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RESEARCH

Hammocks for Non-Domestic String Algebras

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Abstract
We show that the order type of the simplest version of a hammock for string algebras lies in the class of *finite description* linear orders–the smallest class of linear orders containing **0**, **1**, and that is closed under isomorphisms, finite order sum, anti-lexicographic product with ω and ω^*, and shuffle of finite subsets–using condensation (localization) of linear orders as a tool. We also introduce two finite subsets of the set of bands and use them to describe the location of left $\mathbb{N}$-strings in the completion of hammocks.

Keywords String algebra · non-domestic · hammock · finite description linear order

Mathematics Subject Classification (2010) 16G20

1 Introduction

Let Λ be a string algebra over an algebraically closed field $\mathcal{K}$. Brenner [2] introduced certain partially ordered sets known as hammocks to study factorization of maps between finite dimensional indecomposable right Λ-modules. The simplest version of hammocks introduced by Schröer [10, § 3] in the context of string algebras are bounded discrete linear orders–this is the only type of hammock we will deal with in this paper. We compute the order type of a hammock for Λ in terms of some standard order types, thus generalizing (one direction of) the main result of Sardar and the second author from [11] that only dealt with the case when Λ is domestic.

The algebra Λ is domestic if and only if there are only finitely many bands for it. These bands are vertices of a finitary combinatorial gadget known as the bridge quiver [10, § 4]–its slight modification was used for the explicit computation of the order type in the domestic

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case. To extend the concept of a (finitary) bridge quiver to the non-domestic setting, a finite subset [3, Theorem 3.1.6] of the set of bands was introduced in [3, Definition 3.1.1], whose elements are called *prime bands*. We partition the set of bands using an equivalence relation in such a way that each equivalence class contains at least one prime band to obtain a finite set $\mathcal{Q}^{\mathrm{Ba}}$ of equivalence classes that is equipped with a natural reachability partial order $\preceq$. We classify the elements of $\mathcal{Q}^{\mathrm{Ba}}$ as domestic or non-domestic depending on whether the equivalence class is finite or infinite. The existence of a non-domestic element in $\mathcal{Q}^{\mathrm{Ba}}$ characterizes the non-domesticity of the algebra Λ. In this regard, the study of non-domestic string algebras is a combination of domestic and meta-$\bigcup$-cyclic string algebras ([3, § 3.4]), where the latter type of algebras are characterized as those with $\mathcal{Q}^{\mathrm{Ba}}$ consisting only of non-domestic elements such that no two distinct elements are $\preceq$-related.

Let rad_Λ denote the radical of the category of finite length right Λ-modules. Schröer characterized [10, Theorem 2] domestic string algebras as those whose radical is nilpotent. In fact, when Λ is domestic, he showed that $\mathrm{rad}_\Lambda^{\omega\cdot(n+2)} = 0$, where n is the maximum length of a path in its bridge quiver, bypassing the computation of the order types of hammocks. It is conjectured [3, Conjecture 4.4.1] that the stable rank of a non-domestic string algebra is strictly bounded above by ω^2. The results in this paper, especially those in § 11,12, will be used in a future work to settle this conjecture in the affirmative.

Yet another characterization of a non-domestic (string) algebra was given by Prest [6, Proposition 0.6] in terms of the existence of a *factorizable system* in its radical–such a factorizable system is indexed by a bounded interval in rationals with non-empty interior.

To explain the main result of the paper, we need to set up some order-theoretic notations and conventions, for which we mostly follow Rosenstein [8]. The notations $\mathbb{N}$ and $\mathbb{N}^+$ stand for the sets of natural numbers and positive natural numbers respectively. For $n \in \mathbb{N}$, the notation $\mathbf{n}$ stands for the order type of the finite linear order with n elements. The notation ω stands for the order type of $\mathbb{N}$, ω^* for its dual, ζ for the order type of the set of integers, η for the order type of rationals, λ for the order type of reals and ϑ for the order type of irrationals. For linear orders L_1 and L_2, the notations $L_1 + L_2$ and $L_1 \cdot L_2$ stand for their order sum and their anti-lexicographic product respectively. The notation $\sum_{i\in(I,<)} L_i$ stands for the order sum of linear orders L_i indexed by a (possibly infinite) linear order $(I, <)$. A linear order $(L, <)$ is said to be *scattered* if there is no embedding of η in it. An element a in a linear order $(L, <)$ is said to be an immediate predecessor (resp. successor) of $b \in L$ if $a < b$ (resp. $b < a$) and there is no element $c \in L$ such that $a < c < b$ (resp. $b < c < a$). A linear order $(L, <)$ is said to be *discrete* if each element except the minimum, if it exists, has an immediate predecessor and each element except the maximum, if it exists, has an immediate successor.

In a model-theoretic study of linear orders, Läuchli and Leonard [5] introduced two classes $\mathcal{M}_0 \subset \mathcal{M}$ of linear orders (see [8, Definitions 7.6, 7.19]) to understand graded versions of elementary equivalence described via Ehrenfeucht-Fraïssé games. The class $\mathcal{M}_0$ is a subclass of the class of scattered linear orders whereas each linear order in $\mathcal{M}\setminus\mathcal{M}_0$ is not scattered. The class $\mathcal{M}_0$ appeared in [11] as the class $\mathrm{LO_{fp}}$ of *finitely presented linear orders*. Its subclass $\mathrm{dLO}_{\mathrm{fp}}^{11}$ consisting of bounded discrete finitely presented linear orders was characterized as the class of order types of hammocks for domestic string algebras [11, Theorem 12.15].

The main result of this paper (Theorem 11.9) shows that the order type of a hammock for a (non-domestic) string algebra lies in a subclass of the class $\mathcal{M}$ of Läuchli and Leonard. We will refer to the orders in $\mathcal{M}$ as *finite description linear orders*, and thus use a more suggestive notation $\mathrm{LO_{fd}}$ instead of $\mathcal{M}$; its subclass consisting of bounded discrete orders will be denoted by $\mathrm{dLO}_{\mathrm{fd}}^{11}$. Each finite description order is constructed using only finitely many order-theoretic operations on a fixed finite set of linear orders (Definition 2.3). However, in contrast to the domestic case, we do not expect that every linear order in $\mathrm{dLO}^{11}_{\mathrm{fd}}$ is isomorphic to a hammock for a string algebra (Question 11.13).

The technique used to prove the main theorem is "recursive reconstruction" (Lemma 2.4) of a hammock–to explain this method better we need the concept of condensation. A *condensation* of a linear order L is a monotone surjective map $c : L \twoheadrightarrow L'$. When a linear order is thought of as a (small) category then each of its condensations is equivalent to its localization with respect to an appropriate choice of weak equivalences. It is possible to reconstruct L from the knowledge of L' and all fibers of the map c as

$$L \cong \sum_{x \in L'} c^{-1}(x).$$

If L is the hammock under consideration, we choose a suitable $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ to define a split condensation $c_{\mathsf{B}} : L \twoheadrightarrow L_{\mathsf{B}}$ in such a way that $L_{\mathsf{B}} \in \mathrm{dLO}^{11}_{\mathrm{fd}}$, each fiber of c_{B} is itself a hammock (Lemma 7.8) and there are only finitely many distinct order types of fibers. Depending on whether B is domestic or non-domestic, the condensed order L_{B} is isomorphic to a finite order sum of copies of $\omega + \omega^*$ or $\omega + \zeta \cdot \eta + \omega^*$ respectively (Corollary 11.8). Finiteness of $\mathcal{Q}^{\mathrm{Ba}}$ helps to inductively prove that the fibers of c_{B} are indeed in $\mathrm{dLO}^{11}_{\mathrm{fd}}$. The "finite description" of the order type of hammock needs several other supporting finiteness results sprinkled throughout the paper (Corollaries 4.7, 6.7, 6.10, Remark 10.2 and Propositions 11.1, 11.4).

We show in Corollary 9.4 that for an element in L_{B}, the condensation of its immediate successor (resp. predecessor) in L is its immediate successor (resp. predecessor) in L_{B}. We also identify a subset of L_{B} that is in bijection with its finite condensation (see [8, § 4.2])–such a subset is finite if and only if B is domestic. In case B is non-domestic, we further identify its cofinite subset, the elements of which will be called B-centers, whose order type (as a suborder of L) is η (Corollary 11.5).

The completion of the hammock is obtained by adding to it the so-called left $\mathbb{N}$-strings (Proposition 4.11). Recall that every left $\mathbb{N}$-string in a domestic string algebra is almost periodic [7, Proposition 2]; this statement fails in a non-domestic string algebra. Each interval in the hammock isomorphic to ω or ω^* contributes to the completion an almost periodic left $\mathbb{N}$-string of the form ${}^{\infty}\mathfrak{b}\mathfrak{u}$, where $\mathfrak{u}$ is a string and $\mathfrak{b}$ is a special type of prime band (Definition 8.4). The remaining left $\mathbb{N}$-strings, which also include some almost periodic left $\mathbb{N}$-strings, occupy irrational locations in λ (Proposition 12.8).

Though Theorem 11.9 generalizes the backward direction of [11, Theorem 12.15], which computes the order type of hammocks for domestic string algebras, the former employs a recursive algorithm and thus is computationally more complex than the latter.

A finite dimensional $\mathcal{K}$-algebra that is presented as a bound quiver algebra has only a finite amount of data. We believe that for such algebras if one identifies a finite poset that plays the same role as $\mathcal{Q}^{\mathrm{Ba}}$ for string algebras then the method of "condensation away from" elements of this poset could be used to recursively reconstruct hammocks. The class $\mathrm{LO}_{\mathrm{fd}}$ seems to be the natural class of countably infinite linear orders which admit description using only a finite amount of data.

Theorem 11.9 is the key ingredient in the proof of [13, Theorem 1] where Srivastava and the first two authors show that the stable rank of a special biserial algebra–an ordinal that measures the complexity of factorizations in the module category–is strictly bounded above by ω^2, thereby settling [3, Conjecture 4.4.1] in the affirmative. An interested reader is also referred to [12, Theorem α] for an alternate shorter proof of Theorem 11.9 by Srivastava and the second author that uses finite automata.

The rest of the paper is organized as follows. The class $\mathrm{LO}_{\mathrm{fd}}$ of finite description linear orders is defined in § 2; the highlight of this section is the recursive reconstruction lemma

(Lemma 2.4). After describing the completion of a linear order in § 3, completions of some orders in $\mathrm{dLO}^{11}_{\mathrm{fd}}$ are computed. The notations and conventions for string algebras are set up in § 4 and the finiteness of the poset $(\mathcal{Q}^{\mathrm{Ba}}, \preceq)$ is proved in § 5. Before describing the condensation operator c_{B} in § 7, the condensed hammock is introduced in § 6. Lemma 7.8 helps to decompose a hammock as an order sum of smaller hammocks, which enables setting up the induction for the computation of the order type. After introducing two special subsets of the set of prime bands in § 8, the description of the immediate neighbours of strings in the condensed hammock is completed in § 9. The definition and classification of B-centers into finitely many classes is achieved in § 10. Combining all tools gathered thus far, the main result is proved in § 11, where the potential impossibility of its converse is also discussed. Finally, in § 12, based on the description of the completion of the hammock from Proposition 4.11, the set of left $\mathbb{N}$-strings is classified into three classes and their locations in the completion are described.

2 Finite Description Linear Orders

We introduced basic notations, conventions and operations on linear orders in § 1. We need two more finitary operations defined on linear orders. If L_1 and L_2 are non-empty linear orders, define $L_1 \dot{+} L_2$ to be the linear order obtained by identifying in $L_1 + L_2$ the maximum element of L_1 with the minimum element of L_2, if they both exist; otherwise setting $L_1 \dot{+} L_2 := L_1 + L_2$.

The other finitary operation is that of the shuffle of a finite set of linear orders, which we recall below from [8] for the convenience of the reader. This operation will be used in the construction of finite description linear orders.

Cantor proved that η is the only countably infinite dense linear ordering without maximum and minimum elements up to isomorphism (see [8, Theorem 2.8]). The technique used to prove this result is known as the back-and-forth method, which can also be used to prove the following.

Proposition 2.1 *[8, Theorems 7.11, 7.13] For each $n \in \omega$, there is a partition of η into sets $\{D_i \mid 1 \leq i \leq n\}$ such that each D_i is dense in η. Such a partition is unique in the following sense: If $(A, <)$ and $(A', <')$ are countable, unbounded, dense linear orders, A is partitioned into n subsets $\{D_i \mid 1 \leq i \leq n\}$ each of which is dense in A, and A' is partitioned into n subsets $\{D'_i \mid 1 \leq i \leq n\}$ each of which is dense in A' then there is an order isomorphism $f : (A, <) \to (A', <')$ such that $f(D_i) = D'_i$ for each $1 \leq i \leq n$.*

The uniqueness of the partition described in the above result enables us to combine the data of a finite set of linear orders into a single linear order as described below.

Definition 2.2 [8, Definition 7.14] Let $n \in \mathbb{N}$. Suppose $L_1, \cdots, L_n$ is a finite set of linear orders. If $n \in \mathbb{N}^+$, let $\{D_1, \cdots, D_n\}$ be a partition of η guaranteed by Proposition 2.1, where each D_i is dense in η. Define the *shuffle*, denoted Ξ, of linear orders $L_1, \cdots, L_n$ as

$$\Xi(L_1, \cdots, L_n) := \begin{cases} \mathbf{0} & \text{if } n = 0, \\ \sum_{i \in \eta} L'_i, \text{ where } L'_i = L_j \text{ when } i \in D_j & \text{otherwise.} \end{cases}$$

The shuffle operator is a generalization of anti-lexicographic product with η, i.e., $\Xi(L_1) \cong L_1 \cdot \eta$, and it ignores repetitions, $\mathbf{0}$ and permutations, i.e.,

$$\Xi(L_1, L_1, L_2, \cdots, L_n) \cong \Xi(L_1, L_2, \cdots, L_n) \cong \Xi(L_1, L_2, \cdots, L_n, \mathbf{0}) \cong \Xi(L_{\pi(1)}, L_{\pi(2)}, \cdots, L_{\pi(n)})$$

for a permutation π of $\{1, 2, \cdots, n\}$.

Since $\eta + \mathbf{1} + \eta = \eta$, for any $1 \leq j \leq n$, we also get

$$\Xi(L_1, L_2, \cdots, L_n) + L_j + \Xi(L_1, L_2, \cdots, L_n) \cong \Xi(L_1, L_2, \cdots, L_n). \tag{2.1}$$

If $L_j = L_{j1} + L_{j2}$ then using $\eta \cdot \zeta = \eta$ and the above identity we get

$$(L_{j2} + \Xi(L_1, L_2, \cdots, L_n) + L_{j1}) \cdot \zeta \cong \Xi(L_1, L_2, \cdots, L_n). \tag{2.2}$$

Definition 2.3 [8, Definition 7.19] The class $\mathrm{LO_{fd}}$ of *finite description* linear orders is defined as the smallest subclass of linear orders closed under isomorphisms such that

1. $\mathbf{0}, \mathbf{1} \in \mathrm{LO_{fd}}$;
2. if $L_1, L_2 \in \mathrm{LO_{fd}}$ then $L_1 + L_2 \in \mathrm{LO_{fd}}$;
3. if $L \in \mathrm{LO_{fd}}$ then $L \cdot \omega, L \cdot \omega^* \in \mathrm{LO_{fd}}$;
4. if $L_1, L_2, \cdots, L_n \in \mathrm{LO_{fd}}$ for $n \in \mathbb{N}^+$ then $\Xi(L_1, L_2, \cdots, L_n) \in \mathrm{LO_{fd}}$.

The class $\mathrm{LO_{fp}}$ of finitely presented linear orders is the subclass of $\mathrm{LO_{fd}}$ whose definition omits clause (4) in the above. The notation $\mathrm{dLO_{fd}}$ denotes the subclass of $\mathrm{LO_{fd}}$ containing only discrete linear orders. The class $\mathrm{dLO_{fd}}$ of discrete finite description linear orders can be further partitioned into four subclasses, viz. $\mathrm{dLO}^{ij}_{\mathrm{fd}}$ for $i, j \in \{0, 1\}$, where $L \in \mathrm{dLO}^{ij}_{\mathrm{fd}}$ only if it has i minimum elements and j maximum elements. In particular, $\mathrm{dLO}^{11}_{\mathrm{fd}}$ is the class of *bounded discrete finite description linear orders*. The orders $(\omega + \Xi(\underbrace{\zeta, \zeta, \cdots \zeta}_{n \text{ times}}) + \omega^*)$ for $n \in \mathbb{N}$ form a simple family of examples of orders in $\mathrm{dLO}^{11}_{\mathrm{fd}}$. We similarly partition $\mathrm{dLO_{fp}}$ into four subclasses.

We will use the method of recursive reconstruction described in the introduction to construct complex orders in $\mathrm{dLO}^{11}_{\mathrm{fd}}$. An indispensable tool to prove the main result (Theorem 11.9) is the following lemma which shows that, under suitable conditions, if L admits a condensation $c : L \twoheadrightarrow (\omega + \Xi(\underbrace{\zeta, \zeta, \cdots \zeta}_{n \text{ times}}) + \omega^*)$ with fibers in $\mathrm{dLO}^{11}_{\mathrm{fd}}$ then $L \in \mathrm{dLO}^{11}_{\mathrm{fd}}$.

Lemma 2.4 *Fix $n \in \mathbb{N}$. Given any $(n+2)$ functions $L_j : \zeta \to \mathrm{dLO}^{11}_{\mathrm{fd}}$, for $j \in \{0, 1, \cdots, n+1\}$, satisfying*

- $L_0(-k) = L_{n+1}(k) = \mathbf{0}$ *for every $k > 0$;*
- *for each $j \in \{0, 1, \cdots, n\}$, there exist $s_j \geq 0$ and $p_j > 0$ such that $L_j(s_j + p_j + k) \cong L_j(s_j + k)$ for every $k \in \mathbb{N}$;*
- *for each $j \in \{1, \cdots, n+1\}$, there exist $s'_j \leq 0$ and $p'_j > 0$ such that $L_j(s'_j - p'_j - k) \cong L_j(s'_j - k)$ for every $k \in \mathbb{N}$,*

we have

$$L := \sum_{k \in \zeta} L_0(k) + \Xi(\sum_{k \in \zeta} L_1(k), \cdots, \sum_{k \in \zeta} L_n(k)) + \sum_{k \in \zeta} L_{n+1}(k) \in \mathrm{dLO}^{11}_{\mathrm{fd}}.$$

Proof Set

$$H := \sum_{k \in \zeta} L_0(k) \cong L_0(0) + \cdots + L_0(s_0 - 1) + (L_0(s_0) + \cdots + L_0(s_0 + p_0 - 1)) \cdot \omega,$$

$$R := \sum_{k \in \zeta} L_{n+1}(k) \cong (L_{n+1}(s'_{n+1} - p'_{n+1} + 1) + \cdots + L_{n+1}(s'_{n+1})) \cdot \omega^*$$
$$+ L_{n+1}(s'_{n+1} + 1) + \cdots + L_{n+1}(0),$$

and for each $1 \le j \le n$,

$$M_j := \sum_{k \in \zeta} L_j(k) \cong (L_j(s'_j - p'_j + 1) \cdot \omega^* + \cdots + L_j(s'_j))$$
$$+ L_j(s'_j + 1) + \cdots + L_j(s_j - 1) + (L_j(s_j) + \cdots + L_j(s_j + p_j - 1)) \cdot \omega.$$

It is trivially seen that $H \in \mathrm{dLO}^{10}_{\mathrm{fd}}$, $R \in \mathrm{dLO}^{01}_{\mathrm{fd}}$ and $M_j \in \mathrm{dLO}^{00}_{\mathrm{fd}}$ for each $1 \le j \le n$. Hence it follows that $L = H + \Xi(M_1, \cdots, M_n) + R \in \mathrm{dLO}^{11}_{\mathrm{fd}}$. □

Corollary 2.5 *Using the notations of the above proposition, if we have $n = 0$ and the images of L_0 and L_1 lie in $\mathrm{dLO}^{11}_{\mathrm{fp}}$ then $L \in \mathrm{dLO}^{11}_{\mathrm{fp}}$.*

3 Completions of Linear Orders

Recall from [8, Definition 2.19] that a linear order L is *complete* if each of its suborders that is bounded above has a least upper bound. Completeness of a linear order is an order-theoretic property, i.e., it is preserved and reflected by order isomorphisms [8, Lemma 2.21]. A *Dedekind cut* [8, Definition 2.22] of a linear order L is a pair (X, Y) of non-empty intervals of L whose union is L such that each element of X precedes every element of Y. A Dedekind cut (X, Y) is called a *gap* in L if X does not have a maximum element and Y does not have a minimum element. Denote the set of all gaps of L by $\mathcal{G}(L)$. An equivalent criterion [8, Lemma 2.23] for a linear order L to be complete is that L is Dedekind complete, i.e., $\mathcal{G}(L) = \emptyset$.

A *completion* of L [8, Definition 2.31], denoted $\mathcal{C}(L)$, is a complete linear order containing L such that no proper suborder of $\mathcal{C}(L)$ containing L is complete. A completion $\mathcal{C}(L)$ of L exists, the construction of one involves "filling up" its gaps (see the proof of [8, Theorem 2.32(1)]), and is unique up to order isomorphism [8, Theorem 2.32(2)]. The set $\mathcal{G}(L)$ being a subset of $\mathcal{C}(L)$ inherits an order structure from $\mathcal{C}(L)$.

In order to identify a gap (X, Y) of a linear order L, it suffices to find a cofinal sequence of elements of X and a coinitial sequence of elements of Y, where $X' \subseteq X$ is *cofinal* in X if for every $a \in X$, there is $b \in X'$ such that $a \le b$, and dually, Y' is said to be *coinitial* in Y if for every $a \in Y$, there is $b \in Y'$ such that $b \le a$.

Example 3.1 It is trivial to note that $\mathcal{C}(\omega) \cong \omega$, $\mathcal{C}(\omega^*) \cong \omega^*$, $\mathcal{C}(\zeta) \cong \zeta$. Moreover, reals are constructed as the completion of η using cofinal/coinitial sequences, i.e., $\mathcal{C}(\eta) \cong \lambda$.

We will use the technique of finding cofinal/coinitial sequences to compute the completion of certain linear orders in Propositions 4.11 and 12.4.

The main goal of this section is to compute the completions of two classes of order types in $\mathrm{dLO}^{11}_{\mathrm{fd}}$ which are important in the context of this paper, namely $(\omega + \omega^*) \cdot \mathbf{n}$ and $(\omega + \Xi(\zeta) + \omega^*) \cdot \mathbf{n} \cong (\omega + \zeta \cdot \eta + \omega^*)$. The computation of the completion of the former class of order types is easy.

Example 3.2 $\mathcal{C}((\omega + \omega^*) \cdot \mathbf{n}) \cong (\omega + \mathbf{1} + \omega^*) \cdot \mathbf{n}$.

Given $n \in \mathbb{N}^+$ and non-empty linear orders $L_1, \cdots, L_n$, recall the construction of the shuffle $\Xi(L_1, \cdots, L_n)$ from Definition 2.2. Using those notations, it is easily verified that the following four types of Dedekind cuts are elements in $\mathcal{G}(\Xi(L_1, \cdots, L_n))$.

G1. $(\sum_{r\in(-\infty,r_0)\cap\mathbb{Q}} L'_r, \sum_{r\in(r_0,\infty)\cap\mathbb{Q}} L'_r)$ for $r_0 \in \mathbb{R}\setminus\mathbb{Q}$;
G2. $(\sum_{r\in(-\infty,r_0)\cap\mathbb{Q}} L'_r, \sum_{r\in[r_0,\infty)\cap\mathbb{Q}} L'_r)$ for $r_0 \in D_j$ if L_j does not have a minimum;
G3. $(\sum_{r\in(-\infty,r_0]\cap\mathbb{Q}} L'_r, \sum_{r\in(r_0,\infty)\cap\mathbb{Q}} L'_r)$ for $r_0 \in D_j$ if L_j does not have a maximum;
G4. $(\sum_{r\in(-\infty,r_0)\cap\mathbb{Q}} L'_r + L^1_j, L^2_j + \sum_{r\in(r_0,\infty)\cap\mathbb{Q}} L'_r)$ for $r_0 \in D_j$ if $(L^1_j, L^2_j) \in \mathcal{G}(L_j)$.

The following result says that in fact these are all the gaps.

Proposition 3.3 *Given $n \in \mathbb{N}^+$ and non-empty linear orders $L_1, \cdots, L_n$, if $(X, Y) \in \mathcal{G}(\Xi(L_1, \cdots, L_n))$ then (X, Y) is of one of the four types listed above.*

Proof Define a map proj : $\sum_{r\in\mathbb{Q}} L'_r \to \mathbb{Q}$ by $\mathrm{proj}(x) = r$ if $x \in L'_r$. Thus if $(X, Y) \in \mathcal{G}(\Xi(L_1, \cdots, L_n))$ then $\mathrm{proj}(X) \cup \mathrm{proj}(Y) = \mathbb{Q}$ and $r_1 \leq r_2$ whenever $r_1 \in \mathrm{proj}(X)$ and $r_2 \in \mathrm{proj}(Y)$. Hence $\mathrm{proj}(X) \cap \mathrm{proj}(Y)$ is either empty or singleton. If $\mathrm{proj}(X) \cap \mathrm{proj}(Y) = \{r_0\}$ for some $r_0 \in \mathbb{Q}$ then $(X \cap L'_{r_0}, Y \cap L'_{r_0})$ is a gap in L'_{r_0}; this gap is of the form described in G4.

Now assume that $\mathrm{proj}(X) \cap \mathrm{proj}(Y) = \emptyset$. There are three cases.

- If $\mathrm{proj}(X)$ does not have a maximum element and $\mathrm{proj}(Y)$ does not have a minimum element then there exists $r_0 \in \mathbb{R}\setminus\mathbb{Q}$ such that $r_1 < r_0 < r_2$ for every $r_1 \in \mathrm{proj}(X)$ and every $r_2 \in \mathrm{proj}(Y)$. This gap is of the form described in G1.
- If $\mathrm{proj}(X)$ has a maximum element, say r_0, then (X, Y) is a gap if and only if L'_{r_0} does not have a maximum element. This gap is of the form described in G2.
- If $\mathrm{proj}(Y)$ has a minimum element, say r_0, then (X, Y) is a gap if and only if L'_{r_0} does not have a minimum element. This gap is of the form described in G3. □

As a consequence, we have the following result, which computes the completion of $\Xi(L_1, \cdots, L_n)$.

Corollary 3.4 *Given $n \in \mathbb{N}^+$ and non-empty linear orders $L_1, \cdots, L_n$, using notations of Definition 2.2,*

$$\mathcal{C}(\Xi(L_1, \cdots, L_n)) \cong \sum_{r\in\mathbb{R}} T_r, \text{ where } T_r := \begin{cases} \mathbf{1} \dotplus \mathcal{C}(L_j) \dotplus \mathbf{1} & \text{if } r \in D_j \text{ for some } j \in \{1, 2, \cdots, n\}, \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

Using the standard embedding of η in λ, we compute the completion of a standard order type in $\mathrm{dLO}^{11}_{\mathrm{fd}}$.

Corollary 3.5 *Suppose $\mathcal{O} := \omega + \zeta \cdot \eta + \omega^*$. Then*

$$\mathcal{C}(\mathcal{O}) \cong \omega + \mathbf{1} + \left(\sum_{r\in\lambda} T_r\right) + \mathbf{1} + \omega^*, \text{ where } T_r = \begin{cases} \mathbf{1} + \zeta + \mathbf{1} & \text{if } r \in \eta, \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

This result will be useful in §12 along with the partition of $\mathcal{G}(\mathcal{O})$ into three subsets given below.

$$\mathcal{G}^+(\mathcal{O}) := \{x \in \mathcal{G}(\mathcal{O}) \mid x \text{ is minimum of } \mathcal{G}(\mathcal{O}) \text{ or } x \text{ is maximum of } T_r \text{ when } r \in \eta\},$$
$$\mathcal{G}^-(\mathcal{O}) := \{x \in \mathcal{G}(\mathcal{O}) \mid x \text{ is maximum of } \mathcal{G}(\mathcal{O}) \text{ or } x \text{ is minimum of } T_r \text{ when } r \in \eta\},$$
$$\mathcal{G}^0(\mathcal{O}) := \mathcal{G}(\mathcal{O}) \setminus (\mathcal{G}^+(\mathcal{O}) \cup \mathcal{G}^-(\mathcal{O})) = \{T_r \mid r \in \vartheta\}.$$

4 Fundamentals of String Algebras

Fix an algebraically closed field $\mathcal{K}$. A string algebra is a $\mathcal{K}$-algebra $\Lambda := \mathcal{K}\mathcal{Q}/\langle\rho\rangle$ presented as a certain quotient of the path algebra of a finite quiver $\mathcal{Q} = (Q_0, Q_1, s, t)$, where Q_0 is a finite set of vertices, Q_1 is a finite set of arrows, and $s, t : Q_1 \to Q_0$ are the source and target functions respectively, by the ideal generated by a set ρ of monomial relations. For technical reasons, we also choose and fix a pair of maps $\sigma, \varepsilon : Q_1 \to \{1, -1\}$ satisfying certain conditions. We will always use small roman letters v, w possibly with numerical subscripts to denote the vertices and $a, b, c, d, \cdots$ possibly with numerical subscripts to denote arrows of the quiver. Let us denote by Q_1^-, the collection, for each $b_i \in Q_1$, the corresponding capital roman letter B_i with the same subscript. We will use Greek letters $\alpha, \beta, \gamma, \cdots$ with numerical subscripts to denote syllables, i.e., elements of $Q_1 \cup Q_1^-$. We treat the syllable B_i as the inverse of the arrow b_i, and set $s(B_i) := t(b_i)$ and $t(B_i) := s(b_i)$. The reader is referred to [3, § 2.1] for the definition of a string algebra as well as for notations and conventions associated with certain combinatorial entities called strings and bands.

We use the notation $\mathsf{St}(\Lambda)$ to denote the set of strings for Λ. Strings are read from right to left. For example, if $\mathfrak{x} = \alpha_3\alpha_2\alpha_1$ then α_1 is the first syllable of $\mathfrak{x}$ and α_3 is the last syllable of $\mathfrak{x}$. For strings $\mathfrak{x}$ and $\mathfrak{y}$, we say that $\mathfrak{x}$ is a *left substring* (resp. *proper left substring*) of $\mathfrak{y}$, denoted $\mathfrak{x} \sqsubseteq_l \mathfrak{y}$ (resp. $\mathfrak{x} \sqsubset_l \mathfrak{y}$) if $\mathfrak{y} = \mathfrak{u}\mathfrak{x}$ for some (resp. positive length) string $\mathfrak{u}$. Dually say that $\mathfrak{x}$ is a *right substring* (resp. *proper right substring*) of $\mathfrak{y}$, denoted $\mathfrak{x} \sqsubseteq_r \mathfrak{y}$ (resp. $\mathfrak{x} \sqsubset_r \mathfrak{y}$) if $\mathfrak{y} = \mathfrak{x}\mathfrak{u}$ for some (resp. positive length) string $\mathfrak{u}$. Given a string $\mathfrak{x}$, denote by $|\mathfrak{x}|$ the length of the string $\mathfrak{x}$, i.e., the number of syllables in it.

Suppose $\mathfrak{x} \in \mathsf{St}(\Lambda)$ and $|\mathfrak{x}| > 0$. Its *sign*, denoted $\theta(\mathfrak{x}) \in \{1, -1\}$, is defined by $\theta(\mathfrak{x}) = 1$ if and only if the first syllable of $\mathfrak{x}$ is inverse. To identify if $\mathfrak{x}$ has any sign changes, we define

$$\delta(\mathfrak{x}) := \begin{cases} 1 & \text{if all syllables of } \mathfrak{x} \text{ are inverse,} \\ -1 & \text{if all syllables of } \mathfrak{x} \text{ are direct,} \\ 0 & \text{otherwise.} \end{cases}$$

We use the notation $\mathsf{Ba}(\Lambda)$ to denote the set of bands up to a cyclic permutation of its syllables. Note that we work with the convention that the first syllable of a band is inverse while the last syllable is direct. Let $\mathcal{Q}_0^{\mathrm{Ba}}$ be a fixed set of representatives in $\mathsf{Ba}(\Lambda)$. Call a cyclic permutation of an element in $\mathcal{Q}_0^{\mathrm{Ba}}$ a cycle. Denote the set of all cycles in Λ by $\mathsf{Cyc}(\Lambda)$.

Associated to $\mathfrak{x} \in \mathsf{St}(\Lambda)$ there is a finite-dimensional indecomposable module $M(\mathfrak{x})$, called the string module, such that for distinct strings $\mathfrak{x}, \mathfrak{y}$, we have $M(\mathfrak{x}) \cong M(\mathfrak{y})$ if and only if $\mathfrak{x} = \mathfrak{y}^{-1}$.

For any $v \in Q_0$, let $S(v) := M(1_{(v,1)})$ be the associated simple module. Further, let $P(v)$ and $I(v)$ be the projective cover and the injective envelope of $S(v)$ respectively. Let f_v denote the composition $P(v) \twoheadrightarrow S(v) \hookrightarrow I(v)$. Motivated by [2], Schröer [10, § 3] introduced the hammock poset $H(v)$ whose underlying set consists of (isomorphism classes of) triples (N, g, h), where $P(v) \xrightarrow{g} N \xrightarrow{h} I(v)$ is a factorization of f_v through a string module N. The order $\leq$ on $H(v)$ is defined by $(N, g, h) \leq (N', g', h')$ if and only if g' factors through g. Dropping the reference to the maps from $(N, g, h) \in H(v)$, the element $N = M(\mathfrak{x}) \cong M(\mathfrak{x}^{-1})$ of $H(v)$ can be thought of as the pair $(\mathfrak{x}_1, \mathfrak{x}_2)$ of strings, where $\mathfrak{x}_1\mathfrak{x}_2 \in \{\mathfrak{x}, \mathfrak{x}^{-1}\}, s(\mathfrak{x}_1) = t(\mathfrak{x}_2) = v$ and $\varepsilon(\mathfrak{x}_2) = 1$. The image of the left (resp. right) projection map $(\mathfrak{x}_1, \mathfrak{x}_2) \mapsto \mathfrak{x}_1$ (resp. $(\mathfrak{x}_1, \mathfrak{x}_2) \mapsto \mathfrak{x}_2$) on $H(v)$ is a linear order, denoted $(H_l(v), <_l)$ (resp. $(H_r(v), <_r)$). More generally, replacing $S(v) := 1_{(v,1)}$ by $M(\mathfrak{x}_0)$ for a string $\mathfrak{x}_0$ in the above discussion, we can define left and right hammocks of the string $\mathfrak{x}_0$–these are the central objects of study in this paper.

Definition 4.1 The left and right hammock sets of the string $\mathfrak{x}_0$ are defined as

$H_l(\mathfrak{x}_0):=\{\mathfrak{x} \in \mathsf{St}(\Lambda) \mid \mathfrak{x}=\mathfrak{u}\mathfrak{x}_0 \text{ for some string } \mathfrak{u}\},\ H_r(\mathfrak{x}_0):=\{\mathfrak{x} \in \mathsf{St}(\Lambda) \mid \mathfrak{x}=\mathfrak{x}_0\mathfrak{u} \text{ for some string } \mathfrak{u}\}.$

The left hammock $H_l(\mathfrak{x}_0)$ can be equipped with a linear order $<_l$, where for $\mathfrak{x}, \mathfrak{y} \in H_l(\mathfrak{x}_0)$ we have $\mathfrak{x} <_l \mathfrak{y}$ if one of the following holds:

- $\mathfrak{y} = \mathfrak{u}\alpha\mathfrak{x}$ for some string $\mathfrak{u}$ and $\alpha \in Q_1^-$;
- $\mathfrak{x} = \mathfrak{v}\beta\mathfrak{y}$ for some string $\mathfrak{v}$ and $\beta \in Q_1$;
- $\mathfrak{x} = \mathfrak{v}\beta\mathfrak{w}$ and $\mathfrak{y} = \mathfrak{u}\alpha\mathfrak{w}$ for some $\alpha \in Q_1^-$, $\beta \in Q_1$ and strings $\mathfrak{u}, \mathfrak{v}, \mathfrak{w}$.

The ordering $<_r$ on $H_r(\mathfrak{x}_0)$ is defined as $\mathfrak{x} <_r \mathfrak{y}$ if and only if $\mathfrak{x}^{-1} <_l \mathfrak{y}^{-1}$ in $(H_l(\mathfrak{x}_0^{-1}), <_l)$.

We will only study the left hammock in this paper–the dual results will hold for the right hammock.

For $\mathfrak{x}, \mathfrak{y} \in H_l(\mathfrak{x}_0)$, denote by $\mathfrak{x} \sqcap_l \mathfrak{y}$ the maximal common left substring of $\mathfrak{x}$ and $\mathfrak{y}$. If $\mathfrak{x} = \mathfrak{w}(\mathfrak{x} \sqcap_l \mathfrak{y})$ with $|\mathfrak{w}| > 0$ then define $\theta(\mathfrak{x} \mid \mathfrak{y}) := \theta(\mathfrak{w})$ and $\delta(\mathfrak{x} \mid \mathfrak{y}) := \delta(\mathfrak{w})$.

Almost all strings in the left hammock have an immediate successor as well as an immediate predecessor.

Proposition 4.2 *[9, § 2.5] The linear order $(H_l(\mathfrak{x}_0), <_l)$ is a bounded discrete linear order. Its minimum element, denoted $\mathfrak{m}_{-1}(\mathfrak{x}_0)$, is the longest string in $H_l(\mathfrak{x}_0)$ satisfying either $\delta(\mathfrak{m}_{-1}(\mathfrak{x}_0) \mid \mathfrak{x}_0) = -1$ or $\mathfrak{m}_{-1}(\mathfrak{x}_0) = \mathfrak{x}_0$, whereas its maximum element is the longest string, denoted $\mathfrak{M}_1(\mathfrak{x}_0)$, satisfying either $\delta(\mathfrak{M}_1(\mathfrak{x}_0) \mid \mathfrak{x}_0) = 1$ or $\mathfrak{M}_1(\mathfrak{x}_0) = \mathfrak{x}_0$.*

For $\mathfrak{x} \in H_l(\mathfrak{x}_0)$, the notations $l(\mathfrak{x})$ and $\bar{l}(\mathfrak{x})$ were introduced in [3, § 2.4] by comparing the length of $\mathfrak{x}$ with that of its immediate successor and predecessor. If the immediate successor of $\mathfrak{x}$ is longer than $\mathfrak{x}$ then there exists an inverse syllable α such that $\alpha\mathfrak{x}$ is a string, and the immediate successor of $\mathfrak{x}$ is the string $l(\mathfrak{x}) := \mathfrak{w}\alpha\mathfrak{x}$, where $\mathfrak{w}$ is the longest string satisfying either $|\mathfrak{w}| = 0$ or $\delta(\mathfrak{w}) = -1$ such that $\mathfrak{w}\alpha\mathfrak{x}$ is a string. On the other hand, if the immediate predecessor of $\mathfrak{x}$ is longer than $\mathfrak{x}$ then there exists a direct syllable β such that $\beta\mathfrak{x}$ is a string, and the immediate predecessor of $\mathfrak{x}$ is the string $\bar{l}(\mathfrak{x}) := \mathfrak{w}'\beta\mathfrak{x}$, where $\mathfrak{w}'$ is the longest string satisfying either $|\mathfrak{w}'| = 0$ or $\delta(\mathfrak{w}') = 1$ such that $\mathfrak{w}'\beta\mathfrak{x}$ is a string.

The next result shows that intervals in hammocks contain a unique "pivotal" string.

Proposition 4.3 *Given a non-empty interval I in $(H_l(\mathfrak{x}_0), <_l)$, there is a unique string $\mathfrak{u}$ in I with minimal length. Moreover, $I \subseteq H_l(\mathfrak{u})$.*

Proof Since $\{|\mathfrak{x}| : \mathfrak{x} \in I\}$ is a non-empty subset of $\mathbb{N}$, it has a minimum, say m. If possible, let $\mathfrak{u}_1 <_l \mathfrak{u}_2$ be strings in I such that $|\mathfrak{u}_1| = |\mathfrak{u}_2| = m$. Then $\mathfrak{u}_1 <_l \mathfrak{u}_1 \sqcap_l \mathfrak{u}_2 <_l \mathfrak{u}_2$ and $|\mathfrak{u}_1 \sqcap_l \mathfrak{u}_2| < m$. Since I is an interval, $\mathfrak{u}_1 \sqcap_l \mathfrak{u}_2 \in I$, a contradiction to the minimality of m, thus showing that I contains a unique string $\mathfrak{u}$ with $|\mathfrak{u}| = m$.

For $\mathfrak{x} \in I$, the string $\mathfrak{x} \sqcap_l \mathfrak{u}$ lies between $\mathfrak{u}$ and $\mathfrak{x}$, and hence in I. Therefore, $m \leq |\mathfrak{x} \sqcap_l \mathfrak{u}| \leq |\mathfrak{u}| = m$. Since $\mathfrak{u}$ is the unique string with minimal length in I, we conclude that $\mathfrak{x} \sqcap_l \mathfrak{u} = \mathfrak{u}$, i.e., $\mathfrak{u} \sqsubseteq_l \mathfrak{x}$. □

The hammock $H_l(\mathfrak{x}_0)$ can be expressed as $H_l(\mathfrak{x}_0) = H_l^{-1}(\mathfrak{x}_0) \dotplus H_l^1(\mathfrak{x}_0)$, where

$$H_l^i(\mathfrak{x}_0) := \{\mathfrak{x} \in H_l(\mathfrak{x}_0) \mid \text{ either } \mathfrak{x} = \mathfrak{x}_0 \text{ or } \theta(\mathfrak{x} \mid \mathfrak{x}_0) = i\}$$

is a bounded discrete linear suborder of $H_l(\mathfrak{x}_0)$ with minimum element $\mathfrak{m}_i(\mathfrak{x}_0)$ and maximum element $\mathfrak{M}_i(\mathfrak{x}_0)$. It is easily noted that $\mathfrak{M}_{-1}(\mathfrak{x}_0) = \mathfrak{m}_1(\mathfrak{x}_0) = \mathfrak{x}_0$.

Smaller left hammocks can be embedded in bigger hammocks as intervals.

Remark 4.4 If $\mathfrak{y}(\neq \mathfrak{x}_0) \in H_l^i(\mathfrak{x}_0)$ then $H_l(\mathfrak{y})$ is an interval in $H_l^i(\mathfrak{x}_0)$.

The concept of H-equivalence was introduced in [11, § 4] to identify when two left hammocks are isomorphic. Say that two strings $\mathfrak{x}$ and $\mathfrak{y}$ are H-equivalent, denoted $\mathfrak{x} \equiv_H \mathfrak{y}$, if for every string $\mathfrak{u}$, $\mathfrak{u}\mathfrak{x} \in \mathsf{St}(\Lambda)$ if and only if $\mathfrak{u}\mathfrak{y} \in \mathsf{St}(\Lambda)$. Indeed, if $\mathfrak{x} \equiv_H \mathfrak{y}$ then $(H_l(\mathfrak{x}), <_l) \cong (H_l(\mathfrak{y}), <_l)$. As a consequence of [11, Proposition 4.4], a criterion for testing H-equivalence, we note a useful observation that we will use without mention.

Remark 4.5 If $\mathfrak{x}, \mathfrak{y}$ are strings with $\delta(\mathfrak{y}) = 0$ such that $\mathfrak{y}\mathfrak{x}$ is a string then $\mathfrak{y} \equiv_H \mathfrak{y}\mathfrak{x}$. As a consequence, if $\mathfrak{z}$ is a string such that $\mathfrak{z}\mathfrak{y}$ is a string then $\mathfrak{z}\mathfrak{y}\mathfrak{x}$ is a string.

Some finiteness results are the key to the proof of the main theorem which states that $(H_l^i(\mathfrak{x}_0), <_l) \in \mathrm{dLO}_{\mathrm{fd}}^{11}$. Recall the definition of a prime band.

Definition 4.6 [3, Definition 3.1.1] A band $\mathfrak{b} \in \mathsf{Ba}(\Lambda)$ is called a *prime band* if none of its cyclic permutations can be written $\mathfrak{b}_k \cdots \mathfrak{b}_2\mathfrak{b}_1$ for some $k > 1$ where each $\mathfrak{b}_i \in \mathsf{Cyc}(\Lambda)$; otherwise it is called *composite*.

It was shown in [3, Theorem 3.1.6] that there are only finitely many prime bands in $\mathsf{Ba}(\Lambda)$. Also recall from [3, Proposition 3.1.7] that there are only finitely many *band-free strings* in any string algebra, i.e., those which do not contain a cycle as a substring. Call a string $\mathfrak{y} = \mathfrak{z}\mathfrak{x}_0 \in H_l^i(\mathfrak{x}_0)$ *band-free relative to* $(\mathfrak{x}_0, i)$ if $\mathfrak{z}$ is band-free. The following is an immediate consequence of [3, Proposition 3.1.7].

Corollary 4.7 *There are only finitely many band-free strings relative to* $(\mathfrak{x}_0, i)$.

We end this section by mentioning a basic result about bands, which shows that a band $\mathfrak{b}$ has exactly $|\mathfrak{b}|$ distinct cyclic permutations. If $\mathfrak{x} = \alpha_n \cdots \alpha_2\alpha_1$ is a finite power of cyclic permutation of a band, call $1 \leq k \leq n$ a *period* of $\mathfrak{x}$ if $k = n$ or $\mathfrak{x} = \alpha_k \cdots \alpha_1\alpha_n \cdots \alpha_{k+1}$.

Proposition 4.8 *[4, Lemma 1] Let $\mathfrak{x}$ be a finite power of a cyclic permutation of a band and p and q be periods of $\mathfrak{x}$ such that $p + q \leq |\mathfrak{x}| + gcd(p, q)$. Then $gcd(p, q)$ is a period of $\mathfrak{x}$.*

Corollary 4.9 *If $\mathfrak{b} \in \mathsf{Cyc}(\Lambda)$ is such that $\mathfrak{b} = \alpha_n \ldots \alpha_1$ then $\alpha_r \cdots \alpha_1\alpha_n \cdots \alpha_{r+1} \neq \alpha_n \cdots \alpha_1$ for any $1 \leq r < n$.*

Proof If not, then r is a period of $\mathfrak{b}$. Since $|\mathfrak{b}|$ is also a period of $\mathfrak{b}$, we have that $gcd(|\mathfrak{b}|, r) =: t$ is a period of $\mathfrak{b}$ by Proposition 4.8. Therefore, we get that $\alpha_n \cdots \alpha_1 = (\alpha_t \cdots \alpha_1)^{|\mathfrak{b}|/t}$, a contradiction to the fact that $\mathfrak{b}$ is a primitive cyclic string. □

Recall from [3, § 2.1] that a left $\mathbb{N}$-string is a sequence of syllables $\cdots \alpha_3\alpha_2\alpha_1$ such that each $\alpha_n \cdots \alpha_2\alpha_1$ is a string. Call α_i the i^{th} syllable of $\mathfrak{x}$. Denote the set of left $\mathbb{N}$-strings by $\mathbb{N}\text{-}\mathsf{St}(\Lambda)$. A left $\mathbb{N}$-string of the form ${}^{\infty}\mathfrak{b}\mathfrak{u}$ for some cyclic string $\mathfrak{b}$ and some finite string $\mathfrak{u}$ is called an *almost periodic string*.

Definition 4.10 Say a sequence $(\mathfrak{x}_n)_{n\geq 1}$ of strings in $H_l(\mathfrak{x}_0)$ is *convergent* if there is $\mathfrak{y} \in \mathbb{N}\text{-}\mathsf{St}(\Lambda)$ such that

(1) $|\mathfrak{x}_n| \to \infty$ as $n \to \infty$;
(2) there is a sequence $\{n_k \mid k \in \mathbb{N}^+\}$ such that the k^{th} syllables of $\mathfrak{y}$ and $\mathfrak{x}_n$ are identical for $n \geq n_k$.

Clearly the limit of a convergent sequence is unique, and we write $\lim_{n\to\infty} \mathfrak{x}_n := \mathfrak{y}$.

If $\widehat{H}_l(\mathfrak{x}_0)$ is the extension of $H_l(\mathfrak{x}_0)$ by all the left $\mathbb{N}$-strings containing $\mathfrak{x}_0$ as a left substring, it is readily noted that the ordering $<_l$ can be extended to a linear order on $\widehat{H}_l(\mathfrak{x}_0)$.

Proposition 4.11 *The linear order* $(\widehat{H}_l(\mathfrak{x}_0), <_l)$ *is the completion of* $(H_l(\mathfrak{x}_0), <_l)$.

Proof Suppose $\mathfrak{z}$ is a left $\mathbb{N}$-string in $\widehat{H}_l(\mathfrak{x}_0)$. Set $X := \{\mathfrak{x} \in H_l(\mathfrak{x}_0) \mid \mathfrak{x} <_l \mathfrak{z}\}$ and $Y := \{\mathfrak{y} \in H_l(\mathfrak{x}_0) \mid \mathfrak{z} <_l \mathfrak{y}\}$. Then clearly $\mathfrak{m}_{-1}(\mathfrak{x}_0) \in X, \mathfrak{M}_1(\mathfrak{x}_0) \in Y$ and $H_l(\mathfrak{x}_0) = X \sqcup Y$. Moreover, $\mathfrak{x} <_l \mathfrak{y}$ for each $\mathfrak{x} \in X$ and $\mathfrak{y} \in Y$. Thus (X, Y) is a Dedekind cut in $H_l(\mathfrak{x}_0)$. We show that (X, Y) is actually a gap in $H_l(\mathfrak{x}_0)$.

Since the set $\{\mathfrak{v} \in \mathsf{St}(\Lambda) : \delta(\mathfrak{v}) \neq 0\}$ is finite, there are infinitely many inverse as well as direct syllables in $\mathfrak{z}$. Hence $X_1 := \{\mathfrak{x} \in H_l^i(\mathfrak{x}_0) \mid \mathfrak{x} \sqsubset_l \mathfrak{z}, \theta(\mathfrak{z} \mid \mathfrak{x}) = 1\}$ and $Y_1 := \{\mathfrak{y} \in H_l^i(\mathfrak{x}_0) \mid \mathfrak{y} \sqsubset_l \mathfrak{z}, \theta(\mathfrak{z} \mid \mathfrak{y}) = -1\}$ are infinite subsets of X and Y respectively.

Let $\mathfrak{x} \in X$ and $\mathfrak{w} := \mathfrak{x} \sqcap_l \mathfrak{z}$. Since $\mathfrak{x} <_l \mathfrak{z}$, we have $\mathfrak{x} \leq_l \mathfrak{w} <_l \mathfrak{z}$. Hence $\theta(\mathfrak{z} \mid \mathfrak{w}) = 1$ which implies $\mathfrak{w} \in X_1$. This shows that X_1 is an infinite cofinal subset of X, which implies that X does not have a maximum. A dual argument shows that Y_1 is a coinitial subset of Y, and thus Y does not have a minimum. This completes the proof that (X, Y) is a gap in $H_l(\mathfrak{x}_0)$.

Conversely, suppose (X, Y) is a gap in $H_l(\mathfrak{x}_0)$. Since $X \neq \emptyset$ and X is unbounded above, [8, Theorem 3.36] guarantees the existence of a countably infinite monotone increasing sequence $(\mathfrak{x}_n)_{n\in\omega}$ in X that is cofinal in X. Dually we can argue the existence of a countably infinite monotone decreasing sequence $(\mathfrak{y}_n)_{n\in\omega}$ in Y that is coinitial in Y. Then $[\mathfrak{x}_n, \mathfrak{y}_n] \supseteq [\mathfrak{x}_{n+1}, \mathfrak{y}_{n+1}]$ and $\bigcap_{n\in\omega}[\mathfrak{x}_n, \mathfrak{y}_n] = \emptyset$.

For each $n \in \omega$, Proposition 4.3 guarantees the existence of the unique minimal length string $\mathfrak{z}_n \in [\mathfrak{x}_n, \mathfrak{y}_n]$ such that $\mathfrak{z}_n \sqsubseteq_l \mathfrak{z}_{n+1}$. Since $\bigcap_{n\in\omega}[\mathfrak{x}_n, \mathfrak{y}_n] = \emptyset$, for each $n_k \in \omega$ there is a least $n_{k+1} > n_k$ such that $\mathfrak{z}_{n_k} \notin [\mathfrak{x}_{n_{k+1}}, \mathfrak{y}_{n_{k+1}}]$. Hence $\mathfrak{z}_{n_k} \sqsubset_l \mathfrak{z}_{n_{k+1}}$. Thus $|\mathfrak{z}_n| \to \infty$ as $n \to \infty$, which together with $\mathfrak{z}_n \sqsubseteq_l \mathfrak{z}_{n+1}$ for each $n \in \omega$ ensures that $(\mathfrak{z}_n)_{n\in\omega}$ is a convergent sequence with a left $\mathbb{N}$-string, say $\mathfrak{z}$, as its limit. Clearly $\mathfrak{x}_n <_l \mathfrak{z} <_l \mathfrak{y}_n$ for each $n \in \omega$.

Let $\mathfrak{x} \in X$. Since $\bigcap_{n\in\omega}[\mathfrak{x}_n, \mathfrak{y}_n] = \emptyset$, there is some $k \in \omega$ such that $\mathfrak{x} \notin [\mathfrak{x}_k, \mathfrak{y}_k]$. Thus $\mathfrak{x} <_l \mathfrak{x}_k <_l \mathfrak{z}$. Dually we can show that $\mathfrak{z} <_l \mathfrak{y}$ for each $\mathfrak{y} \in Y$.

If $\mathfrak{z}' <_l \mathfrak{z}''$ are two distinct left $\mathbb{N}$-strings satisfying $\mathfrak{x} <_l \mathfrak{z}' <_l \mathfrak{y}$ and $\mathfrak{x} <_l \mathfrak{z}'' <_l \mathfrak{y}$ for each $\mathfrak{x} \in X$ and $\mathfrak{y} \in Y$, then $\mathfrak{z}' <_l \mathfrak{z}' \sqcap_l \mathfrak{z}'' <_l \mathfrak{z}''$, and hence $\mathfrak{z}' \sqcap_l \mathfrak{z}'' \in H_l(\mathfrak{x}_0) \setminus (X \cup Y)$, a contradiction. Thus associated to each gap (X, Y) in $H_l(\mathfrak{x}_0)$ there is a unique left $\mathbb{N}$-string $\mathfrak{z}$ satisfying $\mathfrak{x} <_l \mathfrak{z} <_l \mathfrak{y}$ for each $\mathfrak{x} \in X$ and $\mathfrak{y} \in Y$. If (X_1, Y_1) and (X_2, Y_2) are distinct gaps then it is routine to verify that the left $\mathbb{N}$-strings associated to these gaps are distinct. ◻

5 A Finite Poset

A finite combinatorial gadget known as the *bridge quiver* was introduced in [3, § 3.2] for all string algebras. The finite set of prime bands plays the role of the vertex set for the bridge quiver.

Definition 5.1 [3, Definition 3.2.1] For prime bands $\mathfrak{b}_1, \mathfrak{b}_2$, say that a finite string $\mathfrak{u}$ is a *weak bridge* $\mathfrak{b}_1 \to \mathfrak{b}_2$ if it is band-free and if the word $\mathfrak{b}_2\mathfrak{u}\mathfrak{b}_1$ is a string. Say that a weak bridge $\mathfrak{b}_1 \xrightarrow{\mathfrak{u}} \mathfrak{b}_2$ is a *bridge* if there is no prime band $\mathfrak{b}$ and weak bridges $\mathfrak{b}_1 \xrightarrow{\mathfrak{u}_1} \mathfrak{b}$ and $\mathfrak{b} \xrightarrow{\mathfrak{u}_2} \mathfrak{b}_2$ such that one of the following holds:

- $\mathfrak{u} = \mathfrak{u}_2\mathfrak{u}_1$, $|\mathfrak{u}_1| > 0$, $\mathfrak{u}_2 > 0$;
- $\mathfrak{u} = \mathfrak{u}_2'\mathfrak{u}_1'$, $|\mathfrak{u}_1'| > 0$, $\mathfrak{u}_2' > 0$, $\mathfrak{u}_2 = \mathfrak{u}_2'\mathfrak{u}_2''$, $\mathfrak{u}_1 = \mathfrak{u}_1''\mathfrak{u}_1'$ and $\mathfrak{b} = \mathfrak{u}_1''\mathfrak{u}_2''$.

Denote by $\mathcal{Q}_1^{\mathsf{Ba}}$ and $\widetilde{\mathcal{Q}}_1^{\mathsf{Ba}}$ the set of all bridges and weak bridges between prime bands in $\mathcal{Q}_0^{\mathsf{Ba}}$ respectively. The quiver with vertex set consisting of only prime bands in $\mathcal{Q}_0^{\mathsf{Ba}}$ and arrow set $\mathcal{Q}_1^{\mathsf{Ba}}$ (resp. $\widetilde{\mathcal{Q}}_1^{\mathsf{Ba}}$) is called the *bridge quiver* (resp. the *weak bridge quiver*).

Recall from [3, Lemma 3.3.4] that all strings can be generated by paths in (an appropriate extended) bridge quiver. The property which distinguishes a non-domestic string algebra from a domestic string algebra is the existence of a *meta-band*, i.e., a directed cycle in its bridge quiver [3, Proposition 3.4.2]. A generalised meta-band defined below captures the complete essence of a building block of a string algebra.

Definition 5.2 A *generalised meta-band* (*GMB*, for short), denoted B, is a strongly connected component of the bridge quiver.

Call a GMB B *domestic* if B has only one vertex and *non-domestic* otherwise. Note that a string algebra Λ is non-domestic if and only if there is a non-domestic GMB in its bridge quiver (cf. [3, Proposition 3.4.2]).

Recall the definition of generation of strings from paths in the bridge quiver from [3, § 3.3]. Call a string B-*cycle* if it lies in $\mathsf{Cyc}(\Lambda)$ and is generated by a path in B. Denote the set of all B-cycles by $\mathsf{Cyc}(\mathsf{B})$ and set $\mathsf{Ba}(\mathsf{B}) := \mathcal{Q}_0^{\mathrm{Ba}} \cap \mathsf{Cyc}(\mathsf{B})$. For $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$, call a string $\mathfrak{y}$ to be *band-free with respect to* B if there does not exist $\mathfrak{b} \in \mathsf{Cyc}(\mathsf{B})$ and strings $\mathfrak{y}_1, \mathfrak{y}_2$ such that $\mathfrak{y} = \mathfrak{y}_2\mathfrak{b}\mathfrak{y}_1$.

Define a relation $\preceq$ on the set $\mathcal{Q}_0^{\mathrm{Ba}}$ of bands by declaring $\mathfrak{b}_1 \preceq \mathfrak{b}_2$ if there is a string $\mathfrak{u}$ such that $\mathfrak{b}_2\mathfrak{u}\mathfrak{b}_1$ is a string. This relation is clearly reflexive and transitive.

Proposition 5.3 *The relation $\preceq$ defined above is anti-symmetric if and only if the string algebra Λ is domestic.*

Proof Suppose the relation $\preceq$ is anti-symmetric. If possible, let the string algebra be non-domestic. By [3, Proposition 3.4.2], Λ contains a meta-band. There are two cases.

If the length of the meta-band exceeds 1 then consider two distinct prime bands $\mathfrak{b}_1$ and $\mathfrak{b}_2$ in that meta-band. The definition of a meta-band ensures the existence of strings $\mathfrak{u}$ and $\mathfrak{v}$ such that $\mathfrak{b}_2\mathfrak{u}\mathfrak{b}_1$ and $\mathfrak{b}_1\mathfrak{v}\mathfrak{b}_2$ are strings. This violates that $\preceq$ is anti-symmetric.

On the other hand, if the meta-band is a non-trivial bridge $\mathfrak{b} \xrightarrow{\mathfrak{u}} \mathfrak{b}$ then, by [3, Proposition 3.4.1], we have that $\mathfrak{b}$ is a vertex of a meta-band containing at least two prime bands. The rest of the argument is similar to that in the previous paragraph.

Conversely, suppose $\preceq$ is not anti-symmetric. Then there are distinct bands $\mathfrak{b}_1$ and $\mathfrak{b}_2$ and strings $\mathfrak{u}$ and $\mathfrak{v}$ such that $\mathfrak{b}_2\mathfrak{u}\mathfrak{b}_1$ and $\mathfrak{b}_1\mathfrak{v}\mathfrak{b}_2$ are strings. Now the strings $\mathfrak{v}\mathfrak{b}_2\mathfrak{u}\mathfrak{b}_1^k$ for $k \geq 1$ contain an infinite family of cyclic permutations of distinct bands, proving that Λ is non-domestic. □

Say that $\mathfrak{b}_1 \approx \mathfrak{b}_2$ if $\mathfrak{b}_1 \preceq \mathfrak{b}_2$ and $\mathfrak{b}_2 \preceq \mathfrak{b}_1$, and set $\mathcal{Q}^{\mathrm{Ba}} := \mathcal{Q}_0^{\mathrm{Ba}}/\approx$. Note that $\mathfrak{b}_1 \approx \mathfrak{b}_2$ if and only if there is a GMB B such that $\mathfrak{b}_1, \mathfrak{b}_2 \in \mathsf{Ba}(\mathsf{B})$. Hence we will denote the elements of $\mathcal{Q}^{\mathrm{Ba}}$ using B, possibly with decoration. Borrowing the adjectives for a GMB, if $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ and $\mathrm{card}(\mathsf{B}) = 1$, then say that B is *domestic*, otherwise say that it is *non-domestic*.

By appropriate manipulation of cyclic permutations, it is trivial to note the following. Let $\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}'_1, \mathfrak{b}'_2 \in \mathsf{Cyc}(\Lambda)$ such that $\mathfrak{b}'_1$ and $\mathfrak{b}'_2$ are cyclic permutations of $\mathfrak{b}_1$ and $\mathfrak{b}_2$ respectively. If $\mathfrak{b}_2\mathfrak{u}\mathfrak{b}_1$ is a string for some string $\mathfrak{u}$ then $\mathfrak{b}'_2\mathfrak{v}\mathfrak{b}'_1$ is a string for some string $\mathfrak{v}$. Therefore we can extend the relation $\preceq$ on the set $\mathsf{Cyc}(\Lambda)$ such that for any $\mathfrak{b}_1, \mathfrak{b}_2 \in \mathcal{Q}_0^{\mathrm{Ba}}$, we have $\mathfrak{b}_1 \preceq \mathfrak{b}_2$ if and only if $\mathfrak{b}'_1 \preceq \mathfrak{b}'_2$ where $\mathfrak{b}'_1$ and $\mathfrak{b}'_2$ are cyclic permutations of $\mathfrak{b}_1$ and $\mathfrak{b}_2$ respectively.

Call a string B-*extendable* if it is a substring of a power of a B-cycle. Denote the set of all B-extendable strings by $\mathsf{Ext}(\mathsf{B})$. Any B-extendable string is reachable from another one–we will use the next remark stating this without mention.

Remark 5.4 Let $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathsf{Ext}(\mathsf{B})$. Then there exists a string $\mathfrak{u}$ such that $\mathfrak{x}_2\mathfrak{u}\mathfrak{x}_1 \in \mathsf{Ext}(\mathsf{B})$. To see this, note from the definition of a B-extendable string, there exist strings $\mathfrak{u}_1, \mathfrak{u}_2$ such that

$\mathfrak{u}_1\mathfrak{x}_1 = \mathfrak{b}_1^m$ and $\mathfrak{u}_2\mathfrak{x}_2 = \mathfrak{b}_2^n$ for some B-cycles $\mathfrak{b}_1$ and $\mathfrak{b}_2$. Since $\mathfrak{b}_1 \approx \mathfrak{b}_2$, there exist strings $\mathfrak{v}_1, \mathfrak{v}_2$ such that $\mathfrak{b}_1^m\mathfrak{v}_2\mathfrak{b}_2^n\mathfrak{v}_1 = \mathfrak{u}_1\mathfrak{x}_1\mathfrak{v}_2\mathfrak{u}_2\mathfrak{x}_2\mathfrak{v}_1$ is a power of a B-cycle.

Proposition 5.5 *If* $\mathfrak{b} \in \mathcal{Q}_0^{\mathrm{Ba}}$ *is composite then there is* $\mathfrak{b}_1 \in \mathcal{Q}_0^{\mathrm{Ba}}$ *with* $|\mathfrak{b}_1| < |\mathfrak{b}|$ *such that* $\mathfrak{b} \approx \mathfrak{b}_1$.

Proof If $\mathfrak{b} \in \mathcal{Q}_0^{\mathrm{Ba}}$ is composite then there is a cyclic permutation $\mathfrak{b}'$ of $\mathfrak{b}$ which can be written as $\mathfrak{b}' = \mathfrak{b}'_k \cdots \mathfrak{b}'_1$ for some $k > 1$ and cyclic permutations $\mathfrak{b}'_j$ of $\mathfrak{b}_j \in \mathcal{Q}_0^{\mathrm{Ba}}$. It is then straightforward to note that $|\mathfrak{b}'_1| < |\mathfrak{b}'| = |\mathfrak{b}|$ and $\mathfrak{b}_1 \approx \mathfrak{b}'_1 \approx \mathfrak{b}' \approx \mathfrak{b}$. □

This simple result has an immediate consequence.

Corollary 5.6 *If* $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ *then* B *contains a prime band.*

Proof Let $\mathfrak{b} \in \mathsf{Ba}(\mathsf{B})$. If $\mathfrak{b}$ is prime then we are done. Otherwise, by Proposition 5.5, there exists $\mathfrak{b}_1 \in \mathcal{Q}_0^{\mathrm{Ba}}$ such that $|\mathfrak{b}_1| < |\mathfrak{b}|$ and $\mathfrak{b}_1 \approx \mathfrak{b}$. If $\mathfrak{b}_1$ is a prime band then we are done. Otherwise, we repeat the process on $\mathfrak{b}_1$ to get $\mathfrak{b}_2 \in \mathcal{Q}_0^{\mathrm{Ba}}$ such that $|\mathfrak{b}_2| < |\mathfrak{b}_1|$ and $\mathfrak{b}_2 \approx \mathfrak{b}_1$. Thus we get a sequence of bands $\mathfrak{b}_1, \mathfrak{b}_2, \cdots$ such that $\mathfrak{b} \approx \mathfrak{b}_1 \approx \mathfrak{b}_2 \approx \cdots$ and $|\mathfrak{b}| > |\mathfrak{b}_1| > |\mathfrak{b}_2| > \cdots$. Since $|\mathfrak{b}|$ is finite, this process has to terminate after finitely many steps, thereby giving us a prime band in B. □

Since [3, Theorem 3.1.6] gives that there are only finitely many prime bands, the above corollary yields the following finiteness result.

Proposition 5.7 *The poset* $(\mathcal{Q}^{\mathrm{Ba}}, \preceq)$ *is a finite poset.*

For a fixed string $\mathfrak{x}_0$ and parity $i \in \{1, -1\}$, say that a band $\mathfrak{b}$ is *reachable* from $(\mathfrak{x}_0, i)$ if there is a string $\mathfrak{u}$ such that $\mathfrak{b}\mathfrak{u}\mathfrak{x}_0 \in H_l^i(\mathfrak{x}_0)$. If $\mathfrak{b}_1 \approx \mathfrak{b}_2$ then $\mathfrak{b}_1$ is reachable from $(\mathfrak{x}_0, i)$ if and only if $\mathfrak{b}_2$ is reachable from $(\mathfrak{x}_0, i)$. Hence the subset $\mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0)$ of $\mathcal{Q}^{\mathrm{Ba}}$ of elements reachable from $(\mathfrak{x}_0, i)$ is also finite. Say that $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ is *minimal for* $(\mathfrak{x}_0, i)$ if it is a minimal element of $(\mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0), \preceq)$. Since every finite poset contains a minimal element, the existence of a minimal B for the pair $(\mathfrak{x}_0, i)$ is guaranteed.

Example 5.8 Consider the string algebra Γ_0 in Fig. 1. There are four elements in $\mathcal{Q}_1^{\mathrm{Ba}}(a_0)$, namely $\mathsf{B}_1 = \{b_1B_4b_3B_2\}$, B_2 containing bands d_1D_2 and d_3D_4, B_3 containing bands $e_3E_2E_1$, $g_4G_3g_2G_1$ and k_1K_2, and $\mathsf{B}_4 = \{m_1M_2\}$. Here B_1 and B_4 are domestic; whereas B_2 and B_3 are non-domestic. We have $\mathsf{B}_1 \prec \mathsf{B}_2$ and $\mathsf{B}_3 \prec \mathsf{B}_4$ as the only order relations in $(\mathcal{Q}_1^{\mathrm{Ba}}(a_0), \prec)$. Only B_1 and B_3 are minimal for $(a_0, 1)$.

6 Some Finiteness Results

We introduce some sets of strings in a hammock that are close to an element of $\mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0)$ and show under a suitable condition that some subsets of those sets are finite.

Given $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ and $j \in \{-1, 1\}$, the following set captures the strings which "touch" B with sign j.

$$\mathsf{St}_j(\mathsf{B}) := \{\mathfrak{x} \in \mathsf{St}(\Lambda) \mid \mathfrak{b}\mathfrak{x} \text{ is a string for some } \mathfrak{b} \in \mathsf{Cyc}(\mathsf{B}) \text{ such that } \theta(\mathfrak{b}) = j\}.$$

Further set

$$\mathsf{St}(\mathsf{B}) := \mathsf{St}_1(\mathsf{B}) \cup \mathsf{St}_{-1}(\mathsf{B}),\ \ \mathsf{St}_{\pm 1}(\mathsf{B}) := \mathsf{St}_1(\mathsf{B}) \cap \mathsf{St}_{-1}(\mathsf{B}).$$

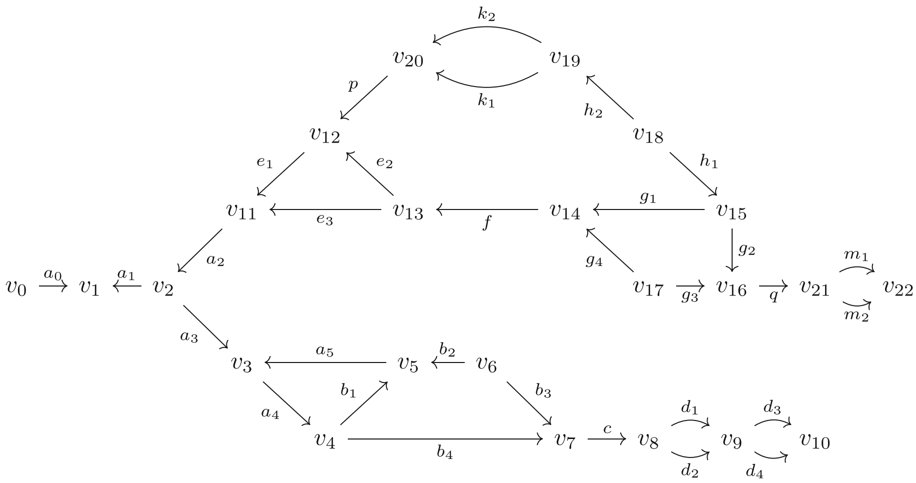

Fig. 1 Γ_0 with $\rho = \{a_3a_2, a_4a_5, b_4a_4, a_5b_1, cb_4, d_2c, d_4d_1, d_3d_2, a_2e_3, e_3f, fg_4, g_2h_1, k_2h_2, qg_3, m_2q, pk_2, e_1p\}$

If $\mathsf{B} \in \mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0)$ then set

$\mathsf{St}_j(\mathfrak{x}_0, i; \mathsf{B}) := \mathsf{St}_j(\mathsf{B}) \cap H_l^i(\mathfrak{x}_0)$, $\mathsf{St}(\mathfrak{x}_0, i; \mathsf{B}) := \mathsf{St}(\mathsf{B}) \cap H_l^i(\mathfrak{x}_0)$ and $\mathsf{St}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B}) := \mathsf{St}_{\pm 1}(\mathsf{B}) \cap H_l^i(\mathfrak{x}_0)$.

Now we close the above sets of strings under left substrings. For $j \in \{1, -1\}$, define

$$\overline{\mathsf{St}}_j(\mathsf{B}) := \{\mathfrak{x} \in \mathsf{St}(\Lambda) \mid \mathfrak{u}\mathfrak{x} \in \mathsf{St}(\mathsf{B}) \text{ for some string } \mathfrak{u} \text{ with } \theta(\mathfrak{u}) = j\},$$

$$\overline{\mathsf{St}}(\mathsf{B}) := \overline{\mathsf{St}}_1(\mathsf{B}) \cup \overline{\mathsf{St}}_{-1}(\mathsf{B}) \text{ and } \overline{\mathsf{St}}_{\pm 1}(\mathsf{B}) := \overline{\mathsf{St}}_1(\mathsf{B}) \cap \overline{\mathsf{St}}_{-1}(\mathsf{B}).$$

Further if $\mathsf{B} \in \mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0)$ then set

$$\overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B}) := \{\mathfrak{x} \in H_l^i(\mathfrak{x}_0) \mid \mathfrak{u}\mathfrak{x} \in \mathsf{St}(\mathsf{B}) \cup \{\mathfrak{M}_i(\mathfrak{x}_0)\} \text{ for some string } \mathfrak{u} \text{ with } \theta(\mathfrak{u}) = 1\} \cup \{\mathfrak{M}_i(\mathfrak{x}_0)\},$$
$$\overline{\mathsf{St}}_{-1}(\mathfrak{x}_0, i; \mathsf{B}) := \{\mathfrak{x} \in H_l^i(\mathfrak{x}_0) \mid \mathfrak{u}\mathfrak{x} \in \mathsf{St}(\mathsf{B}) \cup \{\mathfrak{m}_i(\mathfrak{x}_0)\} \text{ for some string } \mathfrak{u} \text{ with } \theta(\mathfrak{u}) = -1\} \cup \{\mathfrak{m}_i(\mathfrak{x}_0)\},$$
$$\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B}) := \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B}) \cup \overline{\mathsf{St}}_{-1}(\mathfrak{x}_0, i; \mathsf{B}) \text{ and } \overline{\mathsf{St}}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B}) := \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B}) \cap \overline{\mathsf{St}}_{-1}(\mathfrak{x}_0, i; \mathsf{B}).$$

When we use the notations $\mathsf{St}_j(\mathfrak{x}_0, i; \mathsf{B})$ and $\overline{\mathsf{St}}_j(\mathfrak{x}_0, i; \mathsf{B})$ then we implicitly assume that $\mathsf{B} \in \mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0)$.

It is trivial to note that $\mathsf{St}_j(\mathsf{B}) \subseteq \overline{\mathsf{St}}_j(\mathsf{B})$ for any $j \in \{-1, 1\}$. The following remarks are straightforward yet useful.

Remark 6.1 For any $\mathfrak{x} \in H_l^i(\mathfrak{x}_0)$, $\mathfrak{x} \in \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B})$ if and only if $\mathfrak{x} \in \overline{\mathsf{St}}_1(\mathsf{B})$ or $\mathfrak{x} \sqsubseteq_l \mathfrak{M}_i(\mathfrak{x}_0)$. Dually, for any $\mathfrak{x} \in H_l^i(\mathfrak{x}_0)$, $\mathfrak{x} \in \overline{\mathsf{St}}_{-1}(\mathfrak{x}_0, i; \mathsf{B})$ if and only if $\mathfrak{x} \in \overline{\mathsf{St}}_{-1}(\mathsf{B})$ or $\mathfrak{x} \sqsubseteq_l \mathfrak{m}_i(\mathfrak{x}_0)$.

Remark 6.2 If $\mathfrak{x} \in \overline{\mathsf{St}}(\mathsf{B})$ and $\mathfrak{y} \sqsubset_l \mathfrak{x}$ then $\mathfrak{y} \in \overline{\mathsf{St}}_{\theta(\mathfrak{x}|\mathfrak{y})}(\mathsf{B})$.

Remark 6.3 If $\mathfrak{x} \in \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$, $\mathfrak{y} \sqsubset_l \mathfrak{x}$ and $\mathfrak{y} \in H_l^i(\mathfrak{x}_0)$ then $\mathfrak{y} \in \overline{\mathsf{St}}_{\theta(\mathfrak{x}|\mathfrak{y})}(\mathfrak{x}_0, i; \mathsf{B})$.

Proposition 6.4 *If* $\mathfrak{x} \in \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B})$, $\mathfrak{y} \in \overline{\mathsf{St}}_{-1}(\mathfrak{x}_0, i; \mathsf{B})$ *and* $\mathfrak{x} <_l \mathfrak{y}$ *then* $\mathfrak{x} \in \overline{\mathsf{St}}_1(\mathsf{B})$ *and* $\mathfrak{y} \in \overline{\mathsf{St}}_{-1}(\mathsf{B})$.

Proof If possible, let $\mathfrak{x} \notin \overline{\mathsf{St}}_1(\mathsf{B})$. Then Remark 6.1 implies $\mathfrak{x} \sqsubseteq_l \mathfrak{M}_i(\mathfrak{x}_0)$.

If $\mathfrak{x}_0 = \mathfrak{M}_i(\mathfrak{x}_0)$ then $i = -1$ and $\mathfrak{x} = \mathfrak{x}_0 = \mathfrak{M}_i(\mathfrak{x}_0)$, which contradicts the existence of $\mathfrak{y} >_l \mathfrak{x}$.

Therefore assume that $\mathfrak{x}_0 \sqsubset_l \mathfrak{M}_i(\mathfrak{x}_0)$. This together with $\mathfrak{x} <_l \mathfrak{y}$ implies that $\mathfrak{y} = \mathfrak{w}\mathfrak{v}\mathfrak{x}$ for some strings $\mathfrak{w}, \mathfrak{v}$ with $\delta(\mathfrak{v}) = 1$. Since $\mathfrak{y} \in \overline{\mathsf{St}}_{-1}(\mathfrak{x}_0, i; \mathsf{B})$ and $\mathfrak{x} \sqsubset_l \mathfrak{y}$, Remark 6.3 implies that $\mathfrak{x} \in \overline{\mathsf{St}}_{\theta(\mathfrak{y}|\mathfrak{x})}(\mathfrak{x}_0, i; \mathsf{B}) = \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B})$. Since $\mathfrak{x}_0 = \mathfrak{m}_i(\mathfrak{x}_0) \sqsubset_l \mathfrak{y}$, we have $\mathfrak{y} \in \overline{\mathsf{St}}(\mathsf{B})$ by Remark 6.1. Now $\mathfrak{x} \sqsubset_l \mathfrak{y}$ together with Remark 6.2 implies that $\mathfrak{x} \in \overline{\mathsf{St}}_{\theta(\mathfrak{y}|\mathfrak{x})}(\mathsf{B}) = \overline{\mathsf{St}}_1(\mathsf{B})$, a contradiction to our assumption. Therefore $\mathfrak{x} \in \overline{\mathsf{St}}_1(\mathsf{B})$. Similarly, we can show that $\mathfrak{y} \in \overline{\mathsf{St}}_{-1}(\mathsf{B})$. □

In general, $\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$ could be very large compared to $\mathsf{St}(\mathfrak{x}_0, i; \mathsf{B})$, but it is possible to control this difference when B is minimal for $(\mathfrak{x}_0, i)$.

Proposition 6.5 *For* $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ *and* $j \in \{-1, 1\}$, *if* $\mathfrak{x} \in \overline{\mathsf{St}}_j(\mathsf{B}) \setminus \mathsf{St}_j(\mathsf{B})$ *then* $\mathfrak{x}$ *is band-free with respect to* B.

Proof If possible, let $\mathfrak{x} = \mathfrak{x}_2\mathfrak{b}\mathfrak{x}_1$ for some strings $\mathfrak{x}_1, \mathfrak{x}_2$ and $\mathfrak{b} \in \mathsf{Cyc}(\mathsf{B})$. Since $\mathfrak{x} \in \overline{\mathsf{St}}_j(\mathsf{B})$, there exist a string $\mathfrak{u}$ and $\mathfrak{b}_1 \in \mathsf{Cyc}(\mathsf{B})$ with $\theta(\mathfrak{b}_1\mathfrak{u}) = j$ such that $\mathfrak{b}_1\mathfrak{u}\mathfrak{x}_2\mathfrak{b}\mathfrak{x}_1$ is a string. Now $\mathfrak{b}, \mathfrak{b}_1 \in \mathsf{Cyc}(\mathsf{B})$ implies that there is a string $\mathfrak{v}$ such that $\mathfrak{b}\mathfrak{v}\mathfrak{b}_1$ is a string. Since $\delta(\mathfrak{b}) = \delta(\mathfrak{b}_1) = 0$, we have that $\mathfrak{b}\mathfrak{v}\mathfrak{b}_1\mathfrak{u}\mathfrak{x}_2\mathfrak{b}$ is a string implying that $\mathfrak{x}_2\mathfrak{b}\mathfrak{v}\mathfrak{b}_1\mathfrak{u}$ is a power of a B-cycle. This gives $\mathfrak{x} = \mathfrak{x}_2\mathfrak{b}\mathfrak{x}_1 \in \mathsf{St}_j(\mathsf{B})$, which is a contradiction. □

Corollary 6.6 *If* $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ *is minimal for* $(\mathfrak{x}_0, i)$, $j \in \{-1, 1\}$ *and* $\mathfrak{x} \in \overline{\mathsf{St}}_j(\mathfrak{x}_0, i; \mathsf{B}) \setminus \mathsf{St}_j(\mathfrak{x}_0, i; \mathsf{B})$ *then* $\mathfrak{x}$ *is band-free relative to* $(\mathfrak{x}_0, i)$.

Proof Let $\mathfrak{x} \in \overline{\mathsf{St}}_j(\mathfrak{x}_0, i; \mathsf{B}) \setminus \mathsf{St}_j(\mathfrak{x}_0, i; \mathsf{B})$. In view of Remark 6.1, there are three cases. Since $\delta(\mathfrak{M}_i(\mathfrak{x}_0) \mid \mathfrak{x}_0) \neq 0$, if $\mathfrak{x} \sqsubseteq_l \mathfrak{M}_i(\mathfrak{x}_0)$ then $\mathfrak{x}$ is band-free relative to $(\mathfrak{x}_0, i)$. A dual argument works when $\mathfrak{x} \sqsubseteq_l \mathfrak{m}_i(\mathfrak{x}_0)$.

Finally, if $\mathfrak{x} \in \overline{\mathsf{St}}_j(\mathsf{B})$ then there exist a string $\mathfrak{u}$ and $\mathfrak{b}_1 \in \mathsf{Cyc}(\mathsf{B})$ with $\theta(\mathfrak{b}_1\mathfrak{u}) = j$ such that $\mathfrak{b}_1\mathfrak{u}\mathfrak{x}$ is a string. If possible, let $\mathfrak{x} = \mathfrak{x}_2\mathfrak{b}\mathfrak{x}_1$ for some strings $\mathfrak{x}_1, \mathfrak{x}_2$ and $\mathfrak{b} \in \mathsf{Cyc}(\Lambda)$ such that $\mathfrak{x}_0 \sqsubseteq_l \mathfrak{x}_1$. Then $\mathfrak{b}_1\mathfrak{u}\mathfrak{x}_2\mathfrak{b}\mathfrak{x}_1$ is a string, which gives $\mathfrak{b} \preceq \mathfrak{b}_1$. Since B is minimal for $(\mathfrak{x}_0, i)$, we get $\mathfrak{b} \approx \mathfrak{b}_1$, a contradiction to Proposition 6.5. □

Recall from Corollary 4.7 that there are finitely many strings in $H_l^i(\mathfrak{x}_0)$ which are band-free relative to $(\mathfrak{x}_0, i)$. A simple set theoretic manipulation yields

$$\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B}) \setminus \mathsf{St}(\mathfrak{x}_0, i; \mathsf{B}) \subseteq (\overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B}) \setminus \mathsf{St}_1(\mathfrak{x}_0, i; \mathsf{B})) \cup (\overline{\mathsf{St}}_{-1}(\mathfrak{x}_0, i; \mathsf{B}) \setminus \mathsf{St}_{-1}(\mathfrak{x}_0, i; \mathsf{B})).$$

Therefore we get the following consequence of Corollary 6.6.

Corollary 6.7 *If* $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ *is minimal for* $(\mathfrak{x}_0, i)$ *then the set* $\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B}) \setminus \mathsf{St}(\mathfrak{x}_0, i; \mathsf{B})$ *is finite.*

Example 6.8 Continuing with Example 5.8, recall that B_1 and B_3 are minimal for $(a_0, 1)$. We have $\overline{\mathsf{St}}(a_0, 1; \mathsf{B}_1) \setminus \mathsf{St}(a_0, 1; \mathsf{B}_1) = \{a_0, A_1a_0, a_3A_1a_0\}$ and $\overline{\mathsf{St}}(a_0, 1; \mathsf{B}_3) \setminus \mathsf{St}(a_0, 1; \mathsf{B}_3) = \{a_0, A_1a_0\}$.

We prove yet one more conditional finiteness result.

Proposition 6.9 *Suppose* $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ *is domestic and minimal for* $(\mathfrak{x}_0, i)$. *If* $\mathfrak{x} \in \overline{\mathsf{St}}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B})$ *then* $\mathfrak{x}$ *is band-free relative to* $(\mathfrak{x}_0, i)$.

Proof Since $\mathfrak{M}_i(\mathfrak{x}_0)$ and $\mathfrak{m}_i(\mathfrak{x}_0)$ are band-free relative to $(\mathfrak{x}_0, i)$, in view of Remark 6.1, it is enough to assume that $\mathfrak{x} \in \mathsf{St}_{\pm1}(\mathsf{B}) \cap H_l^i(\mathfrak{x}_0)$.

If possible, let $\mathfrak{x} = \mathfrak{x}_2\mathfrak{b}\mathfrak{x}_1$ for $\mathfrak{b} \in \mathsf{Cyc}(\Lambda)$ and strings $\mathfrak{x}_1, \mathfrak{x}_2$ such that $\mathfrak{x}_0 \sqsubseteq_l \mathfrak{x}_1$. Since $\mathfrak{x} \in \overline{\mathsf{St}}_{\pm1}(\mathsf{B})$, there exist strings $\mathfrak{u}$ and $\mathfrak{v}$ with $\theta(\mathfrak{b}_1\mathfrak{u}) = -\theta(\mathfrak{b}_1\mathfrak{v})$ such that $\mathfrak{b}_1\mathfrak{u}\mathfrak{x}_2\mathfrak{b}\mathfrak{x}_1$ and $\mathfrak{b}_1\mathfrak{v}\mathfrak{x}_2\mathfrak{b}\mathfrak{x}_1$ are strings, where $\mathfrak{b}_1$ is the unique element in B since B is domestic. This gives $\mathfrak{b} \preceq \mathfrak{b}_1$, which further implies that $\mathfrak{b} \approx \mathfrak{b}_1$ since B is minimal for $(\mathfrak{x}_0, i)$. Moreover, B is domestic implies that $\mathfrak{b}$ is a cyclic permutation of $\mathfrak{b}_1$. Then ${}^{\infty}\mathfrak{b}_1\mathfrak{u}\mathfrak{x}_2\mathfrak{b} = {}^{\infty}\mathfrak{b} = {}^{\infty}\mathfrak{b}_2\mathfrak{v}\mathfrak{x}_2\mathfrak{b}$, which implies $\theta(\mathfrak{b}_1\mathfrak{u}) = \theta(\mathfrak{b}_1\mathfrak{v})$, a contradiction. □

Again by the finiteness of the set of band-free strings, we have the following corollary.

Corollary 6.10 *If* $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ *is domestic and minimal for* $(\mathfrak{x}_0, i)$ *then the set* $\overline{\mathsf{St}}_{\pm1}(\mathfrak{x}_0, i; \mathsf{B})$ *is finite.*

Example 6.11 Continuing with Example 5.8, recall that B_1 is domestic and minimal for $(a_0, 1)$. The string $a_3A_1a_0$ is the only element in $\overline{\mathsf{St}}_{\pm1}(a_0, 1; \mathsf{B}_1)$. On the other hand, B_3 is non-domestic and minimal for $(a_0, 1)$, and we have $E_2E_1(e_3E_2E_1)^nA_2A_1a_0 \in \overline{\mathsf{St}}_{\pm1}(a_0, 1; \mathsf{B}_3)$ for every $n \in \mathbb{N}$.

Proposition 6.12 *Let* $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ *be minimal for* $(\mathfrak{x}_0, i)$. *Then* $\overline{\mathsf{St}}_{\pm1}(\mathfrak{x}_0, i; \mathsf{B})$ *is bounded as a suborder of* $(H_l^i(\mathfrak{x}_0), <_l)$.

Proof First note that $\mathfrak{x}_0 \in \{\mathfrak{M}_i(\mathfrak{x}_0), \mathfrak{m}_i(\mathfrak{x}_0)\}$. Without loss of generality, assume that $\mathfrak{m}_i(\mathfrak{x}_0) <_l \mathfrak{M}_i(\mathfrak{x}_0)$. Then $\mathfrak{x}_0 = \mathfrak{M}_i(\mathfrak{x}_0)$ if and only if $i = -1$.

If $i = -1$ then clearly $\mathfrak{x}_0 = \mathfrak{M}_i(\mathfrak{x}_0) \in \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B})$. On the other hand, since B is minimal for $(\mathfrak{x}_0, i)$, there is $\mathfrak{b} \in \mathsf{Ba}(\mathsf{B})$ and a string $\mathfrak{u}$ such that $\mathfrak{b}\mathfrak{u}\mathfrak{x}_0 \in H_l^i(\mathfrak{x}_0)$, which gives $\mathfrak{x}_0 \in \overline{\mathsf{St}}_{-1}(\mathfrak{x}_0, i; \mathsf{B})$. Hence $\mathfrak{x}_0 \in \overline{\mathsf{St}}_{\pm1}(\mathfrak{x}_0, i; \mathsf{B})$. On the other hand, we have $\theta(\mathfrak{b}\mathfrak{u}\mathfrak{x}_0 \mid \mathfrak{m}_i(\mathfrak{x}_0)) = 1$. Hence there is a left substring of $\mathfrak{m}_i(\mathfrak{x}_0)$ that lies in $\overline{\mathsf{St}}_{\pm1}(\mathfrak{x}_0, i; \mathsf{B})$. Since $\delta(\mathfrak{m}_i(\mathfrak{x}_0) \mid \mathfrak{x}_0) = -1$, the longest such left substring will be the least element of $\overline{\mathsf{St}}_{\pm1}(\mathfrak{x}_0, i; \mathsf{B})$.

If $i = 1$ then $\mathfrak{x}_0 = \mathfrak{m}_i(\mathfrak{x}_0)$ is the lower bound. An argument dual to the above paragraph provides the upper bound. □

7 The Condensation Operator c_{B}

Recall the concept of condensation from § 2. In this section, we define a specific condensation operator c_{B} on a hammock which helps in breaking it into smaller hammocks.

Note that $\mathfrak{x}_0 \in \{\mathfrak{m}_i(\mathfrak{x}_0), \mathfrak{M}_i(\mathfrak{x}_0)\} \subseteq \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$ and $\mathfrak{x}_0$ appears as a left substring of every string in $H_l^i(\mathfrak{x}_0)$. Therefore every string in $H_l^i(\mathfrak{x}_0)$ has a left substring in $\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$. Now we use this observation to define the localization/condensation of a string in a hammock with respect to B.

Definition 7.1 If $\mathsf{B} \in \mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0)$ then define the B-*condensation* map

$$c_{\mathsf{B}} : H_l^i(\mathfrak{x}_0) \to \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$$

by associating to each $\mathfrak{x} \in H_l^i(\mathfrak{x}_0)$ its longest left substring in $\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$.

Remark 7.2 Note that if $\mathfrak{x}, \mathfrak{y} \in \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$ then $\{\mathfrak{x}, \mathfrak{y}\} \subseteq c_{\mathsf{B}}([\mathfrak{x}, \mathfrak{y}]) = [\mathfrak{x}, \mathfrak{y}] \cap \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$. As a consequence, the map c_{B} is surjective. Also for any $\mathfrak{x} \in H_l^i(\mathfrak{x}_0)$, we have $c_{\mathsf{B}}(\mathfrak{x}) = \mathfrak{x}$ if and only if $\mathfrak{x} \in \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$.

Now define a function $\varphi_{\mathsf{B}} : H_l^i(\mathfrak{x}_0) \to \{-1, 0, 1\}$ by

$$\varphi_{\mathsf{B}}(\mathfrak{x}) := \begin{cases} 0 & \text{if } c_{\mathsf{B}}(\mathfrak{x}) \in \overline{\mathsf{St}}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B}), \\ 1 & \text{if } c_{\mathsf{B}}(\mathfrak{x}) \in \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B}) \setminus \overline{\mathsf{St}}_{-1}(\mathfrak{x}_0, i; \mathsf{B}), \\ -1 & \text{if } c_{\mathsf{B}}(\mathfrak{x}) \in \overline{\mathsf{St}}_{-1}(\mathfrak{x}_0, i; \mathsf{B}) \setminus \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B}). \end{cases}$$

Remark 7.3 For each $\mathfrak{x} \in H_l^i(\mathfrak{x}_0) \setminus \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$, we have $c_{\mathsf{B}}(\mathfrak{x}) \sqsubset_l \mathfrak{x}$. If $\varphi_{\mathsf{B}}(\mathfrak{x}) = 0$ and $\alpha c_{\mathsf{B}}(\mathfrak{x}) \sqsubseteq_l \mathfrak{x}$ for some syllable α then $\alpha c_{\mathsf{B}}(\mathfrak{x}) \in \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$ since $c_{\mathsf{B}}(\mathfrak{x}) \in \overline{\mathsf{St}}_{\theta(\alpha)}(\mathfrak{x}_0, i; \mathsf{B})$. This is a contradiction to the definition of $c_{\mathsf{B}}(\mathfrak{x})$, and hence $\varphi_{\mathsf{B}}(\mathfrak{x}) \neq 0$.

Take the convention $H_l^0(\mathfrak{y}) := \{\mathfrak{y}\}$.

If $\mathfrak{y} \in \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$ then $\mathfrak{y} \in H_l^{-\varphi_{\mathsf{B}}(\mathfrak{y})}(\mathfrak{y})$. On the other hand, if $\mathfrak{x} \in H_l^i(\mathfrak{x}_0) \setminus \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$ and $j := \theta(\mathfrak{x} \mid c_{\mathsf{B}}(\mathfrak{x}))$ then the definition of $c_{\mathsf{B}}(\mathfrak{x})$ ensures that $c_{\mathsf{B}}(\mathfrak{x}) \notin \overline{\mathsf{St}}_j(\mathfrak{x}_0, i; \mathsf{B})$. Thus $\varphi_{\mathsf{B}}(\mathfrak{x}) = \varphi_{\mathsf{B}}(c_{\mathsf{B}}(\mathfrak{x})) = -j$. We document this observation in the following result.

Proposition 7.4 *If* $\mathfrak{x} \in H_l^i(\mathfrak{x}_0)$ *then* $\mathfrak{x} \in H_l^{-\varphi_{\mathsf{B}}(\mathfrak{x})}(c_{\mathsf{B}}(\mathfrak{x}))$.

The function φ_{B} is defined in such a way that the following statement is true. This will be the key to showing that the algorithm to compute the order type of a hammock terminates after finitely many steps.

Remark 7.5 For each $\mathfrak{y} \in \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$, we have $H_l^{-\varphi_{\mathsf{B}}(\mathfrak{y})}(\mathfrak{y}) \cap \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B}) = \{\mathfrak{y}\}$.

Proposition 7.6 *If* $\mathfrak{x}, \mathfrak{y} \in \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$ *and* $\mathfrak{x} <_l \mathfrak{y}$ *then for each* $\mathfrak{x}' \in H_l^{-\varphi_{\mathsf{B}}(\mathfrak{x})}(\mathfrak{x})$ *and* $\mathfrak{y}' \in H_l^{-\varphi_{\mathsf{B}}(\mathfrak{y})}(\mathfrak{y})$ *we have* $\mathfrak{x}' <_l \mathfrak{y}'$.

Proof If $\mathfrak{x} \sqsubset_l \mathfrak{y}$ then $\mathfrak{x} \sqsubset_l \mathfrak{y} \sqsubseteq_l \mathfrak{y}'$ for each $\mathfrak{y}' \in H_l^{-\varphi_{\mathsf{B}}(\mathfrak{y})}(\mathfrak{y})$. Hence $\theta(\mathfrak{y}' \mid \mathfrak{x}) = \theta(\mathfrak{y} \mid \mathfrak{x}) = 1$. Moreover, since $\mathfrak{x} \sqsubset_l \mathfrak{y} \in \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$, we conclude that $\varphi_{\mathsf{B}}(\mathfrak{x}) \neq -1$. If $\varphi_{\mathsf{B}}(\mathfrak{x}) = 0$ then the conclusion holds. On the other hand, if $\varphi_{\mathsf{B}}(\mathfrak{x}) = 1$ then $\theta(\mathfrak{x}' \mid \mathfrak{x}) = -1$ for each $\mathfrak{x}' \in H_l^{-\varphi_{\mathsf{B}}(\mathfrak{x})}(\mathfrak{x}) \setminus \{\mathfrak{x}\}$. Hence $\theta(\mathfrak{y}' \mid \mathfrak{x}') = \theta(\mathfrak{y}' \mid \mathfrak{x}) = 1$, and hence the conclusion.

A dual argument can be given when $\mathfrak{x} \sqsupset_l \mathfrak{y}$.

Finally when $\mathfrak{x}$ and $\mathfrak{y}$ are incomparable then $\mathfrak{x} \sqcap_l \mathfrak{y} \in \overline{\mathsf{St}}_{\pm 1}(\mathsf{B})$. The arguments in the above two paragraphs then give that $\mathfrak{x}' <_l \mathfrak{x} \sqcap_l \mathfrak{y} <_l \mathfrak{y}'$ for each $\mathfrak{x}' \in H_l^{-\varphi_{\mathsf{B}}(\mathfrak{x})}(\mathfrak{x})$ and $\mathfrak{y}' \in H_l^{-\varphi_{\mathsf{B}}(\mathfrak{y})}(\mathfrak{y})$, and thus the conclusion follows. □

As a consequence, we get that certain hammocks are disjoint.

Corollary 7.7 *If* $\mathfrak{x}, \mathfrak{y} \in \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$ *and* $\mathfrak{x} <_l \mathfrak{y}$ *then* $H_l^{-\varphi_{\mathsf{B}}(\mathfrak{x})}(\mathfrak{x}) \cap H_l^{-\varphi_{\mathsf{B}}(\mathfrak{y})}(\mathfrak{y}) = \emptyset$.

The following is the main result of this section which serves as an ingredient for the main theorem of this paper (Theorem 11.9). Loosely speaking, this result states that any hammock can be broken down into smaller hammocks when we localize/condense the hammock away from $\mathsf{B} \in \mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0)$. This result gives a recursive algorithm to compute the order type of a hammock.

Lemma 7.8 *Suppose* $\mathsf{B} \in \mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0)$. *Then*

$$(H_l^i(\mathfrak{x}_0), <_l) \cong \sum_{\mathfrak{x} \in c_{\mathsf{B}}(H_l^i(\mathfrak{x}_0))} (H_l^{-\varphi_{\mathsf{B}}(\mathfrak{x})}(\mathfrak{x}), <_l).$$

Proof Recall from Remark 7.2 that $c_{\mathsf{B}}(H_l^i(\mathfrak{x}_0)) = \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$. For any $\mathfrak{x} \in H_l^i(\mathfrak{x}_0)$ and $j \in \{1, 0, -1\}$, Remark 4.4 gives that $H_l^j(\mathfrak{x})$ is an interval in $H_l^i(\mathfrak{x}_0)$. Hence

$$H_l^i(\mathfrak{x}_0) \supseteq \bigcup_{\mathfrak{x} \in c_{\mathsf{B}}(H_l^i(\mathfrak{x}_0))} H_l^{-\varphi_{\mathsf{B}}(\mathfrak{x})}(\mathfrak{x}).$$

The inclusion in the other direction is provided by Proposition 7.4 while Corollary 7.7 ensures that the union on the right-hand side is disjoint. Finally, Proposition 7.6 ensures that the above bijection is indeed an order isomorphism. □

8 Neighbours of Strings in B-condensation

This section is devoted to defining operators ℓ_{B} and $\bar{\ell}_{\mathsf{B}}$ on $\mathsf{St}(\mathsf{B})$, which when restricted to $\mathsf{St}(\mathfrak{x}_0, i; \mathsf{B})$ help us to find the immediate neighbours of strings in it. En route, we define two subsets $\mathsf{Ba}_l(\mathsf{B})$ and $\mathsf{Ba}_{\bar{l}}(\mathsf{B})$ of the set of prime bands in B and see that the limit of the sequence of such iterated immediate successors (resp. predecessors) are almost periodic strings of the form ${}^{\infty}\mathfrak{b}\mathfrak{u}\mathfrak{x}_0$, where $\mathfrak{b} \in \mathsf{Ba}_l(\mathsf{B})$ (resp. $\mathsf{Ba}_{\bar{l}}(\mathsf{B})$).

Recall from [3, § 3] that a syllable α is an *exit syllable* of a band $\mathfrak{b}$ if there is a cyclic permutation $\mathfrak{b}'$ of $\mathfrak{b}$ such that $\alpha\mathfrak{b}'$ is a string but α is distinct from the first syllable of $\mathfrak{b}'$. Also recall that *exit* of a bridge $\mathfrak{b}_1 \xrightarrow{\mathfrak{u}} \mathfrak{b}_2$ is the first syllable in ${}^{\infty}\mathfrak{b}_2\mathfrak{u}\mathfrak{b}_1$ from the right where the strings ${}^{\infty}\mathfrak{b}_2\mathfrak{u}\mathfrak{b}_1$ and ${}^{\infty}\mathfrak{b}_1$ differ. Slightly modifying the former, we introduce an *exit* of a band below.

Definition 8.1 Given a band $\mathfrak{b}$, say that a pair $(\beta, \mathfrak{b}')$ is an *exit* of $\mathfrak{b}$ if β is a syllable and $\mathfrak{b}'$ is a cyclic permutation of $\mathfrak{b}$ such that $\beta\mathfrak{b}'$ is a string but $\beta\mathfrak{b}' \not\sqsubseteq_l \mathfrak{b}'^2$.

It is trivial to note that if $(\beta, \mathfrak{b}')$ is an exit of a band $\mathfrak{b}$ then β is an exit syllable of $\mathfrak{b}$. There are some exits of a B-band for a non-domestic $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$; the signs of the corresponding exit syllables are important in the computation of the order type of hammocks.

Definition 8.2 If $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ and $\mathfrak{b} \in \mathsf{Ba}(\mathsf{B})$, say that an exit $(\beta, \mathfrak{b}')$ of $\mathfrak{b}$ is a *non-domestic exit* if $\beta\mathfrak{b}' \in \mathsf{Ext}(\mathsf{B})$.

Remark 8.3 For non-domestic $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ and $\mathfrak{b} \in \mathsf{Ba}(\mathsf{B})$ there is $\mathfrak{b}' \in \mathsf{Ba}(\mathsf{B})$ such that $\mathfrak{b} \neq \mathfrak{b}'$. Let $\mathfrak{u}$ be a string such that $\mathfrak{b}'\mathfrak{u}\mathfrak{b} \in \mathsf{Ext}(\mathsf{B})$. Then ${}^{\infty}\mathfrak{b}'\mathfrak{u}\mathfrak{b} \neq {}^{\infty}\mathfrak{b}$, and hence $\mathfrak{b}$ has a non-domestic exit.

Definition 8.4 Denote by $\mathsf{Ba}_l(\mathsf{B})$ the set of all B-bands having no non-domestic exit $(\beta, \mathfrak{b}')$ with $\beta \in Q_1$. Dually, denote by $\mathsf{Ba}_{\bar{l}}(\mathsf{B})$ the set of all B-bands having no non-domestic exits $(\beta, \mathfrak{b}')$ with $\beta \in Q_1^-$.

Example 8.5 Continuing from Example 5.8, we have $\mathsf{Ba}_l(\mathsf{B}_3) = \{e_3e_2E_1, g_4G_3g_2G_1\}$ and $\mathsf{Ba}_{\bar{l}}(\mathsf{B}_3) = \{k_1K_2\}$.

In view of Remark 8.3, it is trivial to note that if B is non-domestic then $\mathsf{Ba}_l(\mathsf{B}) \cap \mathsf{Ba}_{\bar{l}}(\mathsf{B}) = \emptyset$. We show in Corollary 8.26 and Corollary 8.28 that the sets $\mathsf{Ba}_l(\mathsf{B})$ and $\mathsf{Ba}_{\bar{l}}(\mathsf{B})$ are non-empty and finite.

The following proposition is key to defining the operator ℓ_{B}.

Proposition 8.6 *If* $\mathfrak{x} \in \mathsf{St}_1(\mathsf{B})$ *then there exists* $\mathfrak{y} \in \mathsf{St}_1(\mathsf{B})$ *such that* $\mathfrak{x} \sqsubset_l \mathfrak{y} \sqsubseteq_l l(\mathfrak{x})$.

Proof If possible, assume that for each $\mathfrak{x} \sqsubset_l \mathfrak{y} \sqsubseteq_l l(\mathfrak{x})$ we have $\mathfrak{y} \notin \mathsf{St}_1(\mathsf{B})$. Let $\mathfrak{b} \in \mathsf{Cyc}(\mathsf{B})$ such that $\mathfrak{b}\mathfrak{x}$ is a string and $\theta(\mathfrak{b}) = 1$. Let $\mathfrak{z} := \mathfrak{b}\mathfrak{x} \sqcap_l l(\mathfrak{x})$. As $\theta(\mathfrak{b}) = 1$, we get $\mathfrak{x} \sqsubset_l \mathfrak{z} \sqsubseteq_l l(\mathfrak{x})$. Thus by our assumption $\mathfrak{z} \notin \mathsf{St}_1(\mathsf{B})$. This implies that $\mathfrak{z} \neq \mathfrak{b}\mathfrak{x}$. Since $\mathfrak{x} \sqsubset_l \mathfrak{z} \sqsubset_l \mathfrak{b}\mathfrak{x}$, for an appropriate cyclic permutation $\mathfrak{b}'$ of $\mathfrak{b}$, $\mathfrak{b}'\mathfrak{z}$ is a string. Moreover, $\mathfrak{z} \notin \mathsf{St}_1(\mathsf{B})$ implies that $\theta(\mathfrak{b}') = -1$. Therefore $\alpha\mathfrak{z} \sqsubseteq_l \mathfrak{b}\mathfrak{x}$, where $\alpha \in Q_1$ is the first syllable of $\mathfrak{b}'$. Since $\mathfrak{x} \sqsubset_l \mathfrak{z} \sqsubseteq_l l(\mathfrak{x})$ we get $\alpha\mathfrak{z} \sqsubseteq_l l(\mathfrak{x})$, which contradicts that $\mathfrak{z} = \mathfrak{b}\mathfrak{x} \sqcap_l l(\mathfrak{x})$. □

Definition 8.7 Define $\ell_{\mathsf{B}} : \mathsf{St}_1(\mathsf{B}) \to \mathsf{St}_1(\mathsf{B})$ by choosing $\ell_{\mathsf{B}}(\mathfrak{x})$ to be the maximal (possibly equal) left substring of $l(\mathfrak{x})$ such that $\ell_{\mathsf{B}}(\mathfrak{x}) \in \mathsf{St}_1(\mathsf{B})$.

Remark 8.8 For any $\mathfrak{x} \in \mathsf{St}_1(\mathsf{B})$, we have $\ell_{\mathsf{B}}(\mathfrak{x}) \notin \mathsf{St}_{-1}(\mathsf{B})$.

For $\mathfrak{x} \in \mathsf{St}_1(\mathsf{B})$ we inductively define the powers of the function ℓ_{B} by $\ell^0_{\mathsf{B}}(\mathfrak{x}) := \mathfrak{x}$ and $\ell^{n+1}_{\mathsf{B}}(\mathfrak{x}) := \ell_{\mathsf{B}}(\ell^n_{\mathsf{B}}(\mathfrak{x}))$ for $n \in \mathbb{N}$. Since $\ell^n_{\mathsf{B}}(\mathfrak{x}) \sqsubset_l \ell^{n+1}_{\mathsf{B}}(\mathfrak{x})$ for each n we get that $\lim_{n\to\infty} \ell^n_{\mathsf{B}}(\mathfrak{x})$ is a left $\mathbb{N}$-string. Denote this limit by $\langle 1, \ell_{\mathsf{B}}\rangle(\mathfrak{x})$.

The following remark notes that if $|\mathfrak{x}| > 0$ then $\ell_{\mathsf{B}}(\mathfrak{x})$ depends only on the last syllable of $\mathfrak{x}$. As a consequence, the image of the function ℓ_{B} restricted to $\mathsf{St}_1(\mathfrak{x}_0, i; \mathsf{B})$ lies in $\mathsf{St}_1(\mathfrak{x}_0, i; \mathsf{B})$.

Remark 8.9 If $\alpha\mathfrak{x}, \alpha\mathfrak{y} \in \mathsf{St}_1(\mathsf{B})$ for some $\alpha \in Q_1 \cup Q_1^-$, then $\ell^n_{\mathsf{B}}(\alpha\mathfrak{x}) = \mathfrak{u}\alpha\mathfrak{x}$ if and only if $\ell^n_{\mathsf{B}}(\alpha\mathfrak{y}) = \mathfrak{u}\alpha\mathfrak{y}$. Furthermore, if $\ell^n_{\mathsf{B}}(\alpha)$ exists then $\ell^n_{\mathsf{B}}(1_{(t(\alpha),\varepsilon(\alpha))})$ exists and $\ell^n_{\mathsf{B}}(\alpha) = \ell^n_{\mathsf{B}}(1_{(t(\alpha),\varepsilon(\alpha))})\alpha$.

Proposition 8.10 *For $\mathfrak{x} \in \mathsf{St}_1(\mathsf{B})$, we have $\langle 1, \ell_{\mathsf{B}}\rangle(\mathfrak{x}) = {}^{\infty}\mathfrak{b}\mathfrak{u}\mathfrak{x}$ for some band $\mathfrak{b}$ and string $\mathfrak{u}$.*

Proof Define a function $f : \mathbb{N}^+ \to Q_1 \cup Q_1^-$ such that $f(k)$ is the last syllable of $\ell^k_{\mathsf{B}}(\mathfrak{x})$. As $Q_1 \cup Q_1^-$ is finite, there exist $m, n \in \mathbb{N}^+$ such that $f(m) = f(m+n)$. In view of the fact that $\ell^k_{\mathsf{B}}(\mathfrak{w}) \sqsubset_l \ell^{k+1}_{\mathsf{B}}(\mathfrak{w})$ for any $\mathfrak{w} \in \mathsf{St}_1(\mathsf{B})$, let $\mathfrak{y}\mathfrak{x} := \ell^n_{\mathsf{B}}(\mathfrak{x})$ and $\mathfrak{z}\mathfrak{y}\mathfrak{x} := \ell^{m+n}_{\mathsf{B}}(\mathfrak{x})$, where $\mathfrak{z}$ is a string with $|\mathfrak{z}| > 0$. As $\ell^n_{\mathsf{B}}(\mathfrak{x})$ and $\ell^{m+n}_{\mathsf{B}}(\mathfrak{x})$ have the same last syllable, Remark 8.9 together with induction yields that $\ell^{m+kn}_{\mathsf{B}}(\mathfrak{x}) = \mathfrak{z}^k\mathfrak{y}\mathfrak{x}$ for every $k \in \mathbb{N}$. Since $\mathfrak{z}^k$ is a string for every $k \in \mathbb{N}$, $\mathfrak{z}$ is a finite power of cyclic permutation of a band, say $\mathfrak{b}$. Since $\langle 1, \ell_{\mathsf{B}}\rangle(\mathfrak{x}) = {}^{\infty}\mathfrak{z}\mathfrak{y}\mathfrak{x}$, we get $\langle 1, \ell_{\mathsf{B}}\rangle(\mathfrak{x}) = {}^{\infty}\mathfrak{b}\mathfrak{y}'\mathfrak{x}$ for some string $\mathfrak{y}'$. □

Example 8.11 Recall from Example 5.8 that $e_3E_2E_1$ is a band that lies in B_3. For $A_2A_1a_0 \in \mathsf{St}(\mathsf{B}_3)$, a routine computation yields $\langle 1, \ell_{\mathsf{B}_3}\rangle(A_2A_1a_0) = {}^{\infty}(e_3E_2E_1)A_2A_1a_0$.

The conclusion of Proposition 8.10 is similar to the hypothesis of [3, Proposition 3.4.5], whose proof used the concept of l-strings. However, a statement about l-strings [3, Remark 3.4.4] that was used in the proof is erroneous as demonstrated by Example 8.12. Nevertheless, it does not render [3, Proposition 3.4.5] false, as it can still be proven using techniques similar to those in the proof of Proposition 8.21.

Example 8.12 Consider the string algebra Γ from Fig. 2. For appropriate $j \in \{1, -1\}$, we have

$$\langle 1, l\rangle(1_{(v,j)}) = {}^{\infty}(cbaEbafcbD).$$

Here $Ebafcb$, $DcbaEb$ and $fcbDcb$ are l-strings with the same first syllable and same length.

Motivated by the concept of l-strings introduced in [3, § 3.4], now we define ℓ_{B}-strings to prove similar results where l is replaced with ℓ_{B}.

Definition 8.13 A string $\mathfrak{u}$ is an ℓ_{B}-*string* if $\delta(\mathfrak{u}) = 0$ and $\mathfrak{u} \sqsubseteq \langle 1, \ell_{\mathsf{B}}\rangle(1_{(v,i)})$ for some $1_{(v,i)} \in \mathsf{St}_1(\mathsf{B})$.

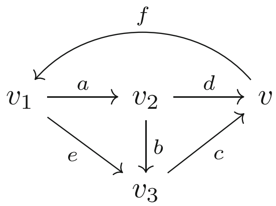

Fig. 2 Γ with $\rho = \{ce, da, ef, fd, cbaf, fcba\}$

Remark 8.14 If $\ell_{\mathsf{B}}^n(\mathfrak{x}) = \mathfrak{u}\mathfrak{y}\mathfrak{x}$, where $\mathfrak{y}$ and $\mathfrak{u}$ are strings with $|\mathfrak{u}| > 0$, then $\theta(\mathfrak{u}) = 1$ if and only if there exists $0 \leq k < n$ such that $\ell_{\mathsf{B}}^k(\mathfrak{x}) = \mathfrak{y}\mathfrak{x}$.

Proposition 8.15 *Let* $\mathfrak{x} \in \mathsf{St}_1(\mathsf{B})$ *and* $\ell_{\mathsf{B}}^n(\mathfrak{x}) = \mathfrak{u}\mathfrak{x}$ *for some string* $\mathfrak{u}$ *and* $n \in \mathbb{N}$*. Then there exist* $\mathfrak{b} \in \mathsf{Cyc}(\mathsf{B})$, $m \in \mathbb{N}^+$ *and* $\alpha \in Q_1^-$ *such that* $\alpha\mathfrak{u} \sqsubseteq_l \mathfrak{b}^m$ *and* $\mathfrak{b}^m\mathfrak{x}$ *is a string.*

Proof For each $n \in \mathbb{N}$, let $\mathfrak{u}_n\mathfrak{x} := \ell_{\mathsf{B}}^n(\mathfrak{x})$. Let $\alpha_n \in Q_1^-$ satisfy $\alpha_n\mathfrak{u}_n\mathfrak{x} \sqsubseteq_l \ell_{\mathsf{B}}^{n+1}(\mathfrak{x})$. We will prove the result by induction on n.

For $n = 0$, we have $\ell_{\mathsf{B}}^n(\mathfrak{x}) = \mathfrak{x}$. Since $\mathfrak{x} \in \mathsf{St}_1(\mathsf{B})$, there is $\mathfrak{b} \in \mathsf{Cyc}(\mathsf{B})$ with $\theta(\mathfrak{b}) = 1$ such that $\mathfrak{b}\mathfrak{x}$ is a string. Taking α to be the first syllable of $\mathfrak{b}$ proves the statement.

For $n > 0$, by induction hypothesis, there exists $\mathfrak{b} \in \mathsf{Cyc}(\mathsf{B})$ and $m \in \mathbb{N}^+$ such that $\alpha_{n-1}\mathfrak{u}_{n-1} \sqsubseteq_l \mathfrak{b}^m$ and $\mathfrak{b}^m\mathfrak{x}$ is a string. Let $\mathfrak{u}_n := \mathfrak{y}\alpha_{n-1}\mathfrak{u}_{n-1}$ such that if $|\mathfrak{y}| > 0$ then $\delta(\mathfrak{y}) = -1$. There are two cases.

- Suppose $|\mathfrak{y}| > 0$. Let $\mathfrak{b}_1$ be a cyclic permutation of $\mathfrak{b}$ such that $\alpha_{n-1}\mathfrak{u}_{n-1} \sqsubseteq_r \mathfrak{b}_1^m$. Since $\mathfrak{u}_n\mathfrak{x} \in \mathsf{St}_1(\mathsf{B})$, there is $\mathfrak{b}_2 \in \mathsf{Cyc}(\mathsf{B})$ such that $\theta(\mathfrak{b}_2) = 1$ and $\mathfrak{b}_2\mathfrak{u}_n\mathfrak{x} = \mathfrak{b}_2\mathfrak{y}\alpha_{n-1}\mathfrak{u}_{n-1}\mathfrak{x}$ is a string. Since $\delta(\mathfrak{y}) = -1$ and $\alpha_{n-1} \in Q_1^-$ we get that $\mathfrak{b}_2\mathfrak{y}\mathfrak{b}_1^m$ is a string. Since $\mathfrak{b}_1, \mathfrak{b}_2 \in \mathsf{Cyc}(\mathsf{B})$ we get that $\mathfrak{b}_1\mathfrak{w}\mathfrak{b}_2$ is a string for some string $\mathfrak{w}$. Thus $\mathfrak{w}\mathfrak{b}_2\mathfrak{y}\mathfrak{b}_1^m$ is a power of a B-cycle and $\alpha_n\mathfrak{u}_n \sqsubseteq \mathfrak{w}\mathfrak{b}_2\mathfrak{y}\mathfrak{b}_1^m$. Let $\mathfrak{b}'$ be a B-cycle such that for some k, $\mathfrak{b}'^k$ is a cyclic permutation of $\mathfrak{w}\mathfrak{b}_2\mathfrak{y}\mathfrak{b}_1^m$ and $\alpha_n\mathfrak{u}_n = \alpha_n\mathfrak{y}\alpha_{n-1}\mathfrak{u}_{n-1} \sqsubseteq_l \mathfrak{b}'^k$. Since $\delta(\mathfrak{y}) = -1$ and $\theta(\alpha_n) = 1$ we get $\delta(\alpha_n\mathfrak{u}_n) = \delta(\alpha_n\mathfrak{y}\alpha_{n-1}\mathfrak{u}_n) = 0$. Thus $\alpha_n\mathfrak{u}_n \sqsubseteq_l \mathfrak{b}'^k$ and $\mathfrak{b}'^k\mathfrak{x}$ is a string.
- If $|\mathfrak{y}| = 0$ then $\mathfrak{u}_n \sqsubseteq_l \mathfrak{b}^m$. Let $\mathfrak{w}\mathfrak{u}_n := \mathfrak{b}^m$ for some string $\mathfrak{w}$. Let α be the first syllable of $\mathfrak{b}\mathfrak{w}$. By Remark 8.8, we have $\mathfrak{u}_n\mathfrak{x} \notin \mathsf{St}_{-1}(\mathsf{B})$. Therefore $\alpha \in Q_1^-$ and $\mathfrak{b}^{m+1}$ satisfy the conditions of the conclusion.

This completes the proof. □

Corollary 8.16 *If* $\mathfrak{x} \in \mathsf{St}_1(\mathsf{B})$ *and* $\mathfrak{u}\mathfrak{y}\mathfrak{x} \sqsubset_l \langle 1, \ell_{\mathsf{B}}\rangle(\mathfrak{x})$ *for some strings* $\mathfrak{u}$ *and* $\mathfrak{y}$ *then there exists* $\mathfrak{b} \in \mathsf{Cyc}(\mathsf{B})$ *such that* $\mathfrak{b}\mathfrak{y}\mathfrak{x}$ *is a string and* $\mathfrak{u} \sqsubseteq_l \mathfrak{b}^m$ *for some* $m \in \mathbb{N}^+$.

Corollary 8.17 *If* $\mathfrak{u}$ *is an* ℓ_{B}*-string then* $\mathfrak{u} \in \mathsf{Ext}(\mathsf{B})$.

Proposition 8.18 *If* $\mathfrak{x}$ *and* $\alpha\mathfrak{x}$ *are* ℓ_{B}*-strings for some* $\alpha \in Q_1 \cup Q_{-1}$ *then* $\mathfrak{x} \in \mathsf{St}_{-1}(\mathsf{B})$ *if and only if* $\alpha \in Q_1$.

Proof Suppose $\mathfrak{x} \in \mathsf{St}_{-1}(\mathsf{B})$. Since $\alpha\mathfrak{x}$ is an ℓ_{B}-string, there exist a string $\mathfrak{y}$, $n \in \mathbb{N}^+$, $v \in Q_0$ and $j \in \{1, -1\}$ such that $\alpha\mathfrak{x}\mathfrak{y} \sqsubseteq_l \ell_{\mathsf{B}}^{n+1}(1_{(v,j)})$ and $\ell_{\mathsf{B}}^{n-1}(1_{(v,j)}) \sqsubset_l \mathfrak{x}\mathfrak{y} \sqsubseteq_l \ell_{\mathsf{B}}^n(1_{(v,j)})$. Let $\mathfrak{u} := \ell_{\mathsf{B}}^{n-1}(1_{(v,j)})$. Hence $\mathfrak{x}\mathfrak{y} \sqsubseteq_l \ell_{\mathsf{B}}(\mathfrak{u})$. Since $\mathfrak{x}\mathfrak{y} \in \mathsf{St}_{-1}(\mathsf{B})$, there exists $\mathfrak{b} \in \mathsf{Cyc}(\mathsf{B})$ with $\theta(\mathfrak{b}) = -1$ such that $\mathfrak{b}\mathfrak{x}\mathfrak{y}$ is a string. Let β be the first syllable of $\mathfrak{b}$. Then $\beta \in Q_1$ gives $\beta\mathfrak{x}\mathfrak{y} \sqsubseteq_l \ell_{\mathsf{B}}(\mathfrak{u})$. Let $\mathfrak{z} := \ell_{\mathsf{B}}(\mathfrak{u}) \sqcap_l \mathfrak{b}\mathfrak{x}\mathfrak{y}$. Since $\mathfrak{u} \sqsubset_l \beta\mathfrak{x}\mathfrak{y} \sqsubseteq_l \mathfrak{z} \sqsubset_l \mathfrak{b}\mathfrak{x}\mathfrak{y}$ and $\mathfrak{u} <_l \ell_{\mathsf{B}}(\mathfrak{u}) <_l \mathfrak{b}\mathfrak{x}\mathfrak{y}$, we get that $\mathfrak{x}\mathfrak{y} \sqsubset_l \gamma\mathfrak{z} \sqsubseteq_l \mathfrak{b}\mathfrak{x}\mathfrak{y}$ for some $\gamma \in Q_1^-$. This shows that $\mathfrak{z} \in \mathsf{St}_1(\mathsf{B})$. Since $\beta\mathfrak{x}\mathfrak{y} \sqsubseteq_l \mathfrak{z}$ we get $\beta\mathfrak{x}\mathfrak{y} \sqsubseteq_l \ell_\mathsf{B}(\mathfrak{u})$. Thus $\beta\mathfrak{x}\mathfrak{y} \sqsubseteq_l \ell_\mathsf{B}^{n+1}(1_{(v,j)})$, which gives $\alpha = \beta \in Q_1$.

Conversely if $\alpha\mathfrak{x}$ is an ℓ_B-string for $\alpha \in Q_1$, Corollary 8.17 yields $\mathfrak{b} \in \mathsf{Cyc}(\mathsf{B})$ such that $\alpha\mathfrak{x} \sqsubseteq_l \mathfrak{b}$. Thus $\mathfrak{x} \in \mathsf{St}_{-1}(\mathsf{B})$. □

Corollary 8.19 *For strings $\mathfrak{x}$, $\mathfrak{u}$, $\mathfrak{v}$, if $\mathfrak{x}$, $\mathfrak{u}\mathfrak{x}$ and $\mathfrak{v}\mathfrak{x}$ are ℓ_B-strings then $\mathfrak{u}\mathfrak{x}$ and $\mathfrak{v}\mathfrak{x}$ do not fork.*

Proof Suppose, if possible, $\mathfrak{u}\mathfrak{x}$ and $\mathfrak{v}\mathfrak{x}$ fork. Let $\mathfrak{z}\mathfrak{x} := \mathfrak{u}\mathfrak{x} \sqcap_l \mathfrak{v}\mathfrak{x}$ with $\theta(\mathfrak{v}\mathfrak{x} \mid \mathfrak{z}\mathfrak{x}) = 1$ and $\theta(\mathfrak{u}\mathfrak{x} \mid \mathfrak{z}\mathfrak{x}) = -1$. Since $\mathfrak{x} \sqsubseteq \mathfrak{z}\mathfrak{x} \sqsubset \mathfrak{u}\mathfrak{x}$, we have that $\mathfrak{z}\mathfrak{x}$ is an ℓ_B-string. By Proposition 8.18, we have that $\mathfrak{z}\mathfrak{x} \in \mathsf{St}_{\pm 1}(\mathsf{B})$, a contradiction to the combination of Remarks 8.8 and 8.14. □

Remark 8.20 If $\mathfrak{x}$ is an ℓ_B-string then there exist $\mathfrak{b} \in \mathsf{Ba}(\mathsf{B})$ and a string $\mathfrak{y}$ such that $\mathfrak{b}\mathfrak{y}\mathfrak{u}$ is a string.

Proposition 8.21 *If every cyclic permutation of a band $\mathfrak{b}$ is an ℓ_B-string, then $\mathfrak{b}$ is a prime band.*

Proof Suppose, if possible, $\mathfrak{b}$ is a composite band. Then there exist $n > 1, a_1, \cdots, a_n \in \mathbb{N}^+$, $\mathfrak{b}_1, ..., \mathfrak{b}_n \in \mathsf{Cyc}(\Lambda)$ satisfying $\mathfrak{b}_n \neq \mathfrak{b}_1$ as well as $\mathfrak{b}_j \neq \mathfrak{b}_{j+1}$ for any $j \in \{1, \cdots, n-1\}$, and a cyclic permutation $\mathfrak{b}'$ of $\mathfrak{b}$ such that $\mathfrak{b}' = \mathfrak{b}_n^{a_n} \cdots \mathfrak{b}_2^{a_2}\mathfrak{b}_1^{a_1}$. The hypothesis implies that $\mathfrak{b}'$ is an ℓ_B-string. Note that for $v = s(\mathfrak{b}')$ and appropriate $j \in \{1, -1\}$, $\mathfrak{b}_1, \mathfrak{b}_2, \cdots, \mathfrak{b}_n \in (H_l(1_{(v,j)}), <_l)$. By Corollary 8.17, $\mathfrak{b}_1, \mathfrak{b}_2, \cdots, \mathfrak{b}_n \in \mathsf{Cyc}(\mathsf{B})$.

Claim. If $j \neq k$ and $\mathfrak{b}_j\mathfrak{b}_k$ is an ℓ_B-string then $\mathfrak{b}_j <_l \mathfrak{b}_k$ in $(H_l(1_{(v,j)}), <_l)$.

Proof of the claim. Let $\mathfrak{y} := \mathfrak{b}_k \sqcap_l \mathfrak{b}_j$. If $\delta(\mathfrak{y}) = 0$ then $\mathfrak{y}$ is an ℓ_B-string. In view of Corollary 4.9, the strings $\mathfrak{b}_{j-1}^{a_{j-1}} \cdots \mathfrak{b}_1^{a_1}\mathfrak{b}_n^{a_n} \cdots \mathfrak{b}_j^{a_j}$ and $\mathfrak{b}_{k-1}^{a_{k-1}} \cdots \mathfrak{b}_1^{a_1}\mathfrak{b}_n^{a_n} \cdots \mathfrak{b}_k^{a_k}$ fork, which contradicts Corollary 8.19. Hence $\delta(\mathfrak{y}) \neq 0$ and, in particular, as $\delta(\mathfrak{b}_i) = \delta(\mathfrak{b}_j) = 0$ but $\delta(\mathfrak{y}) \neq 0$, we get that $\mathfrak{b}_i$ and $\mathfrak{b}_j$ fork.

Assume, if possible, that we have $\mathfrak{b}_k <_l \mathfrak{b}_j$. Let $\mathfrak{x} := \mathfrak{b}_j\mathfrak{b}_k \sqcap_l \mathfrak{b}_k^2$, so that $\mathfrak{x}$ is an ℓ_B-string. Then $\alpha\mathfrak{x} \sqsubseteq_l \mathfrak{b}_j\mathfrak{b}_k$ and $\beta\mathfrak{x} \sqsubseteq_l \mathfrak{b}_k^2$ for some $\alpha \in Q_1^-$ and $\beta \in Q_1$. As $\mathfrak{b}_k^2 \in \mathsf{Ext}(\mathsf{B})$ and $\beta\mathfrak{x} \sqsubseteq_l \mathfrak{b}_k^2$, we have $\mathfrak{x} \in \mathsf{St}_{-1}(\mathsf{B})$. Further since $\alpha\mathfrak{x}$ is an ℓ_B-string, Proposition 8.18 gives that $\alpha \in Q_1$, a contradiction. This completes the proof of the claim. ■

Since $\mathfrak{b}_{j+1}\mathfrak{b}_j$ is an ℓ_B-string, it follows from the claim that $\mathfrak{b}_{j+1} <_l \mathfrak{b}_j$ for every $j \in \{1, 2, \cdots, n-1\}$. Using transitivity of $<_l$, we get $\mathfrak{b}_n <_l \mathfrak{b}_1$. However $\mathfrak{b}_1\mathfrak{b}_n$ being a substring of a cyclic permutation of $\mathfrak{b}$ is also an ℓ_B-string, and hence the claim gives that $\mathfrak{b}_1 <_l \mathfrak{b}_n$, which is a contradiction. Therefore $n = 1$ and $a_1 = 1$, which shows that $\mathfrak{b}$ is prime. □

Corollary 8.22 *If $\mathfrak{x} \in \mathsf{St}_1(\mathsf{B})$ then $\langle 1, \ell_\mathsf{B}\rangle(\mathfrak{x}) = {}^\infty\mathfrak{b}\mathfrak{u}\mathfrak{x}$ for some prime B-band $\mathfrak{b}$.*

Proof By Proposition 8.10, $\langle 1, \ell_\mathsf{B}\rangle(\mathfrak{x}) = {}^\infty\mathfrak{b}\mathfrak{u}\mathfrak{x}$ for some band $\mathfrak{b}$. By Remark 8.9, we have $\langle 1, \ell_\mathsf{B}\rangle(1_{(t(\mathfrak{x}),\varepsilon(\mathfrak{x}))}) = {}^\infty\mathfrak{b}\mathfrak{u}$. Thus every cyclic permutation of $\mathfrak{b}$ is an ℓ_B-string. But then Proposition 8.21 gives that $\mathfrak{b}$ is a prime band. Finally, Corollary 8.17 yields $\mathfrak{b} \in \mathsf{Ba}(\mathsf{B})$. □

In fact, the band appearing in Corollary 8.22 is more than just a prime band. We now show that it lies in $\mathsf{Ba}_l(\mathsf{B})$.

Proposition 8.23 *If $\mathfrak{x} \in \mathsf{St}_1(\mathsf{B})$ and $\langle 1, \ell_\mathsf{B}\rangle(\mathfrak{x}) = {}^\infty\mathfrak{b}\mathfrak{u}\mathfrak{x}$ for some band $\mathfrak{b}$ and a string $\mathfrak{u}$ then $\mathfrak{b} \in \mathsf{Ba}_l(\mathsf{B})$.*

Proof The existence of $\mathfrak{b}$ and $\mathfrak{u}$ is guaranteed by Proposition 8.10, whereas Corollary 8.22 gives that $\mathfrak{b} \in \mathsf{Ba}(\mathsf{B})$. Suppose, if possible, $\mathfrak{b} \notin \mathsf{Ba}_l(\mathsf{B})$. Then there exists an exit $(\beta, \mathfrak{b}')$ of $\mathfrak{b}$ such that $\theta(\beta) = -1$. Rewrite ${}^\infty\mathfrak{b}\mathfrak{u}\mathfrak{x}$ as ${}^\infty\mathfrak{b}'\mathfrak{v}\mathfrak{x}$ for some string $\mathfrak{v}$. Since $\theta(\mathfrak{b}') = 1$, Remark 8.14 yields $n \in \mathbb{N}$ such that $\ell_\mathsf{B}^n(\mathfrak{x}) = \mathfrak{b}'\mathfrak{v}\mathfrak{x}$. Since $\beta\mathfrak{b}'\mathfrak{v}\mathfrak{x} \in \mathsf{Ext}(\mathsf{B}) \subseteq \mathsf{St}(\mathsf{B})$, we have $\mathfrak{b}'\mathfrak{v}\mathfrak{x} \in \mathsf{St}_{-1}(\mathsf{B})$, a contradiction to Remark 8.8. □

Example 8.24 Continuing from Example 8.11, indeed $e_3E_2E_1 \in \mathsf{Ba}_l(\mathsf{B}_3)$ and $e_3E_2E_1$ is a prime band.

Combining Propositions 8.10, 8.23 and Corollary 8.22, we have the following result.

Corollary 8.25 *If* $\mathfrak{x} \in \mathsf{St}_1(\mathsf{B})$ *then* $\langle 1, \ell_{\mathsf{B}}\rangle(\mathfrak{x}) = {}^{\infty}\mathfrak{b}\mathfrak{u}\mathfrak{x}$ *for some string* $\mathfrak{u}$ *and* $\mathfrak{b} \in \mathsf{Ba}_l(\mathsf{B})$.

Since $\mathsf{St}_1(\mathsf{B}) \neq \emptyset$, the above result guarantees the existence of a band in $\mathsf{Ba}_l(\mathsf{B})$.

Corollary 8.26 *If* $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ *then* $\mathsf{Ba}_l(\mathsf{B}) \neq \emptyset$.

In fact, we can guarantee that each $\mathfrak{b} \in \mathsf{Ba}_l(\mathsf{B})$ occurs in the conclusion of Proposition 8.23 for some $\mathfrak{x}$.

Proposition 8.27 *If* $\mathfrak{b} \in \mathsf{Ba}_l(\mathsf{B})$ *then there exist* $\mathfrak{x} \in \mathsf{St}_1(\mathsf{B})$ *and a string* $\mathfrak{u}$ *such that* $\langle 1, \ell_{\mathsf{B}}\rangle(\mathfrak{x}) = {}^{\infty}\mathfrak{b}\mathfrak{u}\mathfrak{x}$.

Proof Since $\mathfrak{b} \in \mathsf{Ba}(\mathsf{B})$, we have $\theta(\mathfrak{b}) = 1$ and $\mathfrak{b} \in \mathsf{St}_1(\mathsf{B})$. Thus by Proposition 8.23, there exist $\mathfrak{b}_1 \in \mathsf{Ba}_l(\mathsf{B})$ and a string $\mathfrak{u}$ such that $\langle 1, \ell_{\mathsf{B}}\rangle(\mathfrak{b}) = {}^{\infty}\mathfrak{b}_1\mathfrak{u}\mathfrak{b}$. Suppose, if possible, ${}^{\infty}\mathfrak{b}_1\mathfrak{u}\mathfrak{b} \neq {}^{\infty}\mathfrak{b}$. Since $\mathfrak{b} \in \mathsf{Ba}_l(\mathsf{B})$, we get that $\theta({}^{\infty}\mathfrak{b}_1\mathfrak{u}\mathfrak{b} \mid {}^{\infty}\mathfrak{b}) = 1$. Let $\mathfrak{z}\mathfrak{b} := {}^{\infty}\mathfrak{b}_1\mathfrak{u}\mathfrak{b} \sqcap_l {}^{\infty}\mathfrak{b}$. By Remark 8.14, there exists $n \geq 0$ such that $\ell_{\mathsf{B}}^n(\mathfrak{b}) = \mathfrak{z}\mathfrak{b}$. Since $\theta({}^{\infty}\mathfrak{b}_1\mathfrak{u}\mathfrak{b}) = \theta(\mathfrak{b}) = 1$, we get that $|\mathfrak{z}| > 0$. Hence $n > 0$ and, in particular, $\mathfrak{z}\mathfrak{b} \in \mathsf{St}_{-1}(\mathsf{B})$, a contradiction to Remark 8.8. □

The finiteness of the set $\mathsf{Ba}_l(\mathsf{B})$ can be concluded from the next result which is obtained by combining Propositions 8.23 and 8.27.

Corollary 8.28 *If* $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ *and* $\mathfrak{b} \in \mathsf{Ba}_l(\mathsf{B})$ *then* $\mathfrak{b}$ *is prime.*

Remark 8.29 If $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ is non-domestic then $\mathsf{Ba}_l(\mathsf{B}) \cap \mathsf{Ba}_{\bar{l}}(\mathsf{B}) = \emptyset$ in view of Remark 8.3. On the other hand, if B is domestic then $\mathsf{Ba}_l(\mathsf{B}) = \mathsf{Ba}_{\bar{l}}(\mathsf{B}) = \mathsf{B}$.

9 Extending the B-neighbour Operators

In Section 6, we defined sets $\overline{\mathsf{St}}_j(\mathfrak{x}_0, i; \mathsf{B})$ for $j \in \{-1, 1\}$ containing strings which eventually reach $\mathsf{B} \in \mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0)$. We would like to extend the function ℓ_{B} on this bigger set, $\overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B})$, and obtain a result similar to Corollary 8.25.

Proposition 9.1 *If* $\mathfrak{x} \in \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B}) \setminus \{\mathfrak{M}_i(\mathfrak{x}_0)\}$ *then there exists* $\mathfrak{z} \in \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B})$ *such that* $\mathfrak{x} \sqsubset_l \mathfrak{z} \sqsubseteq_l l(\mathfrak{x})$.

Proof If $\mathfrak{x} \sqsubset_l \mathfrak{M}_i(\mathfrak{x}_0)$ then the result holds trivially. So assume $\mathfrak{x} \not\sqsubseteq_l \mathfrak{M}_i(\mathfrak{x}_0)$. Then Remark 6.1 yields $\mathfrak{x} \in \overline{\mathsf{St}}_1(\mathsf{B})$. If possible, suppose that $\mathfrak{y} \notin \overline{\mathsf{St}}_1(\mathsf{B})$ for each $\mathfrak{x} \sqsubset_l \mathfrak{y} \sqsubseteq_l l(\mathfrak{x})$.

Since $\mathfrak{x} \in \overline{\mathsf{St}}_1(\mathsf{B})$, there are $\mathfrak{b} \in \mathsf{Cyc}(\mathsf{B})$ and $\mathfrak{u} \in \mathsf{St}(\Lambda)$ such that $\mathfrak{b}\mathfrak{u}\mathfrak{x} \in \mathsf{St}(\Lambda)$ and $\theta(\mathfrak{b}\mathfrak{u}) = 1$. Let $\mathfrak{z} := \mathfrak{b}^2\mathfrak{u}\mathfrak{x} \sqcap_l l(\mathfrak{x})$. Clearly $\mathfrak{z} \sqsubset_l \mathfrak{b}^2\mathfrak{u}\mathfrak{x}$. As $\theta(\mathfrak{b}\mathfrak{u}) = 1$, we have $\mathfrak{x} \sqsubset_l \mathfrak{z} \sqsubseteq_l l(\mathfrak{x})$. By our assumption, $\mathfrak{z} \notin \overline{\mathsf{St}}_1(\mathsf{B})$. Hence $\theta(\mathfrak{b}^2\mathfrak{u}\mathfrak{x} \mid \mathfrak{z}) = -1$. Let $\alpha \in Q_1$ be such that $\alpha\mathfrak{z} \sqsubseteq_l \mathfrak{b}^2\mathfrak{u}\mathfrak{x}$. Since $\mathfrak{x} \sqsubset_l \mathfrak{z} \sqsubseteq_l l(\mathfrak{x})$, we get $\alpha\mathfrak{z} \sqsubseteq_l l(\mathfrak{x})$, which contradicts that $\mathfrak{z} = \mathfrak{b}\mathfrak{u}\mathfrak{x} \sqcap_l l(\mathfrak{x})$. □

It immediately follows from Proposition 9.1 that $\mathfrak{x} \sqsubset_l c_{\mathsf{B}}(l(\mathfrak{x}))$, which motivates the following definition.

Definition 9.2 Define $l_{\mathsf{B}} : \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B}) \setminus \{\mathfrak{M}_i(\mathfrak{x}_0)\} \to \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B})$ by $l_{\mathsf{B}}(\mathfrak{x}) := c_{\mathsf{B}}(l(\mathfrak{x}))$, i.e, by choosing $l_{\mathsf{B}}(\mathfrak{x})$ to be the maximal (possibly equal) left substring of $l(\mathfrak{x})$ such that $l_{\mathsf{B}}(\mathfrak{x}) \in \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B})$.

Similarly, define $\bar{l}_{\mathsf{B}} : \overline{\mathsf{St}}_{-1}(\mathfrak{x}_0, i; \mathsf{B}) \setminus \{\mathfrak{m}_i(\mathfrak{x}_0)\} \to \overline{\mathsf{St}}_{-1}(\mathfrak{x}_0, i; \mathsf{B})$ by $\bar{l}_{\mathsf{B}}(\mathfrak{x}) := c_{\mathsf{B}}(\bar{l}(\mathfrak{x}))$.

Proposition 9.3 *If* $\mathfrak{x} \in \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B}) \setminus \{\mathfrak{M}_i(\mathfrak{x}_0)\}$ *then* $\mathfrak{x} <_l l_{\mathsf{B}}(\mathfrak{x})$*,* $\varphi_{\mathsf{B}}(l_{\mathsf{B}}(\mathfrak{x})) = 1$ *and* $H_l^{-1}(l_{\mathsf{B}}(\mathfrak{x})) = (\mathfrak{x}, l_{\mathsf{B}}(\mathfrak{x})]$.

Proof By the definition of $l_{\mathsf{B}}(\mathfrak{x})$, it is clear that $\mathfrak{x} <_l l_{\mathsf{B}}(\mathfrak{x})$, $l_{\mathsf{B}}(\mathfrak{x}) \notin \overline{\mathsf{St}}_{-1}(\mathfrak{x}_0, i; \mathsf{B})$, and hence $\varphi_{\mathsf{B}}(l_{\mathsf{B}}(\mathfrak{x})) = 1$.

Let $\mathfrak{z} \in (\mathfrak{x}, l_{\mathsf{B}}(\mathfrak{x}))$. If $\mathfrak{x} \not\sqsubseteq_l \mathfrak{z}$ then $\mathfrak{z} \sqcap_l \mathfrak{x} = \mathfrak{z} \sqcap_l l_{\mathsf{B}}(\mathfrak{x})$. This implies that both $\mathfrak{x}$ and $l_{\mathsf{B}}(\mathfrak{x})$ lie on the same side of $\mathfrak{z}$ in the hammock, a contradiction. Hence $\mathfrak{x} \sqsubset_l \mathfrak{z}$. Since $\theta(\mathfrak{z} \mid \mathfrak{x}) = 1$, we get $a \in Q_1$ such that $A\mathfrak{x} \sqsubseteq_l \mathfrak{z}$. By definition, $A\mathfrak{x} \sqsubseteq_l l_{\mathsf{B}}(\mathfrak{x})$. Hence $A\mathfrak{x} \sqsubseteq_l l_{\mathsf{B}}(\mathfrak{x}) \sqcap_l \mathfrak{z}$ and $\theta(l_{\mathsf{B}}(\mathfrak{x}) \mid \mathfrak{z}) = 1$. If $l_{\mathsf{B}}(\mathfrak{x}) = \mathfrak{x}'A\mathfrak{x}$ then either $|\mathfrak{x}'| = 0$ or $\delta(\mathfrak{x}') = -1$. Therefore we conclude that $l_{\mathsf{B}}(\mathfrak{x}) \sqsubset_l \mathfrak{z}$, which together with $\theta(\mathfrak{z} \mid l_{\mathsf{B}}(\mathfrak{x})) = -1$ gives $\mathfrak{z} \in H_l^{-1}(l_{\mathsf{B}}(\mathfrak{x}))$ as required. □

Combining the above with Remark 7.5, we conclude that $l_{\mathsf{B}}(\mathfrak{x})$ is the immediate successor of $\mathfrak{x}$ in $\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$.

Corollary 9.4 *If* $\mathfrak{x} \in \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B}) \setminus \{\mathfrak{M}_i(\mathfrak{x}_0)\}$ *then* $(\mathfrak{x}, l_{\mathsf{B}}(\mathfrak{x})) \cap \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B}) = \emptyset$.

Now we show that the function l_{B} is indeed an extension of the function ℓ_{B} on a larger domain.

Proposition 9.5 *If* $\mathfrak{x} \in \mathsf{St}_1(\mathfrak{x}_0, i; \mathsf{B})$ *then* $\ell_{\mathsf{B}}(\mathfrak{x}) = l_{\mathsf{B}}(\mathfrak{x})$.

Proof Let $\ell_{\mathsf{B}}(\mathfrak{x}) =: \mathfrak{z}'\mathfrak{x}$ and $l_{\mathsf{B}}(\mathfrak{x}) =: \mathfrak{z}\mathfrak{x}$. Since $\ell_{\mathsf{B}}(\mathfrak{x}) \in \mathsf{St}_1(\mathfrak{x}_0, i; \mathsf{B}) \subseteq \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B})$, we have $\ell_{\mathsf{B}}(\mathfrak{x}) \sqsubseteq_l \mathfrak{z}\mathfrak{x} \sqsubseteq_l l(\mathfrak{x})$. If $\mathfrak{z} = \mathfrak{z}'$ then there is nothing to prove. Therefore assume that $\mathfrak{z}' \sqsubset_l \mathfrak{z}$. By Remark 8.9, $1_{(t(\mathfrak{x}),\varepsilon(\mathfrak{x}))} \in \mathsf{St}_1(\mathsf{B})$ and $\ell_{\mathsf{B}}(1_{(t(\mathfrak{x}),\varepsilon(\mathfrak{x}))}) = \mathfrak{z}'$. Then Corollary 8.17 yields $\mathfrak{b} \in \mathsf{Cyc}(\mathsf{B})$ such that $\mathfrak{z}' \sqsubseteq_l \mathfrak{b}^n$ for some $n \in \mathbb{N}^+$.

Since $\mathfrak{z}'\mathfrak{b}$ is a string, $\theta(\mathfrak{z}') = 1$ and $\theta(\mathfrak{z} \mid \mathfrak{z}') = -1$, we get that $\mathfrak{z}\mathfrak{b}$ is a string. Since $\mathfrak{z}' \sqsubset_l \mathfrak{z} \sqsubseteq_l l(1_{(t(\mathfrak{x}),\varepsilon(\mathfrak{x}))})$ we get $\delta(\mathfrak{z}) = 0$. As $\mathfrak{z}\mathfrak{x} \in \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B})$, there is $\mathfrak{b}' \in \mathsf{Cyc}(\mathsf{B})$ and a string $\mathfrak{u}$ such that $\mathfrak{b}'\mathfrak{u}\mathfrak{z}\mathfrak{x} \in \mathsf{St}(\Lambda)$ and $\theta(\mathfrak{b}'\mathfrak{u}) = 1$. Furthermore, $\delta(\mathfrak{z}) = 0$ gives that $\mathfrak{b}'\mathfrak{u}\mathfrak{z}\mathfrak{b}$ is a string. Thus $\mathfrak{z}\mathfrak{b}\mathfrak{u}\mathfrak{z}\mathfrak{b}'\mathfrak{u}$ is a power of a B-cycle, $\theta(\mathfrak{z}\mathfrak{b}\mathfrak{y}\mathfrak{b}'\mathfrak{u}) = 1$ and $(\mathfrak{z}\mathfrak{b}\mathfrak{y}\mathfrak{b}'\mathfrak{u})\mathfrak{z}$ is a string. Hence $\mathfrak{z} \in \mathsf{St}_1(\mathsf{B})$. Since $\delta(\mathfrak{z}) = 0$, we have $\mathfrak{z}\mathfrak{x} \in \mathsf{St}_1(\mathsf{B})$. Since $\mathfrak{z}\mathfrak{x} \sqsubseteq_l l(\mathfrak{x})$, we get a contradiction to the definition of $\ell_{\mathsf{B}}(\mathfrak{x})$. □

For $\mathfrak{x} \in \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B})$ we inductively define the powers of the function l_{B} by $l_{\mathsf{B}}^0(\mathfrak{x}) := \mathfrak{x}$ and $l_{\mathsf{B}}^{n+1}(\mathfrak{x}) := l_{\mathsf{B}}(l_{\mathsf{B}}^n(\mathfrak{x}))$ for $n \in \mathbb{N}$, if $l_{\mathsf{B}}^n(\mathfrak{x}) \neq \mathfrak{M}_i(\mathfrak{x}_0)$. Note that $l_{\mathsf{B}}^n(\mathfrak{x})$ exists for each n if and only if $\mathfrak{x} \in \overline{\mathsf{St}}_1(\mathsf{B})$. Whenever this happens, using $l_{\mathsf{B}}^n(\mathfrak{x}) \sqsubset_l l_{\mathsf{B}}^{n+1}(\mathfrak{x})$ we get that $\lim_{n\to\infty} l_{\mathsf{B}}^n(\mathfrak{x})$ is a left $\mathbb{N}$-string; denote this limit by $\langle 1, l_{\mathsf{B}}\rangle(\mathfrak{x})$.

Remark 9.6 If $l_{\mathsf{B}}^n(\mathfrak{x}) = \mathfrak{u}\mathfrak{y}\mathfrak{x}$, where $\mathfrak{y}$ and $\mathfrak{u}$ are strings with $|\mathfrak{u}| > 0$, then $\theta(\mathfrak{u}) = 1$ if and only if there exists $0 \leq k < n$ such that $l_{\mathsf{B}}^k(\mathfrak{x}) = \mathfrak{y}\mathfrak{x}$.

The proof of the following result is along similar lines as the proof of Proposition 8.10.

Proposition 9.7 *If* $\mathfrak{x} \in \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B}) \cap \overline{\mathsf{St}}_1(\mathsf{B})$ *then* $\langle 1, l_{\mathsf{B}}\rangle(\mathfrak{x}) = {}^\infty\mathfrak{b}\mathfrak{u}\mathfrak{x}$ *for some band* $\mathfrak{b}$ *and some string* $\mathfrak{u}$*.*

However the band $\mathfrak{b}$ obtained above might not be a B-band as is evident from the following example.

Example 9.8 Continuing from Example 5.8, for $a_0 \in \overline{\mathsf{St}}_1(a_0, 1; \mathsf{B}_2) \cap \overline{\mathsf{St}}_1(\mathsf{B}_2)$, we have $\langle 1, l_{\mathsf{B}_2}\rangle(a_0) = {}^\infty(b_1B_4b_3B_2)b_1a_4a_3A_1a_0$, but $b_1B_4b_3B_2 \notin \mathsf{B}_2$.

This issue is resolved if B is minimal for $(\mathfrak{x}_0, i)$.

Proposition 9.9 *Suppose* B *is minimal for* $(\mathfrak{x}_0, i)$*. If* $\mathfrak{x} \in \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B}) \cap \overline{\mathsf{St}}_1(\mathsf{B})$ *then* $\langle 1, l_{\mathsf{B}} \rangle(\mathfrak{x}) = {}^{\infty}\mathfrak{b}\mathfrak{u}\mathfrak{x}$ *for some* $\mathfrak{b} \in \mathsf{Ba}_l(\mathsf{B})$ *and some string* $\mathfrak{u}$.

Proof By Corollary 6.7, we have that $l_{\mathsf{B}}^n(\mathfrak{x}) \in \mathsf{St}_1(\mathfrak{x}_0, i; \mathsf{B})$ for some $n \in \mathbb{N}$. Therefore $\langle 1, l_{\mathsf{B}} \rangle(\mathfrak{x}) = \langle 1, l_{\mathsf{B}} \rangle(l_{\mathsf{B}}^n(\mathfrak{x})) = \langle 1, \ell_{\mathsf{B}} \rangle(l_{\mathsf{B}}^n(\mathfrak{x}))$, where the last equality follows from Proposition 9.5. The conclusion is then immediate from Corollary 8.25.

We end this section with a useful result that will be used to prove the density of some special strings called B-centers in Proposition 10.12.

Proposition 9.10 *Suppose* $\mathsf{B} \in \mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0)$ *is non-domestic and minimal for* $(\mathfrak{x}_0, i)$*. If* $\mathfrak{x} \in \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B})$ *and* $\mathfrak{y} \in \overline{\mathsf{St}}_{-1}(\mathfrak{x}_0, i; \mathsf{B})$ *such that* $\mathfrak{x} <_l \mathfrak{y}$ *then* $\langle 1, l_{\mathsf{B}} \rangle(\mathfrak{x}) <_l \langle 1, \bar{l}_{\mathsf{B}} \rangle(\mathfrak{y})$.

Proof Proposition 6.4 gives $\mathfrak{x} \in \overline{\mathsf{St}}_1(\mathsf{B})$ and $\mathfrak{y} \in \overline{\mathsf{St}}_{-1}(\mathsf{B})$. In view of Proposition 9.9, let $\langle 1, l_{\mathsf{B}} \rangle(\mathfrak{x}) =: {}^{\infty}\mathfrak{b}_1\mathfrak{u}_1\mathfrak{x}$ and $\langle 1, \bar{l}_{\mathsf{B}} \rangle(\mathfrak{y}) =: {}^{\infty}\mathfrak{b}_2\mathfrak{u}_2\mathfrak{y}$, where $\mathfrak{b}_1 \in \mathsf{Ba}_l(\mathsf{B})$, $\mathfrak{b}_2 \in \mathsf{Ba}_{\bar{l}}(\mathsf{B})$ and $\mathfrak{u}_1, \mathfrak{u}_2$ are strings. Since $\mathsf{Ba}_l(\mathsf{B}) \cap \mathsf{Ba}_{\bar{l}}(\mathsf{B}) = \emptyset$, we get that $\mathfrak{b}_1$ is not a cyclic permutation of $\mathfrak{b}_2$, which implies ${}^{\infty}\mathfrak{b}_1\mathfrak{u}_1\mathfrak{x} \neq {}^{\infty}\mathfrak{b}_2\mathfrak{u}_2\mathfrak{y}$. Let $\mathfrak{w} := {}^{\infty}\mathfrak{b}_1\mathfrak{u}_1\mathfrak{x} \sqcap_l {}^{\infty}\mathfrak{b}_2\mathfrak{u}_2\mathfrak{y}$ so that $\mathfrak{w} \in \overline{\mathsf{St}}_{\pm 1}(\mathsf{B})$. If $\theta({}^{\infty}\mathfrak{b}_1\mathfrak{u}_1\mathfrak{x} \mid \mathfrak{w}) = 1$ then Remark 9.6 yields $n \in \mathbb{N}^+$ such that $l_{\mathsf{B}}^n(\mathfrak{x}) = \mathfrak{w}$, a contradiction to Proposition 9.3. Hence $\theta({}^{\infty}\mathfrak{b}_1\mathfrak{u}_1\mathfrak{x} \mid \mathfrak{w}) = -1$, which completes the proof. □

10 B-centers

Given $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$, the presence of some special strings in $\mathsf{St}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B})$, which we shall call B-*centers*, characterizes non-domesticity of B. The suborder of such strings is a dense linear order, and it is responsible for the shuffle structure (see clause (4) of Definition 2.3) in the hammocks for non-domestic string algebras. The absence of B-centers in domestic string algebras (Proposition 10.11) thus prohibits domestic string algebras to have a shuffle structure.

To define B-centers, we need a notion called B-*equivalence* which guarantees that the maximal scattered intervals in $\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$ around B-centers are canonically isomorphic.

Definition 10.1 Let $\mathsf{B} \in \mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0)$. Two strings $\mathfrak{x}$ and $\mathfrak{y}$ in $\mathsf{St}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B})$ are said to be B-*equivalent*, denoted $\mathfrak{x} \equiv_{\mathsf{B}} \mathfrak{y}$, if there exist distinct syllables α and β such that $\alpha\mathfrak{x}$, $\beta\mathfrak{x}$, $\alpha\mathfrak{y}$ and $\beta\mathfrak{y}$ are strings.

It is trivial to note that $\equiv_{\mathsf{B}}$ is an equivalence relation on $\mathsf{St}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B})$. The following remark notes that there are finitely many $\equiv_{\mathsf{B}}$-classes.

Remark 10.2 Associated to each string $\mathfrak{x}$ of a B-equivalence class there is a unique pair $(\alpha, \beta) \in Q_1 \times Q_1^-$ for which $\alpha\mathfrak{x}$ and $\beta\mathfrak{x}$ are strings. Therefore the assignment of each B-equivalence class to its corresponding pair in $Q_1 \times Q_1^-$ is injective. Since $Q_1 \times Q_1^-$ is finite in every string algebra, we have that there are finitely many B-equivalence classes.

Example 10.3 Recall from Example 5.8 that $\mathsf{B}_3 \in \mathcal{Q}_1^{\mathrm{Ba}}(a_0)$ is non-domestic. There are three B_3-equivalence classes in $\overline{\mathsf{St}}_{\pm 1}(a_0, 1; \mathsf{B})$ with strings $E_1A_2A_1a_0$, $G_1FE_2E_1A_2A_1a_0$, $k_1h_2H_1G_1FE_2E_1A_2A_1a_0$ as their representatives.

The notion of B-equivalence is strictly weaker than H-equivalence as demonstrated in Example 10.4. However, these notions coincide in the case of gentle string algebras.

Example 10.4 Consider the string algebra Γ' from Fig. 3. Choose $j \in \{1, -1\}$ such that $f 1_{(v_5,j)}$ is a string. There is only one non-domestic $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}_{-1}(1_{(v_5,j)})$. Consider the strings $f,\ feDf \in \mathsf{St}_{\pm 1}(1_{(v_5,j)}, -1; \mathsf{B})$. Since $cf, cfeDf, Df, DfeDf$ are strings, we have $f \equiv_{\mathsf{B}} feDf$. On the other hand, $f \not\equiv_H feDf$ since acf is a string but $acfeDf$ is not.

Proposition 10.5 *Let* $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}_i(\mathfrak{x}_0)$. *The following statements are equivalent for a string* $\mathfrak{x} \in \mathsf{St}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B})$.

(1) There exists $\mathfrak{b}' \in \mathsf{Cyc}(\mathsf{B})$ *such that* $\mathfrak{b}'\mathfrak{x} \in \mathsf{St}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B})$ *and* $\mathfrak{b}'\mathfrak{x} \equiv_{\mathsf{B}} \mathfrak{x}$.
(2) There exist strings $\mathfrak{x}_1, \mathfrak{x}_2$ *and* $\mathfrak{b}' \in \mathsf{Cyc}(\mathsf{B})$ *such that* $\mathfrak{x} = \mathfrak{x}_2\mathfrak{x}_1$, $\mathfrak{x}_2\mathfrak{b}'\mathfrak{x}_1 \in \mathsf{St}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B})$, $\mathfrak{x}_2\mathfrak{b}'\mathfrak{x}_1 \equiv_{\mathsf{B}} \mathfrak{x}$ *and* $\mathfrak{x}_2\mathfrak{b}' \in \mathsf{Ext}(\mathsf{B})$.
(3) There exist strings $\mathfrak{z}, \mathfrak{u}$ *such that* $\delta(\mathfrak{z}) = 0$, $\mathfrak{z} \in \mathsf{Ext}(\mathsf{B})$, $\mathfrak{z}\mathfrak{u} \in \mathsf{St}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B})$ *and* $\mathfrak{z}\mathfrak{u} \equiv_{\mathsf{B}} \mathfrak{x}$.
(4) There exists a syllable $\gamma \in \mathsf{St}_{\pm 1}(\mathsf{B})$ *such that* $1_{(t(\mathfrak{x}),\varepsilon(\mathfrak{x}))}\gamma$ *is a string,* $\alpha\gamma, \beta\gamma \in \mathsf{Ext}(\mathsf{B})$ *for distinct syllables* α, β *such that* $\alpha\mathfrak{x}, \beta\mathfrak{x} \in \mathsf{St}(\Lambda)$.

Proof (1) $\Longrightarrow$ (2): This is immediate as we can take $\mathfrak{x}_1 = \mathfrak{x}$ and $\mathfrak{x}_2 = 1_{(v,j)}$ for appropriate (v, j) such that $1_{(v,j)}\mathfrak{x}_1$ is a string.

(2) $\Longrightarrow$ (3): Take $\mathfrak{z} = \mathfrak{x}_2\mathfrak{b}'$ and $\mathfrak{u} = \mathfrak{x}_1$. Since $\mathfrak{b}'$ is a cyclic permutation of a band and $\mathfrak{b}' \sqsubseteq_l \mathfrak{z}$, we have $\delta(\mathfrak{z}) = 0$. Moreover, $\mathfrak{z}\mathfrak{u} = \mathfrak{x}_2\mathfrak{b}'\mathfrak{x}_1 \equiv_{\mathsf{B}} \mathfrak{x}$.

(3) $\Longrightarrow$ (4): Since $\delta(\mathfrak{z}) = 0$, we have $|\mathfrak{z}| > 0$. Take γ to be the last syllable of $\mathfrak{z}$. Since $\mathfrak{z}\mathfrak{u} \in \mathsf{St}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B})$, there exist $\mathfrak{b}'_1, \mathfrak{b}'_2 \in \mathsf{Cyc}(\mathsf{B})$ with $\theta(\mathfrak{b}'_1) = 1 = -\theta(\mathfrak{b}'_2)$ such that $\mathfrak{b}'_1\mathfrak{z}\mathfrak{u}$ and $\mathfrak{b}'_2\mathfrak{z}\mathfrak{u}$ are strings. Let the first syllables of $\mathfrak{b}'_1$ and $\mathfrak{b}'_2$ be α and β respectively. Since $\mathfrak{z} \in \mathsf{Ext}(\mathsf{B})$, there exists $\mathfrak{b}' \in \mathsf{Cyc}(\mathsf{B})$ such that $\mathfrak{b}'^n = \mathfrak{w}\mathfrak{z}$ for some string $\mathfrak{w}$ and $n \in \mathbb{N}$. Let $\mathfrak{v}$ be a string such that $\mathfrak{b}'\mathfrak{v}\mathfrak{b}'_1$ is a string. Since $\delta(\mathfrak{z}) = 0$ and $\mathfrak{z}\mathfrak{b}'$ and $\mathfrak{b}'_1\mathfrak{z}$ are strings, we conclude that $\mathfrak{z}\mathfrak{b}'\mathfrak{v}\mathfrak{b}'_1$ is a power of a cyclic permutation of a B-band. Since the last syllable of $\mathfrak{z}$ is γ and the first syllable of $\mathfrak{b}'_1$ is α, it follows that $\alpha\gamma \in \mathsf{Ext}(\mathsf{B})$. Similarly we can show that $\beta\gamma \in \mathsf{Ext}(\mathsf{B})$.

(4) $\Longrightarrow$ (1): Without loss of generality, let $\theta(\gamma) = \theta(\alpha) = -\theta(\beta)$. There are two cases.

Case 1: Either $|\mathfrak{x}| = 0$ or $\theta(\gamma') = \theta(\gamma)$, where γ' is the last syllable of $\mathfrak{x}$.
Since $\beta\gamma \in \mathsf{Ext}(\mathsf{B})$, there is $\mathfrak{b}' \in \mathsf{Cyc}(\mathsf{B})$ and $n \in \mathbb{N}^+$ such that $\gamma\mathfrak{w}\beta = \mathfrak{b}'^n$ for some string $\mathfrak{w}$. Further, since $\alpha\gamma \in \mathsf{Ext}(\mathsf{B})$, there exists a string $\mathfrak{w}'$ such that $\alpha\gamma\mathfrak{w}'\mathfrak{b}'$ is a string. Since the first syllable of $\mathfrak{b}'$ is β and $\theta(\mathfrak{b}') = -\theta(\gamma)$, we get that $\gamma\mathfrak{w}'\mathfrak{b}'$ is a power of a B-cycle, say $\mathfrak{b}''$, such that $\alpha\mathfrak{b}''\mathfrak{x}$ and $\beta\mathfrak{b}''\mathfrak{x}$ are strings.

Case 2: $\theta(\gamma') = -\theta(\gamma)$, where γ' is the last syllable of $\mathfrak{x}$.
Since $\alpha\gamma \in \mathsf{Ext}(\mathsf{B})$, there exists $\mathfrak{b}' \in \mathsf{Cyc}(\mathsf{B})$ and $n \in \mathbb{N}^+$ such that $\gamma\mathfrak{w}\alpha = \mathfrak{b}'^n$ for some string $\mathfrak{w}$. Since the first syllable of $\mathfrak{b}'$ is α, and thus $\theta(\mathfrak{b}') = -\theta(\gamma')$, we get that $\mathfrak{b}'\mathfrak{x}$ is a string. Further since the last syllable of $\mathfrak{b}'$ is γ, and $\theta(\gamma) = -\theta(\beta)$ we get that $\beta\mathfrak{b}'$ is a string. Since $\delta(\mathfrak{b}') = 0$, it follows that $\beta\mathfrak{b}'\mathfrak{x}$ is a string too, thus completing the proof. □

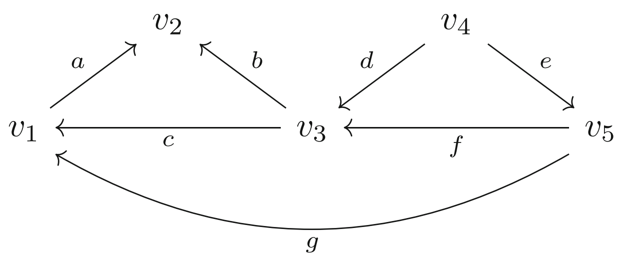

Fig. 3 Γ' with $\rho = \{bf, cd, ge, ag, acfe\}$

Definition 10.6 Say that $\mathfrak{x} \in \mathsf{St}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B})$ is a B-*center* if it satisfies one of the equivalent conditions of Proposition 10.5. Denote the set of all B-centers by $\mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B})$.

The following example shows that $\mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B})$ can be a proper subset of $\mathsf{St}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B})$.

Example 10.7 Consider the string algebra Γ'' from Fig. 4. There is a unique non-domestic $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}_{-1}(a_1)$ with $a_2a_1 \in \mathsf{St}_{\pm 1}(a_1, -1; \mathsf{B}) \setminus \mathrm{Cent}(a_1, -1; \mathsf{B})$.

Remark 10.8 In view of Remark 8.9, if $\mathfrak{x} \equiv_{\mathsf{B}} \mathfrak{y}$ and $\langle 1, l_{\mathsf{B}}\rangle(\mathfrak{x}) = {}^{\infty}\mathfrak{b}\mathfrak{u}\mathfrak{x}$ for some $\mathfrak{b} \in \mathsf{Ba}_l(\mathsf{B})$ and string $\mathfrak{u}$ then $\langle 1, l_{\mathsf{B}}\rangle(\mathfrak{y}) = {}^{\infty}\mathfrak{b}\mathfrak{u}\mathfrak{y}$.

Given $\mathfrak{x} \in \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B})$, we say that $\mathfrak{x}$ is the center of the interval $\mathcal{I}_{(\mathfrak{x}_0, i; \mathsf{B})}(\mathfrak{x}) := (\langle 1, \bar{l}_{\mathsf{B}}\rangle(\mathfrak{x}), \langle 1, l_{\mathsf{B}}\rangle(\mathfrak{x}))$ in the hammock $(H^i_l(\mathfrak{x}_0), <_l)$. The following result shows that two intervals of the above form are canonically isomorphic if and only if their centers are B-equivalent.

Proposition 10.9 *Suppose* $\mathfrak{x}, \mathfrak{y} \in \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B})$. *If* $\mathfrak{x} \equiv_{\mathsf{B}} \mathfrak{y}$ *then for any string* $\mathfrak{u}$, $\mathfrak{u}\mathfrak{x} \in \mathcal{I}_{(\mathfrak{x}_0, i; \mathsf{B})}(\mathfrak{x})$ *if and only if* $\mathfrak{u}\mathfrak{y} \in \mathcal{I}_{(\mathfrak{x}_0, i; \mathsf{B})}(\mathfrak{y})$.

Proof Since $\mathfrak{x} \in \mathsf{St}_1(\mathsf{B})$, Corollary 8.25 and Proposition 9.5 together give $\langle 1, l_{\mathsf{B}}\rangle(\mathfrak{x}) = {}^{\infty}\mathfrak{b}\mathfrak{u}_1\mathfrak{x}$ for some string $\mathfrak{u}_1$ and $\mathfrak{b} \in \mathsf{Ba}_l(\mathsf{B})$. Then Remark 10.8 gives $\langle 1, l_{\mathsf{B}}\rangle(\mathfrak{y}) = {}^{\infty}\mathfrak{b}\mathfrak{u}_1\mathfrak{y}$. Suppose $\mathfrak{u}\mathfrak{x} \in \mathcal{I}_{(\mathfrak{x}_0, i; \mathsf{B})}(\mathfrak{x})$. Without loss of generality, we can assume that $|\mathfrak{u}| > 0$ and $\theta(\mathfrak{u}) = 1$ so that $\mathfrak{u}\mathfrak{x} \in (\mathfrak{x}, {}^{\infty}\mathfrak{b}\mathfrak{u}_1\mathfrak{x})$. If $\mathfrak{u}\mathfrak{y}$ is a string then $\mathfrak{u}\mathfrak{x} \in (\mathfrak{x}, {}^{\infty}\mathfrak{b}\mathfrak{u}_1\mathfrak{x})$ immediately implies $\mathfrak{u}\mathfrak{y} \in (\mathfrak{y}, {}^{\infty}\mathfrak{b}\mathfrak{u}_1\mathfrak{y})$. Hence it remains to show that $\mathfrak{u}\mathfrak{y}$ is a string. There are two possibilities.

Since ${}^{\infty}\mathfrak{b}\mathfrak{u}_1\mathfrak{y}$ is a left $\mathbb{N}$-string, if $\mathfrak{u} \sqsubset_l {}^{\infty}\mathfrak{b}\mathfrak{u}_1$ then $\mathfrak{u}\mathfrak{y}$ is a string.

If $\mathfrak{u}$ and ${}^{\infty}\mathfrak{b}\mathfrak{u}_1$ fork then let $\mathfrak{z} := \mathfrak{u} \sqcap_l {}^{\infty}\mathfrak{b}\mathfrak{u}_1$. Since $\theta({}^{\infty}\mathfrak{b}\mathfrak{u}_1) = \theta(\mathfrak{u}) = 1$, we have $|\mathfrak{z}| > 0$. Thus by the above paragraph, $\mathfrak{z}\mathfrak{y}$ is a string. As $\mathfrak{u} <_l {}^{\infty}\mathfrak{b}\mathfrak{u}_1$, we get that $\alpha\mathfrak{z} \sqsubseteq_l \mathfrak{u}$ for some $\alpha \in Q_1$. Since $\theta(\mathfrak{z}) = 1$, $\mathfrak{z}\mathfrak{y}$ is a string and $\delta(\alpha\mathfrak{z}) = 0$, we get that $\mathfrak{u}\mathfrak{y}$ is a string. □

Now we show the existence of B-centers when B is non-domestic using Remark 8.3.

Proposition 10.10 *Let* $\mathfrak{x} \in H^i_l(\mathfrak{x}_0)$, $\mathfrak{b} \in \mathsf{Ba}(\mathsf{B})$ *and* $(\beta, \mathfrak{b}')$ *be a non-domestic exit of* $\mathfrak{b}$ *with* $\theta(\beta) = 1$. *If* $\beta\mathfrak{b}'\mathfrak{x} \in H^i_l(\mathfrak{x}_0)$ *then* $\mathfrak{b}'\mathfrak{x} \in \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B})$.

Proof Since $\theta(\beta) = 1$ we have $\theta((\mathfrak{b}')^2\mathfrak{x} \mid \mathfrak{b}'\mathfrak{x}) = -1$. Since $\beta\mathfrak{b}' \in \mathsf{Ext}(\mathsf{B})$, there exists $\mathfrak{b}'' \in \mathsf{Cyc}(\mathsf{B})$ such that $\beta\mathfrak{b}' \sqsubseteq_l \mathfrak{b}''$. So for an appropriate cyclic permutation $\mathfrak{b}'''$ of $\mathfrak{b}''$, we have that $\mathfrak{b}'''\mathfrak{b}'\mathfrak{x}$ is a string with $\theta(\mathfrak{b}''') = \theta(\beta) = 1$ thus showing $\mathfrak{b}'\mathfrak{x} \in \mathsf{St}_1(\mathfrak{x}_0, i; \mathsf{B})$. Also $\mathfrak{b}'\mathfrak{b}'\mathfrak{x}$ is a string with $\theta(\mathfrak{b}') = -1$ which gives $\mathfrak{b}'\mathfrak{x} \in \mathsf{St}_{-1}(\mathfrak{x}_0, i; \mathsf{B})$. Finally, since $\delta(\mathfrak{b}') = 0$ we have $\mathfrak{b}'\mathfrak{b}'\mathfrak{x} \equiv_{\mathsf{B}} \mathfrak{b}'\mathfrak{x}$. In view of Proposition 10.5(1), we conclude that $\mathfrak{b}'\mathfrak{x} \in \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B})$. □

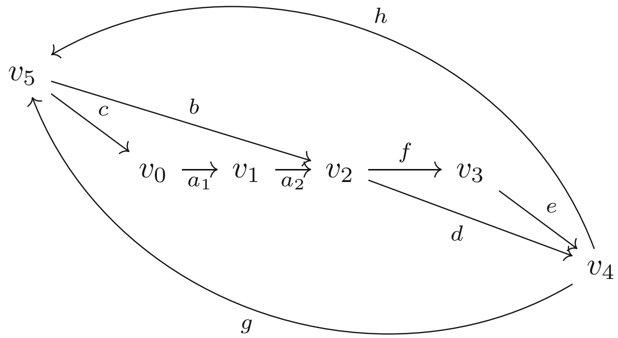

Fig. 4 Γ'' with $\rho = \{fa_2, db, ge, hd, bg, ch, da_2a_1c, efb\}$

Contrary to the above result, there are no B-centers for any domestic $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$.

Proposition 10.11 *If* $\mathsf{B} \in \mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0)$ *is domestic then* $\mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B}) = \emptyset$.

Proof If possible, let $\mathfrak{y} \in \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B})$. Then Proposition 10.5(1) yields $\mathfrak{b}' \in \mathsf{Cyc}(\mathsf{B})$ such that $\mathfrak{b}'\mathfrak{y} \in \mathsf{St}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B})$ and $\mathfrak{b}'\mathfrak{y} \equiv_{\mathsf{B}} \mathfrak{y}$. Since $\mathfrak{b}'\mathfrak{y} \in \mathsf{St}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B})$ there exists $\mathfrak{b}'' \in \mathsf{Cyc}(\mathsf{B})$ with $\theta(\mathfrak{b}') = -\theta(\mathfrak{b}'')$ such that $\mathfrak{b}''\mathfrak{b}'\mathfrak{y}$ is a string. Since $\mathfrak{b}', \mathfrak{b}'' \in \mathsf{Ext}(\mathsf{B})$, $\mathfrak{b}'\mathfrak{u}\mathfrak{b}''$ is a string for some string $\mathfrak{u}$. Since B is domestic and $\mathfrak{u}\mathfrak{b}''\mathfrak{b}'$ is a mixed cyclic string, we get $\mathfrak{u}\mathfrak{b}''\mathfrak{b}' = \mathfrak{b}_1^n$ for some $n \in \mathbb{N}^+$ and a cyclic permutation $\mathfrak{b}_1$ of $\mathfrak{b}'$. In view of Corollary 4.9, we get $\mathfrak{b}' = \mathfrak{b}_1$, which gives a contradiction to $\theta(\mathfrak{b}') = -\theta(\mathfrak{b}'')$. □

We end this section with the following proposition, which is the key to investigating the order type of the set of B-centers.

Proposition 10.12 *Suppose* $\mathsf{B} \in \mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0)$ *is non-domestic and minimal for* $(\mathfrak{x}_0, i)$*, and* $\mathfrak{x} <_l \mathfrak{y}$ *for some* $\mathfrak{x} \in \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B})$ *and* $\mathfrak{y} \in \overline{\mathsf{St}}_{-1}(\mathfrak{x}_0, i; \mathsf{B})$*. Then for any* $\mathfrak{z} \in \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B})$ *there exists* $\mathfrak{z}' \in \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B})$ *such that* $\mathfrak{z}' \equiv_{\mathsf{B}} \mathfrak{z}$ *and* $\langle 1, l_{\mathsf{B}}\rangle(\mathfrak{x}) <_l \mathfrak{z}' <_l \langle 1, \bar{l}_{\mathsf{B}}\rangle(\mathfrak{y})$.

Proof Proposition 9.10 gives $\langle 1, l_{\mathsf{B}}\rangle(\mathfrak{x}) <_l \langle 1, \bar{l}_{\mathsf{B}}\rangle(\mathfrak{y})$. Let $\mathfrak{w} := \langle 1, l_{\mathsf{B}}\rangle(\mathfrak{x}) \sqcap_l \langle 1, \bar{l}_{\mathsf{B}}\rangle(\mathfrak{y})$. Proposition 10.5(3) gives the existence of strings $\mathfrak{z}_1, \mathfrak{z}_2$ such that $\delta(\mathfrak{z}_2) = 0$, $\mathfrak{z}_2 \in \mathsf{Ext}(\mathsf{B})$, $\mathfrak{z}_2\mathfrak{z}_1 \in \mathsf{St}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B})$ and $\mathfrak{z}_2\mathfrak{z}_1 \equiv_{\mathsf{B}} \mathfrak{z}$.

In view of Proposition 9.9, let $\langle 1, l_{\mathsf{B}}\rangle(\mathfrak{x}) = {}^{\infty}\mathfrak{b}\mathfrak{u}_1\mathfrak{x}$ for some $\mathfrak{b} \in \mathsf{Ba}_l(\mathsf{B})$ and some string $\mathfrak{u}_1$. Since $\theta({}^{\infty}\mathfrak{b}\mathfrak{u}_1\mathfrak{x} \mid \mathfrak{w}) = -1$, there exists $n \in \mathbb{N}^+$ such that $\theta(\mathfrak{b}^n\mathfrak{u}_1\mathfrak{x} \mid \mathfrak{w}) = -1$. As $\mathfrak{b} \in \mathsf{Ba}_l(\mathsf{B})$, consider an exit $(\beta, \mathfrak{b}')$ of $\mathfrak{b}$. There exists a string $\mathfrak{u} \sqsubset_l \mathfrak{b}$ such that $\beta\mathfrak{b}'\mathfrak{u}\mathfrak{b}$ is a string. Since $\beta\mathfrak{b}', \mathfrak{z}_2 \in \mathsf{Ext}(\mathsf{B})$, there exists a string $\mathfrak{v}$ such that $\mathfrak{z}_2\mathfrak{v}\beta\mathfrak{b}'$ is a string. Now $\delta(\mathfrak{b}') = 0$ implies $\mathfrak{z}' := \mathfrak{z}_2\mathfrak{v}\beta\mathfrak{b}'\mathfrak{u}\mathfrak{b}^n\mathfrak{u}_1\mathfrak{x}$ is a string, $\mathfrak{z}' \in \mathsf{St}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B})$ and $\mathfrak{z}' \equiv_{\mathsf{B}} \mathfrak{z}$. Since $\theta(\mathfrak{b}^n\mathfrak{u}_1\mathfrak{x} \mid \mathfrak{w}) = -1$, we get $\theta(\mathfrak{z}' \mid \mathfrak{w}) = -1$. Also $\theta(\beta) = 1$ implies $\theta({}^{\infty}\mathfrak{b}\mathfrak{u}_1\mathfrak{x} \mid \mathfrak{z}') = -1$. Therefore we have $\langle 1, l_{\mathsf{B}}\rangle(\mathfrak{x}) <_l \mathfrak{z}' <_l \mathfrak{w} <_l \langle 1, \bar{l}_{\mathsf{B}}\rangle(\mathfrak{y})$ to complete the proof. □

11 Computation of the Order Type of Hammocks

So far we have collected most of the ingredients to prove the main result (Theorem 11.9), whose proof we finish in this section. Furthermore, we prove a partial converse followed by a discussion about the potential impossibility of the converse in its full generality.

Since there are finitely many strings that are band-free relative to $(\mathfrak{x}_0, i)$, recall from Corollary 6.6 that if $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ is minimal for $(\mathfrak{x}_0, i)$ then there are finitely many strings in $\overline{\mathsf{St}}_j(\mathfrak{x}_0, i; \mathsf{B}) \setminus \mathsf{St}_j(\mathfrak{x}_0, i; \mathsf{B})$ for each $j \in \{-1, 1\}$. A simple set theoretic manipulation yields

$$\overline{\mathsf{St}}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B}) \setminus \mathsf{St}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B}) \subseteq (\overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B}) \setminus \mathsf{St}_1(\mathfrak{x}_0, i; \mathsf{B})) \cup (\overline{\mathsf{St}}_{-1}(\mathfrak{x}_0, i; \mathsf{B}) \setminus \mathsf{St}_{-1}(\mathfrak{x}_0, i; \mathsf{B})),$$

and thus $\overline{\mathsf{St}}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B}) \setminus \mathsf{St}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B})$ is finite. The following proposition shows that the set $\mathsf{St}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B}) \setminus \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B})$ is also finite when B is minimal for $(\mathfrak{x}_0, i)$.

Proposition 11.1 *If* $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ *is minimal for* $(\mathfrak{x}_0, i)$ *and* $\mathfrak{x} \in \mathsf{St}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B}) \setminus \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B})$ *then* $\mathfrak{x}$ *is band-free relative to* $(\mathfrak{x}_0, i)$.

Proof If $\mathfrak{x} \in \mathsf{St}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B}) \setminus \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B})$ such that $\mathfrak{x} = \mathfrak{x}_2\mathfrak{b}\mathfrak{x}_1$ for some $\mathfrak{b} \in \mathsf{Cyc}(\mathsf{B})$, then we have $\mathfrak{x} \equiv_{\mathsf{B}} \mathfrak{x}_2\mathfrak{b}^2\mathfrak{x}_1$, which implies $\mathfrak{x} \in \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B})$ by Proposition 10.5(2), a contradiction. This proves that $\mathfrak{x}$ is band-free with respect to B. As a consequence, if $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ is minimal for $(\mathfrak{x}_0, i)$ then $\mathfrak{x}$ is band-free relative to $(\mathfrak{x}_0, i)$. □

Since

$$\overline{\mathsf{St}}_{\pm1}(\mathfrak{x}_0, i; \mathsf{B}) \setminus \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B}) = (\overline{\mathsf{St}}_{\pm1}(\mathfrak{x}_0, i; \mathsf{B}) \setminus \mathsf{St}_{\pm1}(\mathfrak{x}_0, i; \mathsf{B})) \cup (\mathsf{St}_{\pm1}(\mathfrak{x}_0, i; \mathsf{B}) \setminus \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B})),$$

we get that there are finitely many strings in $\overline{\mathsf{St}}_{\pm1}(\mathfrak{x}_0, i; \mathsf{B}) \setminus \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B})$. This observation helps us to break the linear order $(\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B}), <_l)$ into finitely many "irreducible" intervals.

Definition 11.2 Let $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ be minimal for $(\mathfrak{x}_0, i)$. Call an interval $[\mathfrak{x}, \mathfrak{y}]$ in $H_l^i(\mathfrak{x}_0)$ a B-*beam* if $\mathfrak{x}, \mathfrak{y} \in \overline{\mathsf{St}}_{\pm1}(\mathfrak{x}_0, i; \mathsf{B}) \setminus \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B})$, $\mathfrak{x} <_l \mathfrak{y}$ and $(\mathfrak{x}, \mathfrak{y}) \cap \overline{\mathsf{St}}_{\pm1}(\mathfrak{x}_0, i; \mathsf{B}) \subseteq \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B})$.

Since the set $\overline{\mathsf{St}}_{\pm1}(\mathfrak{x}_0, i; \mathsf{B}) \setminus \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B})$ is finite when $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ is minimal for $(\mathfrak{x}_0, i)$, we get that there are finitely many B-beams. Let n_{B} denote the number of B-beams.

If $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ is minimal for $(\mathfrak{x}_0, i)$ and $\mathfrak{y}_0 <_l \mathfrak{y}_1 <_l \cdots <_l \mathfrak{y}_{n_{\mathsf{B}}}$ is the complete list of elements in $\overline{\mathsf{St}}_{\pm1}(\mathfrak{x}_0, i; \mathsf{B}) \setminus \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B})$ then

$$(H_l^i(\mathfrak{x}_0), <_l) = [\mathfrak{m}_i(\mathfrak{x}_0), \mathfrak{y}_0] \dotplus [\mathfrak{y}_0, \mathfrak{y}_1] \dotplus \cdots \dotplus [\mathfrak{y}_{n_{\mathsf{B}}-1}, \mathfrak{y}_{n_{\mathsf{B}}}] \dotplus [\mathfrak{y}_{n_{\mathsf{B}}}, \mathfrak{M}_i(\mathfrak{x}_0)], \tag{11.1}$$

where each $[\mathfrak{y}_j, \mathfrak{y}_{j+1}]$ is a B-beam.

In view of Remark 7.2, we have

$$(c_{\mathsf{B}}(H_l^i(\mathfrak{x}_0)), <_l) = c_{\mathsf{B}}([\mathfrak{m}_i(\mathfrak{x}_0), \mathfrak{y}_0]) \dotplus c_{\mathsf{B}}([\mathfrak{y}_0, \mathfrak{y}_1]) \dotplus \cdots \dotplus c_{\mathsf{B}}([\mathfrak{y}_{n_{\mathsf{B}}-1}, \mathfrak{y}_{n_{\mathsf{B}}}]) \dotplus c_{\mathsf{B}}([\mathfrak{y}_{n_{\mathsf{B}}}, \mathfrak{M}_i(\mathfrak{x}_0)]). \tag{11.2}$$

Example 11.3 Continuing Example 5.8, since B_1 is minimal for $(a_0, 1)$, Equation (11.1) takes the form

$$H_l^1(a_0) = [a_0, a_0] \dotplus [a_0, a_3 A_1 a_0] \dotplus [a_3 A_1 a_0, A_1 a_0] \dotplus [A_1 a_0, H_1 G_1 F E_2 E_1 A_2 A_1 a_0]. \tag{11.3}$$

and Equation (11.2) takes the form

$$c_{\mathsf{B}_1}(H_l^1(a_0)) = c_{\mathsf{B}_1}([a_0, a_3 A_1 a_0]) \dotplus c_{\mathsf{B}_1}([a_3 A_1 a_0, A_1 a_0]) \dotplus c_{\mathsf{B}_1}([A_1 a_0, H_1 G_1 F E_2 E_1 A_2 A_1 a_0]). \tag{11.4}$$

Proposition 11.4 *The sets* $c_{\mathsf{B}}([\mathfrak{m}_i(\mathfrak{x}_0), \mathfrak{y}_0])$ *and* $c_{\mathsf{B}}([\mathfrak{y}_{n_{\mathsf{B}}}, \mathfrak{M}_i(\mathfrak{x}_0)])$ *are finite.*

Proof Without loss of generality, assume that $i = -1$. Recall from the proof of Proposition 6.12 that $\mathfrak{y}_{n_{\mathsf{B}}} = \mathfrak{x}_0 = \mathfrak{M}_i(\mathfrak{x}_0)$, and hence $\mathrm{card}([\mathfrak{y}_{n_{\mathsf{B}}}, \mathfrak{M}_i(\mathfrak{x}_0)]) = 1$. On the other hand, the same proof describes $\mathfrak{y}_0$ as the longest left substring of $\mathfrak{m}_i(\mathfrak{x}_0)$ that lies in $\overline{\mathsf{St}}_{\pm1}(\mathfrak{x}_0, i; \mathsf{B})$. Hence $\mathfrak{z} \in c_{\mathsf{B}}([\mathfrak{m}_i(\mathfrak{x}_0), \mathfrak{y}_0])$ if and only if $\mathfrak{y}_0 \sqsubseteq_l \mathfrak{z} \sqsubseteq_l \mathfrak{m}_i(\mathfrak{x}_0)$. Since there are only finitely left substrings of $\mathfrak{m}_i(\mathfrak{x}_0)$, the proof is complete. □

Recall from Remark 10.2 that the set $\mathsf{St}_{\pm1}(\mathfrak{x}_0, i; \mathsf{B})/{\equiv_{\mathsf{B}}}$ is finite. Let $k_{\mathsf{B}} := \mathrm{card}(\mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B})/{\equiv_{\mathsf{B}}})$. Propositions 10.10 and 10.11 together imply that $k_{\mathsf{B}} = 0$ if and only if B is domestic.

The next result is a consequence of Proposition 10.12, which computes the order type of the suborder of B-centers inside a B-beam.

Corollary 11.5 *If* $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ *is minimal for* $(\mathfrak{x}_0, i)$ *and* $[\mathfrak{x}, \mathfrak{y}]$ *be a* B-*beam then*

$$(\mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B}) \cap [\mathfrak{x}, \mathfrak{y}], <_l) \cong \Xi(\underbrace{1, 1, \cdots, 1}_{k_{\mathsf{B}} \text{ times}}).$$

Proof Since $H_l^i(\mathfrak{x}_0)$ is countable, the set $\mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B}) \cap [\mathfrak{x}, \mathfrak{y}]$ is also countable. If B is domestic then Proposition 10.11 implies that both sides are empty linear orders. On the other hand, if B is non-domestic then Proposition 10.10 implies that $\mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B}) \neq \emptyset$. Therefore it suffices to prove that each B-equivalence class of B-centers intersects the beam $[\mathfrak{x}, \mathfrak{y}]$ in a non-empty, unbounded, and dense fashion. Let $\mathfrak{w} \in \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B})$.

Non-empty: Since $\mathfrak{x} \in \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B})$ and $\mathfrak{y} \in \overline{\mathsf{St}}_{-1}(\mathfrak{x}_0, i; \mathsf{B})$, Proposition 10.12 applied on $\mathfrak{x} <_l \mathfrak{y}$ yields $\mathfrak{z} \in \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B}) \cap (\mathfrak{x}, \mathfrak{y})$ such that $\mathfrak{z} \equiv_{\mathsf{B}} \mathfrak{w}$.

Unbounded: Let $\mathfrak{z} \in \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B}) \cap (\mathfrak{x}, \mathfrak{y})$. Since $\mathfrak{z} \in \mathsf{St}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B})$, Proposition 10.12 applied on $\mathfrak{x} <_l \mathfrak{z}$ and $\mathfrak{z} <_l \mathfrak{y}$ guarantees the existence of $\mathfrak{z}_1 \in \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B}) \cap (\mathfrak{z}, \mathfrak{y})$ and $\mathfrak{z}_2 \in \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B}) \cap (\mathfrak{x}, \mathfrak{z})$ respectively such that $\mathfrak{z}_1 \equiv_{\mathsf{B}} \mathfrak{z}_2 \equiv_{\mathsf{B}} \mathfrak{w}$.

Dense: If $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B}) \cap (\mathfrak{x}, \mathfrak{y})$ then Proposition 10.12 applied on $\mathfrak{z}_1 <_l \mathfrak{z}_2$ yields $\mathfrak{z}_3 \in \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B}) \cap (\mathfrak{z}_1, \mathfrak{z}_2)$ such that $\mathfrak{z}_3 \equiv_{\mathsf{B}} \mathfrak{w}$. □

Proposition 11.6 *Let* $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ *be minimal for* $(\mathfrak{x}_0, i)$. *If* $\mathfrak{x} \in \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B})$ *then there exists* $\mathfrak{y} \in \overline{\mathsf{St}}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B})$ *and* $n \in \mathbb{N}$ *such that* $l_{\mathsf{B}}^n(\mathfrak{y}) = \mathfrak{x}$.

Proof In view of Proposition 9.9, let $\langle 1, l_{\mathsf{B}} \rangle(\mathfrak{x}) =: {}^{\infty}\mathfrak{b}\mathfrak{u}\mathfrak{x}$ for some string $\mathfrak{u}$ and $\mathfrak{b} \in \mathsf{Ba}_l(\mathsf{B})$. Consider the shortest string $\mathfrak{y} \in \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B})$ such that $\langle 1, l_{\mathsf{B}} \rangle(\mathfrak{y}) = {}^{\infty}\mathfrak{b}\mathfrak{u}\mathfrak{x}$. We claim that $\mathfrak{y} \in \overline{\mathsf{St}}_{-1}(\mathfrak{x}_0, i; \mathsf{B})$.

If $\mathfrak{y} \in \overline{\mathsf{St}}_{-1}(\mathsf{B})$ then we are done. Otherwise, in view of Remark 6.1, we need to show that $\mathfrak{y} \sqsubseteq_l \mathfrak{m}_i(\mathfrak{x}_0)$.

If possible, let $\mathfrak{x}_0 \sqsubseteq_l \mathfrak{z} \sqsubset_l \mathfrak{y}$ be a string such that $\theta(\mathfrak{y} \mid \mathfrak{z}) = 1$. Without loss, take $\mathfrak{z}$ to be the longest such string. Then $\mathfrak{y} = \mathfrak{w}\alpha\mathfrak{z}$, where $\alpha \in Q_1^-$ and $\mathfrak{w}$ satisfies either $|\mathfrak{w}| = 0$ or $\delta(\mathfrak{w}) = -1$. Clearly $\mathfrak{z} \in \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B})$ and $\mathfrak{y} \sqsubseteq_l l_{\mathsf{B}}(\mathfrak{z})$.

If $\mathfrak{y} \sqsubset_l l_{\mathsf{B}}(\mathfrak{z})$ then there exists $\beta \in Q_1$ such that $l_{\mathsf{B}}(\mathfrak{z}) \sqsupseteq_l \beta\mathfrak{y} \in \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B})$. Thus $\mathfrak{y} \in \overline{\mathsf{St}}_{-1}(\mathfrak{x}_0, i; \mathsf{B})$. Since $\delta(l_{\mathsf{B}}(\mathfrak{z})) = 0$, in view of Remark 6.1, we get $l_{\mathsf{B}}(\mathfrak{z}) \in \overline{\mathsf{St}}_1(\mathsf{B})$ implying that $\mathfrak{y} \in \overline{\mathsf{St}}_{-1}(\mathsf{B})$, a contradiction to our assumption. On the other hand, if $\mathfrak{y} = l_{\mathsf{B}}(\mathfrak{z})$ then $\langle 1, l_{\mathsf{B}} \rangle(\mathfrak{y}) = \langle 1, l_{\mathsf{B}} \rangle(\mathfrak{z})$, a contradiction to the minimality of $|\mathfrak{y}|$.

Therefore there does not exist any string $\mathfrak{z}$ with $\mathfrak{x}_0 \sqsubseteq_l \mathfrak{z} \sqsubset_l \mathfrak{y}$ and $\theta(\mathfrak{y} \mid \mathfrak{z}) = 1$, which gives $\mathfrak{y} \sqsubseteq_l \mathfrak{m}_i(\mathfrak{x}_0)$.

Finally, since $\mathfrak{y} \sqsubseteq_l \mathfrak{x}$ and $\theta(\langle 1, l_{\mathsf{B}} \rangle(\mathfrak{y}) \mid \mathfrak{x}) = 1$, we have $l_{\mathsf{B}}^n(\mathfrak{y}) = \mathfrak{x}$ by Remark 8.9. This completes the proof. □

The proof of Proposition 11.6 and of its dual can be used to define a further condensation operator

$$C_{\mathsf{B}} : \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B}) \to \overline{\mathsf{St}}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B}).$$

The next result shows that C_{B} is compatible with the partition given by (11.2).

Proposition 11.7 *If* $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ *is minimal for* $(\mathfrak{x}_0, i)$, $[\mathfrak{x}, \mathfrak{y}]$ *is a* B*-beam and* $\mathfrak{z} \in c_{\mathsf{B}}([\mathfrak{x}, \mathfrak{y}])$ *then* $C_{\mathsf{B}}(\mathfrak{z}) \in [\mathfrak{x}, \mathfrak{y}]$.

Proof Without loss of generality, assume that $\mathfrak{z} \in \overline{\mathsf{St}}_1(\mathfrak{x}_0, i; \mathsf{B})$. Then Proposition 11.6 gives some $n \in \mathbb{N}$ such that $l_{\mathsf{B}}^n(C_{\mathsf{B}}(\mathfrak{z})) = \mathfrak{z}$. Let $\mathfrak{w} := C_{\mathsf{B}}(\mathfrak{z})$ for brevity so that $\mathfrak{w} \leq_l \mathfrak{z}$. We claim that $\mathfrak{w} \in [\mathfrak{x}, \mathfrak{y}]$.

Now $\varphi_{\mathsf{B}}(\mathfrak{z}) = 0$ if and only if $\mathfrak{w} = \mathfrak{z}$. On the other hand, if $\varphi_{\mathsf{B}}(\mathfrak{z}) = 1$, then assume the claim fails, i.e., assume $\mathfrak{w} <_l \mathfrak{x} <_l \mathfrak{z}$. Then clearly $\mathfrak{w} \sqsubset_l \mathfrak{z}$. Let $\mathfrak{v} := \mathfrak{w} \sqcap_l \mathfrak{x}$. There are two cases.

If $\mathfrak{v} \sqsubset_l \mathfrak{w}$ then $\mathfrak{w} \sqsubset_l \mathfrak{z}$ gives $\theta(\mathfrak{z} \mid \mathfrak{x}) = \theta(\mathfrak{z} \mid \mathfrak{v}) = -1$, a contradiction to $\mathfrak{x} <_l \mathfrak{z}$.

On the other hand, if $\mathfrak{v} = \mathfrak{w}$ then we have $\mathfrak{w} \sqsubset_l \mathfrak{x}$ and $\mathfrak{w} \sqsubset_l \mathfrak{z}$. Thus $\mathfrak{w} \sqsubset_l \mathfrak{x} \sqcap_l \mathfrak{z}$. If $\mathfrak{x} \sqcap_l \mathfrak{z} = \mathfrak{x}$ then Remark 9.6 yields $l_{\mathsf{B}}^k(\mathfrak{w}) = \mathfrak{x}$ for some $k \in \mathbb{N}^+$, a contradiction to Proposition 9.3 as $\mathfrak{x} \in \overline{\mathsf{St}}_{-1}(\mathfrak{x}_0, i; \mathsf{B})$. Thus $\mathfrak{x} \sqcap_l \mathfrak{z} \sqsubset_l \mathfrak{x}$. If $\mathfrak{x} \sqcap_l \mathfrak{z} = \mathfrak{z}$ then $\mathfrak{z} \sqsubset_l \mathfrak{x}$ together with $\theta(\mathfrak{x} \mid \mathfrak{z}) = -1$ implies that $\mathfrak{z} \in \overline{\mathsf{St}}_{-1}(\mathfrak{x}_0, i; \mathsf{B})$ by Remark 6.3, a contradiction to $\varphi_{\mathsf{B}}(\mathfrak{z}) = 1$. On the other hand, if $\mathfrak{x} \sqcap_l \mathfrak{z} \sqsubset_l \mathfrak{z}$ then $\mathfrak{w} \sqsubset_l \mathfrak{x} \sqcap_l \mathfrak{z}$ together with $\theta(\mathfrak{z} \mid \mathfrak{x} \sqcap_l \mathfrak{z}) = 1$ implies $l_{\mathsf{B}}^k(\mathfrak{w}) = \mathfrak{x} \sqcap_l \mathfrak{z} \in \overline{\mathsf{St}}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B})$ for some $k \in \mathbb{N}^+$ by Remark 9.6, a contradiction to Proposition 9.3.

The definition of a B-beam gives $\mathfrak{w} \in \overline{\mathsf{St}}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B}) \cap [\mathfrak{x}, \mathfrak{y}] = (\mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B}) \cup \{\mathfrak{x}, \mathfrak{y}\}) \cap [\mathfrak{x}, \mathfrak{y}]$.

Dually, we can show that if $\mathfrak{z} \in \overline{\mathsf{St}}_{-1}(\mathfrak{x}_0, i; \mathsf{B})$ we get $\mathfrak{w} \in \overline{\mathsf{St}}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B}) \cap [\mathfrak{x}, \mathfrak{y}] = (\mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B}) \cup \{\mathfrak{x}, \mathfrak{y}\}) \cap [\mathfrak{x}, \mathfrak{y}]$ such that $\bar{l}_{\mathsf{B}}^n(\mathfrak{w}) = \mathfrak{z}$ for some $n \in \mathbb{N}$. ◻

Now let us analyze different summands/intervals on the right hand side of Equation (11.2). The first and the last intervals are finite thanks to Proposition 11.4. Since $c_{\mathsf{B}}([\mathfrak{y}_j, \mathfrak{y}_{j+1}])$ is the condensation of a B-beam, the minimality of B for $(\mathfrak{x}_0, i)$ allows us to use Propositions 11.6 and 11.7 to conclude that given $\mathfrak{z} \in c_{\mathsf{B}}([\mathfrak{y}_j, \mathfrak{y}_{j+1}])$ exactly one of the following is true: $\mathrm{card}([\mathfrak{z}, \mathfrak{z}_0] \cup [\mathfrak{z}_0, \mathfrak{z}])$ is finite for a unique B-center $\mathfrak{z}_0$, $\mathrm{card}([c_{\mathsf{B}}(\mathfrak{y}_j), \mathfrak{z}])$ is finite or $\mathrm{card}([\mathfrak{z}, c_{\mathsf{B}}(\mathfrak{y}_{j+1})])$ is finite. Finally, Corollary 11.5 describes the order type of B-centers in a B-beam–such centers are fixed under c_{B}. Combining this discussion with the results in § 9 about the discreteness of B-condensation of the hammock, we compute its order type in the next result.

Corollary 11.8 *Suppose* $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ *is minimal for* $(\mathfrak{x}_0, i)$. *If* $[\mathfrak{x}, \mathfrak{y}]$ *is a* B-*beam then*

$$(c_{\mathsf{B}}([\mathfrak{x}, \mathfrak{y}]), <_l) \cong \omega + \Xi(\underbrace{\zeta, \zeta, \cdots, \zeta}_{k_{\mathsf{B}}\ \textit{times}}) + \omega^*.$$

As a consequence,

$$(c_{\mathsf{B}}(H_l^i(\mathfrak{x}_0)), <_l) \cong (\omega + \Xi(\underbrace{\zeta, \zeta, \cdots, \zeta}_{k_{\mathsf{B}}\ \textit{times}}) + \omega^*) \cdot \mathbf{n}_{\mathsf{B}}.$$

Recall the definition of $\mathcal{I}_{(\mathfrak{x}_0, i; \mathsf{B})}(\mathfrak{x})$ for $\mathfrak{x} \in \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B})$. If B is minimal for $(\mathfrak{x}_0, i)$, we can extend this definition to all $\mathfrak{x} \in \overline{\mathsf{St}}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B})$ as follows:

$$\mathcal{I}_{(\mathfrak{x}_0, i; \mathsf{B})}(\mathfrak{y}_k) := \begin{cases} [\mathfrak{m}_i(\mathfrak{x}_0), \mathfrak{M}_i(\mathfrak{x}_0)] & \text{if } 0 = k = n_{\mathsf{B}}, \\ [\mathfrak{m}_i(\mathfrak{x}_0), \langle 1, \ell_{\mathsf{B}} \rangle(\mathfrak{y}_0)) & \text{if } 0 = k < n_{\mathsf{B}}, \\ (\langle 1, \bar{\ell}_{\mathsf{B}} \rangle(\mathfrak{y}_0), \mathfrak{M}_i(\mathfrak{x}_0)] & \text{if } 0 < k = n_{\mathsf{B}}, \\ (\langle 1, \bar{\ell}_{\mathsf{B}} \rangle(\mathfrak{y}_k), \langle 1, \ell_{\mathsf{B}} \rangle(\mathfrak{y}_k)) & \text{if } 0 < k < n_{\mathsf{B}}. \end{cases}$$

Let $\mathbf{c}_{\mathsf{B}} := C_{\mathsf{B}} \circ c_{\mathsf{B}}$. It is straightforward to verify that $\mathbf{c}_{\mathsf{B}}^{-1}(\mathfrak{x}) = \mathcal{I}_{(\mathfrak{x}_0, i; \mathsf{B})}(\mathfrak{x})$ for each $\mathfrak{x} \in \overline{\mathsf{St}}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B})$. Thus

$$H_l^i(\mathfrak{x}_0) \cong \sum_{\mathfrak{x} \in \overline{\mathsf{St}}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B})} \mathbf{c}_{\mathsf{B}}^{-1}(\mathfrak{x}) \cong \sum_{\mathfrak{x} \in \overline{\mathsf{St}}_{\pm 1}(\mathfrak{x}_0, i; \mathsf{B})} \mathcal{I}_{(\mathfrak{x}_0, i; \mathsf{B})}(\mathfrak{x}). \tag{11.5}$$

Now we have all the tools necessary for proving the main result of this paper.

Theorem 11.9 *Given a string* $\mathfrak{x}_0$ *and a parity* $i \in \{1, -1\}$, *we have* $(H_l^i(\mathfrak{x}_0), <_l) \in \mathrm{dLO}_{\mathrm{fd}}^{11}$.

Proof The proof is by induction on the size of $\mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0)$, which is a finite poset by Proposition 5.7.

Base Step. If $\mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0) = \emptyset$ then the strings in $H_l^i(\mathfrak{x}_0)$ are band-free relative to $(\mathfrak{x}_0, i)$, implying that $H_l^i(\mathfrak{x}_0)$ is finite in view of Corollary 4.7. Thus $H_l^i(\mathfrak{x}_0) \in \mathrm{dLO}_{\mathrm{fd}}^{11}$.

Inductive Step. Assume that $\mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0) \neq \emptyset$ and for any $\mathfrak{x} \in \mathsf{St}(\Lambda)$ and $j \in \{1, -1\}$ with $\mathrm{card}(\mathcal{Q}_j^{\mathrm{Ba}}(\mathfrak{x})) < \mathrm{card}(\mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0))$, we have $H_l^j(\mathfrak{x}) \in \mathrm{dLO}_{\mathrm{fd}}^{11}$.

Since $\mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0) \neq \emptyset$, choose $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ that is minimal for $(\mathfrak{x}_0, i)$. Then Lemma 7.8 gives

$$(H_l^i(\mathfrak{x}_0), <_l) \cong \sum_{\mathfrak{x} \in c_{\mathsf{B}}(H_l^i(\mathfrak{x}_0))} (H_l^{-\varphi_{\mathsf{B}}(\mathfrak{x})}(\mathfrak{x}), <_l).$$

If $\mathfrak{x} \in c_{\mathsf{B}}(H_l^i(\mathfrak{x}_0)) \subseteq H_l^i(\mathfrak{x}_0)$ then $\mathcal{Q}_{-\varphi_{\mathsf{B}}(\mathfrak{x})}^{\mathrm{Ba}}(\mathfrak{x}) \subseteq \mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0)$. Moreover, for any such $\mathfrak{x}$, we have $\mathsf{B} \notin \mathcal{Q}_{-\varphi_{\mathsf{B}}(\mathfrak{x})}^{\mathrm{Ba}}(\mathfrak{x})$ thanks to Remark 7.5, and hence $\mathcal{Q}_{-\varphi_{\mathsf{B}}(\mathfrak{x})}^{\mathrm{Ba}}(\mathfrak{x}) \subsetneq \mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0)$. Therefore by the induction hypothesis, we get $H_l^{-\varphi_{\mathsf{B}}(\mathfrak{x})}(\mathfrak{x}) \in \mathrm{dLO}_{\mathrm{fd}}^{11}$ for each $\mathfrak{x} \in c_{\mathsf{B}}(H_l^i(\mathfrak{x}_0))$.

In view of Equation (11.2), Proposition 7.6 and the fact that $H_l^{-\varphi_{\mathsf{B}}(\mathfrak{y}_p)}(\mathfrak{y}_p) = H_l^0(\mathfrak{y}_p) = \{\mathfrak{y}_p\}$ for each $0 \leq p \leq n_{\mathsf{B}}$, we can write

$$H_l^i(\mathfrak{x}_0) = \widetilde{L}_0 \dotplus \widetilde{L}_1 \dotplus \cdots \dotplus \widetilde{L}_{n_{\mathsf{B}}+1},$$

where

$$\widetilde{L}_k := \begin{cases} \sum_{\mathfrak{x} \in [\mathfrak{m}_i(\mathfrak{x}_0), \mathfrak{y}_0]} (H_l^{-\varphi_{\mathsf{B}}(\mathfrak{x})}(\mathfrak{x}), <_l) & \text{if } k = 0, \\ \sum_{\mathfrak{x} \in [\mathfrak{y}_{k-1}, \mathfrak{y}_k]} (H_l^{-\varphi_{\mathsf{B}}(\mathfrak{x})}(\mathfrak{x}), <_l) & \text{if } 1 \leq k \leq n_{\mathsf{B}}, \\ \sum_{\mathfrak{x} \in [\mathfrak{y}_{n_{\mathsf{B}}}, \mathfrak{M}_i(\mathfrak{x}_0)]} (H_l^{-\varphi_{\mathsf{B}}(\mathfrak{x})}(\mathfrak{x}), <_l) & \text{if } k = n_{\mathsf{B}} + 1. \end{cases}$$

Since a finite order sum of linear orders in $\mathrm{dLO}_{\mathrm{fd}}^{11}$ lies in $\mathrm{dLO}_{\mathrm{fd}}^{11}$, using the induction hypothesis and Proposition 11.4, we see that $\widetilde{L}_0, \widetilde{L}_{n_{\mathsf{B}}+1} \in \mathrm{dLO}_{\mathrm{fd}}^{11}$. We will use Lemma 2.4 to show that $\widetilde{L}_k \in \mathrm{dLO}_{\mathrm{fd}}^{11}$ for $1 \leq k \leq n_{\mathsf{B}}$.

Proposition 11.7 showed $c_{\mathsf{B}}([\mathfrak{y}_0, \mathfrak{y}_1]) = [\mathfrak{y}_0, \mathfrak{y}_1] \cap \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B}) \cong (\omega + \Xi(\underbrace{\zeta, \zeta, \cdots, \zeta}_{k_{\mathsf{B}} \text{ times}}) + \omega^*)$. Its proof together with Equation (11.5) helps us to write

$$\widetilde{L}_1 = \overline{L}_1 + \overline{L}_2 + \overline{L}_3,$$

where

$$\overline{L}_1 := \mathcal{I}_{(\mathfrak{x}_0, i; \mathsf{B})}(\mathfrak{y}_0) \cap [\mathfrak{y}_0, \mathfrak{y}_1] = \sum_{n \in \omega} H_l^{-\varphi_{\mathsf{B}}(l_{\mathsf{B}}^n(\mathfrak{y}_0))}(l_{\mathsf{B}}^n(\mathfrak{y}_0)),$$

$$\overline{L}_2 := \sum_{\mathfrak{x} \in \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B}) \cap [\mathfrak{y}_0, \mathfrak{y}_1]} \mathcal{I}_{(\mathfrak{x}_0, i; \mathsf{B})}(\mathfrak{x})$$

$$= \sum_{\mathfrak{x} \in \mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B}) \cap [\mathfrak{y}_0, \mathfrak{y}_1]} \left(\sum_{n \in \omega^*,\, n \neq 0} H_l^{-\varphi_{\mathsf{B}}(\bar{l}_{\mathsf{B}}^n(\mathfrak{x}))}(\bar{l}_{\mathsf{B}}^n(\mathfrak{x})) + \sum_{n \in \omega} H_l^{-\varphi_{\mathsf{B}}(l_{\mathsf{B}}^n(\mathfrak{x}))}(l_{\mathsf{B}}^n(\mathfrak{x})) \right),$$

$$\overline{L}_3 := \mathcal{I}_{(\mathfrak{x}_0, i; \mathsf{B})}(\mathfrak{y}_1) \cap [\mathfrak{y}_0, \mathfrak{y}_1] = \sum_{n \in \omega^*} H_l^{-\varphi_{\mathsf{B}}(\bar{l}_{\mathsf{B}}^n(\mathfrak{y}_1))}(\bar{l}_{\mathsf{B}}^n(\mathfrak{y}_1)).$$

Let $\{\mathfrak{x}_1, \cdots, \mathfrak{x}_{k_{\mathsf{B}}}\}$ be a set of representatives of distinct B-equivalence classes of $\mathrm{Cent}(\mathfrak{x}_0, i; \mathsf{B}) \cap [\mathfrak{y}_0, \mathfrak{y}_1]$. Proposition 10.9 states that the order type of the interval $\mathcal{I}_{(\mathfrak{x}_0, i; \mathsf{B})}(\mathfrak{x}_j)$

in $H_l^i(\mathfrak{x}_0)$ is independent of the choice of the representative $\mathfrak{x}_j$ for each $j \in \{1, \cdots, k_{\mathsf{B}}\}$. Thus if L_j denotes the order type of $\mathcal{I}_{(\mathfrak{x}_0,i;\mathsf{B})}(\mathfrak{x}_j)$, then

$$\overline{L}_2 \cong \Xi(L_1, \cdots, L_{k_{\mathsf{B}}}).$$

For any string $\mathfrak{z} \in c_{\mathsf{B}}([\mathfrak{y}_0, \mathfrak{y}_1])$ with $\varphi_{\mathsf{B}}(\mathfrak{z}) \neq -1$, Proposition 9.9 ensures the existence of a string $\mathfrak{u}$ and $\mathfrak{b} \in \mathsf{Ba}_l(\mathsf{B})$ such that $\langle 1, l_{\mathsf{B}}\rangle(\mathfrak{z}) = {}^{\infty}\mathfrak{b}\mathfrak{u}\mathfrak{z}$. Since $\theta(\mathfrak{b}) = 1$, Remark 9.6 yields $s' \in \mathbb{N}$ and $p \in \mathbb{N}^+$ such that $l_{\mathsf{B}}^{s'}(\mathfrak{z}) = \mathfrak{b}\mathfrak{u}\mathfrak{z}$ and $l_{\mathsf{B}}^{s'+p}(\mathfrak{z}) = \mathfrak{b}^2\mathfrak{u}\mathfrak{z} = \mathfrak{b}l_{\mathsf{B}}^{s'}(\mathfrak{z})$. Since $\delta(\mathfrak{b}) = 0$, we have $l_{\mathsf{B}}^{s'+p}(\mathfrak{z}) \equiv_H l_{\mathsf{B}}^{s'}(\mathfrak{z})$. Since $\mathfrak{b} \in \mathsf{St}_1(\mathsf{B})$, Remark 8.9 and Proposition 9.5 together imply $l_{\mathsf{B}}^{s'+p+k}(\mathfrak{z}) \equiv_H l_{\mathsf{B}}^{s'+k}(\mathfrak{z})$ for each $k \in \mathbb{N}$. Since $\varphi_{\mathsf{B}}(l_{\mathsf{B}}^q(\mathfrak{z})) = 1$ for each $q \in \mathbb{N}^+$, we get $H_l^{-\varphi_{\mathsf{B}}(l_{\mathsf{B}}^{s'+p+k}(\mathfrak{z}))}(l_{\mathsf{B}}^{s'+p+k}(\mathfrak{z})) \cong H_l^{-\varphi_{\mathsf{B}}(l_{\mathsf{B}}^{s'+k}(\mathfrak{z}))}(l_{\mathsf{B}}^{s'+k}(\mathfrak{z}))$. A dual result can be shown for $\mathfrak{z} \in c_{\mathsf{B}}([\mathfrak{y}_0, \mathfrak{y}_1])$ with $\varphi_{\mathsf{B}}(\mathfrak{z}) \neq 1$. Thus we have shown that all the hypotheses of Lemma 2.4 are satisfied, and hence we get that $\widetilde{L}_1 = \overline{L}_1 + \overline{L}_2 + \overline{L}_3 \in \mathrm{dLO}^{11}_{\mathrm{fd}}$. A similar argument shows that $\widetilde{L}_k \in \mathrm{dLO}^{11}_{\mathrm{fd}}$ for each $1 \leq k \leq n_{\mathsf{B}}$, and this completes the proof. □

Example 11.10 We compute the order type of $H_l^1(a_0)$ from Example 5.8. Continuing from Example 11.3, recall that B_1 is minimal and domestic for $(a_0, 1)$. By domesticity of B_1, we have $k_{\mathsf{B}_1} = 0$, and thus by Corollary 11.8 we have

$$c_{\mathsf{B}_1}([a_0, a_3A_1a_0]) = \sum_{k\in\omega}\{l_{\mathsf{B}_1}^k(a_0)\} + \sum_{k\in\omega^*}\{\bar{l}_{\mathsf{B}_1}^k(a_3A_1a_0)\} \cong \omega + \omega^*. \tag{11.6}$$

Using Lemma 7.8, we obtain

$$[a_0, a_3A_1a_0] = H_l^0(a_0) + \sum_{k\in\omega,\ k\neq 0} H_l^{-1}(l_{\mathsf{B}_1}^k(a_0)) + \sum_{k\in\omega^*,\ k\neq 0} H_l^1(\bar{l}_{\mathsf{B}_1}^k(a_3A_1a_0)) + H_l^0(a_3A_1a_0). \tag{11.7}$$

Note that $l_{\mathsf{B}_1}^{2k+r}(a_0) \equiv_H l_{\mathsf{B}_1}^{2+r}(a_0)$ for every $k \geq 0$ and $0 \leq r \leq 1$ with $(k, r) \neq (0, 0)$. Moreover, $H_l^1(\bar{l}_{\mathsf{B}_1}^k(a_3A_1a_0)) = \{\bar{l}_{\mathsf{B}_1}^k(a_3A_1a_0)\}$ for every $k \in \omega^*$, $k \neq 0$, and $H_l^{-1}(l_{\mathsf{B}_1}(a_0)) = \{l_{\mathsf{B}_1}(a_0)\}$. Plugging these in Equation (11.7), we get

$$[a_0, a_3A_1a_0] \cong \mathbf{1} + (\mathbf{1} + H_l^{-1}(l_{\mathsf{B}_1}^2(a_0))) \cdot \omega + \omega^*. \tag{11.8}$$

To compute $H_l^{-1}(l_{\mathsf{B}_1}^2(a_0))$, note that non-domestic B_2 is the unique element of $\mathcal{Q}_{-1}^{\mathrm{Ba}}(l_{\mathsf{B}_1}^2(a_0))$, and thus minimal for $(l_{\mathsf{B}_1}^2(a_0), -1)$. Since $k_{\mathsf{B}_2} = 1$, applying Lemma 7.8 to Corollary 11.8 with the help of the base case of the proof of Theorem 11.9, we get $H_l^{-1}(l_{\mathsf{B}_1}^2(a_0)) \cong \omega + \Xi(\zeta) + \omega^*$, so that Equation (11.7) takes the form

$$[a_0, a_3A_1a_0] \cong \mathbf{1} + (\mathbf{1} + \omega + \Xi(\zeta) + \omega^*) \cdot \omega + \omega^* \cong (\omega + \Xi(\zeta) + \omega^*) \cdot \omega + \omega^*. \tag{11.9}$$

Similarly we can obtain

$$[a_3A_1a_0, A_1a_0] \cong (\omega + \Xi(\zeta) + \omega^*) \cdot \omega + \omega^*. \tag{11.10}$$

Again applying Lemma 7.8 to the last term of the right-hand side of Equation (11.4), we obtain

$$\begin{aligned}[A_1a_0, H_1G_1FE_2E_1A_2A_1a_0] = {} & \{A_1a_0\} + \{A_2A_1a_0\} + \{E_1A_2A_1a_0\} + H_l^{-1}(E_2E_1A_2A_1a_0) \\ & + \{FE_2E_1A_2A_1a_0\} + H_l^{-1}(G_1FE_2E_1A_2A_1a_0) + H_l^{-1}(H_1G_1FE_2E_1A_2A_1a_0).\end{aligned} \tag{11.11}$$

To compute $H_l^{-1}(E_2E_1A_2A_1a_0)$, note that B_3 is non-domestic and minimal for $(E_2E_1A_2A_1a_0, -1)$. Recall from Example 10.3 that $k_{\mathsf{B}_3} = 3$. It is easy to verify that

$$\mathcal{I}_{(E_2E_1A_2A_1a_0,-1;\mathsf{B}_3)}(G_1FE_2E_1A_2A_1a_0) \cong \mathcal{I}_{(E_2E_1A_2A_1a_0,-1;\mathsf{B}_3)}(k_1h_2H_1G_1FE_2E_1A_2A_1a_0) \cong \zeta,$$
$$\mathcal{I}_{(E_2E_1A_2A_1a_0,-1;\mathsf{B}_3)}(E_2E_1e_3E_2E_1A_2A_1a_0) \cong (\omega^* + (\omega+\omega^*)\cdot\omega). \tag{11.12}$$

Using Equations (11.12), Corollary 11.8, and Lemma 7.8, we obtain

$$\begin{aligned} H_l^{-1}(E_2E_1A_2A_1a_0) &\cong \omega + \Xi(\zeta,\zeta,\omega^* + (\omega+\omega^*)\cdot\omega) + \omega^*,\\ H_l^{-1}(G_1FE_2E_1A_2A_1a_0) &\cong (\omega+\omega^*)\cdot\omega + \Xi(\zeta,\zeta,\omega^* + (\omega+\omega^*)\cdot\omega) + \omega^*,\\ H_l^{-1}(H_1G_1FE_2E_1A_2A_1a_0) &\cong \omega + \Xi(\zeta,\zeta,\omega^* + (\omega+\omega^*)\cdot\omega) + \omega^*. \end{aligned} \tag{11.13}$$

Plugging Equations (11.13) in Equation (11.11) while using Equation (2.1) we get

$$\begin{aligned} [A_1a_0, H_1G_1FE_2E_1A_2A_1a_0] \cong\ & \mathbf{3} + \omega + \Xi(\zeta,\zeta,\omega^* + (\omega+\omega^*)\cdot\omega) + \omega^*\\ & + \mathbf{1} + (\omega+\omega^*)\cdot\omega + \Xi(\zeta,\zeta,\omega^* + (\omega+\omega^*)\cdot\omega) + \omega^*\\ & + \omega + \Xi(\zeta,\zeta,\omega^* + (\omega+\omega^*)\cdot\omega) + \omega^*\\ \cong\ & \omega + \Xi(\zeta,\zeta,\omega^* + (\omega+\omega^*)\cdot\omega) + \omega^*. \end{aligned} \tag{11.14}$$

Plugging Equations (11.9), (11.10) and (11.14) in Equation (11.3), we obtain

$$H_l^1(a_0) \cong ((\omega + \Xi(\zeta) + \omega^*)\cdot\omega + \omega^*)\cdot\mathbf{2} + \omega + \Xi(\zeta,\zeta,\omega^* + (\omega+\omega^*)\cdot\omega) + \omega^*. \tag{11.15}$$

Since Theorem 11.9 generalizes the backward direction [11, Theorem 12.15] and the latter has a converse for linear orders in $\mathrm{dLO}_{\mathrm{fp}}^{11}$, it is natural to ask if the converse to the former is true. Proposition 11.12 proves a special case of the converse for which the next result is essential.

Proposition 11.11 *Suppose* $L(\neq \mathbf{0}) \in \mathrm{dLO}_{\mathrm{fp}}$. *Then*

- *if* $L \in \mathrm{dLO}_{\mathrm{fp}}^{01}$, *then there exist* $L_1 \in \mathrm{dLO}_{\mathrm{fp}}^{11}$ *and* $L_2 \in \mathrm{dLO}_{\mathrm{fp}}^{11} \cup \{\mathbf{0}\}$ *such that* $L \cong L_1 \cdot \omega^* + L_2$;
- *if* $L \in \mathrm{dLO}_{\mathrm{fp}}^{10}$, *then there exist* $L_2 \in \mathrm{dLO}_{\mathrm{fp}}^{11}$ *and* $L_1 \in \mathrm{dLO}_{\mathrm{fp}}^{11} \cup \{\mathbf{0}\}$ *such that* $L \cong L_1 + L_2 \cdot \omega$; *and*
- *if* $L \in \mathrm{dLO}_{\mathrm{fp}}^{00}$, *then there are* $L_1, L_3 \in \mathrm{dLO}_{\mathrm{fp}}^{11}$ *and* $L_2 \in \mathrm{dLO}_{\mathrm{fp}}^{11} \cup \{\mathbf{0}\}$ *such that* $L \cong L_1 \cdot \omega^* + L_2 + L_3 \cdot \omega$.

Proof We use the notations and results from [1] to prove the first result; the proofs of the rest are similar.

Recall from [1, Proposition 5.6] that for any $L \in \mathrm{LO}_{\mathrm{fp}}$ there is $(T, s_T) \in 3\mathbf{ST}_\omega$ such that $\mathrm{LIN}(T, s_T) \cong L$. Suppose for $n \in \mathbb{N}^+$ the notation 0^n denotes $\underbrace{00\cdots0}_{n \text{ times}}$. It is easy to note that L has a minimum if and only if whenever $0^n \in T$ for some $n \in \mathbb{N}^+$ then $s_T(0^n) \neq -$.

Now if $L \in \mathrm{dLO}_{\mathrm{fp}}^{01}$ then choose the least $N(T) \in \mathbb{N}^+$ such that $0^{N(T)} \in T$ and $s_T(0^{N(T)}) = -$. We use induction on $N(T)$ to obtain required L_1 and L_2.

Base step ($N(T) = 1$): Let w be the width of (T, s_T). Then $L \cong L_1' \cdot \omega^* + L_2'$, where $L_1' := \mathrm{LIN}(\widehat{T}_0, s_{\widehat{T}_0})$ and $L_2' := \sum_{1\leq k<w} \mathrm{LIN}(T_k, s_{T_k})$. If $L_2' \neq \mathbf{0}$ then it has a maximum.

If L'_1 has a maximum then by discreteness of L, it also has a minimum. Furthermore, if $L'_2 \neq \mathbf{0}$ then it also has a minimum. Thus irrespective of whether $L'_2 = \mathbf{0}$ or not, we can choose $L_1 := L'_1$ and $L_2 := L'_2$.

On the other hand, if $L'_2 \neq \mathbf{0}$ and L'_1 does not have a maximum then let $x \in L'_1$ be any element. Since $L'_1 \cdot \omega^*$ is discrete, x has an immediate successor, say y. Thus we can write $L'_1 = L'_{11} + L'_{12}$, where x is the maximum of L'_{11} and y is the minimum of L'_{12}. Then $L \cong L'_1 \cdot \omega^* + L_2 \cong (L'_{12} + L'_{11}) \cdot \omega^* + (L'_{12} + L'_2)$, so that $L_1 := L'_{12} + L'_{11}$ and $L_2 := L'_{12} + L'_2$ are as required.

Inductive step $(N(T) > 1)$: Here $s_T(0^{N(T)-1}) = +$. Note that $N(\mathsf{EXUDE}((T, s_T); 0^{N(T)-1})) < N(T)$. Moreover, recall from [1, Proposition 6.5] that $\mathrm{LIN}(\mathsf{EXUDE}((T, s_T); 0^{N(T)-1})) \cong L$. Thus the induction hypothesis applied to $\mathsf{EXUDE}((T, s_T); 0^{N(T)-1})$ produces the required orders L_1 and L_2. □

Proposition 11.12 *If $L_0 \in \mathrm{dLO}^{10}_{\mathrm{fp}}$, $L_1 \in \mathrm{dLO}^{00}_{\mathrm{fp}}$ and $L_2 \in \mathrm{dLO}^{01}_{\mathrm{fp}}$ then there is a non-domestic string algebra Λ, a string $\mathfrak{x}_0$ for Λ and a parity $i \in \{1, -1\}$ such that $(H^i_l(\mathfrak{x}_0), <_l) \cong L_0 + \Xi(L_1) + L_2$.*

Proof Proposition 11.11 yields $L_{00}, L_{01}, L_{10}, L_{11}, L_{12}, L_{20}, L_{21} \in \mathrm{dLO}^{11}_{\mathrm{fp}}$ such that

$$L_0 = L_{00} + L_{01} \cdot \omega,\ L_1 = L_{11} \cdot \omega^* + L_{10} + L_{12} \cdot \omega,\ \text{and } L_2 = L_{21} \cdot \omega^* + L_{20}.$$

Consider the string algebra Γ''' from Fig. 5, where the forward direction of [11, Theorem 12.15] allows us to choose quivers with relations (Q_{00}, ρ_{00}), (Q_{01}, ρ_{01}), (Q_{20}, ρ_{20}), (Q_{21}, ρ_{21}), (Q_{10}, ρ_{10}), (Q_{10}, ρ_{10}), (Q_{11}, ρ_{11}) and (Q_{12}, ρ_{12}) for domestic gentle algebras such that

$$\begin{gathered} H_l^{-1}(a) \cong L_{00},\ H_l^{-1}(l_1) \cong L_{01}, \\ H_l^{-1}(q) \cong L_{10},\ H_l^{1}(R) \cong L_{11},\ H_l^{-1}(y) \cong L_{12}, \\ H_l^{1}(B) \cong L_{20}, \text{ and } H_l^{1}(j_2) \cong L_{21}. \end{gathered} \tag{11.16}$$

Since the algebras presented by (Q_{mn}, ρ_{mn}) are gentle and ρ for Γ''' consists only of paths of length 2, we conclude that Γ''' is a gentle algebra.

We will show for an appropriate $j \in \{-1, 1\}$ that $H_l^{-1}(1_{(v_0,j)}) \cong L_0 + \Xi(L_1) + L_2$.

Since Γ''' is gentle, for any string $\mathfrak{u} \in H_l^{-1}(1_{(v_0,j)})$ with $|\mathfrak{u}| > 0$, we have $\mathfrak{u} \equiv_H \alpha$, where α is the last syllable of $\mathfrak{u}$. Consequently, $H_l(\mathfrak{u}) \cong H_l(\alpha)$.

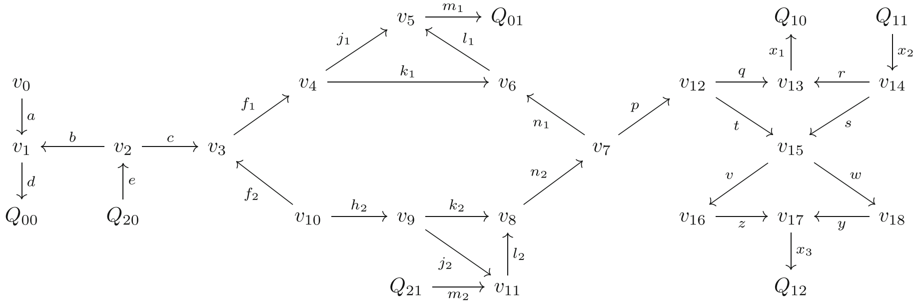

Fig. 5 Γ''' with $\rho = \{db, ce, f_1f_2, j_1f_1, l_1n_1, n_1n_2, m_1j_1, k_2h_2, l_2m_2, n_2k_2, tp, x_1r, sx_2, wt, vs, x_3z\}$

We have that $\mathsf{B}_1 = \{l_1 k_1 J_1\}$ and $\mathsf{B}_2 = \{k_2 J_2 L_2\}$ are minimal for $(1_{(v_0,j)}, -1)$. Choosing B_1 as minimal for $(1_{(v_0,j)}, -1)$, we have

$$H_l^{-1}(1_{(v_0,j)}) = [\mathfrak{m}_{-1}(1_{(v_0,j)}), a] \dot{+} [a, (1_{(v_0,j)})] \cong H_l^{-1}(a) \dot{+} [a, (1_{(v_0,j)})] \cong L_{00} \dot{+} [a, (1_{(v_0,j)})]. \tag{11.17}$$

Since B_1 is domestic, using Corollary 11.8, we have

$$c_{\mathsf{B}_1}([a, 1_{(v_0,j)}]) = \sum_{k\in\omega} \{l^k_{\mathsf{B}_1}(a)\} + \sum_{k\in\omega^*} \{\bar{l}^k_{\mathsf{B}_1}(1_{v_0,j})\} \cong \omega + \omega^*. \tag{11.18}$$

Applying Lemma 7.8 to the above equation,

$$[a, 1_{(v_0,j)}] \cong \sum_{k\in\omega, k\neq 0} H_l^{-1}(l^k_{\mathsf{B}_1}(a)) + \sum_{k\in\omega^*, k\neq 0} H_l^1(\bar{l}^k_{\mathsf{B}_1}(1_{v_0,j})). \tag{11.19}$$

Using appropriate H-equivalences, we obtain

$$\begin{aligned}[a, 1_{(v_0,j)}] &\cong \{j_1 f_1 c B a\} + H_l^{-1}(l_1)\cdot\omega + H_l^1(k_1)\cdot\omega^* + H_l^1(c) + H_l^1(B) + \{a\} \\ &\cong L_{01}\cdot\omega + H_l(N_1)\cdot\omega^* + H_l^1(c) + L_{20} + \mathbf{1}.\end{aligned} \tag{11.20}$$

To compute $H_l^1(c)$, note that B_2 is minimal for $(c, 1)$. Using appropriate H-equivalences, we have

$$H_l^1(c) \cong \{cBa\} + H_l^{-1}(l_2)\cdot\omega + H_l^1(j_2)\cdot\omega^* \cong \mathbf{1} + H_l(n_2)\cdot\omega + L_{21}\cdot\omega^*. \tag{11.21}$$

Note that $N_1 \equiv_H n_1$ and therefore $H_l(N_1) \cong H_l(n_2) \cong H_l^{-1}(N_1)$, which we now compute.

Let $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}_{-1}(1_{(v_0,j)})$ be the only non-domestic element with $\mathsf{Ba}_l(\mathsf{B}) = \{ywVZ\}$ and $\mathsf{Ba}_{\bar{l}}(\mathsf{B}) = \{qTsR\}$. Since $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ is minimal for $(N_1, -1)$ and $k_{\mathsf{B}} = 1$ with $\mathfrak{z} := sRqpN_1$ as a representative for the unique B-equivalence class, it is easy to verify that

$$\mathcal{I}_{(N_1,-1;\mathsf{B})}(\mathfrak{z}) \cong H_l^1(R)\cdot\omega^* + H_l^{-1}(q) + H_l^{-1}(y)\cdot\omega \cong L_{11}\cdot\omega^* + L_{10} + L_{12}\cdot\omega. \tag{11.22}$$

Using Corollary 11.8 and Lemma 7.8, we get

$$\begin{aligned}H_l(N_1) &\cong H_l^{-1}(q) + H_l^{-1}(y)\cdot\omega + \Xi(\mathcal{I}_{(N_1,-1;\mathsf{B})}(\mathfrak{z})) + H_l^1(r)\cdot\omega^* \\ &\cong L_{10} + L_{12}\cdot\omega + \Xi(L_{11}\cdot\omega^* + L_{10} + L_{12}\cdot\omega) + L_{11}\cdot\omega^*.\end{aligned} \tag{11.23}$$

Using the isomorphism $H_l(N_1) \cong H_l(n_1)$ and Equation 2.2 while plugging Equations (11.20), (11.21) and (11.23) in Equation (11.17), we get

$$\begin{aligned}H_l(1_{(v_0,j)}) &\cong L_{00} + L_{01}\cdot\omega + H_l(N_1)\cdot\omega^* + \mathbf{1} + H_l(n_2)\cdot\omega + L_{21}\cdot\omega^* + L_{20} + \mathbf{1} \\ &\cong L_{00} + L_{01}\cdot\omega + H_l(N_1)\cdot(\omega^* + \omega) + L_{21}\cdot\omega^* + L_{20} \\ &\cong L_0 + (L_{10} + L_{12}\cdot\omega + \Xi(L_{11}\cdot\omega^* + L_{10} + L_{12}\cdot\omega) + L_{11}\cdot\omega^*)\cdot(\omega^* + \omega) + L_2 \\ &\cong L_0 + \Xi(L_{11}\cdot\omega^* + L_{10} + L_{12}\cdot\omega) + L_2 \\ &\cong L_0 + \Xi(L_1) + L_2.\end{aligned} \tag{11.24}$$

This completes the proof. □

Question 11.13 Does there exist a non-domestic string algebra Λ, a string $\mathfrak{x}_0$ for Λ and a parity $i \in \{1, -1\}$ such that $(H_l^i(\mathfrak{x}_0), <_l) \cong \omega + \Xi((\omega+\omega^*)\cdot\omega^* + \omega, \omega^* + (\omega+\omega^*)\cdot\omega) + \omega^*$?

We believe that the answer to the above question is negative; however, currently, we do not have any methods to show this. In fact, it is an interesting problem to determine the subclass of $\mathrm{dLO}^{11}_{\mathrm{fd}}$ which consists only of the order types of hammocks for string algebras.

12 Locating left $\mathbb{N}$-strings in the Completion of a Hammock

This last section is devoted to computing the order type of the completion of the B-condensation of a hammock and describing the location of some left $\mathbb{N}$-strings therein. As a consequence, we characterize (Proposition 12.7) some almost periodic left $\mathbb{N}$-strings in terms of scattered subintervals of the condensations.

Given $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ we identify two subsets of $\mathbb{N}\text{-}\mathsf{St}(\Lambda)$ associated to B.

$$\mathbb{N}\text{-}\mathsf{St}(\mathsf{B}) := \{\mathfrak{x} \in \mathbb{N}\text{-}\mathsf{St}(\Lambda) \mid \text{all but finitely many left substrings of } \mathfrak{x} \text{ are in } \mathsf{St}(\mathsf{B})\}.$$

$$\mathbb{N}\text{-}\overline{\mathsf{St}}(\mathsf{B}) := \{\mathfrak{x} \in \mathbb{N}\text{-}\mathsf{St}(\Lambda) \mid \text{all proper left substrings are in } \overline{\mathsf{St}}(\mathsf{B})\}.$$

Further set $\mathbb{N}\text{-}\mathsf{St}(\mathfrak{x}_0, i; \mathsf{B}) := \mathbb{N}\text{-}\mathsf{St}(\mathsf{B}) \cap \widehat{H}_l^i(\mathfrak{x}_0)$ and $\mathbb{N}\text{-}\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B}) := \mathbb{N}\text{-}\overline{\mathsf{St}}(\mathsf{B}) \cap \widehat{H}_l^i(\mathfrak{x}_0)$.

The following remarks are straightforward.

Remark 12.1 Note that $\mathbb{N}\text{-}\mathsf{St}(\mathsf{B}) \subseteq \mathbb{N}\text{-}\overline{\mathsf{St}}(\mathsf{B})$, and therefore $\mathbb{N}\text{-}\mathsf{St}(\mathfrak{x}_0, i; \mathsf{B}) \subseteq \mathbb{N}\text{-}\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$. Moreover, it follows from Corollary 6.7 that $\mathbb{N}\text{-}\mathsf{St}(\mathfrak{x}_0, i; \mathsf{B}) = \mathbb{N}\text{-}\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$ if and only if B is minimal for $(\mathfrak{x}_0, i)$.

Remark 12.2 If $\mathfrak{x} \in H_l^i(\mathfrak{x}_0) \setminus \{\mathfrak{x}_0\}$ and $\mathsf{B} \in \mathcal{Q}_j^{\mathrm{Ba}}(\mathfrak{x})$ then for any $j \in \{-1, 1\}$, we have $\mathbb{N}\text{-}\mathsf{St}(\mathfrak{x}, j; \mathsf{B}) \subseteq \mathbb{N}\text{-}\mathsf{St}(\mathfrak{x}_0, i; \mathsf{B})$ and $\mathbb{N}\text{-}\overline{\mathsf{St}}(\mathfrak{x}, j; \mathsf{B}) \subseteq \mathbb{N}\text{-}\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$.

Now we show that every left $\mathbb{N}$-string in $\widehat{H}_l^i(\mathfrak{x}_0)$ lies in $\mathbb{N}\text{-}\mathsf{St}(\mathfrak{x}_0, i; \mathsf{B})$ for some $\mathsf{B} \in \mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0)$.

Proposition 12.3 *Given* $\mathfrak{x} \in \widehat{H}_l^i(\mathfrak{x}_0) \setminus H_l^i(\mathfrak{x}_0)$*, there exists* $\mathfrak{z} \in H_l^i(\mathfrak{x}_0)$ *with* $\mathfrak{z} \sqsubset_l \mathfrak{x}$ *and* $\mathsf{B} \in \mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0)$ *minimal for* $(\mathfrak{z}, \theta(\mathfrak{x} \mid \mathfrak{z}))$ *such that* $\mathfrak{x} \in \mathbb{N}\text{-}\mathsf{St}(\mathfrak{z}, \theta(\mathfrak{x} \mid \mathfrak{z}); \mathsf{B})$.

Proof Since $\mathfrak{x}_0 \sqsubset_l \mathfrak{x}$ and $\mathfrak{x}$ is a left $\mathbb{N}$-string, there are infinitely many strings $\mathfrak{v}$ such that $\mathfrak{v}\mathfrak{x}_0$ is a string. Therefore by the observation in the base case of the proof of Theorem 11.9, we have $\mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0) \neq \emptyset$.

It is trivial to note that if $\mathfrak{v} \sqsubset_l \mathfrak{v}'$ then $\mathcal{Q}_j^{\mathrm{Ba}}(\mathfrak{v}') \subseteq \mathcal{Q}_{\theta(\mathfrak{v}'|\mathfrak{v})}^{\mathrm{Ba}}(\mathfrak{v})$ for each $j \in \{-1, 1\}$. For each $n \in \mathbb{N}$, let $\mathfrak{x}_n \sqsubset_l \mathfrak{x}$ satisfy $|\mathfrak{x}_n| = |\mathfrak{x}_0| + n$. Therefore the sequence of sets $\mathcal{B}_n := \mathcal{Q}_{\theta(\mathfrak{x}|\mathfrak{x}_n)}^{\mathrm{Ba}}(\mathfrak{x}_n)$ satisfies $\mathcal{B}_{n+1} \subseteq \mathcal{B}_n$ for every $n \in \mathbb{N}$. Since $\mathcal{B}_0$ is finite, thanks to Proposition 5.7, we get that there is $N \in \mathbb{N}$ such that $\mathcal{B}_n = \mathcal{B}_N$ for every $n \geq N$. Choose a minimal element B of $\mathcal{B}_N$ with respect to $\preceq$. This implies that B is minimal for $(\mathfrak{x}_N, \theta(\mathfrak{x} \mid \mathfrak{x}_N))$ and $\mathfrak{x}_n \in \overline{\mathsf{St}}(\mathfrak{x}_N, \theta(\mathfrak{x} \mid \mathfrak{x}_N); \mathsf{B})$ for every $n \geq N$. Thus $\mathfrak{x} \in \mathbb{N}\text{-}\overline{\mathsf{St}}(\mathfrak{x}_N, \theta(\mathfrak{x} \mid \mathfrak{x}_N); \mathsf{B})$. Finally, since B is minimal for $(\mathfrak{x}_N, \theta(\mathfrak{x} \mid \mathfrak{x}_N))$, it follows from Remark 12.1 that $\mathfrak{x} \in \mathbb{N}\text{-}\mathsf{St}(\mathfrak{x}_N, \theta(\mathfrak{x} \mid \mathfrak{x}_N); \mathsf{B})$. □

Recall from Proposition 4.11 that $\mathcal{G}(H_l^i(\mathfrak{x}_0)) \cong \widehat{H}_l^i(\mathfrak{x}_0) \setminus H_l^i(\mathfrak{x}_0)$. Proposition 12.3 together with Remark 12.2 allows us to describe the latter set in two possible ways.

$$\widehat{H}_l^i(\mathfrak{x}_0) \setminus H_l^i(\mathfrak{x}_0) = \bigcup_{\substack{\mathfrak{x} \in H_l^i(\mathfrak{x}_0) \setminus \{\mathfrak{x}_0\},\ j \in \{-1,1\}, \\ \mathsf{B} \text{ is minimal for } (\mathfrak{x}, j) \\ j = i \text{ if } \mathfrak{x} = \mathfrak{x}_0}} \mathbb{N}\text{-}\mathsf{St}(\mathfrak{x}, j; \mathsf{B}), \tag{12.1}$$

$$\widehat{H}_l^i(\mathfrak{x}_0) \setminus H_l^i(\mathfrak{x}_0) = \bigsqcup_{\mathsf{B} \in \mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0)} \mathbb{N}\text{-}\mathsf{St}(\mathfrak{x}_0, i; \mathsf{B}). \tag{12.2}$$

Equation (12.1) will be used to locate the position of a left $\mathbb{N}$-string in the extended hammock $\widehat{H}_l^i(\mathfrak{x}_0)$, whereas Equation (12.2) is a generalization of [7, Proposition 2] which states that every left $\mathbb{N}$-string in a domestic string algebra is almost periodic.

As a consequence of Proposition 12.3, we show that each gap in the hammock $H_l^i(\mathfrak{x}_0)$ corresponds to a gap in $\overline{\mathsf{St}}(\mathfrak{x}, j; \mathsf{B})$ for some $\mathfrak{x} \in H_l^i(\mathfrak{x}_0)$, a parity $j \in \{-1, 1\}$ and $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ minimal for $(\mathfrak{x}, j)$.

Proposition 12.4 *If* (X, Y) *is a gap in* $H_l^i(\mathfrak{x}_0)$ *then there is a string* $\mathfrak{x} \in H_l^i(\mathfrak{x}_0)$*, a parity* $j \in \{-1, 1\}$ *and* $B \in \mathcal{Q}^{\mathrm{Ba}}$ *minimal for* $(\mathfrak{x}, j)$ *such that* $(X \cap \overline{\mathsf{St}}(\mathfrak{x}, j; \mathsf{B}), Y \cap \overline{\mathsf{St}}(\mathfrak{x}, j; \mathsf{B}))$ *is a gap in* $\overline{\mathsf{St}}(\mathfrak{x}, j; \mathsf{B})$.

Proof The gap (X, Y) in $H_l^i(\mathfrak{x}_0)$ corresponds to a unique $\mathfrak{y} \in \widehat{H}_l^i(\mathfrak{x}_0) \setminus H_l^i(\mathfrak{x}_0)$ by Proposition 4.11. Further Proposition 12.3 yields $\mathfrak{x} \in H_l^i(\mathfrak{x}_0)$ such that $\mathfrak{x} \sqsubset_l \mathfrak{y}$ and B minimal for $(\mathfrak{x}, \theta(\mathfrak{y} \mid \mathfrak{x}))$ and $\mathfrak{x} \in \mathbb{N}\text{-}\mathsf{St}(\mathfrak{x}, \theta(\mathfrak{y} \mid \mathfrak{x}); \mathsf{B})$.

Let $\mathfrak{y} =: \mathfrak{z}\mathfrak{x}$ and $j := \theta(\mathfrak{y} \mid \mathfrak{x})$. Since the set $\{\mathfrak{v} \in \mathsf{St}(\Lambda) : \delta(\mathfrak{v}) \neq 0\}$ is finite, there are infinitely many inverse as well as direct syllables in $\mathfrak{z}$. It is easy to see that the set $\{\mathfrak{v}\mathfrak{x} \in \mathsf{St}(\Lambda) : \mathfrak{v}\mathfrak{x} \sqsubset_l \mathfrak{y}, \theta(\mathfrak{y} \mid \mathfrak{v}\mathfrak{x}) = 1\}$ is a cofinal subset of $X \cap \overline{\mathsf{St}}(\mathfrak{x}, j; \mathsf{B})$ having no maximum element and the set $\{\mathfrak{v}\mathfrak{x} \in \mathsf{St}(\Lambda) : \mathfrak{v}\mathfrak{x} \sqsubset_l \mathfrak{y}, \theta(\mathfrak{y} \mid \mathfrak{v}\mathfrak{x}) = -1\}$ is a coinitial subset of $Y \cap \overline{\mathsf{St}}(\mathfrak{x}, j; \mathsf{B})$ having no minimum element. Therefore we conclude that $(X \cap \overline{\mathsf{St}}(\mathfrak{x}, j; \mathsf{B}), Y \cap \overline{\mathsf{St}}(\mathfrak{x}, j; \mathsf{B}))$ is a gap in $\overline{\mathsf{St}}(\mathfrak{x}, j; \mathsf{B})$. □

Conversely we show that each gap of $\overline{\mathsf{St}}(\mathfrak{x}, j; \mathsf{B})$ corresponds to a gap in $H_l^i(\mathfrak{x}_0)$, where $\mathfrak{x} \in H_l^i(\mathfrak{x}_0)$, $j \in \{-1, 1\}$ and $\mathsf{B} \in \mathcal{Q}_j^{\mathrm{Ba}}(\mathfrak{x})$, with the restriction that $j = i$ if $\mathfrak{x} = \mathfrak{x}_0$. Note that we do not require B to be minimal for $(\mathfrak{x}, j)$.

Proposition 12.5 *Let* $\mathfrak{x} \in H_l^i(\mathfrak{x}_0)$*,* $j \in \{-1, 1\}$ *and* $\mathsf{B} \in \mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0)$*. If* (X, Y) *is a gap in* $\overline{\mathsf{St}}(\mathfrak{x}, j; \mathsf{B})$ *then there exists a unique* $X' \supseteq X$ *and a unique* $Y' \supseteq Y$ *such that* (X', Y') *is a gap in* $H_l^i(\mathfrak{x}_0)$.

Proof In view of Proposition 4.11, it suffices to show that there is a unique left $\mathbb{N}$-string $\mathfrak{y}$ such that $X \subseteq \{\mathfrak{v} \in H_l^i(\mathfrak{x}_0) \mid \mathfrak{v} <_l \mathfrak{y}\}$ and $Y \subseteq \{\mathfrak{v} \in H_l^i(\mathfrak{x}_0) \mid \mathfrak{y} <_l \mathfrak{v}\}$.

The technique to get a left $\mathbb{N}$-string $\mathfrak{y}$ by "filling up" the gap (X, Y) in $\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$ is similar to the proof of the converse part of Proposition 4.11, keeping in mind that $\overline{\mathsf{St}}(\mathfrak{x}, j; \mathsf{B})$ is closed under substrings in $H_l^j(\mathfrak{x})$, thanks to Remark 6.3. The construction of $\mathfrak{y}$ ensures that $\mathfrak{y} \in \mathbb{N}\text{-}\overline{\mathsf{St}}(\mathfrak{x}, j; \mathsf{B})$.

If $\mathfrak{y}_1$ and $\mathfrak{y}_2$ are two distinct left $\mathbb{N}$-strings in $\mathbb{N}\text{-}\overline{\mathsf{St}}(\mathfrak{x}, j; \mathsf{B})$ then the string $\mathfrak{y}_3 := \mathfrak{y}_1 \sqcap_l \mathfrak{y}_2$ lies in $\overline{\mathsf{St}}(\mathfrak{x}, j; \mathsf{B})$ and between $\mathfrak{y}_1$ and $\mathfrak{y}_2$ in $(\widehat{H}_l^i(\mathfrak{x}_0), <_l)$. Therefore $\mathfrak{y}_1$ and $\mathfrak{y}_2$ cannot correspond to the same gap in $\overline{\mathsf{St}}(\mathfrak{x}, j; \mathsf{B})$, thus proving the uniqueness of $\mathfrak{y}$. □

Note that the left $\mathbb{N}$-string produced in Proposition 12.5 corresponding to a gap in $\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$ lies in $\mathbb{N}\text{-}\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$. Conversely, given $\mathfrak{x} \in \mathbb{N}\text{-}\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$, the technique used in Proposition 12.4 produces a gap in $\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$.

Corollary 12.6 *Suppose* $\mathsf{B} \in \mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0)$*. Then* $\mathcal{C}(\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})) \cong \mathbb{N}\text{-}\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B}) \sqcup \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$*. In particular, if* B *is minimal for* $(\mathfrak{x}_0, i)$ *then* $\mathcal{C}(\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})) \cong \mathbb{N}\text{-}\mathsf{St}(\mathfrak{x}_0, i; \mathsf{B}) \sqcup \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$ *thanks to Remark 12.1.*

As a consequence of the above corollary, the map $c_{\mathsf{B}} : H_l^i(\mathfrak{x}_0) \to \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$ can be extended to a map $\widehat{H}_l^i(\mathfrak{x}_0) \to \mathbb{N}\text{-}\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B}) \sqcup \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$, which we again denote by c_{B}, where $c_{\mathsf{B}}(\mathfrak{x})$ is the longest left (possibly left $\mathbb{N}$-) substring of $\mathfrak{x}$ that lies in $\mathbb{N}\text{-}\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B}) \sqcup \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$. As a consequence, $\mathcal{C}(\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B}))$ is a condensation of $\mathcal{C}(H_l^i(\mathfrak{x}_0))$ via the composition

$$\mathcal{C}(H_l^i(\mathfrak{x}_0)) \xrightarrow{\cong} \widehat{H}_l^i(\mathfrak{x}_0) \xrightarrow{c_{\mathsf{B}}} \mathbb{N}\text{-}\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B}) \sqcup \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B}) \xrightarrow{\cong} \mathcal{C}(\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})).$$

Propositions 12.4 and 12.5 yield bijections

$$\mathcal{C}(H_l^i(\mathfrak{x}_0)) \cong \bigcup_{\substack{\mathfrak{x}\in H_l^i(\mathfrak{x}_0),\ j\in\{-1,1\},\\ \mathsf{B}\in\mathcal{Q}_j^{\mathrm{Ba}}(\mathfrak{x}),\\ j=i \text{ if } \mathfrak{x}=\mathfrak{x}_0}} \mathcal{C}(\overline{\mathsf{St}}(\mathfrak{x}, j; \mathsf{B})) \cong \bigcup_{\substack{\mathfrak{x}\in H_l^i(\mathfrak{x}_0),\ j\in\{-1,1\},\\ \mathsf{B} \text{ is minimal for } (\mathfrak{x},j),\\ j=i \text{ if } \mathfrak{x}=\mathfrak{x}_0}} \mathcal{C}(\overline{\mathsf{St}}(\mathfrak{x}, j; \mathsf{B})). \quad (12.3)$$

Equation (12.3) shows that it is sufficient to study the order type of $\mathcal{C}(\overline{\mathsf{St}}(\mathfrak{x}, j; \mathsf{B}))$, where $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ minimal for $(\mathfrak{x}, j)$ to understand the position of left $\mathbb{N}$-strings (or equivalently gaps in $H_l^i(\mathfrak{x}_0)$) in the extended hammock $\widehat{H}_l^i(\mathfrak{x}_0)$. Henceforth we study the order type of $\mathcal{C}(\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B}))$, where $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ is minimal for $(\mathfrak{x}_0, i)$, and subsequently the positions of the left $\mathbb{N}$-strings in it.

The following result is useful in determining the position of an almost periodic left $\mathbb{N}$-string of a certain form in the extended hammock $\widehat{H}_l^i(\mathfrak{x}_0)$.

Proposition 12.7 *Suppose* $\mathsf{B} \in \mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0)$ *and* $\mathfrak{y} \in \mathbb{N}\text{-}\mathsf{St}(\mathfrak{x}_0, i; \mathsf{B})$. *Then the following statements are equivalent.*

(1) There exists $\mathfrak{b} \in \mathsf{Ba}_l(\mathsf{B})$ *and a string* $\mathfrak{u}$ *such that* $\mathfrak{y} = {}^{\infty}\mathfrak{b}\mathfrak{u}\mathfrak{x}_0$.
(2) There exists $\mathfrak{x} \in H_l^i(\mathfrak{x}_0)$ *with* $\mathfrak{x} <_l \mathfrak{y}$ *such that* $c_{\mathsf{B}}([\mathfrak{x}, \mathfrak{y})) \cong \omega$.

Proof By Proposition 12.3, there exists $\mathfrak{y} \sqsupset_l \mathfrak{w} \in H_l^i(\mathfrak{x}_0)$ such that B is minimal for $(\mathfrak{w}, \theta(\mathfrak{y} \mid \mathfrak{w}))$ and $\mathfrak{y} \in \mathbb{N}\text{-}\mathsf{St}(\mathfrak{w}, \theta(\mathfrak{y} \mid \mathfrak{w}); \mathsf{B})$.

(1) $\Rightarrow$ (2). Since $\mathfrak{y} = {}^{\infty}\mathfrak{b}\mathfrak{u}\mathfrak{x}_0$, there exists $N \in \mathbb{N}^+$ such that $\mathfrak{w} \sqsubset_l \mathfrak{b}^N\mathfrak{u}\mathfrak{x}_0 \sqsubset_l \mathfrak{y}$. Since $\theta(\mathfrak{b}) = 1$, Remark 9.6 along with the fact that $\mathfrak{b} \in \mathsf{Ba}_l(\mathsf{B})$ implies that for each $n \geq N$, $l_{\mathsf{B}}^{k_n}(\mathfrak{b}^N\mathfrak{u}\mathfrak{x}_0) = \mathfrak{b}^n\mathfrak{u}\mathfrak{x}_0$ for some $k_n \in \mathbb{N}$. Recall the definition of C_{B} stated before Proposition 11.7. The preceding arguments in this proof imply $C_{\mathsf{B}}(\mathfrak{b}^N\mathfrak{u}\mathfrak{x}_0) = C_{\mathsf{B}}(\mathfrak{b}^n\mathfrak{u}\mathfrak{x}_0)$ for every $n \geq N$. Consequently, $\langle 1, l_{\mathsf{B}}\rangle(C_{\mathsf{B}}(\mathfrak{b}^N\mathfrak{u}\mathfrak{x}_0)) = {}^{\infty}\mathfrak{b}\mathfrak{u}\mathfrak{x}_0$. Remark 7.5 gives that the interval $c_{\mathsf{B}}([C_{\mathsf{B}}(\mathfrak{b}^N\mathfrak{u}\mathfrak{x}_0), {}^{\infty}\mathfrak{b}\mathfrak{u}\mathfrak{x}_0)) \cong \omega$.

(2) $\Rightarrow$ (1). Since $\mathfrak{x} <_l \mathfrak{y}$, we have $\theta(\mathfrak{y} \mid \mathfrak{x}) = 1$. Both $\mathfrak{w}$ and $\mathfrak{x} \sqcap_l \mathfrak{y}$ are left substrings of $\mathfrak{y}$. Choose $\mathfrak{z} \sqsubset_l \mathfrak{y}$ with $\theta(\mathfrak{y} \mid \mathfrak{z}) = 1$ such that $\mathfrak{x} \sqcap_l \mathfrak{y} \sqsubset_l \mathfrak{z}$ and $\mathfrak{w} \sqsubset_l \mathfrak{z}$. Clearly, $\mathfrak{x} <_l \mathfrak{z} <_l \mathfrak{y}$. Since $\mathfrak{y} \in \mathbb{N}\text{-}\mathsf{St}(\mathfrak{w}, \theta(\mathfrak{y} \mid \mathfrak{w}); \mathsf{B})$, we have $\mathfrak{z} \in \overline{\mathsf{St}}(\mathfrak{w}, \theta(\mathfrak{y} \mid \mathfrak{w}); \mathsf{B})$. Therefore the interval $c_{\mathsf{B}}([\mathfrak{z}, \mathfrak{y}))$ being an infinite suborder of $c_{\mathsf{B}}([\mathfrak{x}, \mathfrak{y}))$ is also isomorphic to ω. Note that $l_{\mathsf{B}}^n(\mathfrak{z}) <_l \mathfrak{y}$ for every $n \in \mathbb{N}$, and therefore $\langle 1, l_{\mathsf{B}}\rangle(\mathfrak{z}) \leq_l \mathfrak{y}$. If $\langle 1, l_{\mathsf{B}}\rangle(\mathfrak{z}) \neq \mathfrak{y}$ then the string $\mathfrak{z}' := \langle 1, l_{\mathsf{B}}\rangle(\mathfrak{z}) \sqcap_l \mathfrak{y}$ satisfies $\langle 1, l_{\mathsf{B}}\rangle(\mathfrak{z}) <_l \mathfrak{z}' <_l \mathfrak{y}$, which is a contradiction to $c_{\mathsf{B}}([\mathfrak{z}, \mathfrak{y})) \cong \omega$. Therefore $\mathfrak{y} = \langle 1, l_{\mathsf{B}}\rangle(\mathfrak{z})$, which implies $\mathfrak{y} = {}^{\infty}\mathfrak{b}\mathfrak{v}\mathfrak{z}$ for some $\mathfrak{b} \in \mathsf{Ba}_l(\mathsf{B})$ and a string $\mathfrak{v}$ by Proposition 9.9. Since $\mathfrak{z} \in H_l^i(\mathfrak{x}_0)$, we have $\mathfrak{y} = {}^{\infty}\mathfrak{b}\mathfrak{u}\mathfrak{x}_0$ for some string $\mathfrak{u}$. ◻

Now we compute $\mathcal{C}(\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B}))$ when $\mathsf{B} \in \mathcal{Q}^{\mathrm{Ba}}$ is minimal for $(\mathfrak{x}_0, i)$ and understand the position of gaps and the form of the left $\mathbb{N}$-strings corresponding to them.

When B is non-domestic and minimal for $(\mathfrak{x}_0, i)$, the order type of $\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$ was computed in Corollary 11.8 to be $\mathcal{O} := \omega + \zeta \cdot \eta + \omega^*$. Recall from Corollary 3.5 that

$$\mathcal{C}(\mathcal{O}) \cong \omega + \mathbf{1} + \left(\sum_{r\in\lambda} T_r\right) + \mathbf{1} + \omega^*, \text{ where } T_r := \begin{cases} \mathbf{1} + \zeta + \mathbf{1} & \text{if } r \in \eta, \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

Recall from the end of § 3 that we partitioned the set $\mathcal{G}(\mathcal{O})$ as $\mathcal{G}(\mathcal{O}) = \mathcal{G}^+(\mathcal{O}) \sqcup \mathcal{G}^-(\mathcal{O}) \sqcup \mathcal{G}^0(\mathcal{O})$. Along similar lines, we partition the set $\mathbb{N}\text{-}\mathsf{St}(\mathfrak{x}_0, i; \mathsf{B})$ into three classes as follows.

$$\mathbb{N}\text{-}\mathsf{St}_l(\mathfrak{x}_0, i; \mathsf{B}) := \{\mathfrak{x} \in \mathbb{N}\text{-}\mathsf{St}(\mathfrak{x}_0, i; \mathsf{B}) \mid \mathfrak{x} = {}^{\infty}\mathfrak{b}\mathfrak{u}\mathfrak{x}_0 \text{ for some } \mathfrak{b} \in \mathsf{Ba}_l(\mathsf{B}) \text{ and some string } \mathfrak{u}\},$$
$$\mathbb{N}\text{-}\mathsf{St}_{\bar{l}}(\mathfrak{x}_0, i; \mathsf{B}) := \{\mathfrak{x} \in \mathbb{N}\text{-}\mathsf{St}(\mathfrak{x}_0, i; \mathsf{B}) \mid \mathfrak{x} = {}^{\infty}\mathfrak{b}\mathfrak{u}\mathfrak{x}_0 \text{ for some } \mathfrak{b} \in \mathsf{Ba}_{\bar{l}}(\mathsf{B}) \text{ and some string } \mathfrak{u}\},$$
$$\mathbb{N}\text{-}\mathsf{St}_0(\mathfrak{x}_0, i; \mathsf{B}) := \mathbb{N}\text{-}\mathsf{St}(\mathfrak{x}_0, i; \mathsf{B}) \setminus (\mathbb{N}\text{-}\mathsf{St}_l(\mathfrak{x}_0, i; \mathsf{B}) \cup \mathbb{N}\text{-}\mathsf{St}_{\bar{l}}(\mathfrak{x}_0, i; \mathsf{B})).$$

The following is a straightforward consequence of Corollary 3.5, Proposition 12.7 and its dual, and describes the positions of all left $\mathbb{N}$-strings in $\mathcal{C}(\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B}))$.

Proposition 12.8 *Let* B *be non-domestic and minimal for* $(\mathfrak{x}_0, i)$. *Then using the notations described above, the order isomorphism* $\mathbb{N}\text{-}\mathsf{St}(\mathfrak{x}_0, i; \mathsf{B}) \cup \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B}) \cong \mathcal{C}(\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})) \cong \mathcal{C}(\mathcal{O})$ *restricts to the following order isomorphisms:*

$$\mathbb{N}\text{-}\mathsf{St}_l(\mathfrak{x}_0, i; \mathsf{B}) \cong \mathcal{G}^+(\mathcal{O}),\ \ \mathbb{N}\text{-}\mathsf{St}_{\bar{l}}(\mathfrak{x}_0, i; \mathsf{B}) \cong \mathcal{G}^-(\mathcal{O}) \text{ and } \mathbb{N}\text{-}\mathsf{St}_0(\mathfrak{x}_0, i; \mathsf{B}) \cong \mathcal{G}^0(\mathcal{O}).$$

Remark 12.9 When $\mathsf{B} \in \mathcal{Q}_i^{\mathrm{Ba}}(\mathfrak{x}_0)$ is domestic and $\mathfrak{b}$ is the unique element of B, then $\mathfrak{y} \in \mathbb{N}\text{-}\mathsf{St}(\mathfrak{x}_0, i; \mathsf{B})$ if and only if there exists a string $\mathfrak{u}$ such that $\mathfrak{y} = {}^{\infty}\mathfrak{b}\mathfrak{u}\mathfrak{x}_0$.

Furthermore, if B is minimal for $(\mathfrak{x}_0, i)$, Proposition 6.7 gives that the set $\mathbb{N}\text{-}\mathsf{St}(\mathfrak{x}_0, i; \mathsf{B})$ is finite. Corollary 12.6 gives that $\mathcal{C}(\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})) = \mathbb{N}\text{-}\mathsf{St}(\mathfrak{x}_0, i; \mathsf{B}) \sqcup \overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$, and hence $\mathcal{G}(\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})) \cong \mathbb{N}\text{-}\mathsf{St}(\mathfrak{x}_0, i; \mathsf{B})$. In order to understand the location of finitely many elements of $\mathbb{N}\text{-}\mathsf{St}(\mathfrak{x}_0, i; \mathsf{B})$ in $\mathcal{C}(\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B}))$, recall from Corollary 11.8 that $\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B}) \cong (\omega+\omega^*)\cdot\mathbf{n}_{\mathsf{B}}$. The completion of the latter is $(\omega + \mathbf{1} + \omega^*) \cdot \mathbf{n}_{\mathsf{B}}$ (Example 3.2), which contains only finitely many extra points than in $\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B})$.

Finally, Proposition 12.7 and its dual applied to the explicit form of the order type of $\mathcal{C}(\overline{\mathsf{St}}(\mathfrak{x}_0, i; \mathsf{B}))$ gives that each element $\mathfrak{y} \in \mathbb{N}\text{-}\mathsf{St}(\mathfrak{x}_0, i; \mathsf{B})$ is of the form $\langle 1, l_{\mathsf{B}}\rangle(\mathfrak{z})$ for some $\mathfrak{z} \in \overline{\mathsf{St}}_{\pm1}(\mathfrak{x}_0, i; \mathsf{B})$ as well as of the form $\langle 1, \bar{l}_{\mathsf{B}}\rangle(\mathfrak{z}')$ for some $\mathfrak{z}' \in \overline{\mathsf{St}}_{\pm1}(\mathfrak{x}_0, i; \mathsf{B})$, which is in agreement with the fact that $\mathsf{Ba}_l(\mathsf{B}) = \mathsf{Ba}_{\bar{l}}(\mathsf{B}) = \{\mathfrak{b}\}$ since B is domestic.

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References

1. Agrawal, S., Kuber, A., Gupta, E.: Euclidean algorithm for a class of linear orders. Discrete Mathematics **346**(12), 113639 (2023)
2. Brenner, S.: A combinatorial characterisation of finite Auslander-Reiten quivers. In: Proc. ICRA 4, Ottawa 1984, SLNM 1177, pages 13–49 (1986)
3. Gupta, E., Kuber, A., Sardar, S.: On the stable radical of some non-domestic string algebras. Algebras and Representation Theory **25**(5), 1207–1230 (2022)
4. Knuth, D.E., Morris Jr, J.H., Pratt, V.R.: Fast pattern matching in strings. SIAM J. Comput. **6**(2):323–350 (1977)
5. Läuchli, H., Leonard, J.: On the elementary theory of linear order. Fundamenta Mathematicae **59**(1), 109–116 (1966)

6. Prest, M.: Morphisms between finitely presented modules and infinite-dimensional representations. In Canad. Math. Soc. Conf. Proc **24**, 447–455 (1998)
7. Ringel, C.M.: Some algebraically compact modules I. In: Abelian groups and modules, pages 419–439. Springer (1995)
8. Rosenstein, J.G.: Linear orderings. Academic press (1982)
9. Schröer, J.: Hammocks for string algebras. PhD thesis, Bielefeld University (Germany) (1998)
10. Schröer, J.: On the infinite radical of a module category. Proceedings of the London Mathematical Society **81**(3), 651–674 (2000)
11. Sardar, S., Kuber, A.: On the order types of hammocks for domestic string algebras. Journal of Pure and Applied Algebra, 2024. 107763. https://doi.org/10.1016/j.jpaa.2024.107763
12. Srivastava, S., Kuber, A.: Automating the stable rank computation for special biserial algebras. arXiv:2407.02326 (2024)
13. Srivastava, S., Sinha, V., Kuber, A.: On the stable radical of the module category for special biserial algebras. arXiv:2311.10178 (2023)