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THE PROPERTY OF BEING ALGEBRAIC FOR THE LATTICE OF ALL τ-CLOSED TOTALLY SATURATED FORMATIONS

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It is proved that the lattice of all τ-closed totally saturated formations of finite groups is algebraic.

In the paper we consider finite groups only. Information on all relevant terms is contained in [1-4].

The methods and constructions of general lattice theory used in exploring the inner structure of formations allow of creating simpler schemes of proving both the known facts and new results in formation theory (see [2-5]). General properties of the lattice of totally saturated formations as well as the structure of such formations with prescribed restrictions on lattices of totally saturated subformations are dealt with in [6-13].

At the same time the lattice of totally saturated formations is now one of the least known lattices of group formations. A number of open questions posed in [2-5, 14] testify to this judgement.

In [3], it was proved that for any non-negative integer n, both the lattice of all τ-closed n-multiply saturated formations and the lattice of all soluble totally saturated ones are algebraic. Moreover, the question was posed whether the lattice l_∞^τ of all τ-closed totally saturated formations is algebraic (see [3, Question 4.4.6]). In the present paper, this question is answered in the affirmative.

We recall some of the notation and definitions. A non-empty system θ of formations is called a *complete lattice of formations* if the intersection of any family of formations in θ is again in θ and the set θ contains a formation $\mathfrak{F}$ such that $\mathfrak{H} \subseteq \mathfrak{F}$, for any formation $\mathfrak{H} \in \theta$. The formations in θ are referred to as *θ-formations.*

Let A and B be groups, $\varphi : A \to B$ be an epimorphism, and Ω and Σ be some systems of subgroups in A and B, respectively. We denote by Ω^φ the set $\{H^\varphi \mid H \in \Omega\}$, and by $\Sigma^{\varphi^{-1}}$ the set $\{H^{\varphi^{-1}} \mid H \in \Sigma\}$ of all preimages in A of all groups of Σ. Let $\mathfrak{X}$ be any non-empty class of groups, and let $G \in \mathfrak{X}$ be associated with some system $\tau(G)$ of its subgroups. Following [3] we say that τ is a *subgroup $\mathfrak{X}$-functor* (or else τ is a *subgroup functor on* $\mathfrak{X}$) if $(\tau(A))^\varphi \subseteq \tau(B)$ and $(\tau(B))^{\varphi^{-1}} \subseteq \tau(A)$ for every epimorphism $\varphi : A \to B$, where $A, B \in \mathfrak{X}$; moreover, $G \in \tau(G)$ for any group $G \in \mathfrak{X}$.

The class $\mathfrak{F}$ of groups is said to be *τ-closed* if $\tau(G) \subseteq \mathfrak{F}$, for any group $G \in \mathfrak{F}$. Every formation of finite groups is said to be 0-*multiply saturated.* For $n \geqslant 1$, the formation $\mathfrak{F}$ is *n-multiply saturated* if it has a local screen such that all non-empty values of the screen are $(n-1)$-multiply saturated formations. A formation that is n-multiply saturated for any non-negative integer n is referred to as *totally saturated.* If, in addition, $\mathfrak{F}$ is *τ-closed* then we call $\mathfrak{F}$ a *τ-closed n-multiply saturated*, and accordingly, a *τ-closed totally saturated*, formation.

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Let $\mathfrak{X}$ be some set of groups. By $l_\infty^\tau \mathrm{form}\, \mathfrak{X}$ we denote a τ-closed totally saturated formation generated by the class $\mathfrak{X}$ of groups — that is, the intersection of all τ-closed totally saturated formations containing $\mathfrak{X}$. If, moreover, $\mathfrak{X} = \{G\}$ then $l_\infty^\tau \mathrm{form}\, G$ is conceived of as a *one-generated τ-closed totally saturated* formation. For any $\mathfrak{M}$ and $\mathfrak{H}$ in l_∞^τ, we put $\mathfrak{M} \vee_\infty^\tau \mathfrak{H} = l_\infty^\tau \mathrm{form}\, (\mathfrak{M} \cup \mathfrak{H})$. The set of all τ-closed totally saturated formations l_∞^τ which is partially ordered w.r.t. inclusion $\subseteq$ forms a complete lattice.

A screen all of whose non-empty values are l_∞^τ-formations is said to be l_∞^τ*-valued.* Let $\{f_i \mid i \in I\}$ be some system of l_∞^τ-valued screens. Then $\vee_\infty^\tau(f_i \mid i \in I)$ denotes a screen f such that $f(p) = l_\infty^\tau \mathrm{form} \left(\bigcup_{i \in I} f_i(p) \right)$, if at least one of the formations $f_i(p)$ is not $\varnothing$. Otherwise we put $f(p) = \varnothing$.

Recall that an element a of a lattice L is *compact* if $a \le \vee(x_j \mid j \in F)$ holds for $a \le \vee(x_j \mid j \in J)$ and some finite subset $F \subset J$ [15]. A lattice L is *algebraic* if each element $a \in L$ is a union of compact elements of the lattice L. Below, for every non-empty set π of primes, by $\mathfrak{N}_\pi$ and $\mathfrak{S}_\pi$ we denote the respective classes of all nilpotent and all soluble π-groups.

In order to prove our main result, we need some of the well-known facts from the formation theory of groups.

LEMMA 1 [3]. Let A be a monolithic group with non-Abelian monolith, $\mathfrak{M}$ be some τ-closed semiformation, and $A \in l_n^\tau \mathrm{form}\, \mathfrak{M}$. Then $A \in \mathfrak{M}$.

LEMMA 2 [2]. Let f be the local screen of a formation $\mathfrak{F}$ and G be a finite group. If there is a prime p for which $G/O_p(G) \in f(p) \cap \mathfrak{F}$ then the group G belongs to the formation $\mathfrak{F}$.

LEMMA 3 [2]. Let f_1 be the local screen of $\mathfrak{F}$ and let $\mathfrak{H}$ be a non-empty formation such that $\pi(\mathfrak{H}) \subseteq \pi(\mathfrak{F})$. Then the formation $\mathfrak{F}\mathfrak{H}$ has the local screen f such that for every prime p, the following statements hold:

(1) $f(p) = f_1(p)\mathfrak{H}$ if $p \in \pi(\mathfrak{F})$;

(2) $f(p) = \varnothing$ if $p \notin \pi(\mathfrak{F})$.

LEMMA 4 (see 3, [Lemma 4.1.2]). Let $\mathfrak{F}_i$ be a τ-closed totally saturated formation and f_i be a minimal l_∞^τ-valued local screen for $\mathfrak{F}_i$ $(i \in I)$. Then $\vee_\infty^\tau(f_i \mid i \in I)$ is the minimal l_∞^τ-valued local screen of $\vee_\infty^\tau(\mathfrak{F}_i \mid i \in I)$.

LEMMA 5 [13]. Let $\mathfrak{F}$ be a non-empty τ-closed formation and π be the set of primes such that $\pi(\mathfrak{F}) \subseteq \pi$. Then the product $\mathfrak{S}_\pi \mathfrak{F}$ is a τ-closed totally saturated formation.

LEMMA 6. Let $\mathfrak{H} = l_\infty^\tau \mathrm{form} \left(\bigcup_{i \in I} \mathfrak{F}_i \right)$, where $\mathfrak{F}_i$ $(i \in I)$ is a τ-closed totally saturated formation, and let $A \in \mathfrak{H}$ be a monolithic group. Then $A \in \bigcup_{i \in I} \mathfrak{F}_i$ if $\mathrm{Soc}(A)$ is a non-Abelian group.

Proof. Let A be as in the assumption, with $\pi = \pi(\mathfrak{H})$. By Lemma 5, $\mathfrak{H} = l_\infty^\tau \mathrm{form} \left(\bigcup_{i \in I} \mathfrak{F}_i \right) \subseteq \mathfrak{S}_\pi \tau \mathrm{form} \left(\bigcup_{i \in I} \mathfrak{F}_i \right)$. Hence $A \in \mathfrak{S}_\pi \tau \mathrm{form} \left(\bigcup_{i \in I} \mathfrak{F}_i \right)$. Since $\mathrm{Soc}(A)$ is non-Abelian, $A \in \tau \mathrm{form} \left(\bigcup_{i \in I} \mathfrak{F}_i \right)$. By Lemma 1, $A \in \bigcup_{i \in I} \mathfrak{F}_i$. The proof is complete.

THEOREM. The lattice l_∞^τ of all τ-closed totally saturated formations is algebraic.

Proof. First, we prove that for any group A, a one-generated τ-closed totally saturated formation $\mathfrak{F} = l_\infty^\tau \mathrm{form}\, A$ is a compact element of the lattice l_∞^τ. The proof is by induction on the order of A.

Let A be the counterexample of minimal order, that is,

$$\mathfrak{F} = l_\infty^\tau \mathrm{form}\, A \subseteq \mathfrak{H} = l_\infty^\tau \mathrm{form} \left(\bigcup_{i \in I} \mathfrak{F}_i \right),$$

where $\mathfrak{F}_i$ $(i \in I)$ is an l_∞^τ-formation. We claim that A is a monolithic group. Suppose the contrary. Let N_1 and N_2 be distinct minimal normal subgroups of A. Put $\mathfrak{L} = l_\infty^\tau \mathrm{form}\,(A/N_1)$ and $\mathfrak{M} = l_\infty^\tau \mathrm{form}\,(A/N_2)$.

By induction, the statement is true for the groups A/N_1 and A/N_2. Since

$$\mathfrak{L} = l_\infty^\tau \mathrm{form}\,(A/N_1) \subseteq \mathfrak{H} = l_\infty^\tau \mathrm{form} \left(\bigcup_{i \in I} \mathfrak{F}_i \right),$$

$$\mathfrak{M} = l_\infty^\tau \mathrm{form}\,(A/N_2) \subseteq \mathfrak{H} = l_\infty^\tau \mathrm{form} \left(\bigcup_{i \in I} \mathfrak{F}_i \right),$$

there are tuples of indices $i_1, \ldots, i_k$ and $j_1, \ldots, j_n$ such that

$$\mathfrak{L} \subseteq l_\infty^\tau \mathrm{form} \left(\mathfrak{F}_{i_1} \bigcup \ldots \bigcup \mathfrak{F}_{i_k} \right),$$

$$\mathfrak{M} \subseteq l_\infty^\tau \mathrm{form} \left(\mathfrak{F}_{j_1} \bigcup \ldots \bigcup \mathfrak{F}_{j_n} \right).$$

Hence

$$\mathfrak{F} = \mathfrak{L} \vee_\infty^\tau \mathfrak{M} \subseteq l_\infty^\tau \mathrm{form} \left(\mathfrak{F}_{i_1} \bigcup \ldots \bigcup \mathfrak{F}_{i_k} \bigcup \mathfrak{F}_{j_1} \bigcup \ldots \bigcup \mathfrak{F}_{j_n} \right).$$

Contradiction. Therefore A is a monolithic group.

Let $P = \mathrm{Soc}(A)$. Assume that P is a non-Abelian group. Since $A \in l_\infty^\tau \mathrm{form} \left(\bigcup\limits_{i \in I} \mathfrak{F}_i \right)$, $A \in \bigcup\limits_{i \in I} \mathfrak{F}_i$ by Lemma 6. Therefore there is an index $i \in I$ such that $A \in \mathfrak{F}_i$. Contradiction.

Hence P is an Abelian p-group. Since $l_\infty^\tau \mathrm{form}\,(A/\Phi(A)) = l_\infty^\tau \mathrm{form}\, A$, using induction we obtain $P \nsubseteq \Phi(A)$. Therefore $P = F_p(A) = O_p(A)$ and $A = [P]B$ for some maximal subgroup B of A. Let f_i, f, and h be minimal l_∞^τ-valued local screens for the formations $\mathfrak{F}_i$, $\mathfrak{F}$, and $\mathfrak{H}$, respectively. By Lemma 4, $h = \vee_\infty^\tau (f_i \mid i \in I)$. Since $P = F_p(A)$ and $A \in \mathfrak{H}$, we have

$$B \simeq A/F_p(A) \in h(p) = \vee_\infty^\tau (f_i(p) \mid i \in I).$$

Keeping in mind that $|B| < |A|$ and using induction, we find an index set $J = \{j_1, \ldots, j_t\}$, for which

$$B \simeq A/F_p(A) \in \vee_\infty^\tau (f_j(p) \mid j \in J).$$

By Lemma 4, $l = \vee_\infty^\tau (f_j \mid j \in J)$ is a minimal l_∞^τ-valued local screen of the formation $\mathfrak{L} = \vee_\infty^\tau (\mathfrak{F}_j \mid j \in J)$. Hence

$$A/O_p(A) \simeq B \in l(p) = \vee_\infty^\tau (f_j(p) \mid j \in J).$$

By Lemma 2, A belongs to $\mathfrak{L}$. Consequently $\mathfrak{F} = l_\infty^\tau \mathrm{form}\, A \subseteq \mathfrak{L} = l_\infty^\tau \mathrm{form} \left(\bigcup\limits_{j \in J} \mathfrak{F}_j \right)$. Contradiction. We have thus shown that our assumption is invalid, and so $\mathfrak{F}$ is a compact element of the lattice l_∞^τ. Since every l_∞^τ-formation is obviously a union of its one-generated l_∞^τ-subformations in l_∞^τ, the lattice l_∞^τ is algebraic. The theorem is proved.

Recall that the subgroup $\mathfrak{X}$-functor τ is *trivial* if $\tau(G) = \{G\}$ for any group $G \in \mathfrak{X}$.

Applying the theorem to the case where τ is a trivial subgroup functor, we derive the following:

COROLLARY. The lattice of all totally saturated formations is algebraic.

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