p(x)) = C$.

Case 2. Case 1 fails. We let $\beta(x) = A$ and $\beta(\varphi_p(x)) = B$.

Procedure $\mathbf{P_i}$. Choose the least number $x \in R_i$ such that $\beta(x)$ is still not defined. If $\beta(x^+)$ is also undefined then let $\beta(x^+) = \beta(x) = A$. If $\beta(x^+)\downarrow$ then put $\beta(x) = \beta(x^+)$. Go to Procedure **B**.

Procedure **B**. Choose the least number $y \in H \setminus R$ such that β is still not defined at all nodes of the star with center y, and put $a(t) = y$ and $\beta(y) = \alpha(t)$. Go to the next stage.

Obviously, β is a Σ^0_{n+5}-computable numbering.

Properties of the construction. (1) For every t, functions β^t and a^t are defined only on finite sets.

(2) For every x, there exists a stage t starting with which $\beta(x)$ becomes defined forever. Furthermore, after this stage t, the function β will never change its value in x.

(3) a Is a total function with range$(a) \subseteq H \setminus R$. For every x, $\alpha(x) = \beta(a(x))$.

(4) For every x, at least one of the following holds: $\beta(x) = A$, $\beta(x) = B$, or $\beta(x) = \alpha(y)$ for some y.

(5) For every x and t, if $\beta^t(x)\downarrow$ then $\beta^t(x^+)\downarrow$.

(6) For every x and t, if $x \in$ range(a^t) then $\beta^t(x)\downarrow$.

Properties (1)-(6) are evident, and (2)-(4) imply that β is a numbering for the family $\mathcal{B}$.

(7) For every i, j, p, t, if the conditions (i) and (ii) above hold then there exists $x \in X^t$ satisfying (iii) or (iv).

To prove this, we choose any numbers i, j, p, t for which (i) and (ii) are satisfied, and assume that both (iii) and (iv) fail. Then φ_p is injective on X^t, and

$$(\forall x \in X\,\forall^t)(\varphi_p(x))^+ \in \text{range}(a^t) \;\&\; \exists m(\varphi_p(x) \in R_m \;\&\; m \leqslant t).$$

The set X^t is infinite by Lemma 6 and property (1). By the same property, the set $Y \leftrightharpoons \{y \mid r_m^{-1}(y) \in \text{range}(a^t) \;\&\; m \leqslant t\}$ is finite since all mappings r_k, $k \in \omega$, are injective. Therefore φ_p maps the infinite set X^t into the finite set Y in a one-to-one manner. Contradiction.

(8) For every i, j, p, if (i) and (ii) are satisfied then there exists $z \in V_i$ such that $\beta(z) \neq \beta(\varphi_p(z))$.

Let $V_i - V_j \notin I$, $V_i \subseteq \text{dom}(\varphi_p)$, $\varphi_p[V_i] \subseteq V_j$, and $t = \langle i, j, p\rangle$. If (iii) holds at stage $t+1$ then φ_p maps two different numbers $x, y \in V_i$ into the same number $\varphi_p(x) = \varphi_p(y)$, and by construction, we have $\beta(x) \neq \beta(y)$. Therefore at least one of the inequalities $\beta(x) \neq \beta(\varphi_p(x))$ and $\beta(y) \neq \beta(\varphi_p(y))$ holds.

Suppose now that (iii) fails at stage $t+1$ and let $x \in X^t$ be a number chosen by Procedure $\mathbf{R_{i,j,p}}$ at this stage. By construction, $x \in V_i \setminus (V_j \cup U_t)$ and both $\beta^t(x)$ and $\beta^t(\varphi_p(x))$ are undefined. This implies that $x \notin V_m$ and $r_m(x)$ is undefined for all $m \in \hat{I}$, $m \leqslant t$. Note that $\varphi_p(x) \notin \text{range}(a^t)$ in view of property (6). We consider the following four possibilities:

(a) $\varphi_p(x) \notin R$;

(b) $\varphi_p(x) \in R$, $(\varphi_p(x))^+ = x$;

(c) $\varphi_p(x) \in R$, $(\varphi_p(x))^+ \neq x$, and $(\varphi_p(x))^+ \in \text{range}(a^t)$;

(d) $\varphi_p(x) \in R$, $(\varphi_p(x))^+ \neq x$, and $(\varphi_p(x))^+ \notin \text{range}(a^t)$.

For (a), (b), and (c), we have $\beta(x) = A$ and $\beta(\varphi_p(x)) = B$ by Case 2 of Procedure $\mathbf{R_{i,j,p}}$ at stage $t+1$. For (d), by Case 1 we obtain $\beta(x) = D$ and $\beta(\varphi_p(x)) = C$, where $\{C, D\} = \{A, B\}$.

(9) For every $m \in \hat{I}$ and for almost all $x \in V_m \setminus R$, $\beta(x) = \beta(r_m(x))$.

Let $m \in \hat{I}$. We suppose that $v \in V_m \setminus R$ and $\beta(v) \neq \beta(r_m(v))$ and prove that this inequality is caused by a conflict between the strategy for $\mathbf{P_m}$ and the $\mathbf{R_{i,j,p}}$-strategy for some i, j, p.

Let $s+1$ and $t+1$ be stages at which, respectively, $\beta(v)$ and $\beta(r_m(v))$ are defined. By property (2), $\beta(v) = \beta^{s+1}(v)$ and $\beta(r_m(v)) = \beta^{t+1}(r_m(v))$. Consider a star with center v and denote by $q+1$ the least stage at which β is defined at a node of this star for the first time. By property (5), $q = s$ and $s \leqslant t$ since $(r_m(v))^+ = v$. We consider the cases $s = t$ and $s < t$ separately.

Let $s = t$. This means that β is defined exactly at two nodes of the star with center v by the end of stage $t+1$, and that one of these nodes is v. Therefore $\beta(v)$ cannot be defined at stage $t+1$ in Procedure $\mathbf{B}$, and (iii) does not hold at stage $t+1$. Thus we have to examine the possibilities (a)-(d) specified in the proof of property (8).

Obviously, $\beta(v)$ and $\beta(r_m(v))$ cannot be defined by (a) and (c) since in these cases β is determined from the values in the centers of two disjoint stars. Case (d) is also impossible, for in this instance the values of β at the nodes v and $r_m(v)$ have to be identical by construction. We are therefore left with the only possibility — (b). We adopt the notation used in the description of the procedure for $\mathbf{R_{i,j,p}}$ and in the proof of property (8). In this notation, $v = x$, $\varphi_p(x) = r_m(x)$, and $t = \langle i, j, p\rangle$. We have a conflict of the first type: $\beta(x) = A$ and $\beta(\varphi_p(x)) = \beta(r_m(x)) = B$. Moreover, $x \in V_i \setminus (V_j \cup U_t)$, and so $m > \langle i, j, p\rangle$. This implies that we can have not more than a finite number of β-indices $v \in V_m \setminus R$ for which the inequality $\beta(v) \neq \beta(\varphi_p(v))$ is caused by conflicts of the first type.

Now let $s < t$. In view of property (5), we should handle stage $t+1$ at which β is defined at node $r_m(v)$ provided that β has been defined in the center v before. As with $s = t$, it is easy to see that, at stage $t+1$, $\beta(r_m(v))$ cannot be defined via Procedure **B**, (iii) does not hold, and the inequality $\beta(v) \neq \beta(\varphi_p(v))$ cannot be caused by (a), (b), and (d).

With the above conventions on the notation, we handle possibility (c). In this case $\varphi_p(x) = r_m(x)$, $(\varphi_p(x))^+ = v$, $v \in \operatorname{range}(a^t)$, and hence the inequality $m > \langle i, j, p\rangle$ is a consequence of (iv). We have $\beta(x) = A$, $\beta(r_m(x)) = B$, and $\beta(v) \neq B$. Thus there are not more than finitely many β-indices $v \in V_m \setminus R$ for which the inequality $\beta(v) \neq \beta(\varphi_p(v))$ is caused by conflicts of the second type.

Properties (3), (8), and (9) imply that all the requirements are met.

THEOREM 2. For every n, every non-trivial Σ^0_{n+4}-computable family $\mathcal{B}$, and every Σ^0_{n+1}-computable family $\mathcal{A}$, the Rogers semilattices $\mathcal{R}^0_{n+4}(\mathcal{B})$ and $\mathcal{R}^0_{n+1}(\mathcal{A})$ are not isomorphic.

Proof. Let $\mathcal{B}$ be any Σ^0_{n+4}-computable family, and let $\alpha \in \mathrm{Com}^0_{n+4}(\mathcal{B})$. Instead of working directly with the relation "$V_i - V_j \in I$" (in i, j) and with the set $\hat{I}$, we use their enumerations relative to the oracle $\mathbf{0}^{(n+3)}$. The relation and set are $\mathbf{0}^{(n+3)}$-c.e. by Lemma 5.

Denote the relation "$V_i - V_j \in I$" by $Q(i,j)$. Let $Q^t(i,j)$, $t \in \omega$, be its approximation, that is, $Q^t(i,j)$ is a $\mathbf{0}^{(n+3)}$-computable relation in i, j, t;

$Q^t(i,j) \to Q^{t+1}(i,j)$ for any i, j, and t;

$Q(i,j) \leftrightarrow \exists s\, \forall t \geqslant s Q^t(i,j)$ for any i and j.

Let $\hat{I}^t$, $t \in \omega$, be an enumeration of the set $\hat{I}$ relative to $\mathbf{0}^{(n+3)}$, that is, $\hat{I}^t \subseteq \hat{I}^{t+1}$, $t \in \omega$, and $\hat{I} = \bigcup\limits_{t \in \omega} \hat{I}^t$, and let $\hat{U}_t$ and $\hat{X}^t$ be approximations of the sets U_t and X^t, and namely,

$$\hat{U}_t = \bigcup \{V_k \mid k \leqslant t \ \&\ k \in \hat{I}^t\},$$

$$\hat{X}^t = \{x \mid x \in V_i \setminus (V_j \cup \hat{U}^t) \ \&\ \beta^t(x)\uparrow \ \&\ \beta^t(\varphi_p(x))\uparrow\} \text{ for } t = \langle i, j, p\rangle.$$

Hereinafter, we use the notation adopted in the proof of Theorem 1. The construction below is an ample approximation of the construction of Theorem 1.

Construction

Stage 0. Let $\beta(x)$ and $a(x)$ be undefined for all x. Go to the next stage.

Stage $t+1$. Let $t = \langle i, j, p\rangle$. We carry out the following three procedures, starting with Procedure $\mathbf{R_{i,j,p}}$. (All the instructions of the stage are performed relative to the oracle $\mathbf{0}^{(n+3)}$.)

Procedure $\mathbf{R_{i,j,p}}$. Check whether the following condition holds:

(ii) $V_i \subseteq \operatorname{dom}(\varphi_p)$ and $\varphi_p[V_i] \subseteq V_j$.

If (ii) does not hold then go to Procedure $\mathbf{P_i}$. Otherwise search for the least $s > t$ such that $Q^s(i,j)$ or there exists $x \in \hat{X}^s$ for which at least one of the following two conditions is satisfied:

(iii)′ $\exists y (y \neq x \ \&\ y \in \hat{X}^s \ \&\ \varphi_p(y) = \varphi_p(x))$;

(iv) $(\varphi_p(x))^+ \in \operatorname{range}(a^t) \Rightarrow \forall m (\varphi_p(x) \in R_m \Rightarrow m > t)$.

If $Q^s(i,j)$ then go to Procedure $\mathbf{P_i}$. Otherwise choose the least $x \in \hat{X}^s$ which meets the requirements above. If (iii)′ holds then pick the least y satisfying (iii)′ and put $\beta(x) = A$, $\beta(y) = B$. Go to Procedure P_i.

If (iii)$'$ does not hold (but (iv) does) then denote by z the center $(\varphi_p(x))^+$ and carry out the instructions of the next two cases, and then go to Procedure P_i.

Case 1. We have $\varphi_p(x) \in R$, $z \neq x$, and $z \notin \mathrm{range}(\mathrm{a}^{\mathrm{t}})$. If $\beta^t(z)\uparrow$ then let $\beta(z) = A$. Denote by C the set $\beta(z)$, and by D an element of the set $\{A, B\} \setminus \{C\}$. Now let $\beta(x) = D$ and $\beta(\varphi_p(x)) = C$.

Case 2. Case 1 fails. Let $\beta(x) = A$ and $\beta(\varphi_p(x)) = B$.

Procedure $\mathbf{P_i}$. Choose the least number $x \in R_i$ such that $\beta(x)$ is still not defined. If $\beta(x^+)$ is also undefined then let $\beta(x^+) = \beta(x) = A$. If $\beta(x^+)\downarrow$ then put $\beta(x) = \beta(x^+)$. Go to Procedure **B**.

Procedure **B**. Choose the least number $y \in H \setminus R$ such that β is still not defined at all nodes of the star with center y, and put $a(t) = y$ and $\beta(y) = \alpha(t)$. Go to the next stage.

Properties of the construction. Obviously, β is a Σ^0_{n+4}-computable numbering. It is easy to see that the construction is a slight modification of the construction of Theorem 1. For this reason, we omit unnecessary repetitions in proving the properties of the modified construction. Properties (1)-(6) are exactly the same as in the proof of Theorem 1. These ensure that β is a Σ^0_{n+4}-computable numbering of the family $\mathcal{B}$. Consider the remaining properties.

(7) For any i, j, p, if condition (ii) holds at stage $t + 1$ with $t = \langle i, j, p\rangle$ then there exists $s > t$ such that $Q^s(i, j)$, or there exists $x \in \hat{X}^s$ satisfying at least one of (iii)$'$, (iv).

If $V_i - V_j \in I$ then there evidently exists s such that $Q^s(i, j)$. Suppose now that $V_i - V_j \notin I$ and let s be the least number such that $s > t$ and

$$\{k \mid k \leqslant \langle i, j, p\rangle \ \&\ k \in \hat{I}\} = \{k \mid k \leqslant \langle i, j, p\rangle \ \&\ k \in \hat{I}^s\}.$$

Then $\hat{U}_s = U_t$, $\hat{X}^s = X^t$, and hence the condition (iii)$'$ is identical to (iii). Now, therefore, we can argue as we did in the proof of property (7) of the previous construction.

(8) For any i, j, p, if $V_i - V_j \notin I$ and (ii) holds then there exists $z \in V_i$ such that $\beta(z) \neq \beta(\varphi_p(z))$.

Let $V_i - V_j \notin I$. Keeping in mind that $Q^s(i, j)$ fails for all s, we repeat the proof for property (8) in Theorem 1.

(9) For every $m \in \hat{I}$ and almost all $x \in V_m \setminus R$, $\beta(x) = \beta(r_m(x))$.

Let $m \in \hat{I}$ and let t_0 be the least number such that $m \in \hat{I}^{t_0}$. Then $V_m \subseteq \hat{U}_s$ for all $s \geqslant t_0$, and hence $V_m \cap \hat{X}^t = \varnothing$ for all $t \geqslant t_0$. Repeating the argument used in the proof of property (9) of Theorem 1, we can show that for every $t > \max\{t_0, m\}$ there is no $v \in V_m \setminus R$ such that:

(a) $\beta(v) \neq \beta(r_m(v))$;

(b) at least one of the values $\beta(v)$ or $\beta(r_m(v))$ is defined at stage t;

(c) both $\beta(v)$ and $\beta(r_m(v))$ are defined by the end of stage t.

Property (1) implies that the number of v's which satisfy the three conditions above at stages $t \leqslant \max\{t_0, m\}$ is finite.

Again, properties (3), (8), and (9) imply that all the requirements are met. □

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