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RESEARCH

Rota-Baxter Lie bialgebras, classical Yang-Baxter equations and special L-dendriform bialgebras

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Abstract

This paper extends the well-known fact that a Rota-Baxter operator of weight 0 on a Lie algebra induces a pre-Lie algebra, to the level of bialgebras. We first show that a nondegenerate symmetric bilinear form that is invariant on a Rota-Baxter Lie algebra of weight 0 gives such a form that is left-invariant on the induced pre-Lie algebra and thereby gives a special L-dendriform algebra. This fact is obtained as a special case of Rota-Baxter Lie algebras with an adjoint-admissible condition, for a representation of the Lie algebra to admit a representation of the Rota-Baxter Lie algebra on the dual space. This condition can also be naturally formulated for Manin triples of Rota-Baxter Lie algebras, which can in turn be characterized in terms of bialgebras, thereby extending the Manin triple approach to Lie bialgebras. In the case of weight 0, the resulting Rota-Baxter Lie bialgebras give rise to special L-dendriform bialgebras, lifting the aforementioned connection that a Rota-Baxter Lie algebra induces a pre-Lie algebra to the level of bialgebras. The relationship between these two classes of bialgebras is also studied in terms of the coboundary cases, classical Yang-Baxter equations and $\mathcal{O}$-operators.

Keywords Rota-Baxter operator · Classical Yang-Baxter equation · Pre-Lie algebra · Bialgebra · Special L-dendriform algebra

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1 Introduction

A Rota-Baxter operator of weight 0 on a Lie algebra, as the operator form of the classical Yang-Baxter equation, gives rise to a pre-Lie algebra. Enriching this to the bialgebra level, we introduce Rota-Baxter Lie bialgebras via the Manin triple approach and show that Rota-Baxter Lie bialgebras of weight 0 give rise to spacial L-dendriform bialgebras. Implications to the classical Yang-Baxter equation are also given.

We first recall some background on Rota-Baxter operators, pre-Lie algebras and special L-dendriform algebras.

1.1 Rota-Baxter Lie Algebras, Pre-Lie Algebras and Special L-dendriform Algebras

The notion of Rota-Baxter operators on associative algebras originated from the 1960 work [8] of G. Baxter in a probability study. Motivated by the connections in combinatorics, quantum field theory, number theory and operads, the study of Rota-Baxter operators has experienced a great expansion in recent years. See [13, 19] for introductions and further references.

As a remarkable coincidence, Rota-Baxter operators on Lie algebras also arose independently as the operator forms of the classical Yang-Baxter equation (CYBE), which are closely related to classical integrable systems and quantum groups [10].

Definition 1.1 Let $(\mathfrak{g}, [-,-])$ be a Lie algebra. If a linear operator $P : \mathfrak{g} \to \mathfrak{g}$ satisfies

$$[P(x), P(y)] = P([P(x), y] + [x, P(y)] + \lambda[x, y]), \quad \forall x, y \in \mathfrak{g}, \tag{1}$$

for a fixed scalar λ, then P is called a **Rota-Baxter operator of weight** λ on $(\mathfrak{g}, [-,-])$, or simply a **Rota-Baxter operator** on $(\mathfrak{g}, [-,-])$ if there is no confusion. Moreover, $(\mathfrak{g}, [-,-], P)$ is called a **Rota-Baxter Lie algebra of weight** λ, or simply a **Rota-Baxter Lie algebra**.

Semenov-Tian-Shansky [20] first showed that, if there exists a nondegenerate symmetric invariant bilinear form on a Lie algebra $(\mathfrak{g}, [-,-])$, then an antisymmetric solution $r \in \mathfrak{g}\otimes\mathfrak{g}$ of the CYBE can be equivalently expressed as what is now called a Rota-Baxter operator of weight 0 on the Lie algebra $(\mathfrak{g}, [-,-])$. Then the operator is regarded as an **operator form** of the CYBE. This operator approach was expanded by Semenov-Tian-Shansky using the notion of modified CYBE, which is just a Rota-Baxter operator of weight 1 upon a linear transformation, and by Kupershmidt [15] using the notion of $\mathcal{O}$-operators, later called relative Rota-Baxter operators.

Another important role played by Rota-Baxter Lie algebras of weight 0 is that they produce **pre-Lie algebras**. More precisely, let $(\mathfrak{g}, [-,-], P)$ be a Rota-Baxter Lie algebra of weight 0 and define

$$x \circ y := [P(x), y], \quad \forall x, y \in \mathfrak{g}. \tag{2}$$

Then $(\mathfrak{g}, \circ)$ is a pre-Lie algebra, defined to be a vector space $\mathfrak{g}$ with a binary operation $\circ$ satisfying

$$(x \circ y) \circ z - x \circ (y \circ z) = (y \circ x) \circ z - y \circ (x \circ z), \quad \forall x, y, z \in \mathfrak{g}. \tag{3}$$

This pre-Lie algebra is called the **induced pre-Lie algebra** from $(\mathfrak{g}, [-,-], P)$.

Pre-Lie algebras, also called left-symmetric algebras, arose from studies in diverse areas, including convex homogeneous cones [23], affine manifolds and affine structures on Lie groups [14], deformations of associative algebras [12], and then appear in many other fields in mathematics and mathematical physics. From the operadic viewpoint, the operad of pre-Lie algebras is the splitting (successor) of the operad of Lie algebras [4]. See [9] and the references therein for more details.

On the other hand, nondegenerate symmetric left-invariant bilinear forms on pre-Lie algebras are in correspondence with left-invariant flat pseudo-metrics on Lie groups [5] (see Remark 2.13). The underlying algebra structures of pre-Lie algebras with such bilinear forms are shown to be the special L-dendriform algebras, as a subclass of L-dendriform algebras [7] whose operad is the two-fold splitting (successor) of the operad of Lie algebras (see Remark 2.11).

In this paper, we first give a further supplement on the fact that a Rota-Baxter Lie algebra of weight 0 induces a pre-Lie algebra in terms of bilinear forms. We show that a nondegenerate symmetric invariant bilinear form on a Rota-Baxter Lie algebra of weight 0 is also left-invariant on the induced pre-Lie algebra, and thereby gives a special L-dendriform algebra.

More precisely, from a nondegenerate symmetric invariant bilinear form on a Rota-Baxter Lie algebra, there is a representation of the Rota-Baxter Lie algebra on the dual space, which is equivalent to the adjoint representation. The condition for a Rota-Baxter Lie algebra to admit a representation on the dual space is formulated as the notion of an admissible condition. In particular, the adjoint-admissible condition on a Rota-Baxter Lie algebra of weight 0 induces a special L-dendriform algebra, which is the cause behind the aforementioned phenomenon that a nondegenerate symmetric invariant bilinear form on a Rota-Baxter Lie algebra of weight 0 gives rise to a special L-dendriform algebra.

Our main purpose of this paper is to apply these results to derive special L-dendriform bialgebras from Rota-Baxter Lie bialgebras of weight 0, as we will elaborate next.

1.2 Rota-Baxter Lie Bialgebras and Special L-dendriform Bialgebras

Recall that a Lie bialgebra, which is equivalently characterized as a Manin triple of Lie algebras, is the algebraic structure corresponding to a Poisson-Lie group. It is also the classical structure of a quantized universal enveloping algebra [10, 11]. Furthermore, antisymmetric solutions of the CYBE, or the classical r-matrices, naturally give rise to coboundary Lie bialgebras [10, 20]. See [2, 3, 9, 15] for their relation with $\mathcal{O}$-operators and pre-Lie algebras and other details.

In this paper, we extend the Lie bialgebra theory to a bialgebra theory for Rota-Baxter Lie algebras, following the approach of Rota-Baxter antisymmetric infinitesimal bialgebras in [6]. Explicitly, we introduce the notion of Manin triples of Rota-Baxter Lie algebras as a class of Rota-Baxter Lie algebras with nondegenerate symmetric invariant bilinear forms. It serves as a natural enrichment of Manin triples of Lie algebras [10] by equipping a triple of Lie algebras $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ with a triple of Rota-Baxter operators $(P + Q^*, P, Q^*)$. Although in principle we allow arbitrary Rota-Baxter operators in the notion of Manin triples of Rota-Baxter Lie algebras, they turn out to satisfy the specific adjoint-admissible condition (see Lemma 3.2). The notion of Rota-Baxter Lie bialgebras is thus introduced as an equivalent structure of Manin triples of Rota-Baxter Lie algebras.

The notion of Rota-Baxter Lie bialgebras applies to arbitrary weights of the Rota-Baxter operators. This unified approach enables us to recover previous constructions as special cases: the Rota-Baxter Lie bialgebra developed in [16] corresponds to the case when $Q = -P - \mathrm{id}$ and the weight $\lambda = 1$; while the Rota-Baxter Lie bialgebra introduced in [21] corresponds to the case when $Q = -P$ and $\lambda = 0$.

For the most part, however, we focus on the case of weight 0. Manin triples of Rota-Baxter Lie algebras of weight 0, or equivalently, Rota-Baxter Lie bialgebras of weight 0, give rise to pre-Lie algebras with nondegenerate symmetric left-invariant bilinear forms on the double spaces, which are precisely the Manin triples of pre-Lie algebras associated to such bilinear forms, or equivalently, special L-dendriform bialgebras, as introduced in [5]. This type of induction of special L-dendriform bialgebras from Rota-Baxter Lie bialgebras of weight 0 is consistent with the aforementioned relationship between Rota-Baxter Lie algebras of weight 0 with the adjoint-admissible condition and special L-dendriform algebras. On the other hand, note that the construction of Rota-Baxter Lie bialgebras can also be regarded as imposing Rota-Baxter operators on Lie bialgebras. Therefore such an induction also extends to the context of bialgebras the fact that a Rota-Baxter operator of weight 0 on a Lie algebra induces a pre-Lie algebra.

Moreover, the relationships of Rota-Baxter Lie bialgebras of weight 0 and special L-dendriform bialgebras are interpreted in several aspects, such as the coboundary cases, classical Yang-Baxter equations and $\mathcal{O}$-operators.

1.3 Layout of the Paper

The paper is organized as follows.

In Section 2, we give the notion of representations of Rota-Baxter Lie algebras. An admissibility of a linear operator for a Rota-Baxter Lie algebra is introduced in order to construct a reasonable representation on the dual space. We also observe that an invariant bilinear form on a Rota-Baxter Lie algebra of weight 0 is left-invariant on the induced pre-Lie algebra. Moreover, a Rota-Baxter Lie algebra of weight 0 with a linear operator satisfying the adjoint-admissible condition induces a special L-dendriform algebra.

In Section 3, we introduce the notion of Rota-Baxter Lie bialgebras as characterized by Manin triples of Rota-Baxter Lie algebras. We establish the explicit relationship between Rota-Baxter Lie bialgebras of weight 0 and special L-dendriform bialgebras, both directly and in their respectively equivalent interpretations in terms of Manin triples.

In Section 4, we focus on coboundary Rota-Baxter Lie bialgebras, leading to the introduction of the admissible CYBE in Rota-Baxter Lie algebras whose antisymmetric solutions are used to construct Rota-Baxter Lie bialgebras. The notions of $\mathcal{O}$-operators on Rota-Baxter Lie algebras and Rota-Baxter pre-Lie algebras are introduced to produce antisymmetric solutions of the admissible CYBE. Furthermore, when the weight is 0, we study the induced special L-dendriform bialgebras from these Rota-Baxter Lie bialgebras, in particular their relationship in terms of the coboundary cases, classical Yang-Baxter equations and $\mathcal{O}$-operators.

Notation Unless otherwise specified, all the vector spaces and algebras are finite-dimensional over a field $\mathbb{K}$ of characteristic 0, although many results and notions, in particular that of a Rota-Baxter Lie bialgebra, remain valid in the infinite-dimensional case. Linear maps and tensor products are taken over $\mathbb{K}$.

2 Rota-Baxter Lie algebras, pre-Lie algebras and Special L-dendriform Algebras

This section introduces an admissibility condition for a Rota-Baxter Lie algebra, so that a representation of the Rota-Baxter Lie algebra can be defined on the dual space of a representation of the Lie algebra. An invariant bilinear form on a Rota-Baxter Lie algebra of weight

0 is shown to be left-invariant on the induced pre-Lie algebra. Special L-dendriform algebras are interpreted in terms of the representations of pre-Lie algebras. As a consequence, a Rota-Baxter Lie algebra of weight 0 with a linear operator satisfying the adjoint-admissible condition gives a special L-dendriform algebra which is compatible with the induced pre-Lie algebra.

2.1 Rota-Baxter Lie Algebras and their Representations

We first recall some basic notions and facts on the representations of Lie algebras.

Definition 2.1 A **representation** of a Rota-Baxter Lie algebra $(\mathfrak{g}, [-,-], P)$ is a triple (V, ρ, α), where (V, ρ) is a representation of the Lie algebra $(\mathfrak{g}, [-,-])$ and $\alpha : V \to V$ is a linear map satisfying the equation

$$\rho(P(x))\alpha(v) = \alpha(\rho(P(x))v) + \alpha(\rho(x)\alpha(v)) + \lambda\alpha(\rho(x)v), \quad \forall x \in \mathfrak{g}, v \in V. \tag{4}$$

A linear map $\varphi : V_1 \to V_2$ is called a **homomorphism from** (V_1, ρ_1, α_1) **to** (V_2, ρ_2, α_2) if

$$\varphi(\rho_1(x)v) = \rho_2(x)\varphi(v), \quad \varphi(\alpha_1(v)) = \alpha_2(\varphi(v)), \quad \forall x \in g, v \in V_1. \tag{5}$$

Further, if φ is bijective, then we call φ an **isomorphism from** (V_1, ρ_1, α_1) **to** (V_2, ρ_2, α_2), and in this case, (V_1, ρ_1, α_1) and (V_2, ρ_2, α_2) are called **equivalent**.

Example 2.2 Let $(\mathfrak{g}, [-,-], P)$ be a Rota-Baxter Lie algebra. Then $(\mathfrak{g}, \mathrm{ad}, P)$ is a representation of $(\mathfrak{g}, [-,-], P)$, which is called the **adjoint representation** of $(\mathfrak{g}, [-,-], P)$.

Representations of Rota-Baxter Lie algebras are easily checked to have the following characterization.

Proposition 2.3 *Let* $(\mathfrak{g}, [-,-], P)$ *be a Rota-Baxter Lie algebra of weight* λ *and* V *be a vector space. Let* $\rho : \mathfrak{g} \to \mathrm{End}(V)$ *and* $\alpha : V \to V$ *be linear maps. Then* (V, ρ, α) *is a representation of* $(\mathfrak{g}, [-,-], P)$ *if and only if* $(\mathfrak{g} \oplus V, [-,-]_{\mathfrak{g}\oplus V}, P + \alpha)$ *is a Rota-Baxter Lie algebra of weight* λ*, where the operation* $[-,-]_{\mathfrak{g}\oplus V}$ *on* $\mathfrak{g} \oplus V$ *is given by*

$$[x + u, y + v]_{\mathfrak{g}\oplus V} := [x, y] + \rho(x)v - \rho(y)u, \quad \forall x, y \in \mathfrak{g}, u, v \in V. \tag{6}$$

In this case, the resulting Rota-Baxter Lie algebra is denoted by $(\mathfrak{g} \ltimes_\rho V, P + \alpha)$ *and called the* ***semi-direct product*** *of* $(\mathfrak{g}, [-,-], P)$ *by* (V, ρ, α).

Let $\mathfrak{g}$ and V be vector spaces. For a linear map $\rho : \mathfrak{g} \to \mathrm{End}(V)$, we define $\rho^* : \mathfrak{g} \to \mathrm{End}(V^*)$ by

$$\langle \rho^*(x)u^*, v\rangle = -\langle u^*, \rho(x)v\rangle, \ \forall x \in \mathfrak{g}, u^* \in V^*, v \in V.$$

Now we study representations of Rota-Baxter Lie algebras on the dual spaces.

Lemma 2.4 *Let* $(\mathfrak{g}, [-,-], P)$ *be a Rota-Baxter Lie algebra of weight* λ*,* V *be a vector space and* $\beta : V \to V$ *be a linear map. Then* (V^*, ρ^*, β^*) *is a representation of* $(\mathfrak{g}, [-,-], P)$ *if and only if* (V, ρ) *is a representation of* $(\mathfrak{g}, [-,-])$ *and the following equation holds*:

$$\beta(\rho(P(x))v) - \rho(P(x))\beta(v) - \beta(\rho(x)\beta(v)) - \lambda\rho(x)\beta(v) = 0, \quad \forall x \in \mathfrak{g}, v \in V. \tag{7}$$

In particular, for a linear map $Q : \mathfrak{g} \to \mathfrak{g}$*, the triple* $(\mathfrak{g}^*, \mathrm{ad}^*, Q^*)$ *is a representation of* $(\mathfrak{g}, [-,-], P)$ *if and only if*

$$Q([P(x), y]) - [P(x), Q(y)] - Q([x, Q(y)]) - \lambda[x, Q(y)] = 0, \quad \forall x, y \in \mathfrak{g}. \tag{8}$$

Proof It is basic that (V^*, ρ^*) is a representation of $(\mathfrak{g}, [-,-])$ if and only if (V, ρ) is a representation of $(\mathfrak{g}, [-,-])$. For all $x \in \mathfrak{g}, u^* \in V^*, v \in V$, we have

$$\begin{aligned}&\langle \rho^*(P(x))\beta^*(u^*) - \beta^*(\rho^*(P(x))u^*) - \beta^*(\rho^*(x)\beta^*(u^*)) - \lambda\beta^*(\rho^*(x)u^*), v\rangle \\ &= \langle u^*, -\beta(\rho(P(x))v) + \rho(P(x))\beta(v) + \beta(\rho(x)\beta(v)) + \lambda\rho(x)\beta(v)\rangle.\end{aligned}$$

Thus the triple (V^*, ρ^*, β^*) satisfies Eq. 4 if and only if Eq. 8 holds. Hence the first claim holds. The second claim follows from the first by taking the adjoint representation. □

Definition 2.5 Let $(\mathfrak{g}, [-,-], P)$ be a Rota-Baxter Lie algebra of weight λ, (V, ρ) be a representation of $(\mathfrak{g}, [-,-])$, and $\beta : V \to V$ be a linear map. If (V^*, ρ^*, β^*) is a representation of $(\mathfrak{g}, [-,-], P)$ (that is, Eq. 7 holds), then we say that β is **admissible to the Rota-Baxter Lie algebra** $(\mathfrak{g}, [-,-], P)$ **on** (V, ρ), or $(\mathfrak{g}, [-,-], P)$ is β**-admissible on** (V, ρ). In particular, if there is a linear map $Q : \mathfrak{g} \to \mathfrak{g}$ satisfying Eq. 8 such that $(\mathfrak{g}^*, \mathrm{ad}^*, Q^*)$ is a representation of $(\mathfrak{g}, [-,-], P)$, we simply say that Q **is adjoint-admissible to** $(\mathfrak{g}, [-,-], P)$ or $(\mathfrak{g}, [-,-], P)$ **is** Q**-adjoint-admissible**.

Remark 2.6 By Lemma 2.4, for a Rota-Baxter Lie algebra $(\mathfrak{g}, [-,-], P)$ of weight λ, it is straightforward to check that the linear maps $-P - \lambda\mathrm{id}_{\mathfrak{g}}$, $-\lambda\mathrm{id}_{\mathfrak{g}}$ and 0 are adjoint-admissible. However, later in Example 2.8 we will see that we are not necessarily limited to these choices.

We next give adjoint-admissible operators from invariant bilinear forms on Lie algebras. Recall that a bilinear form $\mathcal{B}$ on a Lie algebra $(\mathfrak{g}, [-,-])$ is called **invariant** if

$$\mathcal{B}([x, y], z) = \mathcal{B}(x, [y, z]), \quad \forall x, y, z \in \mathfrak{g}. \tag{9}$$

Proposition 2.7 *Let* $(\mathfrak{g}, [-,-], P)$ *be a Rota-Baxter Lie algebra of weight* λ *and* $\mathcal{B}$ *be a nondegenerate invariant bilinear form on the Lie algebra* $(\mathfrak{g}, [-,-])$. *Let* $\widehat{P}$ *be the adjoint linear map of* P *with respect to* $\mathcal{B}$, *characterized by*

$$\mathcal{B}(P(x), y) = \mathcal{B}(x, \widehat{P}(y)), \quad \forall x, y \in \mathfrak{g}. \tag{10}$$

Then $\widehat{P}$ *is adjoint-admissible to* $(\mathfrak{g}, [-,-], P)$, *or equivalently,* $(\mathfrak{g}^*, \mathrm{ad}^*, \widehat{P}^*)$ *is a representation of* $(\mathfrak{g}, [-,-], P)$. *Moreover,* $(\mathfrak{g}^*, \mathrm{ad}^*, \widehat{P}^*)$ *is equivalent to* $(\mathfrak{g}, \mathrm{ad}, P)$ *as representations of* $(\mathfrak{g}, [-,-], P)$.

Conversely, let $(\mathfrak{g}, [-,-], P)$ *be a Rota-Baxter Lie algebra of weight* λ *and* $Q : \mathfrak{g} \to \mathfrak{g}$ *be a linear map that is adjoint-admissible to* $(\mathfrak{g}, [-,-], P)$. *If the resulting representation* $(\mathfrak{g}^*, \mathrm{ad}^*, Q^*)$ *of* $(\mathfrak{g}, [-,-], P)$ *is equivalent to* $(\mathfrak{g}, \mathrm{ad}, P)$, *then there exists a nondegenerate invariant bilinear from* $\mathcal{B}$ *on* $(\mathfrak{g}, [-,-], P)$ *such that* $Q = \widehat{P}$.

Proof The proof is similar to that of [6, Proposition 3.9]. Note that in the context of Lie algebras, by the antisymmetry of the Lie bracket, the bilinear form $\mathcal{B}$ no longer needs to be symmetric. □

Example 2.8 Let $(\mathfrak{g}, [-,-])$ be the 3-dimensional simple Lie algebra $\mathfrak{sl}(2, \mathbb{K})$ with a basis $\left\{x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}$ and with the product

$$[h, x] := 2x, \ [h, y] := -2y, \ [x, y] := h. \tag{11}$$

Define a linear operator $P : \mathfrak{g} \to \mathfrak{g}$ by

$$P(x) := x + y, \ P(h) := 2h + 4y, \ P(y) := x - 2h - 3y.$$

Then P is a Rota-Baxter operator of weight 0 on $(\mathfrak{g}, [-,-])$. Moreover, there is a nondegenerate symmetric invariant bilinear form $\mathcal{B}$ on $(\mathfrak{g}, [-,-])$ whose nonzero values are

$$\mathcal{B}(x, y) := \mathcal{B}(y, x) := 1, \quad \mathcal{B}(h, h) := 2.$$

The adjoint linear operator $\widehat{P}$ of P with respect to $\mathcal{B}$ is given by

$$\widehat{P}(x) = -3x + 2h + y, \widehat{P}(h) = -4x + 2h, \widehat{P}(y) = x + y.$$

Then by Proposition 2.7, $\widehat{P}$ is adjoint-admissible to $(\mathfrak{g}, [-,-], P)$, and the corresponding representation $(\mathfrak{g}^*, \mathrm{ad}^*, \widehat{P}^*)$ of $(\mathfrak{g}, [-,-], P)$ is equivalent to $(\mathfrak{g}, \mathrm{ad}, P)$. Also note that $\widehat{P}$ commutes with P.

2.2 Pre-Lie Algebras, their Representations and Special L-dendriform Algebras

We recall some facts on pre-Lie algebras [9]. For a pre-Lie algebra $(A, \circ)$ as defined in Eq. 3, the commutator

$$[x, y] = x \circ y - y \circ x, \quad \forall x, y \in A, \tag{12}$$

defines a Lie algebra $(\mathfrak{g}(A), [-,-])$, called the **sub-adjacent** Lie algebra of $(A, \circ)$, and $(A, \circ)$ is called a **compatible** pre-Lie algebra structure on the Lie algebra $(\mathfrak{g}(A), [-,-])$.

For a vector space A with a binary operation $\circ : A \otimes A \to A$, define linear maps

$$\mathcal{L}_\circ, \mathcal{R}_\circ : A \to \mathrm{End}(A), \quad \mathcal{L}_\circ(x)y := x \circ y =: \mathcal{R}_\circ(y)x, \quad \forall x, y \in A.$$

If $(A, \circ)$ is a pre-Lie algebra, then $(A, \mathcal{L}_\circ)$ is a representation of the sub-adjacent Lie algebra $(\mathfrak{g}(A), [-,-])$.

Definition 2.9 A **representation** of a pre-Lie algebra $(A, \circ)$ is a triple $(V, l_\circ, r_\circ)$, where V is a vector space, and $l_\circ, r_\circ : A \to \mathrm{End}(V)$ are linear maps satisfying

$$l_\circ(x)l_\circ(y)v - l_\circ(x \circ y)v = l_\circ(y)l_\circ(x)v - l_\circ(y \circ x)v, \tag{13}$$

$$l_\circ(x)r_\circ(y)v - r_\circ(y)l_\circ(x)v = r_\circ(x \circ y)v - r_\circ(y)r_\circ(x)v, \quad \forall x, y \in A, v \in V. \tag{14}$$

In fact, $(V, l_\circ, r_\circ)$ is a representation of a pre-Lie algebra $(A, \circ)$ if and only if the direct sum $A \oplus V$ of vector spaces is equipped with a (**semi-direct product**) pre-Lie algebra structure by the operation on $A \oplus V$ defined by

$$(x + u) \circ_{A\oplus V} (y + v) := x \circ y + l_\circ(x)v + r_\circ(y)u, \quad \forall x, y \in A, u, v \in V. \tag{15}$$

We denote the resulting pre-Lie algebra by $A \ltimes_{l_\circ, r_\circ} V$ or simply $A \ltimes V$.

We next recall the notions of L-dendriform algebras and special L-dendriform algebras.

Definition 2.10 [5, 7] An **L-dendriform algebra** is a triple $(A, \triangleright, \triangleleft)$, consisting of a vector space A, and binary operations $\triangleright, \triangleleft : A \otimes A \to A$ satisfying

$$(x \triangleright y) \triangleright z + (x \triangleleft y) \triangleright z + y \triangleright (x \triangleright z) - (y \triangleleft x) \triangleright z - (y \triangleright x) \triangleright z - x \triangleright (y \triangleright z) = 0, \tag{16}$$

$$(x \triangleright y) \triangleleft z + y \triangleleft (x \triangleright z) + y \triangleleft (x \triangleleft z) - (y \triangleleft x) \triangleleft z - x \triangleright (y \triangleleft z) = 0, \quad \forall x, y, z \in A. \tag{17}$$

An L-dendriform algebra is called **special** if $\triangleleft$ is antisymmetric.

Remark 2.11 (a) The operad of L-dendriform algebras is the successor [4] of the operad **pre-Lie** of pre-Lie algebras. Thus it is also the Manin black product **pre-Lie** $\bullet$ **pre-Lie** [4, Corollary 3.5].

(b) For an L-dendriform algebra $(A, \triangleright, \triangleleft)$, there are pre-Lie algebras $(A, \circ)$ and $(A, \star)$ given by

$$x \circ y = x \triangleright y - y \triangleleft x, \quad x \star y = x \triangleright y + x \triangleleft y, \quad \forall x, y \in A, \tag{18}$$

called the **horizontal** and **vertical** pre-Lie algebras respectively [5, 7]. Moreover, if (and only if under our assumption of characteristic 0) $(A, \triangleright, \triangleleft)$ is special, then the horizontal pre-Lie algebra $(A, \circ)$ and the vertical pre-Lie algebra $(A, \star)$ coincide, that is, $x \circ y = x \star y$ for all $x, y \in A$. In this case, $(A, \circ)$ is called the **sub-adjacent** pre-Lie algebra of $(A, \triangleright, \triangleleft)$, and $(A, \triangleright, \triangleleft)$ is called a **compatible** special L-dendriform algebra of $(A, \circ)$.

We now interpret special L-dendriform algebras in terms of representations of pre-Lie algebras.

Proposition 2.12 *Let $(A, \circ)$ be a pre-Lie algebra. Suppose that $\triangleleft : A \otimes A \to A$ is an antisymmetric operation on A. Define an operation $\triangleright$ on A by*

$$x \triangleright y := x \circ y - x \triangleleft y, \quad \forall x, y \in A. \tag{19}$$

Then the following statements are equivalent.

(a) $(A, \triangleright, \triangleleft)$ *is a special L-dendriform algebra;*
(b) *The following equation holds:*

$$x \triangleleft (y \triangleleft z) + y \triangleleft (x \circ z) - z \triangleleft (x \circ y) - x \circ (y \triangleleft z) = 0, \quad \forall x, y, z \in A; \tag{20}$$

(c) $(A, \mathcal{L}_{\triangleright}, -\mathcal{L}_{\triangleleft})$ *is a representation of* $(A, \circ)$*;*
(d) $(A^*, \mathcal{L}_{\circ}^*, \mathcal{L}_{\triangleleft}^*)$ *is a representation of* $(A, \circ)$.

Proof (a) $\iff$ (b). Let $x, y, z \in A$. Then we have

$$\begin{aligned}&(x \triangleright y) \triangleleft z + y \triangleleft (x \triangleright z) + y \triangleleft (x \triangleleft z) - (y \triangleleft x) \triangleleft z - x \triangleright (y \triangleleft z)\\ &= (x \circ y) \triangleleft z + y \triangleleft (x \circ z) - x \circ (y \triangleleft z) + x \triangleleft (y \triangleleft z).\end{aligned}$$

Thus Eq. 17 holds if and only if Eq. 20 holds. Moreover, if this is the case, then we have

$$\begin{aligned}&(x \triangleright y) \triangleright z + (x \triangleleft y) \triangleright z + y \triangleright (x \triangleright z) - (y \triangleleft x) \triangleright z - (y \triangleright x) \triangleright z - x \triangleright (y \triangleright z)\\ &= (x \circ y) \circ z - (x \circ y) \triangleleft z + y \circ (x \circ z) - y \circ (x \triangleleft z) - y \triangleleft (x \circ z) + y \triangleleft (x \triangleleft z)\\ &\quad -(y \circ x) \circ z + (y \circ x) \triangleleft z - x \circ (y \circ z) + x \circ (y \triangleleft z) + x \triangleleft (y \circ z) - x \triangleleft (y \triangleleft z) = 0.\end{aligned}$$

Hence Eq. 16 holds automatically.

(a) $\iff$ (c). It follows from [7, Proposition 3.4].

(c) $\iff$ (d). By [3], if $(V, l_\circ, r_\circ)$ is a representation of a pre-Lie algebra $(A, \circ)$, then $(V^*, l_\circ^* - r_\circ^*, r_\circ^*)$ is also a representation of $(A, \circ)$. Hence the conclusion follows. □

The special L-dendriform algebra also arises naturally on a pre-Lie algebra $(A, \circ)$ with a nondegenerate symmetric bilinear form $\mathcal{B}$ which is **left-invariant** in the sense that

$$\mathcal{B}(x \circ y, z) + \mathcal{B}(y, x \circ z) = 0, \quad \forall x, y, z \in A. \tag{21}$$

Remark 2.13 There is a natural bijection between the set of pre-Lie algebras with a nondegenerate symmetric left-invariant bilinear form and the set of connected and simply-connected Lie groups with a left-invariant flat pseudo-metric [1, 17]. Under this correspondence, the sub-adjacent Lie algebra of a pre-Lie algebra in the former set is precisely the Lie algebra of the corresponding Lie group.

Lemma 2.14 [5] *Let $(A, \circ)$ be a pre-Lie algebra with a nondegenerate symmetric left-invariant bilinear form $\mathcal{B}$. Then there is a compatible special L-dendriform algebra $(A, \rhd, \lhd)$ with the operations $\rhd$ and $\lhd$ defined by*

$$\mathcal{B}(x \lhd y, z) = \mathcal{B}(x, z \circ y), \quad x \rhd y = x \circ y - x \lhd y, \quad \forall x, y, z \in A. \tag{22}$$

2.3 Special L-dendriform Algebras from Rota-Baxter Lie Algebras of Weight 0 with the Adjoint-admissible Condition

A representation of a Rota-Baxter Lie algebra gives rise to a representation of the induced pre-Lie algebra as follows.

Proposition 2.15 *Let (V, ρ, α) be a representation of a Rota-Baxter Lie algebra $(\mathfrak{g}, [-,-], P)$ of weight 0. Define maps $l_{\rho,\alpha}, r_{\rho,\alpha} : \mathfrak{g} \to \mathrm{End}(V)$ by*

$$l_{\rho,\alpha}(x)v := \rho(P(x))v, \;\; r_{\rho,\alpha}(x)v := -\rho(x)\alpha(v), \quad \forall x \in \mathfrak{g}, v \in V. \tag{23}$$

Then $(V, l_{\rho,\alpha}, r_{\rho,\alpha})$ is a representation of the induced pre-Lie algebra $(\mathfrak{g}, \circ)$, called the **induced representation** *of the pre-Lie algebra $(\mathfrak{g}, \circ)$ from (V, ρ, α).*

Proof By Proposition 2.3, there is a Rota-Baxter Lie algebra $(\mathfrak{g} \ltimes_\rho V, P + \alpha)$ of weight 0. Hence by Eq. 2, there is an induced pre-Lie algebra structure on $\mathfrak{g} \oplus V$, defined by

$$\begin{aligned}(x+u) \circ (y+v) &:= [(P+\alpha)(x+u), y+v] = [P(x), y] + \rho(P(x))v - \rho(y)\alpha(u) \\ &= x \circ y + l_{\rho,\alpha}(x)v + r_{\rho,\alpha}(y)u, \quad \forall x, y \in \mathfrak{g}, u, v \in V.\end{aligned}$$

Thus $(V, l_{\rho,\alpha}, r_{\rho,\alpha})$ is a representation of $(\mathfrak{g}, \circ)$. □

Proposition 2.16 *Let $(\mathfrak{g}, [-,-], P)$ be a Rota-Baxter Lie algebra of weight 0 and $(\mathfrak{g}, \circ)$ be the induced pre-Lie algebra. Let $Q : \mathfrak{g} \to \mathfrak{g}$ be a linear map that is adjoint-admissible to $(\mathfrak{g}, [-,-], P)$.*

(a) *The triple $(\mathfrak{g}^*, l_{\mathrm{ad}^*,Q^*}, r_{\mathrm{ad}^*,Q^*})$ is a representation of the pre-Lie algebra $(\mathfrak{g}, \circ)$.*
(b) *Define an operation $\lhd : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ on $\mathfrak{g}$ by*

$$x \lhd y := -Q([x, y]), \quad \forall x, y \in \mathfrak{g}. \tag{24}$$

Then $l_{\mathrm{ad}^*,Q^*} = \mathcal{L}_\circ^*, r_{\mathrm{ad}^*,Q^*} = \mathcal{L}_\lhd^*$.
(c) *Define an operation $\rhd : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ on $\mathfrak{g}$ by Eq. 19. Then $(\mathfrak{g}, \rhd, \lhd)$ is a compatible special L-dendriform algebra of the pre-Lie algebra $(\mathfrak{g}, \circ)$.*

Proof (a) Since Q is adjoint-admissible to $(\mathfrak{g}, [-,-], P)$, $(\mathfrak{g}^*, \mathrm{ad}^*, Q^*)$ is a representation of the Rota-Baxter Lie algebra $(\mathfrak{g}, [-,-], P)$. Hence by Proposition 2.15, $(\mathfrak{g}^*, l_{\mathrm{ad}^*,Q^*}, r_{\mathrm{ad}^*,Q^*})$ is the induced representation of $(\mathfrak{g}, \circ)$ from $(\mathfrak{g}^*, \mathrm{ad}^*, Q^*)$.

(b) For all $x, y \in \mathfrak{g}, a^* \in \mathfrak{g}^*$, we have

$$\begin{aligned}\langle l_{\mathrm{ad}^*,Q^*}(x)a^*, y\rangle &= \langle \mathrm{ad}^*(P(x))a^*, y\rangle = -\langle a^*, [P(x), y]\rangle = -\langle a^*, x \circ y\rangle = \langle \mathcal{L}_\circ^*(x)a^*, y\rangle, \\ \langle r_{\mathrm{ad}^*,Q^*}(x)a^*, y\rangle &= -\langle \mathrm{ad}^*(x)Q^*(a^*), y\rangle = \langle Q^*(a^*), [x, y]\rangle = \langle a^*, Q([x, y])\rangle = \langle \mathcal{L}_\lhd^*(x)a^*, y\rangle.\end{aligned}$$

Hence $l_{\mathrm{ad}^*,Q^*} = \mathcal{L}_\circ^*, r_{\mathrm{ad}^*,Q^*} = \mathcal{L}_\lhd^*$.

(c) By Items (a) and (b), $(\mathfrak{g}^*, \mathcal{L}_\circ^*, \mathcal{L}_\lhd^*)$ is a representation of $(\mathfrak{g}, \circ)$. Thus $(\mathfrak{g}, \rhd, \lhd)$ is a compatible special L-dendriform algebra of $(\mathfrak{g}, \circ)$ by Proposition 2.12. □

As a special case, we obtain

Corollary 2.17 *Let* $(\mathfrak{g}, [-,-], P)$ *be a Rota-Baxter Lie algebra of weight* 0 *and* $(\mathfrak{g}, \circ)$ *be the induced pre-Lie algebra.*

(a) *There is a compatible special L-dendriform algebra* $(\mathfrak{g}, \triangleright, \triangleleft)$ *of the pre-Lie algebra* $(\mathfrak{g}, \circ)$ *given by*

$$x \triangleleft y := P([x, y]), \quad x \triangleright y := x \circ y - x \triangleleft y := [P(x), y] - P([x, y]), \quad \forall x, y \in \mathfrak{g}. \tag{25}$$

(b) *Suppose that* $\mathcal{B}$ *is a nondegenerate symmetric invariant bilinear form on* $(\mathfrak{g}, [-,-])$. *Let* $\widehat{P} : \mathfrak{g} \to \mathfrak{g}$ *be the adjoint linear operator of* P *with respect to* $\mathcal{B}$ *as defined in Eq. 10. Then there is a compatible special L-dendriform algebra* $(\mathfrak{g}, \triangleright, \triangleleft)$ *of the pre-Lie algebra* $(\mathfrak{g}, \circ)$ *defined by*

$$x \triangleleft y := -\widehat{P}([x, y]), \quad x \triangleright y := x \circ y - x \triangleleft y := [P(x), y] + \widehat{P}([x, y]), \quad \forall x, y \in \mathfrak{g}. \tag{26}$$

Proof (a). By Remark 2.6, $-P$ is adjoint-admissible to $(\mathfrak{g}, [-,-], P)$. Hence the conclusion follows from Proposition 2.16.

(b). By Proposition 2.7, $\widehat{P}$ is adjoint-admissible to $(\mathfrak{g}, [-,-], P)$. Then the conclusion follows from Proposition 2.16. □

Proposition 2.18 *Let* $(\mathfrak{g}, [-,-], P)$ *be a Rota-Baxter Lie algebra of weight* 0 *and* $(\mathfrak{g}, \circ)$ *be the induced pre-Lie algebra. If there is an invariant bilinear form* $\mathcal{B}$ *on the Lie algebra* $(\mathfrak{g}, [-,-])$, *then* $\mathcal{B}$ *is left-invariant on the pre-Lie algebra* $(\mathfrak{g}, \circ)$.

Proof For all $x, y, z \in \mathfrak{g}$, we have

$$\mathcal{B}(x \circ y, z) = \mathcal{B}([P(x), y], z) = -\mathcal{B}(y, [P(x), z]) = -\mathcal{B}(y, x \circ z).$$

Hence the conclusion follows. □

Corollary 2.19 *Let* $(\mathfrak{g}, [-,-], P)$ *be a Rota-Baxter Lie algebra of weight* 0 *and* $(\mathfrak{g}, \circ)$ *be the induced pre-Lie algebra. Suppose that* $\mathcal{B}$ *is a nondegenerate symmetric invariant bilinear form on* $(\mathfrak{g}, [-,-])$. *Let* $\widehat{P} : \mathfrak{g} \to \mathfrak{g}$ *be the adjoint linear operator of* P *with respect to* $\mathcal{B}$. *Then the special L-dendriform algebra defined by Eq. 26 agrees with the one defined by Eq. 22.*

Proof On the one hand, by Corollary 2.17 (b), there is a compatible special L-dendriform algebra $(\mathfrak{g}, \triangleright_1, \triangleleft_1)$ of the pre-Lie algebra $(\mathfrak{g}, \circ)$ defined by Eq. 26. On the other hand, by Proposition 2.18, $\mathcal{B}$ is left-invariant on the pre-Lie algebra $(\mathfrak{g}, \circ)$. Hence by Lemma 2.14, there is a compatible special L-dendriform algebra $(\mathfrak{g}, \triangleright_2, \triangleleft_2)$ of the pre-Lie algebra $(\mathfrak{g}, \circ)$ defined by Eq. 22. For all $x, y, z \in \mathfrak{g}$, we have

$$\begin{aligned}\mathcal{B}(x \triangleleft_2 y, z) &= \mathcal{B}(x, z \circ y) = \mathcal{B}(x, [P(z), y]) = -\mathcal{B}([x, y], P(z)) \\ &= -\mathcal{B}(\widehat{P}([x, y]), z) = \mathcal{B}(x \triangleleft_1 y, z).\end{aligned}$$

Thus $x \triangleleft_2 y = x \triangleleft_1 y$. Moreover

$$x \triangleright_2 y = x \circ y - x \triangleleft_2 y = x \circ y - x \triangleleft_1 y = x \triangleright_1 y, \quad \forall x, y \in \mathfrak{g}.$$

Hence the special L-dendriform algebras $(\mathfrak{g}, \triangleright_1, \triangleleft_1)$ and $(\mathfrak{g}, \triangleright_2, \triangleleft_2)$ coincide. □

Example 2.20 Continue with the assumptions in Example 2.8. There is a pre-Lie algebra $(\mathfrak{g}, \circ)$ induced from the Rota-Baxter Lie algebra $(\mathfrak{g}, [-,-], P)$ of weight 0, whose product is explicitly given by

$$x \circ x = -h,\ x \circ h = -2x + 2y,\ x \circ y = h,$$
$$h \circ x = 4x - 4h, \qquad h \circ h = 8y, \qquad h \circ y = -4y,$$
$$y \circ x = 3h - 4x,\ y \circ h = -2x - 6y,\ y \circ y = h + 4y.$$

The sub-adjacent Lie algebra $\mathfrak{g}' = (\mathfrak{g}, [-,-]')$ of $(\mathfrak{g}, \circ)$ is given by

$$[h, x]' = 6x - 4h - 2y,\ [h, y]' = 2x + 2y,\ [x, y]' = 4x - 2h, \tag{27}$$

which is solvable and not isomorphic to $(\mathfrak{g}, [-,-])$. Moreover, there is a compatible special L-dendriform algebra $(\mathfrak{g}, \triangleright, \triangleleft)$ of the pre-Lie algebra $(\mathfrak{g}, \circ)$ given by

$$h \triangleleft x := -\widehat{P}([h, x]) = 6x - 4h - 2y, h \triangleleft y := -\widehat{P}([h, y]) = 2x + 2y,$$
$$x \triangleleft y := -\widehat{P}([x, y]) = 4x - 2h,$$

and $a \triangleright b := a \circ b - a \triangleleft b$ for all $a, b \in \mathfrak{g}$. Also note that $(\mathfrak{g}, \triangleleft)$ coincides with the Lie algebra $\mathfrak{g}'$.

3 Rota-Baxter Lie Bialgebras and Special L-dendriform Bialgebras

In this section, we introduce the notion of Manin triples of Rota-Baxter Lie algebras. When the weight is 0, they give rise to Manin triples of pre-Lie algebras with respect to the symmetric left-invariant bilinear form. Interpreting this relation on the level of bialgebras, we introduce the notion of Rota-Baxter Lie bialgebras, and show that they give rise to special L-dendriform bialgebras when the weight is 0.

3.1 Manin Triples of Rota-Baxter Lie Algebras and Pre-Lie Algebras

Recall [10] that a (standard) **Manin triple of Lie algebras** is a triple $((\mathfrak{g} \oplus \mathfrak{g}^*, [-,-]), \mathfrak{g}, \mathfrak{g}^*)$ of Lie algebras, such that $(\mathfrak{g}, [-,-]_{\mathfrak{g}})$ and $(\mathfrak{g}^*, [-,-]_{\mathfrak{g}^*})$ are Lie subalgebras of the Lie algebra $(\mathfrak{g} \oplus \mathfrak{g}^*, [-,-])$, and the natural nondegenerate symmetric bilinear form $\mathcal{B}_d$ on $(\mathfrak{g} \oplus \mathfrak{g}^*, [-,-])$ given by

$$\mathcal{B}_d(x + a^*, y + b^*) = \langle x, b^* \rangle + \langle a^*, y \rangle, \quad \forall x, y \in \mathfrak{g}, a^*, b^* \in \mathfrak{g}^*, \tag{28}$$

is invariant. We extend this notion to Rota-Baxter Lie algebras.

Definition 3.1 A **Manin triple of Rota-Baxter Lie algebras of weight** λ is a triple $((\mathfrak{g} \oplus \mathfrak{g}^*, [-,-], P_{\mathfrak{g}\oplus\mathfrak{g}^*}), (\mathfrak{g}, P), (\mathfrak{g}^*, Q^*))$ of Rota-Baxter Lie algebras of weight λ such that $((\mathfrak{g} \oplus \mathfrak{g}^*, [-,-]), \mathfrak{g}, \mathfrak{g}^*)$ is a Manin triple of Lie algebras and $P_{\mathfrak{g}\oplus\mathfrak{g}^*} = P + Q^*$. We denote it by $((\mathfrak{g} \oplus \mathfrak{g}^*, [-,-], P + Q^*), (\mathfrak{g}, P), (\mathfrak{g}^*, Q^*))$.

By definition, for a Manin triple of Rota-Baxter Lie algebras $((\mathfrak{g} \oplus \mathfrak{g}^*, [-,-], P + Q^*), (\mathfrak{g}, P), (\mathfrak{g}^*, Q^*))$ of weight λ, the triples $(\mathfrak{g}, [-,-]_{\mathfrak{g}}, P)$ and $(\mathfrak{g}^*, [-,-]_{\mathfrak{g}^*}, Q^*)$ are clearly Rota-Baxter Lie subalgebras of $(\mathfrak{g} \oplus \mathfrak{g}^*, [-,-], P + Q^*)$. Moreover, we have the following conclusion.

Lemma 3.2 *Let* $((\mathfrak{g}\oplus\mathfrak{g}^*, [-,-], P+Q^*), (\mathfrak{g}, P), (\mathfrak{g}^*, Q^*))$ *be a Manin triple of Rota-Baxter Lie algebras of weight* λ.

(a) *The adjoint* $\widehat{P+Q^*}$ *of* $P+Q^*$ *with respect to* $\mathcal{B}_d$ *is* $Q+P^*$. *Further* $Q+P^*$ *is adjoint-admissible to* $(\mathfrak{g}\oplus\mathfrak{g}^*, [-,-], P+Q^*)$.
(b) Q *is adjoint-admissible to* $(\mathfrak{g}, [-,-]_{\mathfrak{g}}, P)$.
(c) P^* *is adjoint-admissible to* $(\mathfrak{g}^*, [-,-]_{\mathfrak{g}^*}, Q^*)$.

Proof The proof is similar to the one of [6, Lemma 3.11]. □

Definition 3.3 [5] Suppose that there are three pre-Lie algebras $(A, \circ_A)$, $(A^*, \circ_{A^*})$ and $(A\oplus A^*, \circ)$ such that $(A, \circ_A)$ and $(A^*, \circ_{A^*})$ are pre-Lie subalgebras of $(A\oplus A^*, \circ)$, and the natural nondegenerate symmetric bilinear form $\mathcal{B}_d$ on $A\oplus A^*$ defined by Eq. 28 is left-invariant. Then the triple $((A\oplus A^*, \circ), A, A^*)$ is called a **Manin triple of pre-Lie algebras with respect to the symmetric left-invariant bilinear form**.

By Corollary 2.17 (b) and Proposition 2.18, we obtain the following result.

Corollary 3.4 *Let* $((\mathfrak{g}\oplus\mathfrak{g}^*, P+Q^*), (\mathfrak{g}, P), (\mathfrak{g}^*, Q^*))$ *be a Manin triple of Rota-Baxter Lie algebras of weight* 0. *Then there is a Manin triple of pre-Lie algebras* $((\mathfrak{g}\oplus\mathfrak{g}^*, \circ), \mathfrak{g}, \mathfrak{g}^*)$ *with respect to the symmetric left-invariant bilinear form, where*

$$(x+a^*)\circ(y+b^*) = [(P+Q^*)(x+a^*), y+b^*], \ \forall x, y\in\mathfrak{g}, a^*, b^*\in\mathfrak{g}^*.$$

Moreover, there is a special L-dendriform algebra $(\mathfrak{g}\oplus\mathfrak{g}^*, \triangleright, \triangleleft)$ *defined by*

$$(x+a^*)\triangleleft(y+b^*) = -(Q+P^*)([x+a^*, y+b^*]), \tag{29}$$
$$(x+a^*)\triangleright(y+b^*) = (x+a^*)\circ(y+b^*) - (x+a^*)\triangleleft(y+b^*), \ \forall x, y\in\mathfrak{g}, a^*, b^*\in\mathfrak{g}^*. \tag{30}$$

It contains $(\mathfrak{g}, \triangleright_{\mathfrak{g}}, \triangleleft_{\mathfrak{g}})$ *and* $(\mathfrak{g}^*, \triangleright_{\mathfrak{g}^*}, \triangleleft_{\mathfrak{g}^*})$ *as special L-dendriform subalgebras, where*

$$x\triangleleft_{\mathfrak{g}} y = -Q([x,y]_{\mathfrak{g}}), \quad x\triangleright_{\mathfrak{g}} y = x\circ_{\mathfrak{g}} y - x\triangleleft_{\mathfrak{g}} y, \tag{31}$$
$$a^*\triangleleft_{\mathfrak{g}^*} b^* = -P^*([a^*, b^*]_{\mathfrak{g}^*}), \ a^*\triangleright_{\mathfrak{g}^*} b^* = a^*\circ_{\mathfrak{g}^*} b^* - a^*\triangleleft_{\mathfrak{g}^*} b^*, \ \forall x, y\in\mathfrak{g}, a^*, b^*\in\mathfrak{g}^*. \tag{32}$$

3.2 Rota-Baxter Lie Bialgebras and Special L-dendriform Bialgebras

Recall that a **Lie bialgebra** [10] is a triple $(\mathfrak{g}, [-,-], \delta)$ consisting of a vector space $\mathfrak{g}$ and linear maps $[-,-]:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$, $\delta:\mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g}$ such that

(a) $(\mathfrak{g}, [-,-])$ is a Lie algebra.
(b) $(\mathfrak{g}, \delta)$ is a Lie coalgebra.
(c) δ is a 1-cocycle of $(\mathfrak{g}, [-,-])$ with values in $\mathfrak{g}\otimes\mathfrak{g}$, that is,

$$\delta([x,y]) = (\mathrm{ad}(x)\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{ad}(x))\delta(y)-(\mathrm{ad}(y)\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{ad}(y))\delta(x), \forall x, y\in\mathfrak{g}. \tag{33}$$

We now extend the notion of Lie bialgebras to Rota-Baxter Lie bialgebras. First we give the notion of Rota-Baxter Lie coalgebras.

Definition 3.5 A **Rota-Baxter Lie coalgebra of weight** λ is a triple $(\mathfrak{g}, \delta, Q)$, where $(\mathfrak{g}, \delta)$ is a Lie coalgebra, and $Q:\mathfrak{g}\to\mathfrak{g}$ is a linear map such that

$$(Q\otimes Q)\delta(x) = (Q\otimes\mathrm{id}+\mathrm{id}\otimes Q)\delta(Q(x)) + \lambda\delta(Q(x)), \quad \forall x\in\mathfrak{g}. \tag{34}$$

Extending the well-known duality between Lie coalgebras and Lie algebras, for a finite-dimensional vector space $\mathfrak{g}$, the pair $(\mathfrak{g}^*, [-,-]_{\mathfrak{g}^*}, Q^*)$ is a Rota-Baxter Lie algebra of weight λ if and only if $(\mathfrak{g}, \delta, Q)$ is a Rota-Baxter Lie coalgebra of weight λ, where $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is the linear dual of $[-,-]_{\mathfrak{g}^*} : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$, that is,

$$\langle \delta(x), a^* \otimes b^* \rangle = \langle x, [a^*, b^*]_{\mathfrak{g}^*} \rangle, \quad \forall x \in \mathfrak{g}, a^*, b^* \in \mathfrak{g}^*. \tag{35}$$

In this case, the Lie algebra structure on $\mathfrak{g}^*$ is also denoted by δ^*, that is,

$$\delta^*(a^* \otimes b^*) := [a^*, b^*]_{\mathfrak{g}^*}, \quad \forall a^*, b^* \in \mathfrak{g}^*.$$

Moreover, for a linear map $P : \mathfrak{g} \to \mathfrak{g}$, the condition that P^* is adjoint-admissible to the Rota-Baxter Lie algebra $(\mathfrak{g}^*, [-,-]_{\mathfrak{g}^*}, Q^*)$ of weight λ, that is, for all $a^*, b^* \in \mathfrak{g}^*$,

$$\begin{aligned} P^*([Q^*(a^*), b^*]_{\mathfrak{g}^*}) - [Q^*(a^*), P^*(b^*)]_{\mathfrak{g}^*} - P^*([a^*, P^*(b^*)]_{\mathfrak{g}^*}) \\ - \lambda[a^*, P^*(b^*)]_{\mathfrak{g}^*} = 0, \end{aligned} \tag{36}$$

can be rewritten in terms of δ as

$$(P \otimes Q)\delta(x) + (P \otimes \mathrm{id} - \mathrm{id} \otimes Q)\delta(P(x)) + \lambda(P \otimes \mathrm{id})\delta(x) = 0, \quad \forall x \in \mathfrak{g}. \tag{37}$$

Definition 3.6 A **Rota-Baxter Lie bialgebra of weight** λ is a vector space $\mathfrak{g}$ together with linear maps

$$[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}, \quad \delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}, \quad P, Q : \mathfrak{g} \to \mathfrak{g},$$

such that

(a) the triple $(\mathfrak{g}, [-,-], \delta)$ is a Lie bialgebra.
(b) the triple $(\mathfrak{g}, [-,-], P)$ is a Rota-Baxter Lie algebra of weight λ.
(c) the triple $(\mathfrak{g}, \delta, Q)$ is a Rota-Baxter Lie coalgebra of weight λ.
(d) Q is adjoint-admissible to $(\mathfrak{g}, [-,-], P)$, that is, Eq. 8 holds.
(e) P^* is adjoint-admissible to $(\mathfrak{g}^*, \delta^*, Q^*)$, that is, Eq. 37 holds.

We denote the Rota-Baxter Lie bialgebra by $(\mathfrak{g}, [-,-], P, \delta, Q)$.

Similar to [6, Theorem 3.13], we have the following result.

Theorem 3.7 *Let* $(\mathfrak{g}, [-,-]_{\mathfrak{g}}, P)$ *be a Rota-Baxter Lie algebra of weight* λ. *Suppose that there is a Rota-Baxter Lie algebra* $(\mathfrak{g}^*, [-,-]_{\mathfrak{g}^*}, Q^*)$ *of weight* λ *on the dual space* $\mathfrak{g}^*$. *Let* $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ *be the linear dual of* $[-,-]_{\mathfrak{g}^*} : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$. *Then there is a Manin triple* $((\mathfrak{g} \oplus \mathfrak{g}^*, [-,-], P + Q^*), (\mathfrak{g}, P), (\mathfrak{g}^*, Q^*))$ *of Rota-Baxter Lie algebras of weight* λ *if and only if* $(\mathfrak{g}, [-,-]_{\mathfrak{g}}, P, \delta, Q)$ *is a Rota-Baxter Lie bialgebra of weight* λ.

On the other hand, there is a bialgebra structure for special L-dendriform algebras introduced in [5].

Definition 3.8 Let A be a vector space and $\Delta, \nabla : A \to A \otimes A$ be linear maps. Suppose that ∇ is co-antisymmetric in the sense that $\nabla + \tau\nabla = 0$ for the flip map $\tau : A \otimes A \to A \otimes A$. Set $\lozenge := \Delta + \nabla$. If the following two equalities hold:

$$(\mathrm{id} \otimes \nabla)\nabla + (\tau \otimes \mathrm{id})(\mathrm{id} \otimes \lozenge)\nabla + (\lozenge \otimes \mathrm{id})\nabla - (\mathrm{id} \otimes \nabla)\lozenge = 0, \tag{38}$$

$$(\lozenge \otimes \mathrm{id})\lozenge - (\mathrm{id} \otimes \lozenge)\lozenge = (\tau \otimes \mathrm{id})(\lozenge \otimes \mathrm{id})\lozenge - (\tau \otimes \mathrm{id})(\mathrm{id} \otimes \lozenge)\lozenge, \tag{39}$$

then (A, Δ, ∇) is called a **special L-dendriform coalgebra**.

Proposition 3.9 *Let A be a finite-dimensional vector space, and $\Delta, \nabla : A \to A \otimes A$ be linear maps. Let $\triangleright_{A^*}, \triangleleft_{A^*} : A^* \otimes A^* \to A^*$ be the linear duals of Δ and ∇ respectively. Then (A, Δ, ∇) is a special L-dendriform coalgebra if and only if $(A^*, \triangleright_{A^*}, \triangleleft_{A^*})$ is a special L-dendriform algebra.*

Proof It is obvious that ∇ is co-antisymmetric if and only if $\triangleleft_{A^*}$ is antisymmetric. Let $\circ_{A^*} : A^* \otimes A^* \to A^*$ be a linear operation given by

$$a^* \circ_{A^*} b^* = a^* \triangleright_{A^*} b^* + a^* \triangleleft_{A^*} b^*, \quad \forall a^*, b^* \in A^*.$$

Then for all $x \in A, a^*, b^*, c^* \in A^*$, we have

$$\begin{aligned}
&\langle (\mathrm{id} \otimes \nabla)\nabla(x) + (\tau \otimes \mathrm{id})(\mathrm{id} \otimes \Diamond)\nabla(x) + (\Diamond \otimes \mathrm{id})\nabla(x) - (\mathrm{id} \otimes \nabla)\Diamond(x), a^* \otimes b^* \otimes c^* \rangle \\
&= \langle x, a^* \triangleleft_{A^*} (b^* \triangleleft_{A^*} c^*) + b^* \triangleleft_{A^*} (a^* \circ_{A^*} c^*) + (a^* \circ_{A^*} b^*) \triangleleft_{A^*} c^* - a^* \circ_{A^*} (b^* \triangleleft_{A^*} c^*) \rangle, \\
&\langle (\Diamond \otimes \mathrm{id})\Diamond(x) - (\mathrm{id} \otimes \Diamond)\Diamond(x) - (\tau \otimes \mathrm{id})(\Diamond \otimes \mathrm{id})\Diamond(x) + (\tau \otimes \mathrm{id})(\mathrm{id} \otimes \Diamond)\Diamond(x), a^* \otimes b^* \otimes c^* \rangle \\
&= \langle x, (a^* \circ_{A^*} b^*) \circ_{A^*} c^* - a^* \circ_{A^*} (b^* \circ_{A^*} c^*) - (b^* \circ_{A^*} a^*) \circ_{A^*} c^* + b^* \circ_{A^*} (a^* \circ_{A^*} c^*) \rangle.
\end{aligned}$$

Hence the conclusion follows by Proposition 2.12. □

Definition 3.10 A **special L-dendriform bialgebra** is a quintuple $(A, \triangleright, \triangleleft, \Delta, \nabla)$ consisting of a special L-dendriform algebra $(A, \triangleright, \triangleleft)$ and a special L-dendriform coalgebra (A, Δ, ∇) such that the following compatibility conditions hold.

$$\Diamond(x \circ y) - (\mathrm{id} \otimes \mathcal{R}_\circ(y))\Delta(x) + (\mathcal{L}_\triangleleft(y) \otimes \mathrm{id})\nabla(x) - (\mathcal{L}_\triangleright(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathcal{L}_\circ(x))\Diamond(y) = 0, \quad (40)$$

$$(\tau - \mathrm{id}^{\otimes 2})((\mathrm{id} \otimes \mathcal{L}_\triangleleft(x))\Diamond(y) - (\mathrm{id} \otimes \mathcal{L}_\triangleleft(y))\Diamond(x) - \Diamond(x \triangleleft y)) = 0, \quad (41)$$

$$\nabla([x, y]) + (\mathcal{L}_\circ(y) \otimes \mathrm{id} + \mathrm{id} \otimes \mathcal{L}_\circ(y))\nabla(x) - (\mathcal{L}_\circ(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathcal{L}_\circ(x))\nabla(y) = 0, \quad (42)$$

for all $x, y \in A$ and $\Diamond := \Delta + \nabla$.

Remark 3.11 This notion of a special L-dendriform bialgebra is equivalent to the one introduced in [5] which is given in terms of the operations on the dual space. Indeed, in [5] the compatible conditions are given by four equations in Theorem 3.16 there. The first and second equations coincide with Eqs. 42 and 40 respectively, while the third and fourth equations are equivalent to Eqs. 41 and 40 respectively.

Theorem 3.12 [5] *Let $(A, \triangleright_A, \triangleleft_A)$ be a special L-dendriform algebra. Suppose that there is a special L-dendriform algebra structure $(A^*, \triangleright_{A^*}, \triangleleft_{A^*})$ on the dual space A^*. Let $(A, \circ_A)$ and $(A^*, \circ_{A^*})$ be the sub-adjacent pre-Lie algebras of $(A, \triangleright_A, \triangleleft_A)$ and $(A^*, \triangleright_{A^*}, \triangleleft_{A^*})$ respectively, and $\Delta, \nabla : A \to A \otimes A$ be the linear duals of $\triangleright_{A^*}, \triangleleft_{A^*}$ respectively. Then there is a Manin triple of pre-Lie algebras $((A \oplus A^*, \circ), A, A^*)$ with respect to the symmetric left-invariant bilinear form such that the induced special L-dendriform algebra $(A \oplus A^*, \triangleright, \triangleleft)$ defined by*

$$\begin{aligned}
&\mathcal{B}_d((x + a^*) \triangleleft (y + b^*), z + c^*) = \mathcal{B}_d(x + a^*, (z + c^*) \circ (y + b^*)), \\
&(x + a^*) \triangleright (y + b^*) = (x + a^*) \circ (y + b^*) - (x + a^*) \triangleleft (y + b^*), \ \forall x, y, z \in A, a^*, b^*, c^* \in A^*,
\end{aligned}$$

includes $(A, \triangleright_A, \triangleleft_A)$ and $(A^*, \triangleright_{A^*}, \triangleleft_{A^*})$ as special L-dendriform subalgebras if and only if $(A, \triangleright_A, \triangleleft_A, \Delta, \nabla)$ is a special L-dendriform bialgebra.

Proposition 3.13 *Let $(\mathfrak{g}, [-,-]_\mathfrak{g}, P)$ and $(\mathfrak{g}^*, [-,-]_{\mathfrak{g}^*}, Q^*)$ be Rota-Baxter Lie algebras of weight 0 such that Q is adjoint-admissible to $(\mathfrak{g}, [-,-]_\mathfrak{g}, P)$ and P^* is adjoint-admissible to $(\mathfrak{g}^*, [-,-]_{\mathfrak{g}^*}, Q^*)$. Let $(\mathfrak{g}, \circ_\mathfrak{g})$ and $(\mathfrak{g}^*, \circ_{\mathfrak{g}^*})$ be the induced pre-Lie algebras defined by Eq. 2, and let $(\mathfrak{g}, \triangleright_\mathfrak{g}, \triangleleft_\mathfrak{g})$ and $(\mathfrak{g}^*, \triangleright_{\mathfrak{g}^*}, \triangleleft_{\mathfrak{g}^*})$ be the compatible special L-dendriform algebras of the pre-Lie algebras $(\mathfrak{g}, \circ_\mathfrak{g})$ and $(\mathfrak{g}^*, \circ_{\mathfrak{g}^*})$ defined by Eqs. 31 and 32 respectively.*

Let $\delta, \Delta, \nabla : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ be the linear duals of $[-,-]_{\mathfrak{g}^*}, \triangleright_{\mathfrak{g}^*}$ and $\triangleleft_{\mathfrak{g}^*}$ respectively. Then $(\mathfrak{g}, \triangleright_{\mathfrak{g}}, \triangleleft_{\mathfrak{g}}, \Delta, \nabla)$ is a special L-dendriform bialgebra if and only if the following equations hold:

$$\begin{aligned}&(Q\otimes \mathrm{id})(\delta([P(x),y]_{\mathfrak{g}} + (\mathrm{id}\otimes \mathrm{ad}_{\mathfrak{g}}(y) + \mathrm{ad}_{\mathfrak{g}}(y)\otimes \mathrm{id})\delta(P(x))\\ &-(\mathrm{id}\otimes \mathrm{ad}_{\mathfrak{g}}(P(x)) + \mathrm{ad}_{\mathfrak{g}}(P(x))\otimes \mathrm{id})\delta(y)) = 0,\end{aligned} \tag{43}$$

$$(Q\otimes Q)(\delta([x,y]_{\mathfrak{g}} + (\mathrm{id}\otimes \mathrm{ad}_{\mathfrak{g}}(y) + \mathrm{ad}_{\mathfrak{g}}(y)\otimes \mathrm{id})\delta(x) - (\mathrm{id}\otimes \mathrm{ad}_{\mathfrak{g}}(x) + \mathrm{ad}_{\mathfrak{g}}(x)\otimes \mathrm{id})\delta(y)) = 0, \tag{44}$$

$$\begin{aligned}\delta([P(x),P(y)]_{\mathfrak{g}}) = &(\mathrm{id}\otimes \mathrm{ad}_{\mathfrak{g}}(P(x)) + \mathrm{ad}_{\mathfrak{g}}(P(x))\otimes \mathrm{id})\delta(P(y))\\ &-(\mathrm{id}\otimes \mathrm{ad}_{\mathfrak{g}}(P(y)) + \mathrm{ad}_{\mathfrak{g}}(P(y))\otimes \mathrm{id})\delta(P(x)),\end{aligned} \tag{45}$$

for all $x, y \in \mathfrak{g}$. In particular, if $(\mathfrak{g}, [-,-]_{\mathfrak{g}}, P, \delta, Q)$ is a Rota-Baxter Lie bialgebra of weight 0, then $(\mathfrak{g}, \triangleright_{\mathfrak{g}}, \triangleleft_{\mathfrak{g}}, \Delta, \nabla)$ is a special L-dendriform bialgebra.

Proof It is obvious that $(\mathfrak{g}, \Delta, \nabla)$ is a special L-dendriform coalgebra. Let $\Diamond$ be the linear dual of $\circ_{\mathfrak{g}^*}$. By Eqs. 2, 31 and 32, for all $x, y \in \mathfrak{g}$, we have

$$\Diamond(x) = (Q\otimes \mathrm{id})\delta(x), \quad \nabla(x) = -\delta(P(x)), \quad \Delta(x) = (Q\otimes \mathrm{id})\delta(x) + \delta(P(x)), \tag{46}$$

$$\begin{aligned}&\mathcal{L}_{\circ_{\mathfrak{g}}}(x) = \mathrm{ad}_{\mathfrak{g}}(P(x)), \mathcal{R}_{\circ_{\mathfrak{g}}}(x) = -\mathrm{ad}_{\mathfrak{g}}(x)P,\\ &\mathcal{L}_{\triangleleft_{\mathfrak{g}}}(x) = -Q\mathrm{ad}_{\mathfrak{g}}(x), \mathcal{L}_{\triangleright_{\mathfrak{g}}}(x) = \mathrm{ad}_{\mathfrak{g}}(P(x)) + Q\mathrm{ad}_{\mathfrak{g}}(x).\end{aligned} \tag{47}$$

Therefore we have

$$\begin{aligned}
\Diamond(x\circ_{\mathfrak{g}} y) &= (Q\otimes \mathrm{id})\delta([P(x),y]_{\mathfrak{g}}),\\
-(\mathrm{id}\otimes \mathcal{R}_{\circ_{\mathfrak{g}}})\Delta(x) &= (\mathrm{id}\otimes \mathrm{ad}_{\mathfrak{g}}(y)P)((Q\otimes \mathrm{id})\delta(x) + \delta(P(x)))\\
&= (\mathrm{id}\otimes \mathrm{ad}_{\mathfrak{g}}(y))((Q\otimes P)\delta(x) + (\mathrm{id}\otimes P)\delta(P(x)))\\
&\overset{(37)}{=} (\mathrm{id}\otimes \mathrm{ad}_{\mathfrak{g}}(y))(Q\otimes \mathrm{id})\delta(P(x)) = (Q\otimes \mathrm{id})(\mathrm{id}\otimes \mathrm{ad}_{\mathfrak{g}}(y))\delta(P(x)),\\
(\mathcal{L}_{\triangleleft_{\mathfrak{g}}}(y)\otimes \mathrm{id})\nabla(x) &= (Q\otimes \mathrm{id})(\mathrm{ad}_{\mathfrak{g}}(y)\otimes \mathrm{id})\delta(P(x)),\\
-(\mathcal{L}_{\triangleright_{\mathfrak{g}}}(x)\otimes \mathrm{id})\Diamond(y) &= -(\mathrm{ad}_{\mathfrak{g}}(P(x))\otimes \mathrm{id})(Q\otimes \mathrm{id})\delta(y) - (Q\mathrm{ad}_{\mathfrak{g}}(x)\otimes \mathrm{id})(Q\otimes \mathrm{id})\delta(y)\\
&\overset{(8)}{=} -(Q\otimes \mathrm{id})(\mathrm{ad}_{\mathfrak{g}}(P(x))\otimes \mathrm{id})\delta(y),\\
-(\mathrm{id}\otimes \mathcal{L}_{\circ_{\mathfrak{g}}})\Diamond(y) &= -(Q\otimes \mathrm{id})(\mathrm{id}\otimes \mathrm{ad}_{\mathfrak{g}}(P(x)))\delta(y).
\end{aligned}$$

Thus Eq. 40 is equivalent to Eq. 43. Similarly, Eqs. 41 and 42 are equivalent to Eqs. 44 and 45 respectively. In particular, if $(\mathfrak{g}, [-,-]_{\mathfrak{g}}, P, \delta, Q)$ is a Rota-Baxter Lie bialgebra of weight 0, then Eq. 33 holds. Therefore Eqs. 43-45 hold. Thus $(\mathfrak{g}, \triangleright_{\mathfrak{g}}, \triangleleft_{\mathfrak{g}}, \Delta, \nabla)$ is a special L-dendriform bialgebra. □

4 Coboundary Rota-Baxter Lie Bialgebras, Admissible CYBEs and the Induced Special L-dendriform Bialgebras

In this section, we study the coboundary Rota-Baxter Lie bialgebras. A class of such bialgebras can be obtained from antisymmetric solutions of a variation of the CYBE, namely the admissible CYBE in Rota-Baxter Lie algebras with the adjoint-admissible condition. Furthermore, such solutions can be obtained from $\mathcal{O}$-operators of Rota-Baxter Lie algebras and Rota-Baxter pre-Lie algebras. Joint with Proposition 3.13, an antisymmetric solution of the admissible CYBE in a Rota-Baxter Lie algebra of weight 0 further gives a special L-dendriform bialgebra. We also provide two actual examples to exhibit the procedure from a Rota-Baxter Lie algebra of weight 0 (respectively, a Rota-Baxter pre-Lie algebra of weight 0) to a special L-dendriform bialgebra.

4.1 Coboundary Rota-Baxter Lie Bialgebras and the Induced Special L-dendriform Bialgebras

Recall [10] that a Lie bialgebra $(\mathfrak{g}, [-,-], \delta)$ is called **coboundary** if there exists an $r \in \mathfrak{g}\otimes\mathfrak{g}$ such that

$$\delta(x) := \delta_r(x) := (\mathrm{ad}(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(x))r, \quad \forall x \in \mathfrak{g}. \tag{48}$$

For Rota-Baxter Lie bialgebras we give a similar notion.

Definition 4.1 A Rota-Baxter Lie bialgebra $(\mathfrak{g}, [-,-], P, \delta, Q)$ is called **coboundary** if there exists an $r \in \mathfrak{g} \otimes \mathfrak{g}$ such that Eq. 48 holds.

Let $(\mathfrak{g}, [-,-])$ be a Lie algebra and $r = \sum_i a_i \otimes b_i \in \mathfrak{g}\otimes\mathfrak{g}$. Let $\delta : \mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g}$ be a linear map defined by Eq. 48. Then δ satisfies the 1-cocycle condition in Eq. 33 automatically. Moreover, by [10], δ makes $(\mathfrak{g}, \delta)$ into a Lie coalgebra such that $(\mathfrak{g}, [-,-], \delta)$ is a Lie bialgebra if and only if for all $x \in \mathfrak{g}$,

$$(\mathrm{ad}(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(x))(r + \tau(r)) = 0, \tag{49}$$

$$\begin{aligned}(\mathrm{ad}(x) \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{ad}(x))([r_{12}, r_{13}] \\ + [r_{12}, r_{23}] + [r_{13}, r_{23}]) = 0,\end{aligned} \tag{50}$$

where

$$[r_{12}, r_{13}] := \sum_{i,j}[a_i, a_j]\otimes b_i \otimes b_j, [r_{12}, r_{23}] := \sum_{i,j} a_i \otimes [b_i, a_j]\otimes b_j, [r_{13}, r_{23}] := \sum_{i,j} a_i \otimes a_j \otimes [b_i, b_j].$$

Suppose that, in addition, $(\mathfrak{g}, [-,-], P)$ is a Q-adjoint-admissible Rota-Baxter Lie algebra of weight λ. In order for $(\mathfrak{g}, [-,-], P, \delta, Q)$ to be a Rota-Baxter Lie bialgebra, we only need to further require that $(\mathfrak{g}^*, \delta^*, Q^*)$ is a P^*-adjoint-admissible Rota-Baxter Lie algebra, that is, $(\mathfrak{g}, \delta, Q)$ is a Rota-Baxter Lie coalgebra and Eq. 37 holds.

Proposition 4.2 *Let $(\mathfrak{g}, [-,-], P)$ be a Q-adjoint-admissible Rota-Baxter Lie algebra of weight λ and $r \in \mathfrak{g} \otimes \mathfrak{g}$. Define a linear map $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ by Eq. 48. Suppose that δ^* defines a Lie algebra structure on $\mathfrak{g}^*$. Then the following conclusions hold.*

(a) *Eq. 34 holds if and only if for all $x \in \mathfrak{g}$,*

$$\begin{aligned}&(\mathrm{id} \otimes Q(\mathrm{ad}(x)) - \mathrm{id} \otimes \mathrm{ad}(Q(x)))(Q \otimes \mathrm{id} - \mathrm{id} \otimes P)(r) \\ &-(Q(\mathrm{ad}(x)) \otimes \mathrm{id} - \mathrm{ad}(Q(x)) \otimes \mathrm{id})(P \otimes \mathrm{id} - \mathrm{id} \otimes Q)(r) = 0.\end{aligned} \tag{51}$$

(b) *Eq. 37 holds if and only if for all $x \in \mathfrak{g}$,*

$$\begin{aligned}&(\mathrm{id} \otimes \mathrm{ad}(P(x)) + \mathrm{ad}(P(x)) \otimes \mathrm{id} + \mathrm{id} \otimes Q(\mathrm{ad}(x)) \\ &-P(\mathrm{ad}(x)) \otimes \mathrm{id} + \lambda\mathrm{id} \otimes \mathrm{ad}(x))(P \otimes \mathrm{id} - \mathrm{id} \otimes Q)(r) = 0.\end{aligned} \tag{52}$$

Proof The conclusion is obtained by following the proof of [6, Theorem 4.3]. □

Theorem 4.3 *Let $(\mathfrak{g}, [-,-], P)$ be a Q-adjoint-admissible Rota-Baxter Lie algebra and $r \in \mathfrak{g} \otimes \mathfrak{g}$. Define a linear map $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ by Eq. 48. Then $(\mathfrak{g}, [-,-], P, \delta, Q)$ is a Rota-Baxter Lie bialgebra if and only if Eqs. 49–52 hold.*

Proof By the assumption, $(\mathfrak{g}, [-,-], P, \delta, Q)$ is a Rota-Baxter Lie bialgebra if and only if $(\mathfrak{g}^*, \delta^*, Q^*)$ is a P^*-adjoint-admissible Rota-Baxter Lie algebra. By Proposition 4.2 and the remarks before it, the latter holds if and only if Eqs. 49–52 hold. □

In particular, we have the following conclusion.

Corollary 4.4 *Let $(\mathfrak{g}, [-,-], P)$ be a Q-adjoint-admissible Rota-Baxter Lie algebra and $r \in \mathfrak{g} \otimes \mathfrak{g}$. Define a linear map $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ by Eq. 48. Then $(\mathfrak{g}, [-,-], P, \delta, Q)$ is a Rota-Baxter Lie bialgebra if Eq. 49 and the following equations hold*:

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \tag{53}$$

$$(P \otimes \mathrm{id} - \mathrm{id} \otimes Q)(r) = 0, \tag{54}$$

$$(Q \otimes \mathrm{id} - \mathrm{id} \otimes P)(r) = 0. \tag{55}$$

Eq. 53 is just the well-known **classical Yang-Baxter equation (CYBE)** in $\mathfrak{g}$ [10]. Moreover, Corollary 4.4 suggests the following variation of the CYBE.

Definition 4.5 Let $(\mathfrak{g}, [-,-], P)$ be a Rota-Baxter Lie algebra. Suppose that $r \in \mathfrak{g} \otimes \mathfrak{g}$ and $Q : \mathfrak{g} \to \mathfrak{g}$ is a linear map. Then Eq. 53 with conditions Eqs. 54 and 55 is called the **Q-admissible classical Yang-Baxter equation** in $(\mathfrak{g}, [-,-], P)$ or simply the **Q-admissible CYBE**.

Note that if r is antisymmetric (that is, $r + \tau(r) = 0$), then Eq. 54 holds if and only if Eq. 55 holds. Then Corollary 4.4 directly gives

Proposition 4.6 *Let $(\mathfrak{g}, [-,-], P)$ be a Q-adjoint-admissible Rota-Baxter Lie algebra and $r \in \mathfrak{g} \otimes \mathfrak{g}$ be an antisymmetric solution of the Q-admissible CYBE in $(\mathfrak{g}, [-,-], P)$. Then $(\mathfrak{g}, [-,-], P, \delta, Q)$ is a coboundary Rota-Baxter Lie bialgebra, where the linear map $\delta = \delta_r$ is defined by Eq. 48.*

On the other hand, there is a similar "coboundary" construction of special L-dendriform bialgebras considered in [5].

Proposition 4.7 [5] *Let $(A, \triangleright, \triangleleft)$ be a special L-dendriform algebra and $(A, \circ)$ be the sub-adjacent pre-Lie algebra. Let $r = \sum_i a_i \otimes b_i \in A \otimes A$ be antisymmetric. Define linear maps $\Delta, \nabla : A \to A \otimes A$ by*

$$\Delta(x) = (\mathcal{L}_{\triangleright}(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(x))(r),\ \nabla(x) = (\mathcal{L}_{\circ}(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathcal{L}_{\circ}(x))(-r),\ \forall x \in A. \tag{56}$$

Then $(A, \triangleright, \triangleleft, \Delta, \nabla)$ is a special L-dendriform bialgebra if r satisfies

$$r_{12} \triangleleft r_{13} = r_{12} \circ r_{23} + r_{13} \circ r_{23}, \tag{57}$$

where

$$r_{12} \triangleleft r_{13} = \sum_{i,j} a_i \triangleleft a_j \otimes b_i \otimes b_j,\ r_{12} \circ r_{23} = \sum_{i,j} a_i \otimes b_i \circ a_j \otimes b_j,\ r_{13} \circ r_{23} = \sum_{i,j} a_i \otimes a_j \otimes b_i \circ b_j.$$

Remark 4.8 In [5], $-r$ instead of r was used to define Δ and ∇ in Eq. 56. We change the sign of r in order to be consistent with another construction given in the following Proposition 4.10.

Proposition 4.9 *Let* $(\mathfrak{g}, [-,-], P)$ *be a Rota-Baxter Lie algebra of weight* 0 *and* $Q : \mathfrak{g} \to \mathfrak{g}$ *be a linear map which is adjoint-admissible to* $(\mathfrak{g}, [-,-], P)$. *Let* $(\mathfrak{g}, \circ)$ *be the induced pre-Lie algebra and* $(\mathfrak{g}, \rhd, \lhd)$ *be the compatible special L-dendriform algebra of* $(\mathfrak{g}, \circ)$, *where* $\circ$, $\lhd$ *and* $\rhd$ *are defined by Eqs.* 2, 24 *and* 19 *respectively, that is,*

$$x \circ y = [P(x), y], \quad x \lhd y = -Q([x, y]), \quad x \rhd y = x \circ y - x \lhd y, \quad \forall x, y \in \mathfrak{g}.$$

If $r \in \mathfrak{g} \otimes \mathfrak{g}$ *is a solution of the* Q*-admissible CYBE in* $(\mathfrak{g}, [-,-], P)$, *then* r *satisfies Eq.* 57.

Proof Let $r = \sum_i a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$. By Eq. 54, we have

$$\begin{aligned}
& r_{12} \circ r_{23} + r_{13} \circ r_{23} - r_{12} \lhd r_{13} \\
&= \sum_{i,j} a_i \otimes [P(b_i), a_j] \otimes b_j + a_i \otimes a_j \otimes [P(b_i), b_j] + Q([a_i, a_j]) \otimes b_i \otimes b_j \\
&= \sum_{i,j} Q(a_i) \otimes [b_i, a_j] \otimes b_j + Q(a_i) \otimes a_j \otimes [b_i, b_j] + Q([a_i, a_j]) \otimes b_i \otimes b_j \\
&= (Q \otimes \mathrm{id} \otimes \mathrm{id})([r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]).
\end{aligned}$$

Hence the conclusion follows. □

Let $(\mathfrak{g}, [-,-], P)$ be a Rota-Baxter Lie algebra of weight 0 and $Q : \mathfrak{g} \to \mathfrak{g}$ be a linear map which is adjoint-admissible to $(\mathfrak{g}, [-,-], P)$. From an antisymmetric solution $r \in \mathfrak{g} \otimes \mathfrak{g}$ of the Q-admissible CYBE in $(\mathfrak{g}, [-,-], P)$, there are two constructions of a special L-dendriform bialgebra. On the one hand, by Proposition 4.6, there is a coboundary Rota-Baxter Lie bialgebra $(\mathfrak{g}, [-,-], P, \delta, Q)$ of weight 0, where the linear map $\delta = \delta_r$ is defined by Eq. 48. Thus by Proposition 3.13, there is a special L-dendriform bialgebra $(\mathfrak{g}, \rhd, \lhd, \Delta, \nabla)$, where $\rhd, \lhd, \Delta, \nabla$ are given by Eqs. 31 and 46 respectively. On the other hand, by Proposition 4.9, r satisfies Eq. 57. Hence by Proposition 4.7, there is a special L-dendriform bialgebra $(\mathfrak{g}, \rhd, \lhd, \Delta', \nabla')$, where $\rhd, \lhd, \Delta', \nabla'$ are given by Eqs. 31 and 56 respectively. The two constructions turn out to be the same.

Proposition 4.10 *With the above notations, the special L-dendriform bialgebras* $(\mathfrak{g}, \rhd, \lhd, \Delta, \nabla)$ *and* $(\mathfrak{g}, \rhd, \lhd, \Delta', \nabla')$ *coincide.*

Proof Let the sub-adjacent Lie algebra of $(\mathfrak{g}, \circ)$ be $\mathfrak{g}' = (\mathfrak{g}, [-,-]')$. Let $r = \sum_i a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$ and $x \in \mathfrak{g}$. Then

$$\begin{aligned}
\nabla(x) &= -\delta(P(x)) = -(\mathrm{ad}(P(x)) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(P(x)))r = -\sum_i ([P(x), a_i] \otimes b_i + a_i \otimes [P(x), b_i]) \\
&= -\sum_i (x \circ a_i \otimes b_i + a_i \otimes x \circ b_i) = -(\mathcal{L}_\circ(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathcal{L}_\circ(x))r = \nabla'(x),
\end{aligned}$$

and similarly, with $\mathrm{ad}'(x)y = [x, y]'$ for $x, y \in \mathfrak{g}$, we have

$$\Delta(x) = (Q \otimes \mathrm{id})\delta(x) + \delta(P(x)) = (\mathcal{L}_\rhd(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}'(x))r = \Delta'(x),$$

showing that the two special L-dendriform bialgebras agree. □

4.2 Admissible CYBEs, $\mathcal{O}$-operators on Rota-Baxter Lie Algebras and Rota-Baxter Pre-Lie Algebras

We first give the operator forms of the antisymmetric solutions of the Q-admissible CYBE. For a vector space V, the isomorphism $V \otimes V \cong \mathrm{Hom}(V^*, V)$ identifies an $r \in V \otimes V$ with a linear map $T_r : V^* \to V$. Thus for $r = \sum_i u_i \otimes v_i$, the corresponding map T_r is

$$T_r : V^* \to V, \quad T_r(u^*) = \sum_i \langle u^*, u_i \rangle v_i, \quad \forall u^* \in V^*. \tag{58}$$

Theorem 4.11 *Let $(\mathfrak{g}, [-,-], P)$ be a Rota-Baxter Lie algebra and $r \in \mathfrak{g} \otimes \mathfrak{g}$ be antisymmetric. Let $Q : \mathfrak{g} \to \mathfrak{g}$ be a linear map. Then r is a solution of the Q-admissible CYBE in $(\mathfrak{g}, [-,-], P)$ if and only if T_r satisfies*

$$[T_r(a^*), T_r(b^*)] = T_r(\mathrm{ad}^*(T_r(a^*))b^* - \mathrm{ad}^*(T_r(b^*))a^*), \quad \forall a^*, b^* \in \mathfrak{g}^*, \tag{59}$$

$$PT_r = T_r Q^*. \tag{60}$$

Proof The proof follows the same argument as in the proof of [6, Theorem 4.12]. □

Then it is natural to introduce the following notion.

Definition 4.12 Let $(\mathfrak{g}, [-,-], P)$ be a Rota-Baxter Lie algebra of weight λ, (V, ρ) be a representation of $(\mathfrak{g}, [-,-])$ and $\alpha : V \to V$ be a linear map. A linear map $T : V \to \mathfrak{g}$ is called a **weak $\mathcal{O}$-operator associated to** (V, ρ) **and** α if T satisfies

$$[T(u), T(v)] = T(\rho(T(u))v - \rho(T(v))u), \quad \forall u, v \in V, \tag{61}$$

$$PT = T\alpha. \tag{62}$$

If in addition, (V, ρ, α) is a representation of $(\mathfrak{g}, [-,-], P)$, then T is called an **$\mathcal{O}$-operator associated to** (V, ρ, α).

Note that for a Lie algebra $(\mathfrak{g}, [-,-])$ and a representation (V, ρ) of $(\mathfrak{g}, [-,-])$, a linear map $T : V \to \mathfrak{g}$ satisfying Eq. 61 is called an **$\mathcal{O}$-operator of** $(\mathfrak{g}, [-,-])$ **associated to** (V, ρ) [15]. The terms relative Rota-Baxter operator and generalized Rota-Baxter operator are also used [18, 22].

Example 4.13 Let $(\mathfrak{g}, [-,-], P)$ be a Rota-Baxter Lie algebra of weight 0. Then P is an $\mathcal{O}$-operator of $(\mathfrak{g}, [-,-], P)$ associated to the adjoint representation $(\mathfrak{g}, \mathrm{ad}, P)$.

Theorem 4.11 can be rewritten in terms of $\mathcal{O}$-operators as follows.

Corollary 4.14 *Let $(\mathfrak{g}, [-,-], P)$ be a Rota-Baxter Lie algebra of weight λ and $r \in \mathfrak{g} \otimes \mathfrak{g}$ be antisymmetric. Let $Q : \mathfrak{g} \to \mathfrak{g}$ be a linear map. Then r is a solution of the Q-admissible CYBE in $(\mathfrak{g}, [-,-], P)$ if and only if T_r is a weak $\mathcal{O}$-operator associated to $(\mathfrak{g}^*, \mathrm{ad}^*)$ and Q^*. If in addition, $(\mathfrak{g}, [-,-], P)$ is a Q-adjoint-admissible Rota-Baxter Lie algebra, then r is a solution of the Q-admissible CYBE in $(\mathfrak{g}, [-,-], P)$ if and only if T_r is an $\mathcal{O}$-operator associated to the representation $(\mathfrak{g}^*, \mathrm{ad}^*, Q^*)$.*

On the other hand, an $\mathcal{O}$-operator of a Lie algebra gives rise to a solution of the CYBE in the semi-direct product Lie algebra.

Lemma 4.15 [2] *Let $(\mathfrak{g}, [-,-])$ be a Lie algebra and (V, ρ) be a representation. Let $T : V \to \mathfrak{g}$ be a linear map which is identified as an element in $(\mathfrak{g} \ltimes_{\rho^*} V^*) \otimes (\mathfrak{g} \ltimes_{\rho^*} V^*)$ through $\mathrm{Hom}(V, \mathfrak{g}) \cong V^* \otimes \mathfrak{g} \subseteq (\mathfrak{g} \ltimes_{\rho^*} V^*) \otimes (\mathfrak{g} \ltimes_{\rho^*} V^*)$. Then $= T - \tau(T)$ is an antisymmetric solution of the CYBE in the Lie algebra $\mathfrak{g} \ltimes_{\rho^*} V^*$ if and only if T is an $\mathcal{O}$-operator of $(\mathfrak{g}, [-,-])$ associated to (V, ρ).*

In order to extend the above construction to the context of Rota-Baxter Lie algebras, we consider the admissibility of linear maps to the semi-direct products of Rota-Baxter Lie algebras.

Theorem 4.16 *Let $(\mathfrak{g}, [-,-], P)$ be a Rota-Baxter Lie algebra of weight λ, and let (V, ρ) be a representation of $(\mathfrak{g}, [-,-])$. Let $Q : \mathfrak{g} \to \mathfrak{g}$ and $\alpha, \beta : V \to V$ be linear maps. Then the following conditions are equivalent.*

(a) *There is a Rota-Baxter Lie algebra $(\mathfrak{g} \ltimes_\rho V, P + \alpha)$ such that the linear map $Q + \beta$ on $\mathfrak{g} \oplus V$ is adjoint-admissible to $(\mathfrak{g} \ltimes_\rho V, P + \alpha)$.*
(b) *There is a Rota-Baxter Lie algebra $(\mathfrak{g} \ltimes_{\rho^*} V^*, P + \beta^*)$ such that the linear map $Q + \alpha^*$ on $\mathfrak{g} \oplus V^*$ is adjoint-admissible to $(\mathfrak{g} \ltimes_{\rho^*} V^*, P + \beta^*)$.*
(c) *The following conditions are satisfied:*

 (i) *(V, ρ, α) is a representation of $(\mathfrak{g}, [-,-], P)$, that is, Eq. 4 holds;*
 (ii) *Q is adjoint-admissible to $(\mathfrak{g}, [-,-], P)$, that is, Eq. 8 holds;*
 (iii) *β is admissible to $(\mathfrak{g}, [-,-], P)$ on (V, ρ), that is, Eq. 7 holds;*
 (iv) *The following equation holds:*

$$\beta(\rho(x)\alpha(v)) = \beta(\rho(Q(x))v) + \rho(Q(x))\alpha(v) + \lambda\rho(Q(x))v, \quad \forall x \in \mathfrak{g}, v \in V. \quad (63)$$

Proof The proof follows the same argument as in the proof of [6, Theorem 4.20]. □

In the following, we apply $\mathcal{O}$-operators to the construction of antisymmetric solutions of the admissible CYBE, and of Rota-Baxter Lie bialgebras.

Theorem 4.17 *Let $(\mathfrak{g}, [-,-]_\mathfrak{g}, P)$ be a Rota-Baxter Lie algebra of weight λ and (V, ρ) be a representation of $(\mathfrak{g}, [-,-]_\mathfrak{g})$. Let $\beta : V \to V$ be a linear map which is admissible to $(\mathfrak{g}, [-,-]_\mathfrak{g}, P)$ on (V, ρ). Let $Q : \mathfrak{g} \to \mathfrak{g}$, $\alpha : V \to V$ and $T : V \to \mathfrak{g}$ be linear maps.*

(a) *$r = T - \tau(T)$ is an antisymmetric solution of the $(Q + \alpha^*)$-admissible CYBE in the Rota-Baxter Lie algebra $(\mathfrak{g} \ltimes_{\rho^*} V^*, P + \beta^*)$ if and only if T is a weak $\mathcal{O}$-operator associated to (V, ρ) and α, and satisfies $T\beta = QT$.*
(b) *Assume that (V, ρ, α) is a representation of $(\mathfrak{g}, [-,-]_\mathfrak{g}, P)$. If T is an $\mathcal{O}$-operator associated to (V, ρ, α) and $T\beta = QT$, then $r = T - \tau(T)$ is an antisymmetric solution of the $(Q + \alpha^*)$-admissible CYBE in the Rota-Baxter Lie algebra $(\mathfrak{g} \ltimes_{\rho^*} V^*, P + \beta^*)$. If in addition, $(\mathfrak{g}, [-,-]_\mathfrak{g}, P)$ is Q-adjoint-admissible and Eq. 63 holds such that the Rota-Baxter algebra $(\mathfrak{g} \ltimes_{\rho^*} V^*, P + \beta^*)$ is $(Q + \alpha^*)$-adjoint-admissible, then there is a Rota-Baxter Lie bialgebra $(\mathfrak{g} \ltimes_{\rho^*} V^*, P + \beta^*, \delta, Q + \alpha^*)$ of weight λ, where the linear map $\delta = \delta_r$ is defined by Eq. 48 with $r = T - \tau(T)$.*

Proof (a). It is the same argument as for [6, Theorem 4.21 (a)].
(b). It follows from Item (a) and Theorem 4.16. □

4.3 Construction of Special L-dendriform Bialgebras from $\mathcal{O}$-operators and Rota-Baxter Pre-Lie Algebras

To complete the paper, we construct special L-dendriform bialgebras from $\mathcal{O}$-operators on Rota-Baxter Lie algebras of weight 0 and from Rota-Baxter pre-Lie algebras.

Proposition 4.18 *Let $(\mathfrak{g}, [-,-]_{\mathfrak{g}}, P)$ be a Rota-Baxter Lie algebra of weight 0. Let $Q : \mathfrak{g} \to \mathfrak{g}$ be a linear map which is adjoint-admissible to $(\mathfrak{g}, [-,-]_{\mathfrak{g}}, P)$. Let (V, ρ, α) be a representation of $(\mathfrak{g}, [-,-]_{\mathfrak{g}}, P)$ and $\beta : V \to V$ be a linear map which is admissible to $(\mathfrak{g}, [-,-]_{\mathfrak{g}}, P)$ on (V, ρ). Suppose that Eq. 63 holds. If T is an $\mathcal{O}$-operator associated to (V, ρ, α) and $T\beta = QT$, then there is a special L-dendriform bialgebra $(\mathfrak{g} \ltimes_{\rho^*} V^*, \triangleright, \triangleleft, \Delta, \nabla)$, where*

$$a \triangleleft b = -(Q + \alpha^*)([a, b]), a \triangleright b = [(P + \beta^*)a, b] + (Q + \alpha^*)[a, b],$$
$$\nabla(a) = -\delta((P + \beta^*)a), \Delta(a) = ((Q + \alpha^*) \otimes \mathrm{id})\delta(a) + \delta((P + \beta^*)a),$$

for all $a, b \in \mathfrak{g} \oplus V^*$. Here $[-,-]$ is the Lie bracket on $\mathfrak{g} \ltimes_{\rho^*} V^*$ and the linear map $\delta = \delta_r$ is defined by Eq. 48 with $r = T - \tau(T)$.

Proof It follows from Theorem 4.17 (b) and Proposition 3.13. □

To illustrate the construction of Rota-Baxter Lie bialgebras by $\mathcal{O}$-operators, we focus on the special case when the $\mathcal{O}$-operators are associated to the adjoint representation of the Rota-Baxter Lie algebra, as given in Example 4.13.

Corollary 4.19 *Let $(\mathfrak{g}, [-,-], P)$ be a Rota-Baxter Lie algebra of weight 0. Suppose that $Q : \mathfrak{g} \to \mathfrak{g}$ is a linear map that is adjoint-admissible to $(\mathfrak{g}, [-,-], P)$ and commutes with P. Then there is a Rota-Baxter Lie bialgebra $(\mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*, P + Q^*, \delta_r, Q + P^*)$ of weight 0, with the linear map δ_r defined by Eq. 48 with $r = P - \tau(P)$. Moreover, there is a special L-dendriform bialgebra $(\mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*, \triangleright, \triangleleft, \Delta, \nabla)$, where*

$$a \triangleleft b = -(Q + P^*)([a, b]), \quad a \triangleright b = [(P + Q^*)a, b] + (Q + P^*)[a, b], \tag{64}$$
$$\nabla(a) = -\delta((P + Q^*)a), \Delta(a) = ((Q + P^*) \otimes \mathrm{id})\delta(a) + \delta((P + Q^*)a), \forall a, b \in \mathfrak{g} \oplus \mathfrak{g}^*. \tag{65}$$

Proof By Example 4.13, P is an $\mathcal{O}$-operator of $(\mathfrak{g}, [-,-], P)$ associated to $(\mathfrak{g}, \mathrm{ad}, P)$. Hence the conclusion follows from Proposition 4.18. □

Some clear choices of Q in Corollary 4.19 are $Q = 0$ and $Q = -P$. In the following construction based on Example 2.20, we use another available choice $Q = \hat{P}$.

Example 4.20 Under the assumptions in Examples 2.8 and 2.20, there is a semi-direct product Lie algebra $\mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*$ given by Eq. 11 and the following nonzero products.

$$[x, x^*]_{\ltimes} = 2h^*, \ [x, h^*]_{\ltimes} = -y^*, \ [h, x^*]_{\ltimes} = -2x^*,$$
$$[h, y^*]_{\ltimes} = 2y^*, \quad [y, h^*]_{\ltimes} = x^*, \quad [y, y^*]_{\ltimes} = -2h^*.$$

The tensor form of P is given by

$$P = x^* \otimes (x + y) + h^* \otimes (2h + 4y) + y^* \otimes (x - 2h - 3y), \tag{66}$$

and thus

$$r = P - \tau(P) = (\mathrm{id}^{\otimes 2} - \tau)\big(x^* \otimes (x + y) + h^* \otimes (2h + 4y) + y^* \otimes (x - 2h - 3y)\big). \tag{67}$$

Then by Corollary 4.19, there is a Rota-Baxter Lie algebra $(\mathfrak{g}\ltimes_{\mathrm{ad}^*}\mathfrak{g}^*, P+\hat{P}^*, \delta, \hat{P}+P^*)$ of weight 0 with δ defined by Eq. 48, that is,

$$\begin{aligned}
\delta(x) &= (\mathrm{id}^{\otimes 2}-\tau)(x^*\otimes h + h^*\otimes(-2x+4h+2y)+y^*\otimes(4x-5h-4y)),\\
\delta(h) &= (\mathrm{id}^{\otimes 2}-\tau)(-4x^*\otimes y - 8h^*\otimes y + 4y^*\otimes(x-h)),\\
\delta(y) &= (\mathrm{id}^{\otimes 2}-\tau)(x^*\otimes(h+4y)+h^*\otimes(-2x+4h+10y)-y^*\otimes(h+4y)),\\
\delta(x^*) &= (\mathrm{id}^{\otimes 2}-\tau)(x^*\otimes(-6h^*+4y^*)+2h^*\otimes y^*),\\
\delta(h^*) &= (\mathrm{id}^{\otimes 2}-\tau)(x^*\otimes(4h^*-2y^*)),\\
\delta(y^*) &= (\mathrm{id}^{\otimes 2}-\tau)(2x^*\otimes h^*+2h^*\otimes y^*).
\end{aligned}$$

Moreover, there is a special L-dendriform bialgebra $(\mathfrak{g}\oplus\mathfrak{g}^*, \triangleright, \triangleleft, \Delta, \nabla)$ given by Eqs. 64 and 65, where $Q=\hat{P}$ and $a, b\in\mathfrak{g}\oplus\mathfrak{g}^*$. Explicitly, the nonzero products are given in Example 2.20 and the following equations:

$$\begin{aligned}
&x\circ x^* = 2h^*, && x\circ h^* = x^*-y^*, && x\circ y^* = -2h^*,\\
&h\circ x^* = -4x^*, && h\circ h^* = 4x^*, && h\circ y^* = -8h^*+4y^*,\\
&y\circ x^* = 4x^*+2h^*, && y\circ h^* = -3x^*-y^*, && y\circ y^* = 6h^*-4y^*,\\
&x^*\circ x = 6h^*-4y^*, && x^*\circ h = -6x^*-2y^*, && x^*\circ y = 4x^*+2h^*,\\
&h^*\circ x = -4h^*+2y^*, && h^*\circ h = 4x^*, && h^*\circ y = -2x^*,\\
&y^*\circ x = -2h^*, && y^*\circ h = 2x^*-2y^*, && y^*\circ y = 2h^*,\\
&x\triangleleft x^* = -4h^*+4y^*, && x\triangleleft h^* = x^*+4h^*-3y^*, && h\triangleleft x^* = 2x^*+2y^*,\\
&h\triangleleft y^* = -2x^*-8h^*+6y^*, && y\triangleleft h^* = -x^*-y^*, && y\triangleleft y^* = 4h^*-4y^*,
\end{aligned}$$

$$\begin{aligned}
\Diamond(x) &= 4x^*\otimes(x-h-y)+12h^*\otimes(x-h-y)+8y^*\otimes(-x+h+y)\\
&\quad +x\otimes(4x^*+8h^*-4y^*)+h\otimes(-2x^*-4h^*+2y^*),\\
\Diamond(h) &= 4x^*\otimes(x-h-y)+16h^*\otimes(x-h-y)+12y^*\otimes(-x+h+y)\\
&\quad +x\otimes(4x^*+8h^*-4y^*)+y\otimes(4x^*+8h^*-4y^*),\\
\Diamond(y) &= h^*\otimes(-4x+4h+4y)+y^*\otimes(4x-4h-4y)\\
&\quad +h\otimes(-2x^*-4h^*+2y^*)+y\otimes(-4x^*-8h^*+4y^*),\\
\Diamond(x^*) &= x^*\otimes(-4x^*-8h^*+4y^*)+h^*\otimes(-4x^*-8h^*+4y^*),\\
\Diamond(h^*) &= x^*\otimes(2x^*+4h^*-2y^*)+y^*\otimes(2x^*+4h^*-2y^*),\\
\Diamond(y^*) &= h^*\otimes(-4x^*-8h^*+4y^*)+y^*\otimes(4x^*+8h^*-4y^*),\\
\nabla(x) &= (\mathrm{id}^{\otimes 2}-\tau)(x^*\otimes(-2h-4y)+h^*\otimes(4x-8h-12y)\\
&\quad +y^*\otimes(-4x+6h+8y)),\\
\nabla(h) &= (\mathrm{id}^{\otimes 2}-\tau)(x^*\otimes(-4h-8y)+h^*\otimes(8x-16h-24y)\\
&\quad +y^*\otimes(-8x+12h+16y)),\\
\nabla(y) &= (\mathrm{id}^{\otimes 2}-\tau)(x^*\otimes(2h+4y)+h^*\otimes(-4x+8h+12y)+y^*\otimes(4x-6h-8y)),\\
\nabla(x^*) &= 4(\mathrm{id}^{\otimes 2}-\tau)(x^*\otimes(-h^*+y^*)+h^*\otimes y^*),\\
\nabla(h^*) &= 8(\mathrm{id}^{\otimes 2}-\tau)(x^*\otimes(h^*-y^*)-h^*\otimes y^*),\\
\nabla(y^*) &= 4(\mathrm{id}^{\otimes 2}-\tau)(x^*\otimes(h^*-y^*)-h^*\otimes y^*),
\end{aligned}$$

and

$$a \triangleleft b := a \circ b - a \triangleright b, \Delta(a) := \Diamond(a) - \nabla(a), \quad \forall a, b \in \mathfrak{g} \oplus \mathfrak{g}^*.$$

On the other hand, $\mathcal{O}$-operators on Rota-Baxter Lie algebras of weight λ can be obtained from Rota-Baxter pre-Lie algebras of weight λ.

Definition 4.21 A **Rota-Baxter pre-Lie algebra of weight** λ is a triple $(A, \circ, P)$, such that $(A, \circ)$ is a pre-Lie algebra, and $P : A \to A$ is a **Rota-Baxter operator of weight** λ on $(A, \circ)$, that is, P satisfies

$$P(x) \circ P(y) = P(P(x) \circ y) + P(x \circ P(y)) + \lambda P(x \circ y), \quad \forall x, y \in A. \tag{68}$$

By a direct verification, we obtain

Proposition 4.22 *Let* $(A, \circ, P)$ *be a Rota-Baxter pre-Lie algebra of weight* λ*. Then* $(\mathfrak{g}(A), [-, -], P)$ *is a Rota-Baxter Lie algebra of weight* λ*, which is called the* **sub-adjacent Rota-Baxter Lie algebra** *of* $(A, \circ, P)$*. Moreover,* $(A, \mathcal{L}_\circ, P)$ *is a representation of the Rota-Baxter Lie algebra* $(\mathfrak{g}(A), [-, -], P)$*, and the identity map* id_A *on* A *is an* $\mathcal{O}$*-operator on the Rota-Baxter Lie algebra* $(\mathfrak{g}(A), [-, -], P)$ *associated to* $(A, \mathcal{L}_\circ, P)$*.*

Proposition 4.23 *Let* $(A, \circ, P)$ *be a Rota-Baxter pre-Lie algebra of weight* λ*. Let* $Q : A \to A$ *be a linear map such that the following equations hold:*

$$Q(P(x) \circ y) = P(x) \circ Q(y) + Q(x \circ Q(y)) + \lambda x \circ Q(y), \tag{69}$$

$$Q(x \circ P(y)) = Q(x) \circ P(y) + Q(Q(x) \circ y) + \lambda Q(x) \circ y, \quad \forall x, y \in A. \tag{70}$$

Then the Rota-Baxter Lie algebra $(\mathfrak{g}(A), [-, -], P)$ *of weight* λ *is* Q*-adjoint-admissible. Moreover,* $Q + P^*$ *is adjoint-admissible to* $(\mathfrak{g}(A) \ltimes_{\mathcal{L}_\circ^*} A^*, P + Q^*)$*. In particular, there is a Rota-Baxter Lie bialgebra* $(\mathfrak{g}(A) \ltimes_{\mathcal{L}_\circ^*} A^*, P + Q^*, \delta, Q + P^*)$ *of weight* λ*, where* $\delta = \delta_r$ *is defined by Eq. 48 with* $r = \mathrm{id_A} - \tau(\mathrm{id_A})$*.*

Proof By the assumption, Eq. 8 holds. Thus $(\mathfrak{g}(A), [-, -], P)$ is Q-adjoint-admissible. Moreover, Eqs. 7 and 63 hold by taking $\rho = \mathcal{L}_\circ, \alpha = P, \beta = Q$. Then by Theorem 4.16, $Q + P^*$ is adjoint-admissible to $(\mathfrak{g}(A) \ltimes_{\mathcal{L}_\circ^*} A^*, P + Q^*)$. Finally, by Theorem 4.17 (b), $(\mathfrak{g}(A) \ltimes_{\mathcal{L}_\circ^*} A^*, P + Q^*, \delta, Q + P^*)$ is a Rota-Baxter Lie bialgebra of weight λ.

Corollary 4.24 *Let* $(\mathfrak{g}, [-, -], P)$ *be a Rota-Baxter Lie algebra of weight 0 and* $(\mathfrak{g}, \circ)$ *be the induced pre-Lie algebra. Let* $Q : \mathfrak{g} \to \mathfrak{g}$ *be a linear map that is adjoint-admissible to* $(\mathfrak{g}, [-, -], P)$ *and commutes with* P*. Then Eqs. 69 and 70 hold for* $\lambda = 0$*. Hence the sub-adjacent Rota-Baxter Lie algebra* $(\mathfrak{g}', P)$ *of* $(\mathfrak{g}, \circ, P)$ *is* Q*-adjoint-admissible. Moreover,* $Q + P^*$ *is adjoint-admissible to* $(\mathfrak{g}' \ltimes_{\mathcal{L}_\circ^*} \mathfrak{g}^*, P + Q^*)$*. Hence there is a Rota-Baxter Lie bialgebra* $(\mathfrak{g}' \ltimes_{\mathcal{L}_\circ^*} \mathfrak{g}^*, P + Q^*, \delta, Q + P^*)$ *of weight* 0*, where* $\delta = \delta_r$ *is defined by Eq. 48 with*

$$r = \mathrm{id}_{\mathfrak{g}} - \tau(\mathrm{id}_{\mathfrak{g}}) = \sum_i (e_i^* \otimes e_i - e_i \otimes e_i^*),$$

where $\{e_1, \cdots, e_n\}$ *is a basis of* $\mathfrak{g}$ *and* $\{e_1^*, \cdots, e_n^*\}$ *is the dual basis. Moreover, there is a special L-dendriform bialgebra* $(\mathfrak{g} \oplus \mathfrak{g}^*, \triangleright, \triangleleft, \Delta, \nabla)$ *given by Eqs. 64 and 65.*

Proof Let $x, y \in \mathfrak{g}$. We have

$$
\begin{aligned}
& Q(P(x) \circ y) - P(x) \circ Q(y) - Q(x \circ Q(y)) \\
& = Q([P(P(x)), y]) - [P(P(x)), Q(y)] - Q([P(x), Q(y)]) \\
& \overset{(8)}{=} 0, \\
& Q(x \circ P(y)) - Q(x) \circ P(y) - Q(Q(x) \circ y) \\
& = Q([P(x), P(y)]) - [PQ(x), P(y)] - Q([PQ(x), y]) \\
& = Q([P(x), P(y)]) - [QP(x), P(y)] - Q([QP(x), y]) \\
& \overset{(8)}{=} 0.
\end{aligned}
$$

Hence Eqs. 69 and 70 hold for $\lambda = 0$. The other statements follow from Propositions 4.18 and 4.23. □

Remark 4.25 Let $(\mathfrak{g}, [-,-], P)$ be a Rota-Baxter Lie algebra of weight 0 and $(\mathfrak{g}, \circ)$ be the induced pre-Lie algebra. Let $Q : \mathfrak{g} \to \mathfrak{g}$ be a linear map that is adjoint-admissible to $(\mathfrak{g}, [-,-], P)$ and commutes with P. Then Q is again adjoint-admissible to the sub-adjacent Rota-Baxter Lie algebra $(\mathfrak{g}', P)$ of $(\mathfrak{g}, \circ, P)$ and commutes with P. Hence Corollaries 4.19 and 4.24 can be applied repeatedly to get a series of Rota-Baxter Lie bialgebras of weight 0 and special L-dendriform bialgebras.

Example 4.26 With the assumptions in Examples 2.8 and 2.20, there is a semi-direct product Lie algebra $\mathfrak{g}' \ltimes_{\mathcal{L}_\circ^*} \mathfrak{g}^*$ given by Eq. 27 and the following nonzero products.

$$
\begin{aligned}
&[x, x^*]'_{\ltimes} = 2h^*, \quad [x, h^*]'_{\ltimes} = x^* - y^*, \quad [x, y^*]'_{\ltimes} = -2h^*, \\
&[h, x^*]'_{\ltimes} = -4x^*, \quad [h, h^*]'_{\ltimes} = 4x^*, \quad [h, y^*]'_{\ltimes} = -8h^* + 4y^*, \\
&[y, x^*]'_{\ltimes} = 4x^* + 2h^*, \; [y, h^*]'_{\ltimes} = -3x^* - y^*, \; [y, y^*]'_{\ltimes} = 6h^* - 4y^*.
\end{aligned}
$$

By Corollary 4.24, there is a Rota-Baxter Lie bialgebra $(\mathfrak{g}' \ltimes_{\mathcal{L}_\circ^*} \mathfrak{g}^*, P + \hat{P}^*, \delta', \hat{P} + P^*)$ of weight 0 with $\delta' = \delta'_r$ defined by Eq. 48 and

$$
r = \mathrm{id}_{\mathfrak{g}} - \tau(\mathrm{id}_{\mathfrak{g}}) = (\mathrm{id}^{\otimes 2} - \tau)(x^* \otimes x + h^* \otimes h + y^* \otimes y).
$$

More precisely, we have

$$
\begin{aligned}
\delta'(x) &= (\mathrm{id}^{\otimes 2} - \tau)(x^* \otimes h + h^* \otimes (-4x + 4h) + y^* \otimes (4x - 3h)), \\
\delta'(h) &= (\mathrm{id}^{\otimes 2} - \tau)(x^* \otimes (2x - 2y) - 8h^* \otimes y + y^* \otimes (2x + 6y)), \\
\delta'(y) &= (\mathrm{id}^{\otimes 2} - \tau)(-x^* \otimes h + 4h^* \otimes y - y^* \otimes (h + 4y)), \\
\delta'(x^*) &= (\mathrm{id}^{\otimes 2} - \tau)(-6x^* \otimes h^* + 4x^* \otimes y^* + 2h^* \otimes y^*), \\
\delta'(h^*) &= (\mathrm{id}^{\otimes 2} - \tau)(4x^* \otimes h^* - 2x^* \otimes y^*), \\
\delta'(y^*) &= (\mathrm{id}^{\otimes 2} - \tau)(2x^* \otimes h^* + 2h^* \otimes y^*).
\end{aligned}
$$

Moreover, there is a special L-dendriform bialgebra $(\mathfrak{g} \oplus \mathfrak{g}^*, \triangleright', \triangleleft', \Delta', \nabla')$ given by Eqs. 64 and 65, where $Q = \hat{P}$ and $a, b \in \mathfrak{g} \oplus \mathfrak{g}^*$. Explicitly, the nonzero products are given in

Example 2.20 and the following equations.

$$\begin{aligned}
x\circ' x &= -4x+2h, & x\circ' h &= -8x+4h, & x\circ' y &= 4x-2h,\\
h\circ' x &= -4x-4y, & h\circ' h &= -8x-8y, & h\circ' y &= 4x+4y,\\
y\circ' x &= 2h+4y, & y\circ' h &= 4h+8y, & y\circ' y &= -2h-4y,\\
x\circ' x^* &= 4x^*+4h^*, & x\circ' h^* &= -2x^*-2y^*, & x\circ' y^* &= 4h^*-4y^*,\\
h\circ' x^* &= 8x^*+8h^*, & h\circ' h^* &= -4x^*-4y^*, & h\circ' y^* &= 8h^*-8y^*,\\
y\circ' x^* &= -4x^*-4h^*, & y\circ' h^* &= 2x^*+2y^*, & y\circ' y^* &= -4h^*+4y^*,\\
x^*\circ' x &= 4x^*+8h^*-4y^*, & x^*\circ' h &= 4x^*+8h^*-4y^*, & x^*\circ' y &= 0,\\
h^*\circ' x &= -2x^*-4h^*+2y^*, & h^*\circ' h &= 0, & h^*\circ' y &= -2x^*-4h^*+2y^*,\\
y^*\circ' x &= 0, & y^*\circ' h &= 4x^*+8h^*-4y^*, & y^*\circ' y &= -4x^*-8h^*+4y^*,\\
x\lhd' h &= -4x+4h+4y, & x\lhd' y &= 4x-4h-4y, & h\lhd' y &= 4x-4h-4y,\\
x\lhd' x^* &= -4x^*-4h^*, & x\lhd' h &= 4x^*+4h^*, & x\lhd' y^* &= 4x^*+4h^*,\\
h\lhd' x^* &= -12x^*-16h^*+4y^*, & h\lhd' h &= 12x^*+16h^*-4y^*, & h\lhd' y^* &= 12x^*+16h^*-4y^*,\\
y\lhd' x^* &= 8x^*+12h^*-4y^*, & y\lhd' h &= -8x^*-12h^*+4y^*, & y\lhd' y^* &= -8x^*-12h^*+4y^*,
\end{aligned}$$

$$\begin{aligned}
\Diamond'(x) &= x^*\otimes(4x-2h)+h^*\otimes(8x-4h)+y^*\otimes(-4x+2h)\\
&\quad +x\otimes(4x^*+4h^*)-h\otimes(2x^*+2y^*)+y\otimes(4h^*-4y^*),\\
\Diamond'(h) &= x^*\otimes(4x+4y)+h^*\otimes(8x+8y)-y^*\otimes(4x+4y)\\
&\quad +x\otimes(8x^*+8h^*)+h\otimes(-4x^*-4y^*)+y\otimes(8h^*-8y^*),\\
\Diamond'(y) &= -x^*\otimes(2h+4y)-h^*\otimes(4h+8y)+y^*\otimes(2h+4y)\\
&\quad -x\otimes(4x^*+4h^*)+h\otimes(2x^*+2y^*)+y\otimes(-4h^*+4y^*),\\
\Diamond'(x^*) &= x^*\otimes(-4x^*-8h^*+4y^*)+h^*\otimes(-4x^*-8h^*+4y^*),\\
\Diamond'(h^*) &= x^*\otimes(2x^*+4h^*-2y^*)+y^*\otimes(2x^*+4h^*-2y^*),\\
\Diamond'(y^*) &= h^*\otimes(-4x^*-8h^*+4y^*)+y^*\otimes(4x^*+8h^*-4y^*),\\
\nabla'(x) &= (\mathrm{id}^{\otimes 2}-\tau)(h^*\otimes(4x-4h-4y)+y^*\otimes(-4x+4h+4y)),\\
\nabla'(h) &= (\mathrm{id}^{\otimes 2}-\tau)(x^*\otimes(-4x+4h+4y)+y^*\otimes(-4x+4h+4y)),\\
\nabla'(y) &= (\mathrm{id}^{\otimes 2}-\tau)(x^*\otimes(4x-4h-4y)+h^*\otimes(4x-4h-4y)),\\
\nabla'(x^*) &= (\mathrm{id}^{\otimes 2}-\tau)(4x^*\otimes(-h^*+y^*)+4h^*\otimes y^*),\\
\nabla'(h^*) &= (\mathrm{id}^{\otimes 2}-\tau)(4x^*\otimes(h^*-y^*)-4h^*\otimes y^*),\\
\nabla'(y^*) &= (\mathrm{id}^{\otimes 2}-\tau)(4x^*\otimes(h^*-y^*)-4h^*\otimes y^*),
\end{aligned}$$

and

$$a\lhd' b := a\circ' b - a\rhd' b,\ \Delta'(a) := \Diamond'(a)-\nabla'(a),\quad \forall a,b\in\mathfrak{g}\oplus\mathfrak{g}^*.$$

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Declarations

Competing interests The authors claim that there is no conflict of interests.

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