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LATTICES OF DOMINIONS IN QUASIVARIETIES OF ABELIAN GROUPS

S. A. Shakhova

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Let $\mathcal{M}$ be any quasivariety of Abelian groups, $\mathrm{dom}_G^{\mathcal{M}}(H)$ be the dominion of a subgroup H of a group G in $\mathcal{M}$, and $L_q(\mathcal{M})$ be the lattice of subquasivarieties of $\mathcal{M}$. It is proved that $\mathrm{dom}_G^{\mathcal{M}}(H)$ coincides with a least normal subgroup of the group G containing H, the factor group with respect to which is in $\mathcal{M}$. Conditions are specified subject to which the set $L(G,H,\mathcal{M}) = \{\mathrm{dom}_G^{\mathcal{N}}(H) \mid \mathcal{N} \in L_q(\mathcal{M})\}$ forms a lattice under set-theoretic inclusion and the map $\varphi : L_q(\mathcal{M}) \to L(G,H,\mathcal{M})$ such that $\varphi(\mathcal{N}) = \mathrm{dom}_G^{\mathcal{N}}(H)$ for any quasivariety $\mathcal{N} \in L_q(\mathcal{M})$ is an antihomomorphism of the lattice $L_q(\mathcal{M})$ onto the lattice $L(G,H,\mathcal{M})$.

INTRODUCTION

The notion of a dominion was introduced in [1] for studying epimorphisms. A *dominion of a subalgebra* H of a universal algebra A in the full category $\mathcal{M}$ ($A \in \mathcal{M}$), denoted $\mathrm{dom}_A^{\mathcal{M}}(H)$, is a set of elements $a \in A$ such that $\varphi(a) = \psi(a)$ for any two morphisms $\varphi, \psi : A \to M$ ($M \in \mathcal{M}$), which coincide on H. It is not hard to see that $\varphi : A \to B$ ($A, B \in \mathcal{M}$) is an epimorphism in $\mathcal{M}$ iff $\mathrm{dom}_B^{\mathcal{M}}(\varphi(A)) = B$.

The notion of a dominion is closely related to the concept of an amalgam (see [2]). An *amalgam* $[A, B; H]$ is a pair of universal algebras A and B with a common subalgebra H. An amalgam $[A, B; H]$ is said to be *special* if there exists an isomorphism between the universal algebras A and B, keeping the elements of H fixed. If, for the special amalgam $[A, B; H]$ in $\mathcal{M}$, there exists a free amalgamated product, denoted $A *_H^{\mathcal{M}} B$, that is, there are canonical injective morphisms $\lambda : A \to A *_H^{\mathcal{M}} B$ and $\rho : B \to A *_H^{\mathcal{M}} B$, with A and B identified with $\lambda(A)$ and $\rho(B)$, respectively, then $\mathrm{dom}_A^{\mathcal{M}}(H) = \lambda(A) \cap \rho(B)$ (see [2, 3]).

The dominions were dealt with in different classes of universal algebras [3-5]. Among axiomatizable classes, however, only quasivarieties were found to enjoy a complete theory of defining relations, which allows of determining a free amalgamated product in these, given any amalgam [6; see also 7]. This was an important argument for launching a study into dominions in quasivarieties of universal algebras, undertaken in [8]. There, the concept of a dominion is extended to the case $A \notin \mathcal{M}$, which turns out useful in dealing with dominions in quasivarieties. There arose a possibility to bring under consideration the set $L(A, H, \mathcal{M}) = \{\mathrm{dom}_A^{\mathcal{N}}(H) \mid \mathcal{N} \in L_q(\mathcal{M})\}$, where $L_q(\mathcal{M})$ is the lattice of subquasivarieties of a quasivariety $\mathcal{M}$. Also, in [8], conditions were specified under which $L(A, H, \mathcal{M})$ forms a lattice under set-theoretic inclusion, and the problem was posed as to the interplay between the lattices $L_q(\mathcal{M})$ and $L(A, H, \mathcal{M})$. In particular, a question was dubbed asking which conditions are necessary for the map $\varphi : L_q(\mathcal{M}) \to L(A, H, \mathcal{M})$, under which $\varphi(\mathcal{N}) = \mathrm{dom}_A^{\mathcal{N}}(H)$ for any $\mathcal{N} \in L_q(\mathcal{M})$, is an antihomomorphism of $L_q(\mathcal{M})$ onto $L(A, H, \mathcal{M})$. For the

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most part of the present paper, we work to resolve this problem for the case of an arbitrary quasivariety of Abelian groups.

We prove that the dominion of a subgroup H of G in an arbitrary quasivariety $\mathcal{M}$ of Abelian groups coincides with a least normal subgroup of the group G containing H, the factor group w.r.t. which belongs to $\mathcal{M}$. It is also stated that if $G/\mathrm{dom}_G^{\mathcal{M}}(H)$ is a finitely generated group then the set $L(G,H,\mathcal{M})$ forms a complete lattice under set-theoretic inclusion. Finally, we specify necessary and sufficient conditions under which $\varphi : L_q(\mathcal{M}) \to L(G,H,\mathcal{M})$, provided that $\varphi(\mathcal{N}) = \mathrm{dom}_G^{\mathcal{N}}(H)$ for any quasivariety $\mathcal{N} \in L_q(\mathcal{M})$, is an antihomomorphism of $L_q(\mathcal{M})$ onto $L(G,H,\mathcal{M})$.

1. PRELIMINARIES

Let $\mathcal{M}$ be a quasivariety of groups, G a group, and H a subgroup of G. Following [8], the *dominion of the subgroup* H of the group G in the quasivariety $\mathcal{M}$ is defined thus:

$$\mathrm{dom}_G^{\mathcal{M}}(H) = \{g \in G \mid \forall M \in \mathcal{M}\, \forall \varphi, \psi : G \to M \ \text{ if } \varphi|_H = \psi|_H \text{ then } \varphi(g) = \psi(g)\},$$

where $\varphi, \psi : G \to M$ are homomorphisms of the group G into the group M; $\varphi|_H$, $\psi|_H$ is the restriction of φ, ψ to H.

Obviously, a dominion is a subgroup of the group G containing H. Moreover, if $\mathcal{M}$ is an arbitrary quasivariety of Abelian groups then $\mathrm{dom}_G^{\mathcal{M}}(H)$ is a normal subgroup containing a derived subgroup of G. Also it is not hard to see that for arbitrary quasivarieties $\mathcal{M}$ and $\mathcal{N}$, $\mathcal{N} \subseteq \mathcal{M}$ implies $\mathrm{dom}_G^{\mathcal{M}}(H) \subseteq \mathrm{dom}_G^{\mathcal{N}}(H)$.

In the paper we adopt the following notation:

$\mathbf{N}$ is the set of natural numbers;

(n, r) is the greatest common divisor of numbers $n, r \in \mathbf{N}$;

$H \leq G$ signifies that H is a subgroup of G;

$H \trianglelefteq G$ signifies that H is a normal subgroup of G;

G/H is the factor subgroup of G w.r.t. a normal subgroup H;

$\overline{g}$ is an element gH of G/H;

$\mathrm{gr}(H)$ is a subgroup of G generated by H;

$E = \{e\}$ is a trivial group;

Z is an infinite cyclic group;

Z_n is cyclic of order n;

Z_{p^∞} is a quasicyclic group of type p^∞, p is a prime;

G' is a derived subgroup of G;

$\ker\varphi$ is the kernel of a homomorphism φ;

$\psi\varphi(g) = \psi(\varphi(g))$ is the image of an element g under the composition of two homomorphisms φ and ψ.

By $\mathcal{M}(G,H)$ we denote the least normal subgroup of the group G containing H, the factor group w.r.t. which belongs to a quasivariety $\mathcal{M}$. It is not hard to show that for any quasivariety $\mathcal{M}$ of groups, $\mathcal{M}(G,H) = \{g \in G \mid \forall M \in \mathcal{M}\, \forall \varphi : G \to M \ \text{ if } H \subseteq \ker\varphi \text{ then } \varphi(g) = e\}$, where φ is a homomorphism of G into M. By $\mathrm{Is}_G(H) = \mathrm{gr}(g \mid g \in G \,\&\, (\exists n)(n \in \mathbf{N} \,\&\, g^n \in H))$ we denote the isolator of a subgroup H in a group G. If $G' \subseteq H$ then $\mathrm{Is}_G(H) = \{g \mid g \in G \,\&\, (\exists n)(n \in \mathbf{N} \,\&\, g^n \in H)\}$ and $\mathrm{Is}_G(H) \trianglelefteq G$. We write $q(G_1, \ldots, G_n)$ to denote a quasivariety generated by the groups $G_1, \ldots, G_n$.

According to [9], two quasivarieties of Abelian groups coincide iff they have equal intersections with a set Q of groups, consisting of groups Z, E and cyclic p-groups, where p runs through the set of all primes.

Relevant results in [9] imply that an arbitrary quasivariety $\mathcal{M}$ of Abelian groups is representable as $\mathcal{M} = q(S)$ for some $S \subseteq Q$, and a cyclic p-group belongs to the quasivariety $q(S)$ iff it is isomorphic to a suitable subgroup of some group in S. Furthermore, if $\mathcal{M} = \bigvee\limits_{i\in I} \mathcal{M}_i$, $\mathcal{M}_i = q(S_i)$ $(S_i \subseteq Q)$ then $\mathcal{M} = q\left(\bigcup\limits_{i\in I} S_i\right)$. It is also worth observing the following: if the group Z does not belong to the quasivariety $\mathcal{M} = q(S)$ then the set S consists of finitely many non-isomorphic cyclic p-groups, and $\mathcal{M}$ is a variety.

The mapping φ of a lattice $(L_1, \wedge, \vee)$ into a lattice $(L_2, \wedge, \vee)$ is called an *antihomomorphism* if $\varphi(a \vee b) = \varphi(a) \wedge \varphi(b)$ and $\varphi(a \wedge b) = \varphi(a) \vee \varphi(b)$ for any $a, b \in L_1$. A 1-1 antihomomorphism is called an *anti-isomorphism*. A lattice is said to be *complete* if for any non-empty subset in that lattice there exist a least upper bound and a greatest lower bound. The mapping of a complete lattice into a complete lattice is called a *complete antihomomorphism* if it sends greatest (least) lower (upper) bounds of non-empty subsets to least (greatest) upper (lower) bounds of their images. Relevant information on the theory of quasivarieties and on lattice theory can be found in [7, 10, 11].

We describe the structure of a dominion in an arbitrary quasivariety of Abelian groups.

THEOREM 1. The dominion of a subgroup H of a group G in any quasivariety $\mathcal{M}$ of Abelian groups coincides with a least normal subgroup of the group G containing H, the factor group w.r.t. which belongs to $\mathcal{M}$, that is, $\mathrm{dom}_G^{\mathcal{M}}(H) = \mathcal{M}(G, H)$.

Proof. Assume that $a \in \mathrm{dom}_G^{\mathcal{M}}(H)$ and $\varphi : G \to M$ $(M \in \mathcal{M})$ is a homomorphism satisfying the condition that $H \subseteq \ker \varphi$. Consider a homomorphism $\psi : G \to M$, under which $\psi(g) = e$ for any element $g \in G$. Since $\varphi|_H = \psi|_H$, $\varphi(a) = \psi(a)$ by the definition of a dominion. Hence $\varphi(a) = e$ and $a \in \mathcal{M}(G, H)$.

We argue for the inverse inclusion. Let $a \in \mathcal{M}(G, H)$, and let $\varphi, \psi : G \to M$ $(M \in \mathcal{M})$ be homomorphisms such that $\varphi|_H = \psi|_H$. Consider a map $\frac{\varphi}{\psi} : G \to M$, defined by setting $\frac{\varphi}{\psi}(g) = \varphi(g)\psi(g)^{-1}$. It is easy to verify that $\frac{\varphi}{\psi}$ is a homomorphism and $H \subseteq \ker \frac{\varphi}{\psi}$. Since $a \in \mathcal{M}(G, H)$, we have $\frac{\varphi}{\psi}(a) = e$. Hence $\varphi(a) = \psi(a)$, and $a \in \mathrm{dom}_G^{\mathcal{M}}(H)$ by the definition of a dominion. $\square$

COROLLARY 1. Let $\mathcal{M}$ be any quasivariety of Abelian groups, G be a group, and $H \leq G$. Then $\mathrm{dom}_G^{\mathcal{M}}(H) = H$ if and only if $H \trianglelefteq G$ and $G/H \in \mathcal{M}$.

COROLLARY 2. Let $\mathcal{M}$ be the quasivariety of torsion-free Abelian groups, G be a group, and $H \leq G$. Then $\mathrm{dom}_G^{\mathcal{M}}(H) = H$ if and only if $H \trianglelefteq G$ and $\mathrm{Is}_G(H) = H$.

COROLLARY 3. Let $\mathcal{M}$ be any variety of Abelian groups, $G \in \mathcal{M}$, and $H \trianglelefteq G$. Then $\mathrm{dom}_G^{\mathcal{M}}(H) = H$.

We note that Theorem 1, for $H = E$, was proved in [8], and Corollary 3 follows from [3, Lemma 2.6]. For non-Abelian quasivarieties of groups, according to [8], Theorem 1 fails; however, $\mathrm{dom}_G^{\mathcal{M}}(H) \subseteq \mathcal{M}(G, H)$ holds in this instance. For example, let $G = S_3$ be a symmetric group of degree 3, whose elements are permutations of the same degree, and let $H = \mathrm{gr}((12)) \leq G$, $\mathcal{M} = qG$. Clearly, $\mathcal{M}(G, H) = G$ and the map $\varphi : G \to G$, under which $\varphi((12)) = (12)$, $\varphi((13)) = (23)$, $\varphi((23)) = (13)$, $\varphi((123)) = (132)$, and $\varphi((132)) = (123)$, is a homomorphism. Since $\varphi|_H = \psi|_H$, where ψ is the identity map of G into itself, we have $\mathrm{dom}_G^{\mathcal{M}}(H) = H$.

2. BASIC RESULT

LEMMA 1. Let $\mathcal{M}$ be any quasivariety of Abelian groups, G be a group, $H \leq G$, and $G/\mathrm{dom}_G^{\mathcal{M}}(H)$ be a finitely generated group. For any set of quasivarieties $\mathcal{N}_i \in L_q(\mathcal{M})$ $(i \in I)$,

$$\mathrm{dom}_G^{\bigvee_{i\in I} \mathcal{N}_i}(H) = \bigcap_{i\in I} \mathrm{dom}_G^{\mathcal{N}_i}(H).$$

Proof. If I is a finite set then the statement of the lemma follows from [8, Lemma 4.2]. Let I be infinite. Put $\mathcal{N} = \bigvee_{i \in I} \mathcal{N}_i$. Since $\mathcal{N}_i \subseteq \mathcal{N}$, we have $\mathrm{dom}_G^{\mathcal{N}}(H) \subseteq \mathrm{dom}_G^{\mathcal{N}_i}(H)$ for any $i \in I$. Hence $\mathrm{dom}_G^{\mathcal{N}}(H) \subseteq \bigcap_{i \in I} \mathrm{dom}_G^{\mathcal{N}_i}(H)$.

Let $a \in \bigcap_{i \in I} \mathrm{dom}_G^{\mathcal{N}_i}(H)$. We claim that $a \in \mathrm{dom}_G^{\mathcal{N}}(H)$. Consider an arbitrary homomorphism $\varphi : G \to N \in \mathcal{N}$ satisfying the condition that $H \subseteq \ker\varphi$. Since $G/\ker\varphi \in \mathcal{N} \subseteq \mathcal{M}$, $\mathrm{dom}_G^{\mathcal{M}}(H) \subseteq \ker\varphi$ in view of Theorem 1. We have $\varphi(G) \cong G/\ker\varphi \cong (G/\mathrm{dom}_G^{\mathcal{M}}(H))/(\ker\varphi/\mathrm{dom}_G^{\mathcal{M}}(H))$. Hence $\varphi(G)$ is a finitely generated group, which factors, according to [12], into a direct product of cyclic p-groups and infinite cyclic groups.

Assume $\varphi(a) \neq e$. We claim that $a \notin \mathrm{dom}_G^{\mathcal{N}_i}(H)$ for some $i \in I$ under this assumption. First, consider the case where there is a projection π of the group $\varphi(G)$ onto one of the cyclic p-groups in the factorization of $\varphi(G)$ for which $\pi\varphi(a) \neq e$. Since $\varphi(G) \in \mathcal{N}$, that cyclic p-group is contained in some quasivariety $\mathcal{N}_i$, $i \in I$. Keeping in mind that $a \notin \ker(\pi\varphi)$ and using Theorem 1, we obtain $a \notin \mathrm{dom}_G^{\mathcal{N}_i}(H)$.

It remains to consider the case where $\pi\varphi(a) \neq e$ only if $\varphi(G)$ is projected onto an infinite cyclic group in the factorization. If $Z \in \mathcal{N}_i$ for some $i \in I$, using a similar argument, we see that $a \notin \mathrm{dom}_G^{\mathcal{N}_i}(H)$. Let $Z \notin \mathcal{N}_i$ for any $i \in I$. Denote by b the generator of a fixed group Z in the factorization of $\varphi(G)$, for which $\pi\varphi(a) = b^n \neq e$ for some $n \in \mathbf{N}$ given a projection π onto that Z. Let $n = p_1^{l_1} \dots p_k^{l_k}$ be the factorization of n into a product of degrees of distinct primes $p_1, \dots, p_k$. Since $Z \in \mathcal{N}$, $q(Z_{p_1^{l_1}}, \dots, Z_{p_k^{l_k}}) \neq \mathcal{N}$ and there exists a group $Z_{p_j^{s_j}} \in \mathcal{N}$ such that $s_j > l_j$ for some j, $1 \leqslant j \leqslant k$, or there exists a $Z_{q^s} \in \mathcal{N}$, where q is a prime, and $q \neq p_i$ for any $i = 1, \dots, k$

Let $Z_{p_j^{s_j}} = \mathrm{gr}(c) \in \mathcal{N}$. Consider a natural homomorphism $\psi : Z \to Z/\mathrm{gr}(b^{p_j^{s_j}}) = Z_{p_j^{s_j}}$. Then $\psi\pi\varphi(a) = \psi(b^n) = c^n = c^{p_j^{l_j} n'} \neq e$, since $s_j > l_j$ and $(p_j, n') = 1$. If $Z_{q^s} = \mathrm{gr}(c) \in \mathcal{N}$, we handle a homomorphism $\psi : Z \to Z/\mathrm{gr}(b^{q^s}) = Z_{q^s}$. Since $(q, n) = 1$, $\psi\pi\varphi(a) = \psi(b^n) = c^n \neq e$. Each of the groups $Z_{p_j^{s_j}}$ and Z_{q^s} belongs to some quasivariety $\mathcal{N}_i$, and so $a \notin \mathrm{dom}_G^{\mathcal{N}_i}(H)$ by Theorem 1. Thus the assumption that $\varphi(a) \neq e$ implies that $a \notin \mathrm{dom}_G^{\mathcal{N}_i}(H)$ for some $i \in I$. Hence $a \notin \bigcap_{i \in I} \mathrm{dom}_G^{\mathcal{N}_i}(H)$, a contradiction with the initial assumption on a. Consequently, $\varphi(a) = e$, and by Theorem 1, $a \in \mathrm{dom}_G^{\mathcal{N}}(H)$. $\square$

The next lemma — an analog of Theorem 4.4 in [8] — follows from Lemma 1.

LEMMA 2. Let $\mathcal{M}$ be any quasivariety of Abelian groups, G be a group, $H \leq G$, and $G/\mathrm{dom}_G^{\mathcal{M}}(H)$ be a finitely generated group. The set $L(G, H, \mathcal{M}) = \{\mathrm{dom}_G^{\mathcal{N}}(H) \mid \mathcal{N} \in L_q(\mathcal{M})\}$ forms a complete lattice under set-theoretic inclusion.

For any quasivariety $\mathcal{M}$ of Abelian groups, we consider the following sublattices of $L_q(\mathcal{M})$:

$$L_q^1(\mathcal{M}) = \{\mathcal{N} \mid \mathcal{N} \in L_q(\mathcal{M}),\ Z \notin \mathcal{N}\},$$

$$L_q^2(\mathcal{M}) = \{\mathcal{N} \mid \mathcal{N} \in L_q(\mathcal{M}),\ Z \in \mathcal{N}\}.$$

For every $i = 1, 2$, we define the set

$$L^i(G, H, \mathcal{M}) = \{\mathrm{dom}_G^{\mathcal{N}}(H) \mid \mathcal{N} \in L_q^i(\mathcal{M})\}.$$

LEMMA 3. Let $\mathcal{M}$ be any quasivariety of Abelian groups, G be a group, $H \leq G$, and $G/\mathrm{dom}_G^{\mathcal{M}}(H)$ be a finitely generated group. For $\mathcal{N}, \mathcal{R} \in L_q^i(\mathcal{M})$, $i = 1, 2$, the equality $\mathrm{dom}_G^{\mathcal{N} \wedge \mathcal{R}}(H) = \mathrm{dom}_G^{\mathcal{N}}(H)\mathrm{dom}_G^{\mathcal{R}}(H)$ holds.

Proof. That $\mathrm{dom}_G^{\mathcal{N}\wedge\mathcal{R}}(H) \supseteq \mathrm{dom}_G^{\mathcal{N}}(H)\mathrm{dom}_G^{\mathcal{R}}(H)$ follows from the definition of a dominion. We argue for the way back. We consider an arbitrary element $a \in \mathrm{dom}_G^{\mathcal{N}\wedge\mathcal{R}}(H)$ and show that there are numbers $n, r \in \mathbf{N}$ such that $a^n \in \mathrm{dom}_G^{\mathcal{N}}(H)$ and $a^r \in \mathrm{dom}_G^{\mathcal{R}}(H)$.

Let $\mathcal{N}, \mathcal{R} \in L_q^1(\mathcal{M})$. Then $G/\mathrm{dom}_G^{\mathcal{N}}(H) \in \mathcal{N}$ and $G/\mathrm{dom}_G^{\mathcal{R}}(H) \in \mathcal{R}$ by Theorem 1. Hence $G/\mathrm{dom}_G^{\mathcal{N}}(H)$ and $G/\mathrm{dom}_G^{\mathcal{R}}(H)$ are periodic groups, and so the required n, r exist.

Let $\mathcal{N}, \mathcal{R} \in L_q^2(\mathcal{M})$. Since $G/\mathrm{Is}_G(\mathrm{gr}(G', H)) \in qZ \subseteq \mathcal{N}\wedge\mathcal{R}$, $\mathrm{dom}_G^{\mathcal{N}\wedge\mathcal{R}}(H) \subseteq \mathrm{Is}_G(\mathrm{gr}(G', H))$ by Theorem 1. The inclusions $\mathrm{gr}(G', H) \subseteq \mathrm{dom}_G^{\mathcal{N}}(H)$ and $\mathrm{gr}(G', H) \subseteq \mathrm{dom}_G^{\mathcal{R}}(H)$ imply that $a^n \in \mathrm{dom}_G^{\mathcal{N}}(H)$ and $a^r \in \mathrm{dom}_G^{\mathcal{R}}(H)$ for some $n, r \in \mathbf{N}$.

Let $n, r \in \mathbf{N}$ be least with the properties $a^n \in \mathrm{dom}_G^{\mathcal{N}}(H)$ and $a^r \in \mathrm{dom}_G^{\mathcal{R}}(H)$. If $(n, r) = 1$ then $a \in \mathrm{dom}_G^{\mathcal{N}}(H)\mathrm{dom}_G^{\mathcal{R}}(H)$, and so the lemma is proved. Assume that $(n, r) \neq 1$, and p is a prime, which is a divisor of (n, r). The group $G/\mathrm{dom}_G^{\mathcal{M}}(H)$ is finitely generated; hence, $G/\mathrm{dom}_G^{\mathcal{N}}(H)$ and $G/\mathrm{dom}_G^{\mathcal{R}}(H)$ too are finitely generated. Consider projections $\pi_1 : G/\mathrm{dom}_G^{\mathcal{N}}(H) \to Z_{p^s} \in \mathcal{N}$ and $\pi_2 : G/\mathrm{dom}_G^{\mathcal{R}}(H) \to Z_{p^t} \in \mathcal{R}$ onto the p-components in the factorizations of $G/\mathrm{dom}_G^{\mathcal{N}}(H)$ and $G/\mathrm{dom}_G^{\mathcal{R}}(H)$ into direct products of cyclic groups such that $\pi_1\theta_1(a) \neq e$ and $\pi_2\theta_2(a) \neq e$, where $\theta_1 : G \to G/\mathrm{dom}_G^{\mathcal{N}}(H)$ and $\theta_2 : G \to G/\mathrm{dom}_G^{\mathcal{R}}(H)$ are natural homomorphisms. Without loss of generality, we may assume that $s \leqslant t$. Then $Z_{p^s} \in \mathcal{N} \wedge \mathcal{R}$. Since $H \subseteq \ker(\pi_1\theta_1)$ and $\pi_1\theta_1(a) \neq e$, by Theorem 1 it follows that $a \notin \mathrm{dom}_G^{\mathcal{N}\wedge\mathcal{R}}(H)$, which is a contradiction. We have $(n, r) = 1$, $a \in \mathrm{dom}_G^{\mathcal{N}}(H)\mathrm{dom}_G^{\mathcal{R}}(H)$. The inverse inclusion is thus proved, that is, $\mathrm{dom}_G^{\mathcal{N}\wedge\mathcal{R}}(H) = \mathrm{dom}_G^{\mathcal{N}}(H)\mathrm{dom}_G^{\mathcal{R}}(H)$. $\square$

LEMMA 4. Let $\mathcal{M}$ be any quasivariety of Abelian groups, G be a group, $H \leq G$, and $G/\mathrm{dom}_G^{\mathcal{M}}(H)$ be a finite group. Then $\mathrm{dom}_G^{\mathcal{N}\wedge\mathcal{R}}(H) = \mathrm{dom}_G^{\mathcal{N}}(H)\mathrm{dom}_G^{\mathcal{R}}(H)$ for $\mathcal{N}, \mathcal{R} \in L_q(\mathcal{M})$.

Proof. It suffices to show that $\mathrm{dom}_G^{\mathcal{N}\wedge\mathcal{R}}(H) \subseteq \mathrm{dom}_G^{\mathcal{N}}(H)\mathrm{dom}_G^{\mathcal{R}}(H)$. Lemma 3 implies that we can limit ourselves to the case $Z \notin \mathcal{N}$, $Z \in \mathcal{R}$. Let m be the order of $G/\mathrm{dom}_G^{\mathcal{M}}(H)$, and let $m = p_1^{m_1} \ldots p_k^{m_k}$ be the factorization of m into a product of degrees of distinct primes $p_1, \ldots, p_k$. It is easy to see that $\mathcal{R} = \mathcal{R}_1 \vee \mathcal{R}_2$, where $\mathcal{R}_1 = q(Z, Z_{q_1^{l_1}}, \ldots, Z_{q_s^{l_s}}, \ldots)$, $\mathcal{R}_2 = q(Z_{p_1^{r_1}}, \ldots, Z_{p_k^{r_k}})$, and $q_1, \ldots, q_s, \ldots$ are primes that are not divisors of m; $l_1, \ldots, l_s, \ldots, r_1, \ldots, r_k \in \mathbf{N} \cup \{\infty\} \cup \{0\}$.

Note that $\mathrm{dom}_G^{\mathcal{R}_1}(H) = G$. Indeed, let $\varphi : G \to R \in \mathcal{R}_1$ be any homomorphism satisfying $H \subseteq \ker\varphi$. Since $G/\ker\varphi \in \mathcal{R}_1 \subseteq \mathcal{M}$, we have $\mathrm{dom}_G^{\mathcal{M}}(H) \subseteq \ker\varphi$ by Theorem 1. The map $\psi : G/\mathrm{dom}_G^{\mathcal{M}}(H) \to R$, given by the rule $\psi(\overline{g}) = \varphi(g)$ for any $g \in G$, is a homomorphism. It is clear that $\varphi = \psi\theta$, where $\theta : G \to G/\mathrm{dom}_G^{\mathcal{M}}(H)$ is a natural homomorphism. The description of the quasivariety $\mathcal{R}_1$ implies $\varphi(G) = \psi\theta(G) = \psi(G/\mathrm{dom}_G^{\mathcal{M}}(H)) = E$. By Theorem 1, $\mathrm{dom}_G^{\mathcal{R}_1}(H) = G$. By Lemma 1, $\mathrm{dom}_G^{\mathcal{R}}(H) = \mathrm{dom}_G^{\mathcal{R}_1\vee\mathcal{R}_2}(H) = \mathrm{dom}_G^{\mathcal{R}_1}(H) \cap \mathrm{dom}_G^{\mathcal{R}_2}(H) = G \cap \mathrm{dom}_G^{\mathcal{R}_2}(H) = \mathrm{dom}_G^{\mathcal{R}_2}(H)$. Using the property of being distributive for the lattice of quasivarieties of Abelian groups, stated in [10], we obtain

$$\mathrm{dom}_G^{\mathcal{N}\wedge\mathcal{R}}(H) = \mathrm{dom}_G^{\mathcal{N}\wedge(\mathcal{R}_1\vee\mathcal{R}_2)}(H) = \mathrm{dom}_G^{(\mathcal{N}\wedge\mathcal{R}_1)\vee(\mathcal{N}\wedge\mathcal{R}_2)}(H) =$$

$$\mathrm{dom}_G^{\mathcal{N}\wedge\mathcal{R}_1}(H) \cap \mathrm{dom}_G^{\mathcal{N}\wedge\mathcal{R}_2}(H) = G \cap \mathrm{dom}_G^{\mathcal{N}\wedge\mathcal{R}_2}(H) = \mathrm{dom}_G^{\mathcal{N}\wedge\mathcal{R}_2}(H).$$

We put $s_1 = \min(m_1, r_1)$, $\ldots$, $s_k = \min(m_k, r_k)$ and consider a quasivariety $\mathcal{R}_2' = q(Z_{p_1^{s_1}}, \ldots, Z_{p_k^{s_k}})$. Obviously, $\mathcal{R}_2' = \mathcal{R}_2 \wedge q(G/\mathrm{dom}_G^{\mathcal{M}}(H))$ and $\mathrm{dom}_G^{\mathcal{R}_2}(H) \subseteq \mathrm{dom}_G^{\mathcal{R}_2'}(H)$. Since $q(G/\mathrm{dom}_G^{\mathcal{M}}(H))$ is a variety, using the isomorphism $G/\mathrm{dom}_G^{\mathcal{R}_2}(H)) \cong (G/\mathrm{dom}_G^{\mathcal{M}}(H))/(\mathrm{dom}_G^{\mathcal{R}_2}(H)/\mathrm{dom}_G^{\mathcal{M}}(H))$, we arrive at $G/\mathrm{dom}_G^{\mathcal{R}_2}(H) \in \mathcal{R}_2 \wedge q(G/\mathrm{dom}_G^{\mathcal{M}}(H)) = \mathcal{R}_2'$. By Theorem 1, $\mathrm{dom}_G^{\mathcal{R}_2'}(H) \subseteq \mathrm{dom}_G^{\mathcal{R}_2}(H)$, whence $\mathrm{dom}_G^{\mathcal{R}_2}(H) = \mathrm{dom}_G^{\mathcal{R}_2'}(H)$. Now, applying Lemma 3, we have

$$\mathrm{dom}_G^{\mathcal{N}\wedge\mathcal{R}}(H) = \mathrm{dom}^{\mathcal{N}\wedge\mathcal{R}_2} \subseteq \mathrm{dom}_G^{\mathcal{N}\wedge\mathcal{R}_2'}(H) = \mathrm{dom}_G^{\mathcal{N}}(H)\mathrm{dom}_G^{\mathcal{R}_2'}(H) =$$

$$\mathrm{dom}_G^{\mathcal{N}}(H)\mathrm{dom}_G^{\mathcal{R}_2}(H) = \mathrm{dom}_G^{\mathcal{N}}(H)\mathrm{dom}_G^{\mathcal{R}}(H). \ \square$$

LEMMA 5. Let $\mathcal{M}$ be any quasivariety of Abelian groups, G be a group, $H \leq G$, and $G/\mathrm{dom}_G^{\mathcal{M}}(H)$ be a finitely generated group. If $\mathcal{M} = q(G/\mathrm{dom}_G^{\mathcal{M}}(H))$ then $\mathcal{N} = q(G/\mathrm{dom}_G^{\mathcal{N}}(H))$ for any quasivarieties $\mathcal{N}, \mathcal{R} \in L_q(\mathcal{M})$, and the following equality holds:

$$\mathrm{dom}_G^{\mathcal{N}}(H) \vee \mathrm{dom}_G^{\mathcal{R}}(H) = \mathrm{dom}_G^{\mathcal{N}\wedge\mathcal{R}}(H).$$

Proof. By Theorem 1, $q(G/\mathrm{dom}_G^{\mathcal{N}}(H)) \subseteq \mathcal{N}$. Suppose $\mathcal{N} \neq q(G/\mathrm{dom}_G^{\mathcal{N}}(H))$. We handle some cases.

Let $Z \notin q(G/\mathrm{dom}_G^{\mathcal{N}}(H))$, $Z \in \mathcal{N}$. Denote by $\overline{a}$ the generator of some group Z in the representation of $G/\mathrm{dom}_G^{\mathcal{M}}(H)$ as a direct product of cyclic groups; $\pi : G/\mathrm{dom}_G^{\mathcal{M}}(H) \to Z \in \mathcal{N}$ is the projection of $G/\mathrm{dom}_G^{\mathcal{M}}(H)$ onto this component; $\theta : G \to G/\mathrm{dom}_G^{\mathcal{M}}(H)$ is a natural homomorphism; a is some preimage of $\overline{a}$ under the natural homomorphism θ. For any $n \in \mathbf{N}$, we have $\pi\theta(a^n) = \pi(\overline{a}^n) = (\pi(\overline{a}))^n \neq e$ and $H \subseteq \ker(\pi\theta)$; hence, $a^n \notin \mathrm{dom}_G^{\mathcal{N}}(H)$ for any $n \in \mathbf{N}$ by Theorem 1. This implies that $Z \in q(G/\mathrm{dom}_G^{\mathcal{N}}(H))$, which is a contradiction with the hypothesis. Therefore the case where $Z \notin q(G/\mathrm{dom}_G^{\mathcal{N}}(H))$ and $Z \in \mathcal{N}$ is an impossibility.

Suppose $Z_{p^l} \notin q(G/\mathrm{dom}_G^{\mathcal{N}}(H))$, $Z_{p^l} \in \mathcal{N}$, and $Z_{p^{l+1}} \notin \mathcal{N}$. Let $\overline{a}$ be the generator for Z_{p^m} $(m \geqslant l)$ in the representation of $G/\mathrm{dom}_G^{\mathcal{M}}(H)$ as a direct product of cyclic groups. Consider a subgroup $(G/\mathrm{dom}_G^{\mathcal{M}}(H))^{p^{m-l}}$ of $G/\mathrm{dom}_G^{\mathcal{M}}(H)$, letting $\overline{a}^{p^{m-l}}$ be the generator for Z_{p^l} in the representation of $(G/\mathrm{dom}_G^{\mathcal{M}}(H))^{p^{m-l}}$ as a direct product of cyclic groups. We construct a chain of homomorphisms where $\theta : G \to G/\mathrm{dom}_G^{\mathcal{M}}(H)$ is a natural homomorphism, $\varphi : G/\mathrm{dom}_G^{\mathcal{M}}(H) \to (G/\mathrm{dom}_G^{\mathcal{M}}(H))^{p^{m-l}}$ is a homomorphism mapping every element into its p^{m-l}th degree, and $\pi : (G/\mathrm{dom}_G^{\mathcal{M}}(H))^{p^{m-l}} \to Z_{p^l} \in \mathcal{N}$ is the projection of the group $(G/\mathrm{dom}_G^{\mathcal{M}}(H))^{p^{m-l}}$ onto the component Z_{p^l}, which is generated by an element $\overline{a}^{p^{m-l}}$, in its factorization.

Let a be some preimage of $\overline{a}$ under the natural homomorphism θ. Since $\pi\varphi\theta(a^{p^{l-1}}) = \pi\varphi(\overline{a}^{p^{l-1}}) = \pi((\overline{a}^{p^{m-l}})^{p^{l-1}}) = \pi(\overline{a}^{p^{m-1}}) = \overline{a}^{p^{m-1}} \neq e$ and $H \subseteq \ker(\pi\varphi\theta)$, we have $a^{p^{l-1}} \notin \mathrm{dom}_G^{\mathcal{N}}(H)$. From $Z_{p^l} \in \mathcal{N}$ and $Z_{p^{l+1}} \notin \mathcal{N}$, it follows that $\psi(a^{p^l}) = e$ under any homomorphism $\psi : G \to N \in \mathcal{N}$. By Theorem 1, $a^{p^l} \in \mathrm{dom}_G^{\mathcal{N}}(H)$, and hence $Z_{p^l} \in q(G/\mathrm{dom}_G^{\mathcal{N}}(H))$, which is a contradiction. This means that the case where $Z_{p^l} \notin q(G/\mathrm{dom}_G^{\mathcal{N}}(H))$, $Z_{p^l} \in \mathcal{N}$, and $Z_{p^{l+1}} \notin \mathcal{N}$ is also impossible. Consequently, $\mathcal{N} = q(G/\mathrm{dom}_G^{\mathcal{N}}(H))$.

We argue to show that $\mathrm{dom}_G^{\mathcal{N}}(H) \vee \mathrm{dom}_G^{\mathcal{R}}(H) = \mathrm{dom}_G^{\mathcal{N}\wedge\mathcal{R}}(H)$ for any quasivarieties $\mathcal{N}, \mathcal{R} \in L_q(\mathcal{M})$. The definition of a dominion maintains that $\mathcal{N} \subseteq \mathcal{R}$ implies $\mathrm{dom}_G^{\mathcal{N}}(H) \supseteq \mathrm{dom}_G^{\mathcal{R}}(H)$. We claim that $\mathcal{N} \subseteq \mathcal{R}$ if $\mathrm{dom}_G^{\mathcal{N}}(H) \supseteq \mathrm{dom}_G^{\mathcal{R}}(H)$. We have $\mathrm{dom}_G^{\mathcal{N}}(H) \cap \mathrm{dom}_G^{\mathcal{R}}(H) = \mathrm{dom}_G^{\mathcal{R}}(H) = \mathrm{dom}_G^{\mathcal{N}\vee\mathcal{R}}(H)$. From the first statement of the present lemma it follows that different subquasivarieties of $\mathcal{M}$ enjoy different dominions. Hence $\mathcal{R} = \mathcal{N} \vee \mathcal{R}$ and $\mathcal{N} \subseteq \mathcal{R}$.

By the definition of a least upper bound, $\mathrm{dom}_G^{\mathcal{N}}(H) \vee \mathrm{dom}_G^{\mathcal{R}}(H) = \mathrm{dom}_G^{\mathcal{K}}(H)$, where $\mathcal{K}$ is a quasivariety generated by the set of all quasivarieties $\mathcal{N}_i \in L_q(\mathcal{M})$ $(i \in I)$ satisfying the condition that $\mathrm{dom}_G^{\mathcal{N}_i}(H) \supseteq \mathrm{dom}_G^{\mathcal{N}}(H) \cup \mathrm{dom}_G^{\mathcal{R}}(H)$. It follows that $\mathcal{N}_i \subseteq \mathcal{N} \wedge \mathcal{R}$, whence $\mathcal{K} = \mathcal{N} \wedge \mathcal{R}$. □

LEMMA 6. Let $\mathcal{M}$ be any quasivariety of Abelian groups, suppose that G is a group, $H \leq G$, and $G/\mathrm{dom}_G^{\mathcal{M}}(H)$ is a finitely generated group, and assume that one of the following conditions holds:

(1) $G/\mathrm{dom}_G^{\mathcal{M}}(H)$ is a finite group;

(2) $\mathcal{M} = q(G/\mathrm{dom}_G^{\mathcal{M}}(H))$.

Then

$$\bigwedge_{i\in I} \mathrm{dom}_G^{\mathcal{N}_i}(H) = \mathrm{dom}_G^{\bigvee_{i\in I}\mathcal{N}_i}(H), \quad \bigvee_{i\in I} \mathrm{dom}_G^{\mathcal{N}_i}(H) = \mathrm{dom}_G^{\bigwedge_{i\in I}\mathcal{N}_i}(H)$$

for any set of quasivarieties $\mathcal{N}_i \in L_q(\mathcal{M})$ $(i \in I)$.

Proof. That the first equality is valid follows from Lemma 1. If $\mathcal{M} = q(G/\mathrm{dom}_G^{\mathcal{M}}(H))$, then the lattice $L_q(\mathcal{M})$ is finite, and so the second equality holds in view of Lemma 5.

Let $G/\mathrm{dom}_G^{\mathcal{M}}(H)$ be a finite group. Put $\mathcal{R} = q(G/\mathrm{dom}_G^{\mathcal{M}}(H))$. Applying Lemma 4 with any quasivariety $\mathcal{N} \in L_q(\mathcal{M})$ yields

$$\mathrm{dom}_G^{\mathcal{N}\wedge\mathcal{R}}(H) = \mathrm{dom}^{\mathcal{N}}\mathrm{dom}_G^{\mathcal{R}}(H) = \mathrm{dom}_G^{\mathcal{N}}(H)\mathrm{dom}_G^{\mathcal{M}}(H) = \mathrm{dom}_G^{\mathcal{N}}(H).$$

The fact that $L_q(\mathcal{R})$ is finite implies that for any set of quasivarieties $\mathcal{N}_i \in L_q(\mathcal{M})$ $(i \in I)$, there exists a finite index subset $J \subseteq I$ such that $\{\mathcal{N}_i \wedge \mathcal{R} \mid i \in I\} = \{\mathcal{N}_i \wedge \mathcal{R} \mid i \in J\}$. By Lemma 4,

$$\bigvee_{i\in I} \mathrm{dom}_G^{\mathcal{N}_i}(H) = \bigvee_{i\in I} \mathrm{dom}_G^{\mathcal{N}_i\wedge\mathcal{R}}(H) = \bigvee_{i\in J} \mathrm{dom}_G^{\mathcal{N}_i\wedge\mathcal{R}}(H) = \mathrm{dom}_G^{\bigwedge_{i\in J}(\mathcal{N}_i\wedge\mathcal{R})}(H) =$$

$$\mathrm{dom}_G^{\bigwedge_{i\in I}(\mathcal{N}_i\wedge\mathcal{R})}(H) = \mathrm{dom}_G^{(\bigwedge_{i\in J}\mathcal{N}_i)\wedge\mathcal{R}}(H) = \mathrm{dom}_G^{\bigwedge_{i\in J}\mathcal{N}_i}(H). \ \square$$

THEOREM 2. Let $\mathcal{M}$ be any quasivariety of Abelian groups, G be a group, $H \leq G$, and $G/\mathrm{dom}_G^{\mathcal{M}}(H)$ be a finitely generated group. Then the map $\varphi : L_q(\mathcal{M}) \to L(G, H, \mathcal{M})$, under which $\varphi(\mathcal{N}) = \mathrm{dom}_G^{\mathcal{N}}(H)$ for any quasivariety $\mathcal{N} \in L_q(\mathcal{M})$, is an antihomomorphism of the lattice $L_q(\mathcal{M})$ onto the lattice $L(G, H, \mathcal{M})$ if and only if one of the following conditions holds:

(1) $G/\mathrm{dom}_G^{\mathcal{M}}(H)$ is a finite group;

(2) $\mathcal{M} = q(G/\mathrm{dom}_G^{\mathcal{M}}(H))$.

If (1) and (2) are satisfied then φ is a complete antihomomorphism.

The map φ is an anti-isomorphism iff $\mathcal{M} = q(G/\mathrm{dom}_G^{\mathcal{M}}(H))$.

Proof. Assume one of (1), (2) holds. By Lemma 6, φ is a complete antihomomorphism of $L_q(\mathcal{M})$ onto $L(G, H, \mathcal{M})$. Now, suppose that none of (1), (2) holds. Put $\mathcal{R} = q(G/\mathrm{dom}_G^{\mathcal{M}}(H))$; then $\mathcal{M} \neq \mathcal{R}$, $Z \in \mathcal{R}$. Hence there is a group $Z_{p^m} \in \mathcal{R}$ such that $Z_{p^{m+1}} \notin \mathcal{R}$ and $Z_{p^{m+1}} \in \mathcal{M}$, where $m \geqslant 0$. The quasivariety $\mathcal{R}$ is representable as $\mathcal{R} = q(Z) \vee \mathcal{R}'$, where $\mathcal{R}'$ is a quasivariety generated by all periodic Abelian groups in the representation of $G/\mathrm{dom}_G^{\mathcal{M}}(H)$ as a direct product of cyclic groups. Let $\mathcal{N} = q(Z_{p^{m+1}}) \vee \mathcal{R}'$. It is not hard to see that $\mathrm{dom}_G^{\mathcal{R}}(H) = \mathrm{dom}_G^{\mathcal{M}}(H) \subseteq \mathrm{dom}_G^{\mathcal{N}}(H)$ and $\mathcal{N} \wedge \mathcal{R} = \mathcal{R}'$.

We argue to show that $\mathrm{dom}_G^{\mathcal{N}}(H) \neq \mathrm{dom}_G^{\mathcal{R}'}(H)$. Indeed, let $\overline{a}$ be one of the generators for a cyclic group of infinite order in the representation of $G/\mathrm{dom}_G^{\mathcal{M}}(H)$ as a direct product of cyclic groups. Since $\mathcal{N}$ and $\mathcal{R}'$ are varieties of Abelian groups, we can choose least numbers $l, n \in \mathbf{N}$ with the properties $a^l \in \mathrm{dom}_G^{\mathcal{R}'}(H)$ and $a^n \in \mathrm{dom}_G^{\mathcal{N}}(H)$. From $Z_{p^{m+1}} \notin \mathcal{R}'$, $Z_{p^{m+1}} \in \mathcal{N}$, and $Z_{p^{m+2}} \notin \mathcal{N}$, it follows that p^{m+1} divides n but does not divide l, whence $\mathrm{dom}_G^{\mathcal{N}}(H) \neq \mathrm{dom}_G^{\mathcal{R}'}(H)$. Therefore $\varphi(\mathcal{N} \wedge \mathcal{R}) = \mathrm{dom}_G^{\mathcal{N}\wedge\mathcal{R}}(H) = \mathrm{dom}_G^{\mathcal{R}'}(H) \neq \mathrm{dom}_G^{\mathcal{N}}(H) = \mathrm{dom}_G^{\mathcal{N}}(H) \vee \mathrm{dom}_G^{\mathcal{M}}(H) = \mathrm{dom}_G^{\mathcal{N}}(H) \vee \mathrm{dom}_G^{\mathcal{R}}(H) = \varphi(\mathcal{N}) \vee \varphi(\mathcal{R})$. Hence φ is not an antihomomorphism.

We finish to prove the last claim of the theorem. Let $\mathcal{M} = q(G/\mathrm{dom}_G^{\mathcal{M}}(H))$. By Lemma 5, $\mathcal{N} \neq \mathcal{R}$ implies $\mathrm{dom}_G^{\mathcal{N}}(H) \neq \mathrm{dom}_G^{\mathcal{R}}(H)$. Hence φ is an anti-isomorphism. Conversely, let φ be an anti-isomorphism. We have $\mathrm{dom}_G^{\mathcal{M}}(H) = \mathrm{dom}_G^{q(G/\mathrm{dom}_G^{\mathcal{M}}(H))}(H)$, and so $\mathcal{M} = q(G/\mathrm{dom}_G^{\mathcal{M}}(H))$. $\square$

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REFERENCES

1. J. R. Isbell, "Epimorphisms and dominions," *Proc. Conf. Cat. Algebra, La Jolla 1965*, Springer-Verlag, New York (1966), pp. 232-246.
2. P. V. Higgins, "Epimorphisms and amalgams," *Coll. Math.*, **56**, No. 1, 1-17 (1988).
3. A. Magidin, "Dominions in varieties of nilpotent groups," *Comm. Alg.*, **28**, No. 3, 1241-1270 (2000).

4. D. Wasserman, "Epimorphisms and dominions in varieties of lattices," Ph. D. Thesis, Univ. California, Berkeley (2001).
5. G. Bergman, "Ordering coproducts of groups and semigroups," *J. Alg.*, **133**, No. 2, 313-339 (1990).
6. A. I. Mal'tsev, *Algebraic Systems* [in Russian], Nauka, Moscow (1970).
7. V. A. Gorbunov, *Algebraic Theory of Quasivarieties, Sib. School Alg. Log.* [in Russian], Nauch. Kniga, Novosibirsk (1999).
8. A. Budkin, "Dominions in quasivarieties of universal algebras," *Stud. Log.*, **78**, Nos. 1/2, 120-127 (2004).
9. A. A. Vinogradov, "Quasivarieties of Abelian groups," *Algebra Logika*, **4**, No. 6, 15-19 (1965).
10. A. I. Budkin, *Quasivarieties of Groups* [in Russian], Altai State University, Barnaul (2002).
11. G. D. Birkhoff, *Lattice Theory*, Am. Math. Soc. (1979).
12. M. I. Kargapolov and Yu. I. Merzlyakov, *Foundations of Group Theory* [in Russian], Nauka, Moscow (1982).