



A Note on Schur-Weyl Dualities for $GL(m)$ and $GL(m|n)$

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Abstract

We use a unified elementary approach to prove the second part of classical, mixed, super, and mixed super Schur-Weyl dualities for general linear groups and supergroups over an infinite ground field of arbitrary characteristic. These dualities describe the endomorphism algebras of the tensor space and mixed tensor space, respectively, over the group algebra of the symmetric group and the Brauer wall algebra, respectively. Our main new results are the second part of the mixed Schur-Weyl dualities and mixed super Schur-Weyl dualities over an infinite ground field of positive characteristic.

Keywords Schur-Weyl dualities · General linear group · General linear supergroup

Mathematics Subject Classification (2010) 20G05 · 16G99

1 Introduction

The classical Schur-Weyl duality is one of the cornerstones of the representation theory of algebraic groups. It connects the representation theory of the general linear and symmetric groups. This duality has been generalized to mixed tensor space settings and established for various groups and supergroups. This paper studies Schur-Weyl dualities over an infinite ground field K of arbitrary characteristic. Its main contribution is a characteristic-free elementary approach that unifies the second parts of the classical, mixed, super, and mixed super Schur-Weyl dualities over infinite ground fields. The results for the mixed and mixed super Schur-Weyl dualities in positive characteristics are new.

Let K be an infinite field, G be the general linear group $GL(m)$ or the general linear supergroup $GL(m|n)$, and $Dist(G)$ the distribution algebra of G . Let V be a natural G -supermodule, $W = V^*$ be its dual, and Σ_r be the symmetric group on r elements. There are commuting actions of the distribution algebra $Dist(G)$ and the group algebra $K\Sigma_r$ on $V^{\otimes r}$. The image of $Dist(GL(m))$ in $End_K(V^{\otimes r})$ is called the Schur algebra and is denoted by

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$S(m, r)$. The image of $Dist(GL(m|n))$ in $End_K(V^{\otimes r})$ is called the Schur superalgebra and is denoted by $S(m|n, r)$. The first part of the Schur-Weyl duality says the image of $K\Sigma_r$ in $End_K(V^{\otimes r})$ is $End_G(V^{\otimes r})^{op}$. The second part states that $End_{K\Sigma_r}(V^{\otimes r})$ is isomorphic to $S(m, r)$ if $G = GL(m)$ and to $S(m|n, r)$ if $G = GL(m|n)$.

Schur first established the Schur-Weyl duality for $GL(m)$ over the ground field \mathbb{C} of complex numbers in [15]. For the survey of Schur-Weyl duality over a field of positive characteristic, please consult [10]. The proof of the first part of Schur-Weyl duality that uses the first fundamental theorem of invariant theory is given in [5, Thm 4.1]. A simple proof of the first part of Schur-Weyl duality in the case $m \geq r$ is provided in [4, p.210].

The super Schur-Weyl duality over fields of characteristic zero was established in [2] and [16]. The second part of super Schur-Weyl duality over fields of positive characteristic was established by N. Muir as described in [9, 2.3 (1)].

Next, consider the mixed tensor product space $V^{\otimes r} \otimes W^{\otimes s}$. Let $\delta = m - n$ and $B_{r,s}(\delta)$ be the walled Brauer algebra.

There are commuting actions of the distribution algebra $Dist(G)$ and the Brauer wall algebra $B_{r,s}^{m,n}(\delta)$ on $V^{\otimes r} \otimes W^{\otimes s}$. The image of $Dist(GL(m))$ in $End_K(V^{\otimes r} \otimes W^{\otimes s})$ is called the rational Schur algebra and is denoted by $S(m, r, s)$. The image of $Dist(GL(m|n))$ in $End_K(V^{\otimes r} \otimes W^{\otimes s})$ is called the rational Schur superalgebra and is denoted by $S(m|n; r, s)$. The first part of the mixed Schur-Weyl duality states that the image of $B_{r,s}^{m,n}(\delta)$ in $End_K(V^{\otimes r} \otimes W^{\otimes s})$ is $End_G(V^{\otimes r} \otimes W^{\otimes s})$. The second part states that $End_{B_{r,s}^{m,n}(\delta)}(V^{\otimes r} \otimes W^{\otimes s})$ is isomorphic to $S(m, r, s)$ if $G = GL(m)$ and to $S(m|n, r, s)$ if $G = GL(m|n)$.

The mixed Schur-Weyl duality for $GL(m)$ over fields of characteristic zero was first established in [8], and an alternative proof was given in [18]. The first part was proved using the first fundamental theorem of invariant theory (see [5, Thm 3.1]), while the second part was called “the hard part” in [18]. We are not aware of analogous results in the positive characteristic case.

The first part of the mixed super Schur-Weyl duality was proved over the ground field K of characteristic zero in [3] (see also [6]) following the approach specified in [13]. In particular, by Theorem 7.8 of [3], if $(m+1)(n+1) > r+s$, then $B_{r,s}^{m,n}(\delta)$ is isomorphic to $End_G(V^{\otimes r} \otimes W^{\otimes s})$. Partial mixed super Schur-Weyl duality results were established earlier in [17]. We have learned recently that the second part of the mixed super Schur-Weyl duality was established in the case when $r+s \leq m-n$ in [14]. We are unaware of a general proof of the second part of the mixed super Schur-Weyl duality in the characteristic zero case and of any evidence in the positive characteristic case.

This paper aims to give elementary proofs of the second part of the classical, super, mixed, and mixed super Schur-Weyl dualities over an infinite ground field of an arbitrary characteristic. We obtain new results for the mixed Schur-Weyl dualities in the positive characteristic case and for the mixed super Schur-Weyl dualities in the case of arbitrary characteristic.

Since we use the algebraic independence of polynomial functions in several variables, we need to assume that the ground field is infinite. In general, this property is not true over finite fields. For Schur-Weyl duality over finite fields, see [1].

In Section 2, we first introduce the Brauer wall algebra, actions of the general linear group $GL(m)$, and the general linear supergroup $GL(m|n)$ on ordinary tensor space $V^{\otimes r}$ and mixed tensor space $V^{\otimes r} \otimes W^{\otimes s}$. We consider induced homomorphisms $\Psi_r, \Phi_r, \Psi_{r,s}$ and $\Phi_{r,s}$ and state classical, mixed, super and mixed super Schur-Weyl dualities.

In Sections 3 through 6, we use elementary methods to investigate the second part of Schur-Weyl dualities. In Section 3, we prove the second part of the classical Schur-Weyl duality, and in Section 4, we demonstrate the second part of the mixed Schur-Weyl duality, both involving the group $GL(m)$. In Section 5, we prove the second part of the super Schur-Weyl duality, and in Section 6, we derive the second part of the mixed super Schur-Weyl duality, both involving the supergroup $GL(m|n)$. In Section 7, we show the limitation of this elementary approach to proving the first part of the classical Schur-Weyl duality.

2 Notation

Throughout the paper, we assume that the ground field K is an infinite field of characteristic zero or positive characteristic p .

2.1 Brauer Wall Algebra $B_{r,s}(\delta)$

Let δ be an element of K . We define generators of $B_{r,s}(\delta)$ by (the isotopy classes of) diagrams. A diagram is a bipartite graph on the set of upper vertices $\overline{1}, \dots, \overline{r+s}$ and lower vertices $\underline{1}, \dots, \underline{r+s}$ such that each vertex is connected to exactly one other vertex by an edge. The edges can be horizontal or vertical. Each vertical edge connects an upper vertex from the set $\overline{1}, \dots, \overline{r}$ to a lower vertex from the set $\underline{1}, \dots, \underline{r}$, or an upper vertex from the set $\overline{r+1}, \dots, \overline{r+s}$ to a lower vertex from the set $\underline{r+1}, \dots, \underline{r+s}$. Each horizontal edge connects an upper vertex from $\overline{1}, \dots, \overline{r}$ to an upper vertex from $\overline{r+1}, \dots, \overline{r+s}$, or a lower vertex from $\underline{1}, \dots, \underline{r}$ to a lower vertex from $\underline{r+1}, \dots, \underline{r+s}$.

To define the multiplication of diagrams σ and τ put the diagram σ under τ and create their concatenation $\sigma \circ \tau$ by removing any internal cycles (appearing in the middle of the concatenation). If there are t internal cycles, then the product $\sigma\tau$ equals $\delta^t \sigma \circ \tau$. We extend this to $(\delta^a \sigma)(\delta^b \tau) = \delta^{a+b+t} \sigma \circ \tau$.

To a transposition $\sigma_i = (i, i+1)$ for $1 \leq i \leq r+s$, denote the diagram τ_i obtained by connecting each upper vertex \overline{j} to a lower vertex $\underline{\sigma_i(j)}$.

For $\sigma_r = (r, r+1)$, denote the diagram τ_r obtained by connecting vertex \overline{j} to \underline{j} for each $j \neq r, r+1$, connecting \overline{r} to $\underline{r+1}$, and $\underline{r+1}$ to \underline{r} .

Then $\tau_1, \dots, \tau_{r+s-1}$ are generators of $B_{r,s}(\delta)$.

The relations between τ_i are described in (2.3)—(2.6) of [3].

The group algebra $K\Sigma_{r+s}$ can be described by permutation diagrams such that there is a vector space isomorphism $\text{flip}_{r,s} : K\Sigma_{r+s} \rightarrow B_{r,s}(\delta)$ as in (2.1) of [3].

The above description of $B_{r,s}(\delta)$ is valid for an arbitrary δ . However, since we work with $G = GL(m|n)$ in this paper, we always assume $\delta = m - n$.

2.2 Actions and Homomorphisms

We consider two cases: the classical case when $G = GL(m)$ and the supercase when $G = GL(m|n)$.

The parity $|j|$ is defined as $|j| = 0$ for $j = 1, \dots, m$ and $|j| = 1$ for $j = m+1, \dots, m+n$.

If $G = GL(m)$, then a matrix unit $e_{ij} \in Dist_1(G)$ acts on basis elements $\{v_1, \dots, v_m\}$ of the natural module V via $e_{ij}v_k = \delta_{jk}v_i$ and on the basis elements $\{v_1^*, \dots, v_m^*\}$ of $W = V^*$ as $e_{ij}v_k^* = -\delta_{ik}v_j^*$.

Analogously, if $G = GL(m|n)$, then a matrix unit $e_{ij} \in Dist_1(G)$ acts on basis elements $\{v_1, \dots, v_{m+n}\}$ of the natural module V via $e_{ij}v_k = \delta_{jk}v_i$ and on the basis elements $\{v_1^*, \dots, v_{m+n}^*\}$ of $W = V^*$ as $e_{ij}v_k^* = -\delta_{ik}(-1)^{|i|(|i|+|j|)}v_j^*$.

2.2.1 Actions on $V^{\otimes r}$

We write generators of $V^{\otimes r}$ as

$$v_{i_1} \otimes \dots \otimes v_{i_r} = v_J.$$

If $G = GL(m)$, then the right action of generators $\tau_j = (j, j + 1)$, for $1 \leq j < r$, of the group algebra $K\Sigma_r$ on generators of the tensor product space $V^{\otimes r}$ is given as $(v_J)\tau_j = v_J$, where the multi-index $J = I.(j, j + 1)$ is obtained from I by transposing entries at the j th and $(j + 1)$ st place.

If $G = GL(m|n)$, the right action of $K\Sigma_r$ on $V^{\otimes r}$ is given as

$$(v_I)\tau_j = (-1)^{|i_j||i_{j+1}|}v_J = (-1)^{|i_j||i_{j+1}|}v_{I.(j,j+1)}.$$

The left action of $GL(m)$ on $V^{\otimes r}$ is given by

$$g.v_I = g.(v_{i_1} \otimes \dots \otimes v_{i_r}) = g.v_{i_1} \otimes \dots \otimes g.v_{i_r},$$

and the left action of the general linear Lie algebra $\mathfrak{gl}(m)$ on $V^{\otimes r}$ is given as

$$\begin{aligned} e_{ij}.v_I &= e_{ij}.(v_{i_1} \otimes \dots \otimes v_{i_r}) = \sum_{a=1}^r v_{i_1} \otimes \dots \otimes e_{ij}v_{i_a} \otimes \dots \otimes v_{i_r} \\ &= \sum_{a=1}^r \delta_{ji_a} v_{i_1} \otimes \dots \otimes v_i \otimes \dots \otimes v_{i_r}. \end{aligned}$$

The left action of $GL(m|n)$ on $V^{\otimes r}$ is given analogously as

$$g.v_I = g.(v_{i_1} \otimes \dots \otimes v_{i_r}) = g.v_{i_1} \otimes \dots \otimes g.v_{i_r},$$

and the left action of the general linear Lie superalgebra $\mathfrak{gl}(m|n)$ on $V^{\otimes r}$ is given as

$$\begin{aligned} e_{ij}.v_I &= e_{ij}.(v_{i_1} \otimes \dots \otimes v_{i_r}) = \sum_{a=1}^r (-1)^{|e_{ij}|(|i_1|+\dots+i_{a-1}|)} v_{i_1} \otimes \dots \otimes e_{ij}v_{i_a} \otimes \dots \otimes v_{i_r} \\ &= \sum_{a=1}^r \delta_{ji_a} (-1)^{|e_{ij}|(|i_1|+\dots+i_{a-1}|)} v_{i_1} \otimes \dots \otimes v_i \otimes \dots \otimes v_{i_r}. \end{aligned}$$

The right action of $K\Sigma_r$ on $V^{\otimes r}$ commutes with the left action of G on $V^{\otimes r}$.

Therefore, there are induced homomorphisms

$$\Psi_r : Dist(G) \rightarrow End_{K\Sigma_r}(V^{\otimes r})^{op} \tag{1}$$

and

$$\Phi_r : K\Sigma_r \rightarrow End_G(V^{\otimes r})^{op}. \tag{2}$$

If $G = GL(m)$, the image of $Dist(G)$ in $End_{K\Sigma_r}(V^{\otimes r})^{op}$ is the Schur algebra $S(m, r)$. If $G = GL(m|n)$, the image of $Dist(G)$ in $End_{K\Sigma_r}(V^{\otimes r})^{op}$ is the Schur superalgebra $S(m|n, r)$.

2.2.2 Actions on $V^{\otimes r} \otimes W^{\otimes s}$

We write the generators of $V^{\otimes r} \otimes W^{\otimes s}$ as

$$v_{i_1} \otimes \dots \otimes v_{i_r} \otimes v_{i_{r+1}}^* \otimes \dots \otimes v_{i_{r+s}}^* = v_I,$$

where the multi-index $I = (i_1, \dots, i_{r+s})$ has entries in the set $\{1, \dots, m+n\}$.

The left actions of $GL(m)$ and $GL(m|n)$ on $V^{\otimes r} \otimes W^{\otimes s}$ are given as

$$\begin{aligned} g.v_I &= g.(v_{i_1} \otimes \dots \otimes v_{i_r} \otimes v_{i_{r+1}}^* \otimes \dots \otimes v_{i_{r+s}}^*) \\ &= g.v_{i_1} \otimes \dots \otimes g.v_{i_r} \otimes g.v_{i_{r+1}}^* \otimes \dots \otimes g.v_{i_{r+s}}^*. \end{aligned}$$

Also, the actions of $\mathfrak{gl}(m)$ and $\mathfrak{gl}(m|n)$ on $V^{\otimes r} \otimes W^{\otimes s}$ are given as

$$\begin{aligned} e_{ij}.v_I &= e_{ij}.(v_{i_1} \otimes \dots \otimes v_{i_r} \otimes v_{i_{r+1}}^* \otimes \dots \otimes v_{i_{r+s}}^*) \\ &= \sum_{a=1}^r v_{i_1} \otimes \dots \otimes e_{ij}v_{i_a} \otimes \dots \otimes v_{i_r} \otimes v_{i_{r+1}}^* \otimes \dots \otimes v_{i_{r+s}}^* \\ &\quad + \sum_{b=1}^s v_{i_1} \otimes \dots \otimes v_{i_r} \otimes v_{i_{r+1}}^* \otimes \dots \otimes e_{ij}v_{i_{r+b}}^* \otimes \dots \otimes v_{i_{r+s}}^* \\ &= \sum_{a=1}^r \delta_{j i_a} v_{i_1} \otimes \dots \otimes v_{i_a} \otimes \dots \otimes v_{i_r} \otimes v_{i_{r+1}}^* \otimes \dots \otimes v_{i_{r+s}}^* \\ &\quad - \sum_{b=1}^s \delta_{i i_{r+b}} v_{i_1} \otimes \dots \otimes v_{i_r} \otimes v_{i_{r+1}}^* \otimes \dots \otimes v_j^* \otimes \dots \otimes v_{i_{r+s}}^* \end{aligned}$$

and

$$\begin{aligned} e_{ij}.v_I &= e_{ij}.(v_{i_1} \otimes \dots \otimes v_{i_r} \otimes v_{i_{r+1}}^* \otimes \dots \otimes v_{i_{r+s}}^*) = \\ &\sum_{a=1}^r (-1)^{|e_{ij}|(|i_1|+\dots+|i_{a-1}|)} v_{i_1} \otimes \dots \otimes e_{ij}v_{i_a} \otimes \dots \otimes v_{i_r} \otimes v_{i_{r+1}}^* \otimes \dots \otimes v_{i_{r+s}}^* \\ &\quad + \sum_{b=1}^s (-1)^{|e_{ij}|(|i_1|+\dots+|i_{r+b-1}|)} v_{i_1} \otimes \dots \otimes v_{i_r} \otimes v_{i_{r+1}}^* \otimes \dots \otimes e_{ij}v_{i_{r+b}}^* \otimes \dots \otimes v_{i_{r+s}}^* \\ &= \sum_{a=1}^r \delta_{j i_a} (-1)^{|e_{ij}|(|i_1|+\dots+|i_{a-1}|)} v_{i_1} \otimes \dots \otimes v_{i_a} \otimes \dots \otimes v_{i_r} \otimes v_{i_{r+1}}^* \otimes \dots \otimes v_{i_{r+s}}^* \\ &\quad - \sum_{b=1}^s \delta_{i i_{r+b}} (-1)^{|i|(|i|+|j|)} (-1)^{|e_{ij}|(|i_1|+\dots+|i_{r+b-1}|)} \\ &\quad v_{i_1} \otimes \dots \otimes v_{i_r} \otimes v_{i_{r+1}}^* \otimes \dots \otimes v_j^* \otimes \dots \otimes v_{i_{r+s}}^*, \end{aligned}$$

respectively.

The right action of generators of the Brauer algebra $B_{r,s}(\delta)$ on generators of the mixed tensor product space $V^{\otimes r} \otimes W^{\otimes s}$ is given as follows.

Assume $G = GL(m)$. If $1 \leq j \leq r + s - 1$ and $j \neq r$, then we define

$$(v_I)\tau_j = v_J = v_{I.(j,j+1)},$$

where the multi-index J is obtained from I by transposing entries at the j th and $(j + 1)$ st place. For $j = r$, we define

$$(v_I)\tau_r = \delta_{i_r,i_{r+1}} \sum_{k=1}^{r+s} v_{i_1} \otimes \dots \otimes v_{i_{r-1}} \otimes v_k \otimes v_k^* \otimes v_{i_{r+2}}^* \otimes \dots \otimes v_{i_{r+s}}^*.$$

Assume $G = GL(m|n)$. If $1 \leq j \leq r + s - 1$ and $j \neq r$, then we define

$$(v_I)\tau_j = (-1)^{|i_j||i_{j+1}|} v_J = (-1)^{|i_j||i_{j+1}|} v_{I.(j,j+1)},$$

where the multi-index J is obtained from I by transposing entries at the j th and $(j + 1)$ st place.

For $j = r$, we define

$$(v_I)\tau_r = -v_{i_{r+1}}^*(v_{i_r})(-1)^{|i_r|} \sum_{k=1}^{r+s} v_{i_1} \otimes \dots \otimes v_{i_{r-1}} \otimes v_k \otimes v_k^* \otimes v_{i_{r+2}}^* \otimes \dots \otimes v_{i_{r+s}}^*.$$

Lemma 2.1 *The element $\tau = \sum_{k=1}^{r+s} v_k \otimes v_k^*$ spans a one-dimensional G -supermodule.*

Proof Compute

$$\begin{aligned} e_{ij}\tau &= \sum_{k=1}^{r+s} [e_{ij}v_k \otimes v_k^* + (-1)^{|e_{ij}||v_k|} v_k \otimes e_{ij}v_k^*] = e_{ij}v_j \otimes v_j^* + (-1)^{|e_{ij}||v_i|} v_i \otimes e_{ij}v_i^* \\ &= v_i \otimes v_j^* + (-1)^{|e_{ij}||v_i|} v_i \otimes -(-1)^{|e_{ij}||v_i^*|} v_j^* = 0 \end{aligned}$$

because $|v_i| = |v_i^*|$. □

Lemma 2.2 *The right action of $B_{r,s}(\delta)$ on $V^{\otimes r} \otimes W^{\otimes s}$ commutes with the left action of G .*

Proof It is clear that the right action of $\Sigma_r \times \Sigma_s$ commutes with the left action of G . Therefore, it is enough to verify that $g(v_I)\tau_r = (gv_I)\tau_r$ for each $g \in G$. We have

$$\begin{aligned} &(gv_I)\tau_r \\ &= \sum_{t=1}^r (-1)^{|g|(|i_1|+\dots+|i_{r-1}|)} (v_{i_1} \otimes \dots \otimes gv_{i_t} \otimes \dots \otimes v_{i_r} \otimes v_{i_{r+1}}^* \otimes v_{i_{r+2}}^* \otimes \dots \otimes v_{i_{r+s}}^*)\tau_r \\ &+ \sum_{t=r+1}^{r+s} (-1)^{|g|(|i_1|+\dots+|i_{r-1}|)} (v_{i_1} \otimes \dots \otimes v_{i_r} \otimes v_{i_{r+1}}^* \otimes \dots \otimes gv_{i_t}^* \otimes \dots \otimes v_{i_{r+s}}^*)\tau_r \\ &= \sum_{t=1}^{r-1} (-1)^{|g|(|i_1|+\dots+|i_{r-1}|)} (-1)^{|i_r|+1} v_{i_{r+1}}^*(v_{i_r}) \\ &\sum_{k=1}^{r+s} v_{i_1} \otimes \dots \otimes gv_{i_t} \otimes \dots \otimes v_{i_{r-1}} \otimes v_k \otimes v_k^* \otimes v_{i_{r+2}}^* \otimes \dots \otimes v_{i_{r+s}}^* \end{aligned}$$

$$\begin{aligned}
 &+ (-1)^{|g|(|i_1|+\dots+|i_{r-1}|)}(-1)^{|g|+|i_r|+1}v_{i_{r+1}}^*(gv_{i_r}) \\
 &\sum_{k=1}^{r+s} v_{i_1} \otimes \dots \otimes v_{i_{r-1}} \otimes v_k \otimes v_k^* \otimes v_{i_{r+2}}^* \otimes \dots \otimes v_{i_{r+s}}^* \\
 &+ (-1)^{|g|(|i_1|+\dots+|i_{r-1}|+|i_r|)}(-1)^{|i_r|+1}gv_{i_{r+1}}^*(v_{i_r}) \\
 &\sum_{k=1}^{r+s} v_{i_1} \otimes \dots \otimes v_{i_{r-1}} \otimes v_k \otimes v_k^* \otimes v_{i_{r+2}}^* \otimes \dots \otimes v_{i_{r+s}}^* \\
 &+ \sum_{t=r+2}^{r+s} (-1)^{|g|(|i_1|+\dots+|i_{t-1}|)}(-1)^{|i_r|+1}v_{i_{r+1}}^*(v_{i_r}) \\
 &\sum_{k=1}^{r+s} v_{i_1} \otimes \dots \otimes v_{i_{r-1}} \otimes v_k \otimes v_k^* \otimes v_{i_{r+2}}^* \otimes \dots \otimes gv_{i_t}^* \otimes \dots \otimes v_{i_{r+s}}^*.
 \end{aligned}$$

Since $gv_{i_{r+1}}^*(v_{i_r}) = (-1)^{|g|+|i_r|+1}v_{i_{r+1}}^*(g^{-1}v_{i_r})$, the expression

$$\begin{aligned}
 &(-1)^{|g|(|i_1|+\dots+|i_{r-1}|+|i_r|)}(-1)^{|i_r|+1}gv_{i_{r+1}}^*(v_{i_r}) \\
 &= (-1)^{|g|(|i_1|+\dots+|i_{r-1}|+|i_r|+|i_{r+1}|)}(-1)^{|i_r|}v_{i_{r+1}}^*(g^{-1}v_{i_r})
 \end{aligned}$$

is the opposite of

$$(-1)^{|g|(|i_1|+\dots+|i_{r-1}|)}(-1)^{|g|+|i_r|+1}v_{i_{r+1}}^*(g^{-1}v_{i_r})$$

because if $v_{i_{r+1}}^*(g^{-1}v_{i_r}) \neq 0$, then $|g| = |i_r| + |i_{r+1}|$.

Therefore, the two middle terms in the above sum offset each other, and

$$\begin{aligned}
 &(gv_I)\tau_r \\
 &= \sum_{t=1}^{r-1} (-1)^{|g|(|i_1|+\dots+|i_{t-1}|)}(-1)^{|i_r|+1}v_{i_{r+1}}^*(v_{i_r}) \\
 &\sum_{k=1}^{r+s} v_{i_1} \otimes \dots \otimes gv_{i_t} \otimes \dots \otimes v_{i_{r-1}} \otimes v_k \otimes v_k^* \otimes v_{i_{r+2}}^* \otimes \dots \otimes v_{i_{r+s}}^* \\
 &+ \sum_{t=r+2}^{r+s} (-1)^{|g|(|i_1|+\dots+|i_{t-1}|)}(-1)^{|i_r|+1}v_{i_{r+1}}^*(v_{i_r}) \\
 &\sum_{k=1}^{r+s} v_{i_1} \otimes \dots \otimes v_{i_{r-1}} \otimes v_k \otimes v_k^* \otimes v_{i_{r+2}}^* \otimes \dots \otimes gv_{i_t}^* \otimes \dots \otimes v_{i_{r+s}}^*.
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 &g(v_I \tau_r) \\
 &= g(v_{i_{r+1}}^* (v_{i_r}) (-1)^{|i_r|+1} \sum_{k=1}^{r+s} v_{i_1} \otimes \dots \otimes v_{i_{r-1}} \otimes v_k \otimes v_k^* \otimes v_{i_{r+2}}^* \otimes \dots \otimes v_{i_{r+s}}^*) \\
 &= \sum_{t=1}^{r-1} (-1)^{|g|(|i_1|+\dots+|i_{r-1}|)} (-1)^{|i_r|+1} v_{i_{r+1}}^* (v_{i_r}) \\
 &\quad \sum_{k=1}^{r+s} v_{i_1} \otimes \dots \otimes g v_{i_t} \otimes \dots \otimes v_{i_{r-1}} \otimes v_k \otimes v_k^* \otimes v_{i_{r+2}}^* \otimes \dots \otimes v_{i_{r+s}}^* \\
 &\quad + \sum_{t=r+2}^{r+s} (-1)^{|g|(|i_1|+\dots+|i_{r-1}|+|i_{r+2}|+\dots+|i_{t-1}|)} (-1)^{|i_r|+1} v_{i_{r+1}}^* (v_{i_r}) \\
 &\quad \sum_{k=1}^{r+s} v_{i_1} \otimes \dots \otimes v_{i_{r-1}} \otimes v_k \otimes v_k^* \otimes v_{i_{r+2}}^* \otimes \dots \otimes g v_{i_t}^* \otimes \dots \otimes v_{i_{r+s}}^*
 \end{aligned}$$

due to Lemma 2.2. If $v_{i_{r+1}}^* (v_{i_r}) \neq 0$, then $|i_{t+1}| = |i_t|$ and

$$(-1)^{|g|(|i_1|+\dots+|i_{r-1}|+|i_{r+2}|+\dots+|i_{t-1}|)} = (-1)^{|g|(|i_1|+\dots+|i_{t-1}|)}.$$

Therefore $g(v_I \tau_r) = (g v_I) \tau_r$. □

Therefore, there are induced homomorphisms

$$\Psi_{r,s} : Dist(G) \rightarrow End_{B_{r,s}(\delta)}(V^{\otimes r} \otimes W^{\otimes s})^{op} \tag{3}$$

and

$$\Phi_{r,s} : B_{r,s}(\delta) \rightarrow End_G(V^{\otimes r} \otimes W^{\otimes s})^{op}. \tag{4}$$

The image of $Dist(GL(m))$ in $End_{B_{r,s}(\delta)}(V^{\otimes r} \otimes W^{\otimes s})^{op}$ is, by definition, the rational Schur algebra $S(m, r, s)$. We denote $A(m, r, s) = S(m, r, s)^*$.

The image of $Dist(GL(m|n))$ in $End_{B_{r,s}(\delta)}(V^{\otimes r} \otimes W^{\otimes s})^{op}$ is, by definition, the rational Schur superalgebra $S(m|n, r, s)$. We denote $A(m|n, r, s) = S(m|n, r, s)^*$.

2.3 Schur-Weyl Dualities

The Schur-Weyl dualities are often expressed as a double centralizer property. Their convenient formulation is stated as follows. The first part of the Schur-Weyl dualities states that the morphism Φ_r , or $\Phi_{r,s}$ respectively, is surjective, while the second part of the Schur-Weyl dualities states that the morphism Ψ_r , or $\Psi_{r,s}$ respectively, is surjective.

The classical Schur-Weyl dualities over \mathbb{C} were established by Schur in [15]. For a survey of results over fields of positive characteristic, consult [10]. If $r \leq m$, then Φ_r is injective and the first part of the classical Schur-Weyl duality can be reformulated as the isomorphism $K \Sigma_r \simeq End_{GL(m)}(V^{\otimes r})^{op}$. The second part of the classical Schur-Weyl duality states that

$$S(m, r) \simeq End_{K \Sigma_r}(V^{\otimes r})^{op}.$$

The super Schur-Weyl duality over fields of characteristic zero were derived in [2]. The second part of the super Schur-Weyl duality states that

$$S(m|n, r) \simeq \text{End}_{K\Sigma_r}(V^{\otimes r})^{op}.$$

The second part of the super Schur-Weyl duality over fields of positive characteristic was determined by N. Muir - see [10].

The mixed Schur-Weyl duality over fields of characteristic zero was proved in [8]. If the characteristic of the ground field K is zero and $r + s < (m + 1)(n + 1)$, then $\Phi_{r,s}$ is injective by [3], and the first part of the mixed Schur-Weyl duality can be reformulated as

$$KB_{r,s} \simeq \text{End}_{GL(m)}(V^{\otimes r} \otimes W^{\otimes s})^{op}.$$

The second part of the mixed Schur-Weyl duality states that

$$S(m, r, s) \simeq \text{End}_{B_{r,s}}(V^{\otimes r} \otimes W^{\otimes s})^{op}.$$

We are unaware of general results on mixed Schur-Weyl duality over fields of positive characteristic. We will show that the second part of mixed Schur-Weyl duality is valid over any infinite field of arbitrary characteristic.

While this paper was under review, we have learned that A. Riesen [14] has established the second part of the mixed super Schur-Weyl duality over the fields of characteristic zero, stating that

$$S(m|n, r, s) \simeq \text{End}_{B_{r,s}(\delta)}(V^{\otimes r} \otimes W^{\otimes s})^{op}$$

under the condition $r + s \leq m - n$. We will show that the second part of mixed super Schur-Weyl duality is valid over any infinite field of arbitrary characteristic.

To derive the second parts of Schur-Weyl dualities, we determine the structure of the duals of $\text{End}_{\Sigma_r}(V^{\otimes r})^{op}$ and $\text{End}_{B_{r,s}(\delta)}(V^{\otimes r} \otimes W^{\otimes s})^{op}$, and compare it with $A(m, r)$, $A(m|n, r)$, $A(m, r, s)$ and $A(m|n, r, s)$, respectively.

3 The Second Half of the Schur-Weyl Duality for $GL(m)$

The second part of Schur-Weyl duality states that $S(m, r)$ is isomorphic to $\text{End}_{K\Sigma_r}(V^{\otimes r})^{op}$.

Let I, J be multi-indices of length r with entries from the set $\{1, \dots, m\}$. Denote by E_{IJ} the matrix unit given by $E_{IJ}(v_K) = \delta_{JK}v_I$. The elements E_{IJ} form a K -basis of the space $\text{End}_K(V^{\otimes r})$. For $\phi \in \text{End}_K(V^{\otimes r})$, we write $\phi = \sum_{IJ} a_{IJ}E_{IJ}$ for appropriate coefficients a_{IJ} .

Lemma 3.1 *If $p > r$, then every morphism in $\text{End}_{K\Sigma_r}(V^{\otimes r})$ is a linear combination of morphisms $\phi = \psi \otimes \dots \otimes \psi$, where $\psi \in \text{End}_K(V)$. If $\psi = \sum_{i,j} a_{ij}e_{ij}$, then $\phi = \sum_{IJ} a_{IJ}E_{IJ}$, where $a_{IJ} = a_{i_1j_1} \dots a_{i_rj_r}$, for $I = i_1 \dots i_r$ and $J = j_1 \dots j_r$. Additionally, a subspace of $V^{\otimes r}$ is $K\Sigma_r$ -submodule if and only if it is invariant under the action of $G = GL(V)$.*

Proof We adapt the classical arguments, e.g., [11, Lemma 6.23].

Write $U = \text{End}_K(V) = V^* \otimes V$. Then $\text{End}_K(V^{\otimes r}) = (V^*)^{\otimes r} \otimes V^{\otimes r} = (V^* \otimes V)^{\otimes r}$ with the compatible action of Σ_r . Therefore $\text{End}_{K\Sigma_r}(V^{\otimes r}) \simeq \text{Sym}^r(\text{End}_K(V))$.

Since $p > r$, the space $\text{Sym}^r(U)$ is spanned by $u^r = r!u \otimes \dots \otimes u$ for $u \in U$. This follows from the polarization identity

$$\sum_{\sigma \in \Sigma_r} f_{\sigma(1)} \otimes f_{\sigma(2)} \otimes \dots \otimes f_{\sigma(r)} = \sum_{I \subset \{1, \dots, r\}} (-1)^{r-|I|} \left(\sum_{i \in I} f_i \right)^{\otimes r}.$$

Therefore when we set $\psi = \frac{1}{\sqrt{r!}}u$, we obtain $a_{IJ} = a_{i_1 j_1} \dots a_{i_r j_r}$.

The second statement follows because $GL(V)$ is dense in $End(V)$ either in Euclidean or Zariski topology. □

Note that even though the coefficients of the map $\phi = \psi \otimes \dots \psi$ are built multiplicatively from the coefficients of ψ , it is difficult to work with the linear combinations of such maps ϕ .

We will follow the approach of [8] based on the following lemma. The multiplicative property we will use will follow from the particular choice of the basis \mathcal{B} .

Lemma 3.2 (Lemma 2.3 of [8]) *Let R be a commutative ring with 1. Let M be a free R -module with basis $\mathcal{B} = \{b_1, \dots, b_l\}$ and U a submodule of M given by a set of linear equations on the coefficients with respect to the basis \mathcal{B} , i.e., $a_{ij} \in R$ such that $U = \{\sum c_j b_j \in M : \sum_j a_{ij} c_j = 0 \text{ for all } i\}$ exist. Let $\{b_1^*, \dots, b_l^*\}$ be the basis of $M^* = Hom_R(M, R)$ dual to \mathcal{B} , and let X be the submodule of M^* generated by all $\sum_j a_{ij} b_j^*$. Then $U \simeq (M^*/X)^*$.*

Proposition 3.3 *There is $S(m, r) \simeq End_{K\Sigma_r}(V^{\otimes r})^{op}$.*

Proof We show that the dual of $End_{K\Sigma_r}(V^{\otimes r})^{op}$ is isomorphic to the coalgebra $A(m, r)$.

We modify and clarify the arguments appearing in the proof of Lemma 3.1 of [8]. We apply Lemma 3.2 for $R = K$, $M = End_K(V^{\otimes r})$, the basis \mathcal{B} consisting of all elements E_{IJ} for multi-indices $I = i_1 \dots i_r$ and $J = j_1 \dots j_r$ of length r , and $U = End_{K\Sigma_r}(V^{\otimes r})$.

Let $\phi \in U$ and $\phi = \sum_{IJ} a_{IJ} E_{IJ}$.

If $1 \leq j < r$, then

$$\begin{aligned} \phi(v_L)\tau_j &= \left(\sum_{IJ} a_{IJ} E_{IJ}(v_L)\right)\tau_j = \left(\sum_I a_{IL} v_I\right)\tau_j = \sum_I a_{IL} v_{I.(j,j+1)} \\ &= \sum_I a_{I.(j,j+1),L} v_I \end{aligned}$$

and

$$\phi(v_L \tau_j) = \phi(v_{L.(j,j+1)}) = \sum_{IJ} a_{IJ} E_{IJ}(v_{L.(j,j+1)}) = \sum_I a_{I,L.(j,j+1)} v_I.$$

Thus ϕ is invariant under the transposition τ_j if and only if $a_{I.(j,j+1),L} = a_{I,L.(j,j+1)}$ for each I, L . Therefore, U is described by the set of equations in the coefficients a_{IJ} of $\phi \in U$ as

$$\phi \in U \text{ if and only if } a_{I.(j,j+1),J} = a_{I,J.(j,j+1)} \text{ for each } I, J \text{ and } 1 \leq j < r.$$

The basis \mathcal{B}^* of M^* , dual to \mathcal{B} , consists of the coefficient functions $E_{IJ}^* = x_{IJ}$ that are constructed multiplicatively from the coefficient functions $x_{ij} = e_{ij}^*$, meaning that $x_{IJ} = x_{i_1 j_1} \dots x_{i_r j_r}$ for $I = i_1 \dots i_r$ and $J = j_1 \dots j_r$.

The submodule X of M^* is generated by equations $x_{I.(j,j+1),J} = x_{I,J.(j,j+1)}$ for all I, J and $1 \leq j < r$. Due to the multiplicativity of the coefficient functions, these equations are identical to the relations generated by the commutativity relations $x_{ij} x_{kl} = x_{kl} x_{ij}$ for all $1 \leq i, j, k, l \leq m$.

If F is the free algebra on generators x_{ij} for $1 \leq i, j \leq m$ and Y the submodule of F generated by elements $x_{ij} x_{kl} - x_{kl} x_{ij}$ for all $1 \leq i, j, k, l \leq m$, then M^*/X is isomorphic to the degree r component D_r of the polynomial algebra F/Y on generators x_{ij} .

Then $D_r \simeq A(m, r)$ shows that $S(m, r) = A(m, r)^*$ has the same dimension as $End_{K\Sigma_r}(V^{\otimes r})^{op}$. We conclude that Ψ_r is surjective and $S(m, r) \simeq End_{K\Sigma_r}(V^{\otimes r})^{op}$. □

4 The Second Half of the Mixed Schur-Weyl Duality for $GL(m)$

Let $r, s \geq 0$ and I, J, K be multi-indices of length $r + s$ with entries from the set $\{1, \dots, m\}$. We write every multi-index $I = (i_1, \dots, i_{r+s})$ as a concatenation $I = I_V I_W$, where $I_V = (i_1 \dots i_r)$ and $I_W = (i_{r+1} \dots i_{r+s})$.

Let v_1, \dots, v_m be a K -basis of V and v_1^*, \dots, v_m^* be the corresponding dual basis of W . Then the elements $v_I = v_{I_V} v_{I_W} = v_{i_1} \dots v_{i_r} v_{i_{r+1}}^* \dots v_{i_{r+s}}^*$ for all multi-indices I of length $r + s$ form a basis of $V^{\otimes r} \otimes W^{\otimes s}$.

Since

$$\begin{aligned} \text{End}_K(V^{\otimes r} \otimes W^{\otimes s}) &\simeq V^{\otimes r} \otimes W^{\otimes s} \otimes (V^{\otimes r} \otimes W^{\otimes s})^* \simeq V^{\otimes r} \otimes W^{\otimes s} \otimes W^{\otimes r} \otimes V^{\otimes s} \\ &\simeq (V^{\otimes r} \otimes W^{\otimes r}) \otimes (W^{\otimes s} \otimes V^{\otimes s}) \simeq \text{End}_K(V^{\otimes r}) \otimes \text{End}_K(W^{\otimes s}) \\ &\simeq \text{End}_K(V)^{\otimes r} \otimes \text{End}_K(W)^{\otimes s}, \end{aligned}$$

every map from $\text{End}_K(V^{\otimes r} \otimes W^{\otimes s})$ is a linear combination of maps $\phi_{V^{\otimes r}} \otimes \phi_{W^{\otimes s}}$, where $\phi_{V^{\otimes r}} \in \text{End}_K(V^{\otimes r})$ and $\phi_{W^{\otimes s}} \in \text{End}_K(W^{\otimes s})$.

Denote by E_{IJ}^V the matrix unit given by $E_{IJ}^V(v_{L_V}) = \delta_{J_V L_V} v_{I_V}$, and by E_{IJ}^W the matrix unit given by $E_{IJ}^W(v_{L_W}) = \delta_{J_W L_W} v_{I_W}$. Then the elements $E_{IJ} = E_{IJ}^V E_{IJ}^W$ form a K -basis of the space $\text{End}_K(V^{\otimes r} \otimes W^{\otimes s})$. For $\phi \in \text{End}_K(V^{\otimes r} \otimes W^{\otimes s})$ we write $\phi = \sum_{IJ} a_{IJ} E_{IJ}$ for appropriate coefficients a_{IJ} .

Lemma 4.1 *Assume $\phi \in \text{End}_K(V^{\otimes r} \otimes W^{\otimes s})$ commutes with the action of Σ_r on the first r components and with the action of Σ_s on the last s components of $V^{\otimes r} \otimes W^{\otimes s}$. If $p > \max\{r, s\}$, then ϕ can be written as a linear combination of maps $(\psi_V \otimes \dots \otimes \psi_V) \otimes (\psi_W \otimes \dots \otimes \psi_W)$ for some $\psi_V \in \text{End}_K(V)$ and $\psi_W \in \text{End}_K(W)$. If $\psi_V = \sum_{ij} a_{ij} E_{ij}$, $\psi_W = \sum_{ij} a_{ij}^* E_{ij}$ and $(\phi_V \otimes \dots \otimes \psi_V) \otimes (\psi_W \otimes \dots \otimes \psi_W) = \sum_{IJ} a_{IJ} E_{IJ}$, then $a_{IJ} = a_{i_1 j_1} \dots a_{i_r j_r} a_{i_{r+1} j_{r+1}}^* \dots a_{i_{r+s} j_{r+s}}^*$.*

Proof Since

$$\text{End}_K(V^{\otimes r} \otimes W^{\otimes s}) \simeq \text{End}_K(V^{\otimes r}) \otimes \text{End}_K(W^{\otimes s}) \simeq \text{End}_K(V)^{\otimes r} \otimes \text{End}_K(W)^{\otimes s},$$

every map from $\text{End}_K(V^{\otimes r} \otimes W^{\otimes s})$ is a linear combination of maps $\phi_{V^{\otimes r}} \otimes \phi_{W^{\otimes s}}$, where $\phi_{V^{\otimes r}} \in \text{End}_K(V^{\otimes r})$ and $\phi_{W^{\otimes s}} \in \text{End}_K(W^{\otimes s})$. If $\phi \in \text{End}_K(V^{\otimes r} \otimes W^{\otimes s})$ commutes with the action of Σ_r on the first r components and with the action of Σ_s on the last s components, then its image in $\text{End}_K(V^{\otimes r}) \otimes \text{End}_K(W^{\otimes s})$ belongs to $\text{Sym}^r(\text{End}_K(V)) \otimes \text{Sym}^s(\text{End}_K(W))$.

If $p > r$ and $p > s$, then by Lemma 3.1 the maps $\phi_{V^{\otimes r}}$ and $\phi_{W^{\otimes s}}$ are linear combinations of maps $\psi_V \otimes \dots \otimes \psi_V$ and $\psi_W \otimes \dots \otimes \psi_W$ for $\psi_V \in \text{End}_K(V)$ and $\psi_W \in \text{End}_K(W)$, respectively. The multiplicativity statement is obvious. \square

To prove the next statement, we will not use the above lemma, but we modify arguments in Section 4 of [8] to the supercase. The partial multiplicativity will follow from the choice of the basis \mathcal{B} in Lemma 3.2.

Proposition 4.2 *There is $S(m, r, s) \simeq \text{End}_{B_{r,s}(\delta)}(V^{\otimes r} \otimes W^{\otimes s})^{op}$.*

Proof We apply Lemma 3.2 for $R = K, M = \text{End}_K(V^{\otimes r} \otimes W^{\otimes s})$, the basis \mathcal{B} consisting of all elements E_{IJ} for multi-indices $I = i_1 \dots i_r i_{r+1} \dots i_{r+s}, J = j_1 \dots j_r j_{r+1} \dots j_{r+s}$, and $U = \text{End}_{B_{r,s}(\delta)}(V^{\otimes r} \otimes W^{\otimes s})$.

Assume $\phi \in U$ and write $\phi = \sum_{IJ} a_{IJ} E_{IJ}$. If $j \neq r$, then

$$\begin{aligned} \phi(v_L)\tau_j &= \left(\sum_{IJ} a_{IJ} E_{IJ}(v_L)\right)\tau_j = \left(\sum_I a_{IL} v_I\right)\tau_j = \sum_I a_{IL} v_{I.(j,j+1)} \\ &= \sum_I a_{I.(j,j+1),L} v_I \end{aligned}$$

and

$$\phi(v_L \tau_j) = \phi(v_{L.(j,j+1)}) = \sum_{IJ} a_{IJ} E_{IJ}(v_{L.(j,j+1)}) = \sum_I a_{I,L.(j,j+1)} v_I.$$

Comparing coefficients at v_I , we obtain that the above expressions coincide if and only if $a_{I.(j,j+1),L} = a_{I,L.(j,j+1)}$ for each $j \neq r$.

It remains to deal with τ_r . For simplicity of writing, assume $r = s = 1$ (similar arguments remain valid for all values of r, s .) Then

$$\phi(v_L)\tau_r = \left(\sum_{IJ} a_{IJ} E_{IJ}(v_L)\right)\tau_r = \left(\sum_I a_{IL} v_I\right)\tau_r = \sum_i a_{(ii)L} \sum_{t=1}^m v_{(it)}$$

and

$$\begin{aligned} \phi(v_L \tau_r) &= \delta_{l_1 l_2} \sum_{t=1}^m \phi(v_{(it)}) = \delta_{l_1 l_2} \sum_{t=1}^m \sum_I a_{I(it)} E_{I(it)}(v_{(it)}) \\ &= \delta_{l_1 l_2} \sum_{t=1}^m \sum_I a_{I(it)} v_I. \end{aligned}$$

The last two expressions coincide if and only if the following conditions are satisfied.

If $l_1 \neq l_2$, then $\sum_i a_{il_1} \otimes a_{il_2}^* = 0$ (comparing coefficients at $v_{(it)}$).

If $l_1 = l_2$ and $i_1 \neq i_2$, then $\sum_{t=1}^m a_{i_1 t} \otimes a_{i_2 t}^* = 0$ (comparing coefficients at $v_{(i_1 i_2)}$).

If $l_1 = l_2 = l$ and $i_1 = i_2$, then $\sum_i a_{il} \otimes a_{il}^* = \sum_{t=1}^m a_{i_1 t} \otimes a_{i_1 t}^*$ (comparing coefficients at $v_{(i_1 i_1)}$).

The basis \mathcal{B}^* of M^* , dual to \mathcal{B} , consists of coefficient functions $E_{IJ}^* = x_{IJ}$ that are constructed multiplicatively from the coefficient functions $x_{ij} = (e_{ij}^V)^*$ and $x_{ij}^* = (e_{ij}^W)^*$ for $1 \leq i, j \leq m$ in the sense that

$$x_{IJ} = x_{i_1 j_1} \cdots x_{i_r j_r} x_{i_{r+1} j_{r+1}}^* \cdots x_{i_{r+s} j_{r+s}}^*$$

for $I = i_1 \dots i_r i_{r+1} \dots i_{r+s}$ and $J = j_1 \dots j_r j_{r+1} \dots j_{r+s}$.

Let us review the generating equations for the submodule X of M^* given by Lemma 3.2. Due to the multiplicativity of the coefficient functions x_{IJ} , the equations $x_{I.(j,j+1),J} = x_{I,J.(j,j+1)}$ for all I, J and $1 \leq j < r$ are identical to the relations generated by the commutativity relations $x_{ij} x_{kl} = x_{kl} x_{ij}$ for all $1 \leq i, j, k, l \leq m$. The same equations for $r \leq j < r + s$ are identical to the relations generated by the commutativity relations $x_{ij}^* x_{kl}^* = x_{kl}^* x_{ij}^*$ for all $1 \leq i, j, k, l \leq m$.

Using the multiplicativity of x_{IJ} , we derive that the remaining equations for X are generated by the equations $\sum_{k=1}^m x_{ki} x_{kj}^* = 0$ for $i \neq j$, $\sum_{k=1}^m x_{ik} x_{jk}^* = 0$ for $i \neq j$ and $\sum_{k=1}^m x_{ki} x_{ki}^* = \sum_{k=1}^m x_{jk} x_{jk}^*$ for all i, j .

Denote by $F = F_1 \otimes_K F_2$, where F_1 the K -submodule of the free algebra on generators x_{ij} generated by monomials of degree r , and F_2 the K -submodule of the free algebra on generators x_{ij}^* generated by monomials of degree s , where $1 \leq i, j \leq m$.

Let Y the submodule of F generated by relations

$$\begin{aligned}
 x_{ij}x_{kl} &= x_{kl}x_{ij} \text{ for all } i, j; \\
 x_{ij}^*x_{kl}^* &= x_{kl}^*x_{ij}^* \text{ for all } i, j; \\
 \sum_{k=1}^m x_{ki}x_{kj}^* &= 0 \text{ for } i \neq j; \\
 \sum_{k=1}^m x_{ik}x_{jk}^* &= 0 \text{ for } i \neq j; \\
 \sum_{k=1}^m x_{ki}x_{ki}^* &= \sum_{k=1}^m x_{jk}x_{jk}^* \text{ for all } i, j.
 \end{aligned}$$

Then by Lemma 3.2, $((\text{End}_{B_{r,s}(\delta)}(V^{\otimes r} \otimes W^{\otimes s}))^{op})^*$, as a K -module, is isomorphic to F/Y .

Recall the definition of $\tilde{A}(m; r, s)$ on p. 62 of [7]. We write $A(m, r, s)$ instead of $\tilde{A}(m; r, s)$ for simplicity. Let c_{ij} be the coefficient functions of the generic matrix C and

$$d_{ij} = (-1)^{l+k} \frac{C[1, \dots, \hat{k}, \dots, m | 1, \dots, \hat{l}, \dots, m]}{C[1, \dots, m | 1, \dots, m]}$$

be the coefficient functions of the matrix C^{-1} . Then $A(m, r, s)$ is the subspace (and a sub-coalgebra) of $K[GL(m)]$ spanned by all products of the form $\prod_{ij} c_{ij}^{a_{ij}} \prod_{ij} d_{ij}^{b_{ij}}$ such that $a_{ij}, b_{ij} \geq 0, \sum_{ij} a_{ij} = r$ and $\sum_{ij} b_{ij} = s$.

The generators c_{ij} and d_{ij} commute and are subject to additional relations $\sum_{j=1}^m c_{ik}d_{kj} = \delta_{ij}$ and $\sum_{k=1}^m c_{ki}d_{jk} = \delta_{ij}$.

In particular,

$$\sum_{k=1}^m c_{ik}d_{kj} = 0 \text{ and } \sum_{k=1}^m c_{ki}d_{jk} = 0 \text{ if } i \neq j,$$

and

$$\sum_{k=1}^m c_{ik}d_{ki} = \sum_{k=1}^m c_{kj}d_{jk} \text{ for all } 1 \leq i, j \leq m.$$

Since we need to have $\sum_{ij} a_{ij} = r$ and $\sum_{ij} b_{ij} = s$, we cannot equate the terms $\sum_{k=1}^m c_{ik}d_{ki} = \sum_{k=1}^m c_{kj}d_{jk}$ to 1 since it would reduce the corresponding degrees to less than r and s .

Therefore, the K -space F/Y is isomorphic to $A(m, r, s)$ via

$$\prod_{ij} x_{ij}^{a_{ij}} \otimes \prod_{ij} (x_{ij}^*)^{b_{ij}} \mapsto \prod_{ij} c_{ji}^{a_{ij}} \prod_{ij} (d_{ij})^{b_{ij}}.$$

Please note the use of the transposition $x_{ij} \mapsto c_{ji}$. Instead, we could have used a transposition in the second component $x_{ij}^* \mapsto d_{ji}$.

In particular, the dimension of $S(m, r, s)$ is the same of that of $A(m, r, s)$ and $\text{End}_{B_{r,s}(\delta)}(V^{\otimes r} \otimes W^{\otimes s})^{op}$.

Therefore, the morphism $\Psi_{r,s} : \text{Dist}(G) \rightarrow \text{End}_{B_{r,s}(\delta)}(V^{\otimes r} \otimes W^{\otimes s})^{op}$ is surjective, and $S(m, r, s) \simeq \text{End}_{B_{r,s}(\delta)}(V^{\otimes r} \otimes W^{\otimes s})^{op}$. □

5 The Second Half of the Super Schur-Weyl Duality

The second part of super Schur-Weyl duality states that $S(m|n, r)$ is isomorphic to $End_{K\Sigma_r}(V^{\otimes r})^{op}$.

Let I, J be multi-indices of length r with entries from the set $\{1, \dots, m + n\}$. Denote by E_{IJ} the matrix unit given by $E_{IJ}(v_K) = \delta_{JK}v_I$. The elements E_{IJ} form a K -basis of the space $End_K(V^{\otimes r})$. For $\phi \in End_K(V^{\otimes r})$ we write $\phi = \sum_{IJ} a_{IJ}E_{IJ}$ for appropriate coefficients a_{IJ} .

Proposition 5.1 *The Schur superalgebra $S(m|n, r)$ is isomorphic to $End_{K\Sigma_r}(V^{\otimes r})^{op}$.*

Proof We show that the dual of $End_{K\Sigma_r}(V^{\otimes r})^{op}$ is isomorphic to the supercoalgebra $A(m|n, r)$.

We modify the arguments from the proof Proposition 3.3. We apply Lemma 3.2 for $R = K$, $M = End_K(V^{\otimes r})$, the basis \mathcal{B} consisting of all elements E_{IJ} for multi-indices $I = i_1 \dots i_r$ and $J = j_1 \dots j_r$ of length r , and $U = End_{K\Sigma_r}(V^{\otimes r})$.

Let $\phi \in U$ and $\phi = \sum_{IJ} a_{IJ}E_{IJ}$.

If $1 \leq j < r$, then

$$\begin{aligned} \phi(v_L)\tau_j &= \left(\sum_{IJ} a_{IJ}E_{IJ}(v_L)\right)\tau_j = \left(\sum_I a_{IL}v_I\right)\tau_j \\ &= \sum_I (-1)^{|j||i_{j+1}|} a_{IL}v_{I.(j,j+1)} = \sum_I (-1)^{|j||i_{j+1}|} a_{I.(j,j+1),L}v_I \end{aligned}$$

and

$$\begin{aligned} \phi(v_L\tau_j) &= \phi((-1)^{|j||i_{j+1}|}v_{L.(j,j+1)}) = (-1)^{|j||i_{j+1}|} \sum_{IJ} a_{IJ}E_{IJ}(v_{L.(j,j+1)}) \\ &= \sum_I (-1)^{|j||i_{j+1}|} a_{I,L.(j,j+1)}v_I. \end{aligned}$$

Thus ϕ is invariant under the transposition τ_j if and only if

$$(-1)^{|j||i_{j+1}|} a_{I.(j,j+1),L} = (-1)^{|j||i_{j+1}|} a_{I,L.(j,j+1)}$$

for each I, L . Therefore, U is described by the set of equations in the coefficients a_{IJ} of $\phi \in U$ as $\phi \in U$ if and only if

$$(-1)^{|j||i_{j+1}|} a_{I.(j,j+1),L} = (-1)^{|j||i_{j+1}|} a_{I,L.(j,j+1)} \text{ for each } I, L \text{ and } 1 \leq j < r.$$

The basis \mathcal{B}^* of M^* , dual to \mathcal{B} , consists of coefficient functions $E_{IJ}^* = x_{IJ}$ that are constructed multiplicatively from the coefficient functions $x_{ij} = e_{ij}^*$, meaning that $x_{IJ} = x_{i_1j_1} \dots x_{i_rj_r}$ for $I = i_1 \dots i_r$ and $J = j_1 \dots j_r$.

The submodule X of M^* is generated by equations

$$(-1)^{|j||i_{j+1}|} x_{I.(j,j+1),L} = (-1)^{|j||i_{j+1}|} x_{I,L.(j,j+1)}$$

for all I, L and $1 \leq j < r$. Due to the multiplicativity of the coefficient functions, these equations are identical to the relations generated by the commutativity relations

$$(-1)^{|i||k|} x_{ij}x_{kl} = (-1)^{|j||l|} x_{kl}x_{ij}$$

for all $1 \leq i, j, k, l \leq m$.

The last relations are not the usual supercommutativity relations between coordinate functions. To get the usual supercommutativity relations, we need to replace the basis \mathcal{B} by a different basis. To motivate this switch, recall that the coefficient functions of $V^{\otimes r}$ for the basis consisting of v_I are given as

$$\chi_{IJ} = (-1)^{\sum_{t=1}^r |i_t|(|i_{t+1}|+|j_{t+1}|+\dots+|i_r|+|j_r|)} c_{IJ},$$

where $c_{IJ} = c_{i_1, j_1} \dots c_{i_r, j_r}$ is the product of the matrix coefficient functions. The functions χ_{IJ} are easier to work with because the comultiplication in terms of χ_{IJ} is nicer than in terms of c_{IJ} .

Define a different basis \mathfrak{B} of $End_K(V^{\otimes r})$ consisting of elements

$$F_{IJ} = (-1)^{\sum_{t=1}^r |i_t|(|i_{t+1}|+|j_{t+1}|+\dots+|i_r|+|j_r|)} E_{IJ}$$

and write $\phi = \sum_{IJ} b_{IJ} F_{IJ}$. Then $b_{IJ} = (-1)^{\sum_{t=1}^r |i_t|(|i_{t+1}|+|j_{t+1}|+\dots+|i_r|+|j_r|)} a_{IJ}$.

If we set $F_{IJ}^* = y_{IJ}$ and $y_{ij} = f_{ij}^*$, the relation $x_{ij}x_{kl} = (-1)^{|i||k|+|j||l|} x_{kl}x_{ij}$ is equivalent to

$$\begin{aligned} y_{ij}y_{kl} &= (-1)^{|i||k|+|i||l|} x_{ij}x_{kl} = (-1)^{|i||k|+|i||l|+|i||k|+|j||l|} x_{kl}x_{ij} = (-1)^{(|i||l|+|j||l|)} x_{kl}x_{ij} \\ &= (-1)^{|i||l|+|j||l|+|k||i|+|k||j|} y_{kl}y_{ij} = (-1)^{(|i|+|j|)(|k|+|l|)} y_{kl}y_{ij} \\ &= (-1)^{|y_{ij}||y_{kl}|} y_{kl}y_{ij}, \end{aligned}$$

which is the usual supercommutativity relation between y_{ij} and y_{kl} .

Let F be the free algebra on generators y_{ij} for $1 \leq i, j \leq m$ and Y the submodule of F generated by elements $y_{ij}y_{kl} - (-1)^{|y_{ij}||y_{kl}|} y_{kl}y_{ij}$ for all $1 \leq i, j, k, l \leq m$. Then M^*/X , as a K -vector space, is isomorphic to the degree r component D_r of the algebra F/Y . Therefore, $(End_{K\Sigma_r}(V^{\otimes r}))^*$, considered as a K -space, is isomorphic to D_r by Lemma 3.2. On the other hand, $D_r \simeq A(m|n, r)$, as K -spaces, via the isomorphism $\prod_{ij} c_{ij}^{a_{ij}} \mapsto \prod_{ij} y_{ij}^{a_{ij}}$. This shows that $S(m|n, r) = A(m|n, r)^*$ has the same dimension as $End_{K\Sigma_r}(V^{\otimes r})^{op}$. We conclude that Ψ_r is surjective and $S(m|n, r) \simeq End_{K\Sigma_r}(V^{\otimes r})^{op}$. \square

6 The Second Half of the Mixed Super Schur-Weyl Duality

Let v_1, \dots, v_{m+n} be a K -basis of V , and v_1^*, \dots, v_{m+n}^* be the corresponding dual basis of W . Recall that the standard matrix units e_{ij} act on V and W as $e_{ij}v_k = \delta_{jk}v_i$ and $e_{ij}v_k^* = -\delta_{ik}(-1)^{|i|(|i|+|j|)}v_j^*$.

Let $r, s \geq 0$ and I, J, K be multi-indices of length $r + s$ with entries from the set $\{1, \dots, m + n\}$. We write every multi-index $I = (i_1, \dots, i_{r+s})$ as a concatenation $I = I_V I_W$, where $I_V = (i_1 \dots i_r)$ and $I_W = (i_{r+1} \dots i_{r+s})$. The elements $v_I = v_{I_V} v_{I_W} = v_{i_1} \dots v_{i_r} v_{i_{r+1}}^* \dots v_{i_{r+s}}^*$ for all multi-indices I of length $r + s$ form a basis of $V^{\otimes r} \otimes W^{\otimes s}$.

Since

$$End_K(V^{\otimes r} \otimes W^{\otimes s}) \simeq End_K(V^{\otimes r}) \otimes End_K(W^{\otimes s}) \simeq End_K(V)^{\otimes r} \otimes End_K(W)^{\otimes s},$$

every map from $End_K(V^{\otimes r} \otimes W^{\otimes s})$ is a linear combination of maps $\phi_{V^{\otimes r}} \otimes \phi_{W^{\otimes s}}$, where $\phi_{V^{\otimes r}} \in End_K(V^{\otimes r})$ and $\phi_{W^{\otimes s}} \in End_K(W^{\otimes s})$.

Denote by E_{IJ}^V the matrix unit given by $E_{IJ}^V(v_{LV}) = \delta_{J_V L_V} v_{IV}$, and by E_{IJ}^W the matrix unit given by $E_{IJ}^W(v_{LW}) = \delta_{J_W L_W} v_{IW}$. Then the elements $E_{IJ} = E_{IJ}^V E_{IJ}^W$ form a K -basis of the space $End_K(V^{\otimes r} \otimes W^{\otimes s})$. For $\phi \in End_K(V^{\otimes r} \otimes W^{\otimes s})$ we write $\phi = \sum_{IJ} a_{IJ} E_{IJ}$ for appropriate coefficients a_{IJ} .

We want to determine the dual of $End_{B_{r,s}(\delta)}(V^{\otimes r} \otimes W^{\otimes s})^{op}$ and compare it with $A(m|n, r, s)$.

6.1 $(End_{B_{r,s}(\delta)}(V^{\otimes r} \otimes W^{\otimes s})^{op})^*$ Expressed by Generators and Relations

We apply Lemma 3.2 for $R = K$, $M = End_K(V^{\otimes r} \otimes W^{\otimes s})$, the basis \mathcal{B} consisting of all elements E_{IJ} for multi-indices $I = i_1 \dots i_r i_{r+1} \dots i_{r+s}$, $J = j_1 \dots j_r j_{r+1} \dots j_{r+s}$, and $U = End_{B_{r,s}(\delta)}(V^{\otimes r} \otimes W^{\otimes s})$.

Assume $\phi \in U$ and write $\phi = \sum_{IJ} a_{IJ} E_{IJ}$. If $j \neq r$, then

$$\begin{aligned} \phi(v_L)\tau_j &= \left(\sum_{IJ} a_{IJ} E_{IJ}(v_L)\right)\tau_j = \left(\sum_I a_{IL} v_I\right)\tau_j \\ &= \sum_I (-1)^{|i_j||i_{j+1}|} a_{IL} v_{I.(j,j+1)} = \sum_I (-1)^{|i_j||i_{j+1}|} a_{I.(j,j+1),L} v_I \end{aligned}$$

and

$$\begin{aligned} \phi(v_L\tau_j) &= \phi((-1)^{|i_j||i_{j+1}|} v_{L.(j,j+1)}) = (-1)^{|i_j||i_{j+1}|} \sum_{IJ} a_{IJ} E_{IJ}(v_{L.(j,j+1)}) \\ &= \sum_I (-1)^{|i_j||i_{j+1}|} a_{I,L.(j,j+1)} v_I. \end{aligned}$$

Comparing coefficients at v_I , we obtain that the above expressions coincide if and only if

$$(-1)^{|i_j||i_{j+1}|} a_{I.(j,j+1),L} = (-1)^{|i_j||i_{j+1}|} a_{I,L.(j,j+1)}$$

for each $j \neq r$.

It remains to deal with τ_r . For simplicity of writing, assume $r = s = 1$ (analogous arguments remain valid for all values of r, s .) Then

$$\phi(v_L)\tau_r = \left(\sum_{IJ} a_{IJ} E_{IJ}(v_L)\right)\tau_r = \left(\sum_I a_{IL} v_I\right)\tau_r = \sum_i (-1)^{|i|+1} a_{(ii)L} \sum_{t=1}^{m+n} v_{(it)}$$

and

$$\begin{aligned} \phi(v_L\tau_r) &= \delta_{l_1 l_2} (-1)^{|l_1|+1} \sum_{t=1}^{m+n} \phi(v_{(tt)}) = \delta_{l_1 l_2} (-1)^{|l_1|+1} \sum_{t=1}^{m+n} \sum_I a_{I(tt)} E_{I(tt)}(v_{(tt)}) \\ &= \delta_{l_1 l_2} (-1)^{|l_1|+1} \sum_{t=1}^{m+n} \sum_I a_{I(tt)} v_I. \end{aligned}$$

The last two expressions coincide if and only if the following conditions are satisfied.

If $l_1 \neq l_2$, then $\sum_i (-1)^{|i|} a_{i l_1} \otimes a_{i l_2}^* = 0$ (comparing coefficients at $v_{(tt)}$).

If $l_1 = l_2 = l$ and $i_1 \neq i_2$, then $(-1)^{|l|} \sum_{t=1}^{m+n} a_{i_1 t} \otimes a_{i_2 t}^* = 0$ (comparing coefficients at $v_{(i_1 i_2)}$).

If $l_1 = l_2 = l$ and $i_1 = i_2$, then $\sum_i (-1)^{|i|} a_{il} \otimes a_{il}^* = (-1)^{|l|} \sum_{t=1}^{m+n} a_{i_1 t} \otimes a_{i_1 t}^*$ (comparing coefficients at $v_{(i_1 i_1)}$).

The basis \mathcal{B}^* of M^* , dual to \mathcal{B} , consists of coefficient functions $E_{IJ}^* = x_{IJ}$ that are constructed multiplicatively from the coefficient functions $x_{ij} = (e_{ij}^V)^*$ and $x_{ij}^* = (e_{ij}^W)^*$ for $1 \leq i, j \leq m+n$ in the sense that

$$x_{IJ} = x_{i_1 j_1} \cdots x_{i_r j_r} x_{i_{r+1} j_{r+1}} \cdots x_{i_{r+s} j_{r+s}}$$

for $I = i_1 \dots i_r i_{r+1} \dots i_{r+s}$ and $J = j_1 \dots j_r j_{r+1} \dots j_{r+s}$.

Next, we look at the generating equations for the submodule X of M^* given by Lemma 3.2. Due to the multiplicativity of the coefficient functions x_{IL} , the equations

$$(-1)^{|i_j||i_{j+1}|} x_{I.(j,j+1),L} = (-1)^{|l_j||l_{j+1}|} x_{I,J.(j,j+1)}$$

for all I, L and $1 \leq j < r$ are identical to the relations generated by the commutativity relations

$$(-1)^{|i_j||i_{j+1}|} x_{i_{j+1} l_j} x_{i_j l_{j+1}} = (-1)^{|l_j||l_{j+1}|} x_{i_j l_{j+1}} x_{i_{j+1} l_j}$$

which are equivalent to

$$x_{ij} x_{kl} = (-1)^{|i||k|+|j||l|} x_{kl} x_{ij}$$

for all $1 \leq i, j, k, l \leq m+n$. The same equations for $r \leq j < r+s$ are identical to the relations generated by the commutativity relations

$$x_{ij}^* x_{kl}^* = (-1)^{|i||k|+|j||l|} x_{kl}^* x_{ij}^*$$

for all $1 \leq i, j, k, l \leq m+n$.

Using the multiplicativity of x_{IL} , we derive that the remaining equations for X are generated by the equations $\sum_{k=1}^{m+n} (-1)^{|k|} x_{ki} x_{kj}^* = 0$ for $i \neq j$, $\sum_{k=1}^{m+n} x_{ik} x_{jk}^* = 0$ for $i \neq j$, and $\sum_{k=1}^{m+n} (-1)^{|k|} x_{ki} x_{ki}^* = (-1)^{|i|} \sum_{k=1}^{m+n} x_{jk} x_{jk}^*$ for all $1 \leq i, j \leq m+n$.

Using Lemma 3.2, we establish the following statement.

Proposition 6.1 Denote by $F = F_1 \otimes_K F_2$, where F_1 is the K -submodule of the free algebra on generators x_{ij} generated by monomials of degree r , and F_2 is the K -submodule of the free algebra on generators x_{ij}^* generated by monomials of degree s , where $1 \leq i, j \leq m+n$.

Let Y the submodule of F generated by relations

$$x_{ij} x_{kl} = (-1)^{|i||k|+|j||l|} x_{kl} x_{ij} \text{ for all } i, j;$$

$$x_{ij}^* x_{kl}^* = (-1)^{|i||k|+|j||l|} x_{kl}^* x_{ij}^* \text{ for all } i, j;$$

$$\sum_{k=1}^{m+n} (-1)^{|k|} x_{ki} x_{kj}^* = 0 \text{ for } i \neq j;$$

$$\sum_{k=1}^{m+n} x_{ik} x_{jk}^* = 0 \text{ for } i \neq j;$$

$$\sum_{k=1}^{m+n} (-1)^{|k|} x_{ki} x_{ki}^* = (-1)^{|i|} \sum_{k=1}^{m+n} x_{jk} x_{jk}^* \text{ for all } i, j.$$

Then $((\text{End}_{B_{r,s}(\delta)}(V^{\otimes r} \otimes W^{\otimes s}))^{op})^*$, as a K -space, is isomorphic to F/Y .

6.2 Mixed Super Schur-Weyl Duality

Proposition 6.2 *There is $S(m|n, r, s) \simeq \text{End}_{B_{r,s}(\delta)}(V^{\otimes r} \otimes W^{\otimes s})^{op}$.*

Proof Analogously to the second half of the proof of Proposition 5.1, we define a different basis of $\text{End}_K(V^{\otimes r})$ consisting of elements

$$F_{IJ}^V = (-1)^{\sum_{t=1}^r |i_t|(|i_{t+1}|+|j_{t+1}|+\dots+|i_r|+|j_r|)} E_{IJ}^V,$$

a different basis of $\text{End}_K(W^{\otimes s})$ consisting of elements

$$F_{IJ}^W = (-1)^{\sum_{t=r+1}^{r+s} |i_t|(|i_{t+1}|+|j_{t+1}|+\dots+|i_{r+s}|+|j_{r+s}|)} E_{IJ}^W,$$

which gives a different basis of $\text{End}_K(V^{\otimes r} \otimes W^{\otimes s})$ consisting of elements $F_{IJ} = F_{IJ}^V F_{IJ}^W$.

If we write $\phi \in \text{End}_K(V^{\otimes r} \otimes W^{\otimes s})$ as $\phi = \sum_{IJ} b_{IJ} F_{IJ}$, then

$$b_{IJ} = (-1)^{\sum_{t=1}^r |i_t|(|i_{t+1}|+|j_{t+1}|+\dots+|i_r|+|j_r|)+\sum_{t=r+1}^{r+s} |i_t|(|i_{t+1}|+|j_{t+1}|+\dots+|i_{r+s}|+|j_{r+s}|)} a_{IJ}.$$

We define $y_{ij} = (f_{ij}^V)^*$, $y_{ij}^* = (f_{ij}^W)^*$ for $1 \leq i, j \leq m+n$ and

$$y_{IJ} = (F_{IJ})^* = y_{i_1 j_1} \dots y_{i_r j_r} y_{i_{r+1} j_{r+1}}^* \dots y_{i_{r+s} j_{r+s}}^*.$$

In the same way as in the second half of the proof of Proposition 5.1, we obtain that the relation $x_{ij} x_{kl} = (-1)^{|i||k|+|j||l|} x_{kl} x_{ij}$ is equivalent to

$$y_{ij} y_{kl} = (-1)^{|y_{ij}||y_{kl}|} y_{kl} y_{ij},$$

which is the usual supercommutativity relations between y_{ij} and y_{kl} . Analogously, the relation $x_{ij}^* x_{kl}^* = (-1)^{|i||k|+|j||l|} x_{kl}^* x_{ij}^*$ is equivalent to

$$y_{ij}^* y_{kl}^* = (-1)^{|y_{ij}^*||y_{kl}^*|} y_{kl}^* y_{ij}^*.$$

The remaining relations are rewritten in terms of y_{ij} 's and y_{ij}^* 's as follows.

$$\begin{aligned} \sum_{k=1}^{m+n} (-1)^{|i|(|k|+1)} y_{ki} y_{kj}^* &= \sum_{k=1}^{m+n} (-1)^{|k|+|i|} (-1)^{|k|(|k|+|i|)} y_{ki} y_{kj}^* \\ &= (-1)^{|i|} \sum_{k=1}^{m+n} (-1)^{|k|} x_{ki} x_{kj}^* = 0 \end{aligned}$$

for $i \neq j$,

$$\sum_{k=1}^{m+n} (-1)^{|j|(|i|+|k|)} y_{ij} y_{jk}^* = \sum_{k=1}^{m+n} x_{ik} x_{jk}^* = 0$$

for $i \neq j$, and

$$\begin{aligned} \sum_{k=1}^{m+n} (-1)^{|i|(|k|+1)} y_{ki} y_{ki}^* &= \sum_{k=1}^{m+n} (-1)^{|k|+|i|} (-1)^{|k|(|k|+|i|)} y_{ki} y_{ki}^* \\ &= (-1)^{|i|} \sum_{k=1}^{m+n} (-1)^k x_{ki} x_{ki}^* = \sum_{k=1}^{m+n} x_{jk} x_{jk}^* = \sum_{k=1}^{m+n} (-1)^{|j|(|j|+|k|)} y_{jk} y_{jk}^* \end{aligned}$$

for every i, j .

Therefore, $((\text{End}_{B_{r,s}(\delta)}(V^{\otimes r} \otimes W^{\otimes s}))^{op})^*$, as a K -space, is isomorphic to F/Y , where F is as above, and Y is generated by the supercommutativity relations

$$y_{ij}y_{kl} = (-1)^{|y_{ij}||y_{kl}|}y_{kl}y_{ij} \text{ for all } i, j;$$

$$y_{ij}^*y_{kl}^* = (-1)^{|y_{ij}^*||y_{kl}^*|}y_{kl}^*y_{ij}^* \text{ for all } i, j,$$

and the additional relations

$$\sum_{k=1}^{m+n} (-1)^{|i|(|k|+1)}y_{ki}y_{kj}^* = 0 \text{ for } i \neq j;$$

$$\sum_{k=1}^{m+n} (-1)^{|j|(|i|+|k|)}y_{ij}y_{jk}^* = 0 \text{ for } i \neq j;$$

$$\sum_{k=1}^{m+n} (-1)^{|i|(|k|+1)}y_{ki}y_{ki}^* = \sum_{k=1}^{m+n} (-1)^{|j|(|j|+|k|)}y_{jk}y_{jk}^* \text{ for all } i, j.$$

We will show that the K -spaces F/Y and $A(m|n, r, s)$ are isomorphic.

Recall that we have denoted by c_{ij} the coefficient functions of the generic matrix C and by

$$d_{ij} = (-1)^{l+k} \frac{C[1, \dots, \widehat{k}, \dots, m|1, \dots, \widehat{l}, \dots, m]}{C[1, \dots, m|1, \dots, m]}$$

the coefficient functions of the matrix C^{-1} . Then $A(m|n, r, s)$ is the subspace (and a sub-coalgebra) of $K[GL(m|n)]$ spanned by all products of the form $\prod_{ij} c_{ij}^{a_{ij}} \prod_{ij} d_{ij}^{b_{ij}}$ such that $a_{ij}, b_{ij} \geq 0, \sum_{ij} a_{ij} = r$ and $\sum_{ij} b_{ij} = s$.

The generators c_{ij} and d_{ij} supercommute and are subject to additional relations $\sum_{j=1}^m c_{ik}d_{kj} = \delta_{ij}$ and $\sum_{k=1}^m d_{jk}c_{ki} = \delta_{ij}$.

In particular,

$$\sum_{k=1}^m c_{ik}d_{kj} = 0 \text{ and } \sum_{k=1}^m (-1)^{|c_{ki}||d_{jk}|}c_{ki}d_{jk} = 0 \text{ if } i \neq j,$$

and

$$\sum_{k=1}^m c_{ik}d_{ki} = \sum_{k=1}^m (-1)^{|c_{kj}||d_{jk}|}c_{kj}d_{jk} \text{ for all } 1 \leq i, j \leq m.$$

Since we need to have $\sum_{ij} a_{ij} = r$ and $\sum_{ij} b_{ij} = s$, we cannot equate the terms $\sum_{k=1}^m c_{ik}d_{ki} = \sum_{k=1}^m c_{kj}d_{jk}$ to 1 since it would reduce the corresponding degrees to less than r and s .

The isomorphism of K -spaces F/Y and $A(m|n, r, s)$ is given using a supertransposition $y_{ij} \mapsto (-1)^{|i|(|j|+1)}c_{ji}$ and $y_{ij}^* \mapsto d_{ij}$.

More explicitly, the above-defined map extends multiplicatively to an isomorphism

$$\prod_{ij} y_{ij}^{a_{ij}} \otimes \prod_{ij} (y_{ij}^*)^{b_{ij}} \mapsto \prod_{ij} ((-1)^{|i|(|j|+1)}c_{ji})^{a_{ij}} \prod_{ij} (d_{ij})^{b_{ij}}.$$

Thus, the dimension of $S(m|n, r, s)$ is the same of that of $A(m|n, r, s)$ and $\text{End}_{B_{r,s}(\delta)}(V^{\otimes r} \otimes W^{\otimes s})^{op}$.

Therefore, the morphism $\Psi_{r,s} : Dist(GL(m|n)) \rightarrow End_{B_{r,s}(\delta)}(V^{\otimes r} \otimes W^{\otimes s})^{op}$ is surjective, and $S(m|n, r, s) \simeq End_{B_{r,s}(\delta)}(V^{\otimes r} \otimes W^{\otimes s})^{op}$. \square

7 The First Half of the Schur-Weyl Duality

Recall the definitions of Φ_r and $\Phi_{r,s}$ from Section 2.

Theorem 7.1 *The morphism Φ_r is surjective. It is injective if and only if $r \leq m$.*

Proof The statement is the first part of the classical Schur-Weyl duality. See Propositions 11 and 15 of [10] for proof. \square

The description of the kernel of Φ_r is given in Lemma 3 of [12].

Theorem 7.2 *(Theorem 7.8 of [3]) Assume the characteristic of the ground field K is zero. Then the map $\Phi_{r,s}$ is surjective. It is injective if and only if $r + s < (m + 1)(n + 1)$.*

To prove the above theorems using elementary methods analogous to those used earlier, we first need to describe $End_G(V^{\otimes r})$ and $End_G(V^{\otimes r} \otimes W^{\otimes s})$. However, even finding the dimensions of these spaces is a nontrivial problem. The action of $K \Sigma_r$ on $V^{\otimes r}$ is faithful if and only if $r \geq m$. Therefore, the dimension of $End_G(V^{\otimes r})$ is $r!$ if $r \geq m$, but it is unclear what this dimension is if $r < m$. For example, if $m = 2$ and $r = 3$, this dimension is 5. If $m \geq r = 3$, the dimension is $r! = 6$. However, even for $m = r = 3$, a direct verification involves a fair amount of computations. Similar obstacles appear in the description of the dimensions of $End_G(V^{\otimes r} \otimes W^{\otimes s})$.

For simplicity, we consider only $End_{GL(m)}(V^{\otimes r})$ and assume that the characteristic of K is zero.

An element of $\phi \in End_{GL(m)}(V^{\otimes r})$ is an element ϕ of $End_K(V^{\otimes r})$ invariant under the action of standard matrix units e_{ij} from the general linear Lie algebra $\mathfrak{gl}(m)$ for all $1 \leq i, j \leq m$.

We write $\phi = \sum_{IJ} a_{IJ} E_{IJ}$ and evaluate

$$\begin{aligned} e_{ij}\phi(v_L) &= e_{ij} \sum_{IJ} a_{IJ} E_{IJ}(v_L) = e_{ij} \sum_I a_{IL} v_I = \sum_I a_{IL} e_{ij} v_I \\ &= \sum_I a_{IL} \sum_{t=1}^r v_{i_1} \otimes \dots \otimes e_{ij} v_{i_t} \otimes \dots \otimes v_{i_r} \\ &= \sum_I a_{IL} \sum_{t=1}^r \delta_{i_t, j} v_{i_1} \otimes \dots \otimes v_{i_t} \otimes \dots \otimes v_{i_r} = \sum_I a_{IL} \sum_{t=1}^r \delta_{i_t, j} v_{i_1 \dots i_{t-1} i_{t+1} \dots i_r} \end{aligned}$$

and

$$\begin{aligned}
 \phi(e_{ij}v_L) &= \phi\left(\sum_{t=1}^r v_{l_1} \otimes \dots \otimes e_{ij}v_{l_t} \otimes \dots \otimes v_{l_r}\right) \\
 &= \phi\left(\sum_{t=1}^r \delta_{l_t,j} v_{l_1} \otimes \dots \otimes v_i \otimes \dots \otimes v_{l_r}\right) \\
 &= \sum_{t=1}^r \delta_{l_t,j} \sum_{IJ} a_{IJ} E_{IJ}(v_{l_1} \otimes \dots \otimes v_i \otimes \dots \otimes v_{l_r}) \\
 &= \sum_{t=1}^r \delta_{l_t,j} \sum_I a_{I(l_1 \dots l_{t-1} i l_{t+1} \dots l_r)} v_I.
 \end{aligned}$$

The equality $e_{ij}\phi(v_L) = \phi(e_{ij}v_L)$ is satisfied if and only if, for each multi-index I , the coefficients at v_I 's in the above two expressions are the same.

The following example shows that the direct comparison of coefficients yields the first half of the Schur-Weyl duality in some simple cases.

Example 7.3 Assume $m = 2, r = 2$. Then $End_{GL(2)}(V^{\otimes 2}) \simeq K \Sigma_2$.

Proof Consider cases

$$i = j = 1$$

$$\overline{l_1} = \overline{l_2} = 1:$$

$$E_1 = 2a_{11|11}v_{11} + a_{12|11}v_{12} + a_{21|11}v_{21}$$

$$E_2 = 2a_{11|11}v_{11} + 2a_{12|11}v_{12} + 2a_{21|11}v_{21} + 2a_{22|11}v_{22}$$

$$\text{implies } a_{12|11} = a_{21|11} = a_{22|11} = 0.$$

$$l_1 = 1, l_2 = 2:$$

$$E_1 = 2a_{11|12}v_{11} + a_{12|12}v_{12} + a_{21|12}v_{21}$$

$$E_2 = a_{11|12}v_{11} + a_{12|12}v_{12} + a_{21|12}v_{21} + a_{22|12}v_{22}$$

$$\text{implies } a_{11|12} = a_{22|12} = 0.$$

$$l_1 = 2, l_2 = 1:$$

$$E_1 = 2a_{11|21}v_{11} + a_{12|21}v_{12} + a_{21|21}v_{21}$$

$$E_2 = a_{11|21}v_{11} + a_{12|21}v_{12} + a_{21|21}v_{21} + a_{22|21}v_{22}$$

$$\text{implies } a_{11|21} = a_{22|21} = 0.$$

$$l_1 = l_2 = 2:$$

$$E_1 = 2a_{11|22}v_{11} + a_{12|22}v_{12} + a_{21|22}v_{21}$$

$$E_2 = 0$$

$$\text{implies } a_{11|22} = a_{12|22} = a_{21|22} = 0.$$

That means only values unassigned are $a_{11|11}, a_{12|12}, a_{21|12}, a_{12|21}, a_{21|21}$ and $a_{22|22}$.

$$i = 1, j = 2$$

$$\overline{l_1} = \overline{l_2} = 1:$$

$$E_1 = a_{12|11}v_{11} + a_{21|11}v_{11} + a_{22|11}v_{12} + a_{22|11}v_{21} = 0$$

$$E_2 = 0$$

$$l_1 = 1, l_2 = 2:$$

$$E_1 = a_{12|12}v_{11} + a_{21|12}v_{11} + a_{22|12}v_{12} + a_{22|12}v_{21}$$

$$E_2 = a_{11|11}v_{11} + a_{12|11}v_{12} + a_{21|11}v_{21} + a_{22|11}v_{22} = a_{11|11}v_{11}$$

$$\text{implies } a_{12|12} + a_{21|12} = a_{11|11}$$

$$l_1 = 2, l_2 = 1:$$

$$E_1 = a_{12|21}v_{11} + a_{21|21}v_{11} + a_{22|21}v_{12} + a_{22|21}v_{21}$$

$$E_2 = a_{11|11}v_{11} + a_{12|11}v_{12} + a_{21|11}v_{21} + a_{22|11}v_{22} = a_{11|11}v_{11}$$

implies $a_{12|21} + a_{21|21} = a_{11|11}$
 $l_1 = l_2 = 2$:
 $E_1 = a_{12|22}v_{11} + a_{21|22}v_{11} + a_{22|22}v_{12} + a_{22|22}v_{21} = a_{22|22}v_{21}$
 $E_2 = a_{11|12}v_{11} + a_{11|21}v_{11} + a_{12|12}v_{12} + a_{12|21}v_{12} + a_{21|12}v_{21} + a_{21|21}v_{21} + a_{22|12}v_{22} + a_{22|21}v_{22}$
implies $a_{12|12} + a_{12|21} = a_{22|22}$ and $a_{21|12} + a_{21|21} = a_{22|22}$.
As a consequence, we also get $a_{11|11} = a_{22|22}$, $a_{21|12} = a_{12|21}$ and $a_{12|12} = a_{21|21}$.
The general solution of $a_{12|12} + a_{12|21} = a_{11|11}$ is a linear combination of two solutions:
 $a_{12|12} = a_{12|21} = 1$ and $a_{11|11} = 2$ (this corresponds to symmetric tensors) and
 $a_{12|12} = 1$, $a_{12|21} = -1$ and $a_{11|11} = 0$ (this corresponds to antisymmetric tensors).
Thus $\text{End}_{GL(2)}(V^{\otimes 2}) \simeq K\Sigma_2$ as expected. \square

For comparison, over arbitrary $GL(m)$, there is a well-known decomposition of $V^{\otimes 2} \simeq \text{Sym}^2(V) \oplus \Lambda^2(V)$ into irreducible symmetric and antisymmetric tensors.

A better approach to the first half of Schur-Weyl duality is to consider the action of specific elements $g \in GL(m)$ on $V^{\otimes r}$ as in Lemma 3.1 of [4] and Theorem 2.1 of [1].

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