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RESEARCH

Stable Unital Bases, Hyperfocal Subalgebras and Basic Morita Equivalences

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Abstract

We investigate Conjecture 1.5 introduced by Barker and Gelvin (J. Gr. Theory **25**, 973–995 2022), which says that any source algebra of a p-block (p is a prime) of a finite group has the unit group containing a basis stabilized by the left and right actions of the defect group. We will reduce this conjecture to a similar statement about the bases of the hyperfocal subalgebras in the source algebras. We will also show that such unital bases of source algebras of two p-blocks, stabilized by the left and right actions of the defect group, are transported through basic Morita equivalences.

Keywords Finite · Group · Source algebra · Hyperfocal subalgebra · Unital basis · Fusion system · Basic · Morita equivalence

Mathematics Subject Classification (2010) 20C20

1 Introduction

Let $\mathcal{O}$ be a complete local noetherian commutative ring with identity element and with an algebraically closed residue field k of prime characteristic p. We understand any $\mathcal{O}$-module and any algebra over $\mathcal{O}$ to be finitely generated and free over $\mathcal{O}$. Any $\mathcal{O}$-algebra $\mathbf{A}$ which we consider has an identity element $1_{\mathbf{A}}$ and the group of units is denoted by $\mathbf{A}^{\times}$. A basis for an algebra over $\mathcal{O}$ is an $\mathcal{O}$-basis as an $\mathcal{O}$-module. Such a basis is said to be *unital* if every element is unital.

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In this paper we consider G to be a finite group such that p divides the order of G and we assume familiarity with: the theory of G-algebras, Brauer maps, Brauer pairs, pointed groups, (almost-)source algebras, as in [7], [8] and [15]. For notations and results with respect to fusion systems we follow [1] or the algebraic approach given by David Craven in his book [2].

Let b be a block idempotent of $\mathcal{O}G$ with defect group D (a p-subgroup of G) and $l \in (\mathcal{O}Gb)^D$ be a primitive idempotent such that $\mathrm{Br}_D^{\mathcal{O}G}(l) \neq 0$. We shall use the notation $A := l\mathcal{O}Gl$; it is an interior D-algebra called the *source algebra* of b. The source algebra A has a $D \times D$-stable basis on which $D \times 1$ and $1 \times D$ act freely. Following [3, Section 6] this means that A is a *bifree bipermutation* D-algebra. In the same paper the authors explore the following conjecture.

Conjecture 1.1 ([3, Conjecture 1.5]) For any block idempotent b of $\mathcal{O}G$, any source algebra A of $\mathcal{O}Gb$ has a unital $D \times D$-stable basis.

We will reduce this conjecture to a conjecture on hyperfocal subalgebras. For any subgroup R of D there is a unique block e_R of $kC_G(R)$ such that $\mathrm{Br}_R^{\mathcal{O}G}(l)e_R \neq 0$. Fixing (D, e_D), a maximal $(\mathcal{O}G, b, G)$-Brauer pair, we define a saturated fusion system which we denote by $\mathcal{F} := \mathcal{F}_{(D,e_D)}(\mathcal{O}G, b, G)$, the saturated fusion system of A on D, associated with b and given by the $(\mathcal{O}G, b, G)$-Brauer pairs included in (D, e_D).

The hyperfocal subgroup of a block was introduced by Puig [11] and, in the language of fusion systems, is defined by

$$\mathrm{hyp}(\mathcal{F}) :=< u\varphi(u^{-1}) | R \leq D, u \in R, \varphi \in O^p(\mathrm{Aut}_{\mathcal{F}}(R)) >$$

It is an important invariant of a block, since a block is nilpotent if and only if its hyperfocal subgroup is trivial. Let $\tilde{D}$ be a normal subgroup of D containing $\mathrm{hyp}(\mathcal{F})$. By [11, Theorem 1.8] there exists a unique (up to $(A^D)^\times$-conjugacy) D-stable, unitary subalgebra $\tilde{A}$ of A (called the *hyperfocal subalgebra* with respect to $\tilde{D}$) such that $A = \tilde{A} \otimes_{\tilde{D}} D$, see 2 for the general case. The hyperfocal subalgebra $\tilde{A}$ is a $\tilde{D}$-interior D-algebra, satisfying $\tilde{A} \cap Dl = \tilde{D}l$ and $1_{\tilde{A}} = 1_A = l$. Set $\bar{D} := D/\tilde{D}$ and consider the subgroup $D \times^{\bar{D}} D$ of $D \times D$, which is recalled in 2 and below Conjecture 1.2. According to [11, 4.1] the hyperfocal subalgebra $\tilde{A}$ is a direct summand of A as $\mathcal{O}[D \times^{\bar{D}} D]$-module, hence $\tilde{A}$ has a $D \times^{\bar{D}} D$-stable basis. With the above notations we launch the following conjecture.

Conjecture 1.2 For any source algebra $A = l\mathcal{O}Gl$ of any block b with defect group D there is a normal subgroup $\tilde{D}$ in D containing $\mathrm{hyp}(\mathcal{F})$, such that the hyperfocal subalgebra $\tilde{A}$ with respect to $\tilde{D}$ has a unital $D \times^{\bar{D}} D$-stable basis.

In Section 2 we extend some results of [3, Sections 2, 4] from G-interior algebras to N-interior G-algebras; in this section we consider N to be a normal subgroup of G, $\mathbf{A}$ is an N-interior G-algebra (see [11, Section 2]), and $\bar{G} := G/N$ is the factor group. The elements of $\bar{G}$ are denoted by $\bar{x} = xN$, $x \in G$. Then $\mathbf{A} \otimes_N G$ is the crossed product, which is also a G-interior algebra, see 2.2 for more details. It follows that $\mathbf{A} \otimes_N G$ is also an $\mathcal{O}[G \times G]$-module. The subgroup

$$G \times^{\bar{G}} G := \{(u, v) | \bar{u} = \bar{v}, u, v \in G\}$$

of $G \times G$ is introduced in [11, 2.5.2] and is the pullback of the natural map $G \to \bar{G}$ with itself. The next proposition is an important technical result which is proved in Proposition 2.2.

Proposition 1.3 *(Proposition 2.2) Let* $\mathbf{A}$ *be an* N*-interior* G*-algebra. If* $\mathbf{A}$ *has a unital* $G \times^{\bar{G}} G$*-stable basis then* $\mathbf{A} \otimes_N G$ *has a unital* $G \times G$*-stable basis.*

Since the source algebra A is a direct summand of $\mathcal{O}G$ as $\mathcal{O}[D \times^{\bar{D}} D]$-module, $\tilde{A}$ inherits a $D \times^{\bar{D}} D$-stable basis on which $\tilde{D} \times 1$ and $1 \times \tilde{D}$ act freely. So $\tilde{A}$ has a basis which we call $\tilde{D}$-bifree, see 2. An immediate consequence of Proposition 1.3 (with $\mathbf{A}$, N, G replaced by $\tilde{A}$, $\tilde{D}$, D, respectively) is the following theorem.

Theorem 1.4 *With the above notations if* b *is a block of* $\mathcal{O}G$ *with source algebra* A *verifying Conjecture 1.2 then* A *verifies Conjecture 1.1.*

Section 3 has two subsections and we start by recalling the concept of a category on a p-group, see [6]. In Subsections 3.1 and 3.2 we introduce some subcategories on a p-group, by considering morphisms which are the identity morphisms on factor groups. These results are needed to give the next main result of our paper, where we obtain some cases of blocks for which Conjecture 1.2 is true. If $\phi \in \mathrm{Mor}(\mathscr{F})$ then $\Delta(\phi)$ is the twisted diagonal subgroup of $D \times D$ with respect to ϕ, that is

$$\Delta(\phi) = \{(\phi(u), u) | u \in \mathrm{dom}(\phi)\}.$$

Applying the results of Subsection 3.1 we define the subcategory $\mathscr{F}_{D \times^{\bar{D}} D}$ of $\mathscr{F}$, on the p-group D, which has as objects all subgroups in D and as morphisms all maps $\phi \in \mathrm{Mor}(\mathscr{F})$ such that $\Delta(\phi) \leq D \times^{\bar{D}} D$. In Subsection 3.2 we will define the categories $N_{\mathscr{F}_{D \times^{\bar{D}} D}}(R)$, on the p-groups $N_D(R)$, where R is a subgroup of D. If $R = D$ the subcategory $N_{\mathscr{F}_{D \times^{\bar{D}} D}}(D)$ of $\mathscr{F}_{D \times^{\bar{D}} D}$ has as objects all subgroups of D. For any $R_1, R_2 \leq D$, a morphism $\phi : R_1 \to R_2$ in $N_{\mathscr{F}_{D \times^{\bar{D}} D}}(D)$ is a morphism $\phi \in \mathrm{Hom}_{\mathscr{F}_{D \times^{\bar{D}} D}}(R_1, R_2)$ which extends to some automorphism $\phi' \in \mathrm{Aut}_{\mathscr{F}_{D \times^{\bar{D}} D}}(D)$. The following theorem extends [3, Proposition 1.6].

Theorem 1.5 *Let* b *be block idempotent* $\mathcal{O}G$ *with defect group* D*, with* $A = l\mathcal{O}Gl$ *the source algebra of* b*, and* $\mathscr{F}$ *be the saturated fusion system of* A *on* D*. Let* $\tilde{D}$ *be a normal subgroup of* D *such that* $\mathrm{hyp}(\mathscr{F}) \leq \tilde{D}$ *and* $\tilde{A}$ *be the hyperfocal subalgebra with respect to* $\tilde{D}$*. If* $\mathscr{F}_{D \times^{\bar{D}} D} = N_{\mathscr{F}_{D \times^{\bar{D}} D}}(D)$ *then* $\tilde{A}$ *has a unital* $D \times^{\bar{D}} D$*-stable basis.*

As a consequence of Theorems 1.4, 1.5 and Proposition 3.2 (iv) we obtain the following corollary.

Corollary 1.6 *Let* b *be block idempotent* $\mathcal{O}G$ *having a defect group* D*, with source algebra* A *and with saturated fusion system* $\mathscr{F}$ *of* A *on* D*.*

(i) If all the assumptions of Theorem 1.5 are satisfied then the source algebra A *verifies Conjecture 1.1;*
(ii) If we choose $\tilde{D} = D$ *then Theorem 1.5 becomes [3, Proposition 1.6];*
(iii) If b *is a nilpotent block then its source algebra* A *verifies Conjecture 1.1.*

Statement (iii) of the above corollary is straightforward, we just want to emphasize a method which is based on the techniques in [3].

For the rest of the section, we consider another finite group H and a block idempotent b' of $\mathcal{O}H$ with defect group E. Let $l' \in (\mathcal{O}Hb')^E$ be a primitive idempotent such that $\mathrm{Br}_E^{\mathcal{O}H}(l') \neq 0$. Similar to the case of the block algebra b, we shall use the notations: $A' := l'\mathcal{O}Hl'$ is the source algebra of b, for any subgroup Q in E the block e'_Q is the unique block of $kC_H(Q)$ such that $\mathrm{Br}_Q^{\mathcal{O}H}(l')e'_Q \neq 0$, $\mathscr{F}'$ is the saturated fusion of A' on E, etc. *Basic*

Morita equivalences between blocks were introduced by Puig in [13], see also [14, Corollary 3.6]. It is a Morita equivalence between the block algebras which respects the local structure of the blocks and can be characterized by the existence of some algebra embedding between interior algebras obtained using the source algebras of the blocks. In Section 5 we will recall more details about basic Morita equivalences and we will prove the second main result of this paper.

Theorem 1.7 *Let b, b' be block idempotents as above such that $\mathcal{O}Gb$ is basic Morita equivalent to $\mathcal{O}Hb'$. If A has a unital $D \times D$-stable basis then A' has a unital $E \times E$-stable basis. Inertial blocks were introduced by Puig in [12]. An inertial block is a block which is basic Morita equivalent to its Brauer correspondent.*

Corollary 1.8 *Any inertial block verifies Conjecture 1.1.*

It is known that e_D is a nilpotent block of $kC_G(D)$ with defect group $Z(D)$ and that e_D remains a block of $kN_G(D, e_D)$ with the same maximal $(N_G(D, e_D), e_D, N_G(D, e_D))$-Brauer pair (D, e_D). The fusion system of e_D is $\mathcal{F}_{(D,e_D)}(kN_G(D, e_D), e_D, N_G(D, e_D))$. By [1, Chapter IV, Proposition 3.8], if we use the language of fusion systems, we can verify that in fact $\mathcal{F}_{(D,e_D)}(kN_G(D, e_D), e_D, N_G(D, e_D))$ is $N_{\mathcal{F}}(D)$, the normalizer fusion subsystem of D in $\mathcal{F}$, see [2, Definition 4.26 (ii)]. In fact, if b is an inertial block then the block algebra $\mathcal{O}Gb$ is basic Morita equivalent with the block algebra $\mathcal{O}N_G(D, e_D)\hat{e}_D$, where $\hat{e}_D$ denotes the block idempotent of $\mathcal{O}N_G(D, e_D)$ lifting the block idempotent e_D of $kN_G(D, e_D)$. Obviously, Corollary 1.8 can be quickly obtained as a consequence of Corollary 1.6 (ii), more precisely [3, Proposition 1.6], since basic Morita equivalences preserve the fusion systems.

In Section 4 we will give the proof of Theorem 1.5. Section 5 is devoted to the proof of the fact that unital stable bases of source algebras are transported between blocks which are basic Morita equivalent, Theorem 1.7.

If L_1, L_2 are two subgroups in some finite group L and $x \in L$ we denote by $c_x : L_1 \to L_2$ the conjugation homomorphism induced by x, when ${}^xL_1 \le L_2$ and ${}^xL_1 := xL_1x^{-1}$. If $h : M \to N$ is a map between two sets and M_1 is a subset of M we denote by $h_{|_{M_1}}$ the restriction map of h to M_1. Sometimes, for the preciseness of our notations we introduce the multiplication "$\cdot$" in the interior of various relations, we omit this most of the time; "$\cdot$" may signify the multiplication in a group or the action of a group on a module.

2 *N*-Interior *G*-Algebras and Stable Unital Bases

In this section we consider N to be a normal subgroup of G and $\mathbf{A}$ is an N-interior G-algebra. This means that $\mathbf{A}$ is an $\mathcal{O}$-algebra endowed with two group homomorphisms $\sigma_{\mathbf{A}} : N \to \mathbf{A}^{\times}$ and $\tau : G \to \mathrm{Aut}_{\mathcal{O}}(\mathbf{A})$, (where $\mathrm{Aut}_{\mathcal{O}}(\mathbf{A})$ is the group of $\mathcal{O}$-algebra automorphisms of $\mathbf{A}$) which satisfy

$$ {}^x(n \cdot a) = {}^xn \cdot {}^xa, {}^na = n \cdot a \cdot n^{-1}; $$

here we used the notations

$$ {}^xa = \tau(x)(a), {}^xn = xnx^{-1}, n \cdot a = \sigma_{\mathbf{A}}(a)n, a \cdot n = n\sigma_{\mathbf{A}}(a), $$

for any $x \in G, n \in N, a \in \mathbf{A}$;

2.1 We denote by $\mathbf{A} \otimes_N G$ the corresponding *crossed product* considered as a G-interior algebra, namely $\mathbf{A} \otimes_N G := \mathbf{A} \otimes_{\mathcal{O}N} \mathcal{O}G$ endowed with the associative operation

$$ (a \otimes x)(a' \otimes x') = a({}^xa') \otimes xx', $$

for any $a, a' \in \mathbf{A}, x, x' \in G$. The structural map of $\mathbf{A} \otimes_N G$ as a G-interior algebra is given by

$$\sigma_{\mathbf{A}\otimes_N G} : G \to (\mathbf{A} \otimes_N G)^{\times},\ \sigma_{\mathbf{A}\otimes_N G}(x) = 1_{\mathbf{A}} \otimes x,$$

for any $x \in G$. Note that if $n \in N$ then

$$\sigma_{\mathbf{A}\otimes_N G}(n) = 1_{\mathbf{A}} \cdot n \otimes 1_G = \sigma_{\mathbf{A}}(n) \otimes 1_G.$$

$\mathbf{A} \otimes_N G$ is a $\bar{G}$-graded algebra, with the identity component $\mathbf{A} \simeq \mathbf{A} \otimes 1$, see [4, 2.1].

2.2 Recall the next subgroup of $G \times G$, which is introduced in [11, 2.5.2],

$$G \times^{\bar{G}} G := \{(u, v) | \bar{u} = \bar{v}, u, v \in G\}.$$

Denoting by $\Delta(G)$ the diagonal subgroup of $G \times G$ it is easy to verify that $(N \times N)\Delta(G)$ is a normal subgroup of $G \times^{\bar{G}} G$. Moreover if $N = 1$ then $G \times^{\bar{G}} G = \Delta(G)$.

Since $\mathbf{A}$ is an N-interior algebra, $\mathbf{A}$ has a structure of an $\mathscr{O}[N \times N]$-module given by

$$(u, v)a = u \cdot a \cdot v^{-1}, \forall u, v \in N, \forall a \in \mathbf{A}.$$

Moreover, $\mathbf{A}$ is an $\mathscr{O}[G \times^{\bar{G}} G]$-module with the action

$$(u, v)a := (uv^{-1}) \cdot {}^{v}a = {}^{u}a \cdot (uv^{-1}),$$

for any $(u, v) \in G \times^{\bar{G}} G$, $a \in \mathbf{A}$. Note that $\mathbf{A} \otimes_N G$ is an $\mathscr{O}[G \times^{\bar{G}} G]$-module and $\mathbf{A}$ is a direct summand of $\mathbf{A} \otimes_N G$ as $\mathscr{O}[G \times^{\bar{G}} G]$-modules.

Remark 2.3 Let L be any finite group. Let B be an $\mathscr{O}[L \times L]$-module with Ω an $L \times L$-stable $\mathscr{O}$-basis and $\omega \in \Omega$. We shall use the notation

$$N_{L\times L}(\omega) := \{(u, v) | (u, v)\omega = \omega, u, v \in B\}.$$

The next theorem has the similar proof to [3, Theorem 2.4] adapted to N-interior G-algebras. We prefer to give some details of the proof, for explicitness, even though our proof is almost a verbatim translation.

Theorem 2.1 *Let* $\mathbf{A}$ *be an* N*-interior* G*-algebra. Then* $\mathbf{A}$ *has a unital* $G \times^{\bar{G}} G$*-stable basis if and only if* $\mathbf{A}$ *has a* $G \times^{\bar{G}} G$*-stable basis* Ω *such that for any* $\omega \in \Omega$ *the group* $N_{G\times^{\bar{G}}G}(\omega)$ *fixes an element of* $\mathbf{A}^{\times}$.

Proof The implication from left to right is clear.

For the other implication, assume Ω is a $G \times^{\bar{G}} G$-stable basis such that, for any $\omega \in \Omega$, the group $N_{G\times^{\bar{G}}G}(\omega)$ fixes an element of $\mathbf{A}^{\times}$. Let T be a system of representatives of the $G \times^{\bar{G}} G$-orbits in Ω. For any $\omega \in T$, we choose $f(\omega) \in \mathbf{A}^{\times} \cap \mathbf{A}^{N_{G\times^{\bar{G}}G}(\omega)}$. We extend this choice to a well-defined map, still denoted by

$$f : \Omega \to \mathbf{A}^{\times}, \quad f((x, y)\omega) := (x, y)f(\omega),$$

for any $(x, y) \in G \times^{\bar{G}} G, \omega \in T$.

For all $\omega' \in \Omega$ the following equality

$$N_{G\times^{\bar{G}}G}(\omega') = N_{G\times^{\bar{G}}G}(f(\omega')) \tag{1}$$

is true, since if $\omega' = (x, y)\omega \in \Omega$, with $(x, y) \in G \times^{\bar{G}} G$, $\omega \in T$, then

$$N_{G\times^{\bar{G}}G}(f(\omega')) = N_{G\times^{\bar{G}}G}(f((x, y)\omega)) = {}^{(x,y)}N_{G\times^{\bar{G}}G}(f(\omega))$$
$$= {}^{(x,y)}N_{G\times^{\bar{G}}G}(\omega) = N_{G\times^{\bar{G}}G}(\omega').$$

Applying (1) and the arguments of [3, Lemma 2.2, Lemma 2.3], we are done. □ We will show that stable unital bases are transported from $\mathbf{A}$ to $\mathbf{A} \otimes_N G$.

Proposition 2.2 *Let $\mathbf{A}$ be an N-interior G-algebra. If $\mathbf{A}$ has a unital $G \times^{\bar{G}} G$-stable basis then $\mathbf{A} \otimes_N G$ has a unital $G \times G$-stable basis.*

Proof Let Ω be a unital $G \times^{\bar{G}} G$-stable basis of $\mathbf{A}$ and $S \subseteq G$ be a system of representatives of the left cosets of N in G. It is clear that

$$\mathscr{B} := \bigcup_{u\in S} \Omega \otimes_N u$$

is an $\mathscr{O}$-basis of basis of $\mathbf{A} \otimes_N G$. If $\omega \otimes u \in \mathscr{B}$ then

$$(\omega \otimes u)({}^{u^{-1}}(\omega^{-1}) \otimes u^{-1}) = 1_{\mathbf{A}} \otimes 1_G = ({}^{u^{-1}}(\omega^{-1}) \otimes u^{-1})(\omega \otimes u).$$

For $G \times G$-stability, we consider the elements

$$(x, y) \in G \times G, \omega \otimes u \in \mathscr{B}, \omega \in \Omega, u \in S.$$

Then, there is a unique $u' \in S$ such that $xuy^{-1} = nu'$, for some $n \in N$. It follows

$$(x, y)(\omega \otimes u) = (1_{\mathbf{A}} \otimes x)(\omega \otimes u)(1_{\mathbf{A}} \otimes y^{-1})$$
$$= {}^{x}\omega \otimes xuy^{-1} = {}^{x}\omega n \otimes u' = ((x, n^{-1}x)\omega \otimes u'),$$

which is in $\mathscr{B}$, since $(x, n^{-1}x) \in G \times^{\bar{G}} G$. The first equality in the above identity is given by the action of $G \times G$ on $\mathbf{A} \otimes_N G$, see 2. Thus, $\mathscr{B}$ is a unital $G \times G$-stable basis of $\mathbf{A} \otimes_N G$. □

For the rest of the section we assume that G is a p-group.

2.4 We adopt the following notations, see [3]. The set $\mathfrak{J}(G)$ is the set of all group isomorphisms ϕ such that $\mathrm{dom}(\phi)$ and $\mathrm{cod}(\phi)$ are subgroups in G. For $\phi \in \mathfrak{J}(G)$ recall that $\Delta(\phi)$, the twisted diagonal subgroup with respect to ϕ, is the subgroup of $G \times G$ formed by the pairs $(\phi(u), u)$ when u runs in $\mathrm{dom}(\phi)$. Moreover

$$\mathfrak{J}_{G\times^{\bar{G}}G}(G) := \{\phi | \phi \in \mathfrak{J}(G), \Delta(\phi) \leq G \times^{\bar{G}} G\},$$

$$\mathscr{L} := \{L | L \leq G \times^{\bar{G}} G, L \cap (N \times 1) = (1, 1) = L \cap (1 \times N)\}.$$

Let $\phi \in \mathfrak{J}_{G\times^{\bar{G}}G}(G)$. We denote by $\mathbf{A}(\phi) := \mathbf{A}^{\Delta(\phi)}/\mathrm{Ker}(\mathrm{Br}^{\mathbf{A}}_{\Delta(\phi)})$ the Brauer quotient with respect to $\Delta(\phi)$. Recall that

$$\mathbf{A}^{\Delta(\phi)} = \{a \in \mathbf{A} | (\phi(u), u)a = a, \text{ for any } u \in \mathrm{dom}(\phi)\}$$

and that

$$\mathrm{Br}^{\mathbf{A}}_{\Delta(\phi)} : \mathbf{A}^{\Delta(\phi)} \to \mathbf{A}^{\Delta(\phi)} / \left(\sum_{L \leq \Delta(\phi)} \mathbf{A}^{\Delta(\phi)}_{L} + J(\mathscr{O})\mathbf{A}^{\Delta(\phi)} \right)$$

is the canonical surjective map, called the Brauer map, see [7, Definition 5.4.10]. The next lemma is straightforward and for its proof we introduce the following notations for the first and second projection of a direct product of groups. For any finite groups L_1, L_2 we denote the first and second projection by

$$\pi : L_1 \times L_2 \to L_1, \pi(x, y) = x, \quad \pi' : L_1 \times L_2 \to L_2, \pi'(x, y) = y,$$

for any $(x, y) \in L_1 \times L_2$.

Lemma 2.3 *There is a bijection $F : \mathfrak{J}_{G\times^{\bar{G}}G}(G) \to \mathscr{L}$ given by $F(\phi) = \Delta(\phi)$, for any $\phi \in \mathfrak{J}_{G\times^{\bar{G}}G}(G)$.*

Proof The above correspondence F is a well-defined map since

$$\Delta(\phi) \cap (G \times 1) = (1 \times G) \cap \Delta(\phi) = (1, 1).$$

Let

$$F_0 : \mathscr{L} \to \mathfrak{J}_{G\times^{\bar{G}}G}(G), F_0(L) = \phi_L,$$

with $\phi_L : \pi'(L) \to \pi(L)$ defined by $\phi_L(u) = v$, where if $u \in \pi'(L)$ there is $(v, u) \in L$ such that $\pi'(v, u) = u$.

$$(F \circ F_0)(L) = F(\phi_L) = \Delta(\phi_L) = \{(\phi_L(u), u) | u \in \mathrm{dom}(\phi_L)\} = \pi(L) \times \pi'(L) = L.$$

$$(F_0 \circ F)(\phi) = F_0(\Delta(\phi)) = \phi_{\Delta(\phi)} = \phi.$$

□

Remark 2.5 Note that the above lemma is true in a more general context. Let K be a subgroup of $G \times G$. We consider the following notations:

$$\mathfrak{J}_K(G) := \{\phi | \phi \in \mathfrak{J}(G), \Delta(\phi) \leq K\},$$

$$\mathscr{L}_K := \{L | L \leq K, L \cap (G \times 1) = (1, 1) = L \cap (1 \times G)\}.$$

Then, similar arguments as above give a bijection between $\mathfrak{J}_K(G)$ and $\mathscr{L}_K$. If $K = G \times^{\bar{G}} G$ then $\mathfrak{J}_K(G) = \mathfrak{J}_{G\times^{\bar{G}}G}(G)$, $K \cap (G \times 1) = N \times 1$ and $\mathscr{L}_K$ is $\mathscr{L}$.

2.6 Let Ω be a $G \times^{\bar{G}} G$-set. We say that Ω is N-*bifree* if $N \times 1$ and $1 \times N$ (which are subgroups of $G \times^{\bar{G}} G$) act freely on Ω.

Lemma 2.4 *Let Ω be a $G \times^{\bar{G}} G$-set. Then Ω is N-bifree if and only if for any $\omega \in \Omega$ there is $\phi \in \mathfrak{J}_{G\times^{\bar{G}}G}(G)$ such that $N_{G\times^{\bar{G}}G}(\omega) = \Delta(\phi)$.*

Proof Fix $\omega \in \Omega$. For the left to right implication, since Ω is N-bifree it follows

$$N_{G\times^{\bar{G}}G}(\omega) \cap (N \times 1) = (1 \times N) \cap N_{G\times^{\bar{G}}G}(\omega) = (1, 1),$$

hence $N_{G\times^{\bar{G}}G}(\omega) \in \mathscr{L}$. By Lemma 2.3, there is $\phi \in \mathfrak{J}_{G\times^{\bar{G}}G}(G)$ such that $N_{G\times^{\bar{G}}G}(\omega) = \Delta(\phi)$.

For the right to left implication, let $(u_1, 1), (u_2, 1) \in N \times 1$ be such that $(u_1, 1)\omega = (u_2, 1)\omega$. It follows that $(u_2^{-1}u_1, 1) \in N_{G\times^{\bar{G}}G}(\omega)$, hence there is $\phi \in \mathfrak{J}_{G\times^{\bar{G}}G}(G)$ such that $(u_2^{-1}u_1, 1) \in \Delta(\phi)$. □

Theorem 2.5 *Let $\mathbf{A}$ be an N-interior G-algebra, admitting an N-bifree $G \times^{\bar{G}} G$-stable basis. The following statements are equivalent:*

(a) *given* $\phi \in \mathfrak{J}_{G\times^{\bar{G}}G}(G)$ *such that* $\mathbf{A}(\phi) \neq 0$ *then* $\mathbf{A}^\times \cap \mathbf{A}^{\Delta(\phi)} \neq \emptyset$;

(b) $\mathbf{A}$ *has a unital* $G \times^{\bar{G}} G$*-stable basis.*

Proof Let Ω be $G \times^{\bar{G}} G$-stable basis.

First we assume (a) is true and that Ω is also N-bifree, with $\omega_0 \in \Omega$. Using Theorem 2.1, we will show that $N_{G\times^{\bar{G}}G}(\omega_0)$ fixes a unit of $\mathbf{A}$. By Lemma 2.4, there is $\phi \in \mathfrak{J}_{G\times^{\bar{G}}G}(G)$ such that $N_{G\times^{\bar{G}}G}(\omega_0) = \Delta(\phi)$, hence

$$\Omega^{\Delta(\phi)} = \Omega^{N_{G\times^{\bar{G}}G}(\omega_0)} \neq \emptyset.$$

By [15, Proposition 27.6 (a)] we obtain that $\mathbf{A}(\phi)$ has as k-basis the set

$$\{\mathrm{Br}^{\mathbf{A}}_{\Delta(\phi)}(\omega) | \omega \in \Omega^{\Delta(\phi)}\}.$$

It follows that $\mathbf{A}(\phi) \neq 0$ and then using statement (a), we obtain $\mathbf{A}^\times \cap \mathbf{A}^{N_{G\times^{\bar{G}}G}(\omega_0)} \neq \emptyset$, which is what we need.

Now, we assume the validity of statement (b) and let $\phi \in \mathfrak{J}_{G\times^{\bar{G}}G}(G)$ satisfy $\mathbf{A}(\phi) \neq 0$. We assume Ω is also unital. It follows

$$\emptyset \neq \Omega^{\Delta(\phi)} = \Omega^\times \cap \Omega^{\Delta(\phi)} \subset \mathbf{A}^\times \cap \mathbf{A}^{\Delta(\phi)}.$$

□

If we take $N = G$ in the next proposition we recover [3, Lemma 4.1].

Proposition 2.6 *Let* $\mathbf{A}$ *be an* N*-interior* G*-algebra. Let* U_μ, V_ν *be pointed groups on* $\mathbf{A}$*, where* $U, V \leq G$*. Let* $\phi \in \mathfrak{J}_{G\times^{\bar{G}}G}(G)$ *be such that* $\mathrm{dom}(\phi) = V$, $\mathrm{cod}(\phi) = U$ *and choose* $i \in \mu$, $j \in \nu$*. The following statements are equivalent:*

(a) *there exists* $r \in \mathbf{A}^\times$ *such that* $(\phi(v)v^{-1}) \cdot {}^v(ir) = rj$*, for any* $v \in V$;

(b) *there exist* $s \in i\mathbf{A}^{\Delta(\phi)}j$ *and* $s' \in j\mathbf{A}^{\Delta(\phi^{-1})}i$ *such that* $i = ss'$ *and* $j = s's$*.*

Moreover, the above equivalent conditions do not depend on the choices of i *and* j*.*

Proof Assume (a) holds and let $s = irj$, $s' = jr^{-1}i$. Since ${}^r j = i$ is equivalent to $rj = ir$ and to $jr^{-1} = r^{-1}i$, we get

$$s = irj = rj = ir, \quad s' = jr^{-1}i = jr^{-1} = r^{-1}i$$

and, moreover

$$ss' = (irj)(jr^{-1}i) = i \cdot ({}^r j) \cdot i = i$$

$$s's = (jr^{-1}i)(irj) = j \cdot ({}^{r^{-1}} i) \cdot j = j.$$

The relation given in (a)

$$(\phi(v)v^{-1}) \cdot {}^v(ir) = rj, \tag{2}$$

for any $v \in V$, is equivalent to $s \in i\mathbf{A}^{\Delta(\phi)}j$. We shall prove that relation (2) is equivalent to $s' \in j\mathbf{A}^{\Delta(\phi^{-1})}i$, that is

$$(\phi^{-1}(u)u^{-1}) \cdot {}^u(jr^{-1}) = r^{-1}i,$$

for all $u \in U$. Indeed, when v runs through V, the element $u = \phi(v)$ runs through U and, we have

$$(\phi(v)v^{-1}) \cdot {}^v(ir) = rj \iff (r^{-1}i) \cdot {}^{\phi(v)}r \cdot (\phi(v)v^{-1}) = j \iff$$

$$(r^{-1}i)\cdot{}^{\phi(v)}r = jv\phi(v^{-1}) \iff r^{-1}i = jv\phi(v^{-1})\cdot{}^{\phi(v)}r^{-1} \iff r^{-1}i = j\cdot({}^{v}r^{-1})\cdot v\phi(v^{-1})$$
$$\iff r^{-1}i = {}^{v}(jr^{-1})\cdot v\phi(v^{-1}) \iff r^{-1}i = v\phi(v^{-1})\cdot{}^{\phi(v)}(jr^{-1})$$
$$\iff r^{-1}i = \phi^{-1}(u)u^{-1}\cdot{}^{u}(jr^{-1}).$$

Assume (b) and since

$$i = ss',\ j = s's,$$

by [15, Exercise 3.2] there is $q \in \mathbf{A}^{\times}$ such that $i = qjq^{-1}$. Then

$$r := s + (1_{\mathbf{A}} - i)q(1_{\mathbf{A}} - j), \quad r' := s' + (1_{\mathbf{A}} - j)q^{-1}(1_{\mathbf{A}} - i)$$

verifies $rr' = 1_{\mathbf{A}}$ and $1_{\mathbf{A}} = r'r$. Furthermore, for any $v \in V$ relation (2) is verified, because

$$ir = s, \quad s = rj, \quad (\phi(v), v)s = s,$$

by our assumptions. □

3 Subcategories of Fusion Systems Induced by Factor Groups

We need the concept of a category $\mathscr{F}$ on a p-group introduced by Linckelmann in [6, Definition 1.1]; that is, $\mathscr{F}$ is a *category on the p-group D* if it is a category whose objects are the subgroups of D and whose morphisms are the injective group homomorphisms satisfying:

- the inclusions are morphisms of $\mathscr{F}$;
- for any $\phi \in \mathrm{Hom}_{\mathscr{F}}(R_1, R_2)$, $R_1, R_2 \leq D$, the induced isomorphism $R_1 \cong \phi(R_1)$ and its inverse are morphisms in $\mathscr{F}$;
- the composition of morphisms in $\mathscr{F}$ is the usual composition of group homomorphisms.

A *subcategory* $\mathscr{E}$ of $\mathscr{F}$ *on the p-subgroup* D_1 of D is a subcategory which is itself a category on the p-group D_1. In this section, if otherwise not stated, we denote by $\mathscr{F}$ a *fusion system* on a p-group D. It is a category on the p-group D, in which the set of objects consists of the subgroups of D and the morphisms are given by the set of certain injective group homomorphisms, such that $\mathscr{F}_D(D) \subseteq \mathscr{F}$ and other axioms are satisfied, see [2, Definition 1.34]. Here $\mathscr{F}_D(D)$ is the subcategory of $\mathscr{F}$ with the same objects as $\mathscr{F}$ and whose morphisms are the group homomorphisms induced by conjugation in D. Any fusion system like $\mathscr{F}$ is called *saturated* [2, Definition 1.37] if other two axioms are satisfied.

Let $\tilde{D}$ be a normal subgroup of D and as in the Introduction we use the notation $\bar{D} := D/\tilde{D}$. The subgroup D' is the commutator subgroup of D.

3.1 Subcategories

In Proposition 3.1 (i) we will show that our next definition makes sense.

Definition 3.1 We define a subcategory $\mathscr{F}_{D\times^{\bar{D}}D}$ of $\mathscr{F}$ by:

(i) $\mathrm{Ob}(\mathscr{F}_{D\times^{\bar{D}}D}) := \mathrm{Ob}(\mathscr{F})$;
(ii) for any $R_1, R_2 \leq D$, the morphisms set is

$$\mathrm{Hom}_{\mathscr{F}_{D\times^{\bar{D}}D}}(R_1, R_2) := \{\phi \in \mathrm{Hom}_{\mathscr{F}}(R_1, R_2) | \Delta(\phi) \leq D \times^{\bar{D}} D\}.$$

In the above definition, the subset of isomorphisms in $\mathscr{F}_{D\times^{\bar{D}}D}$, with domain R_1 and codomain R_2, is clearly

$$\{\phi \in \mathrm{Hom}_{\mathscr{F}}(R_1, R_2) | \phi \in \mathfrak{J}_{D\times^{\bar{D}}D}(D)\}.$$

Recall that the focal subgroup of a fusion system $\mathscr{F}$ is

$$\mathrm{foc}(\mathscr{F}) := < u\varphi(u^{-1}) | R \le D, u \in R, \varphi \in \mathrm{Aut}_{\mathscr{F}}(R) > .$$

Proposition 3.1 *(i)* $\mathscr{F}_{D\times^{\bar{D}}D}$ *is a subcategory of* $\mathscr{F}$ *on the* p*-group* D;

(ii) If the commutator subgroup D' *of* D *is included in* $\tilde{D}$ *then* $\mathscr{F}_{D\times^{\bar{D}}D}$ *is a fusion subsystem of* $\mathscr{F}$;

(iii) Assume $\mathscr{F}$ *is saturated. If* $\mathrm{foc}(\mathscr{F}) \le \tilde{D}$ *then* $\mathscr{F}_{D\times^{\bar{D}}D} = \mathscr{F}$;

(iv) If $\tilde{D} = 1$ *then* $\mathscr{F}_{D\times^{\bar{D}}D} = \mathscr{F}_{\Delta(D)}$, *where* $\mathscr{F}_{\Delta(D)}$ *is the subcategory of* $\mathscr{F}$ *on the* p*-group* D, *whose morphisms consist of inclusion maps.*

Proof (i) It is clear that $\mathscr{F}_{D\times^{\bar{D}}D}$ is included in $\mathscr{F}$ and that the inclusions are in $\mathscr{F}_{D\times^{\bar{D}}D}$. The second axiom of the definition of a category on a p-group, which is reminded at the beginning of this section, is straightforward. We only verify that if we take $\phi \in \mathrm{Hom}_{\mathscr{F}_{D\times^{\bar{D}}D}}(R_2, R_3)$, $\psi \in \mathrm{Hom}_{\mathscr{F}_{D\times^{\bar{D}}D}}(R_1, R_2)$ with $R_1, R_2, R_3 \le D$, then $\phi \circ \psi \in \mathrm{Hom}_{\mathscr{F}_{D\times^{\bar{D}}D}}(R_1, R_3)$. Clearly, $\phi \circ \psi \in \mathrm{Hom}_{\mathscr{F}}(R_1, R_3)$ and, for any $u \in R_1$, we have

$$u^{-1}\phi(\psi(u)) = (u^{-1}\psi(u)) \cdot ((\psi(u))^{-1}\phi(\psi(u))) \in \tilde{D},$$

which is what we need.

(ii) Since (i) holds, we only verify that if $c_x : R_1 \to R_2$, where ${}^xR_1 \le R_2 \le D, x \in D$, is the conjugation map, then $c_x \in \mathrm{Hom}_{\mathscr{F}_{D\times^{\bar{D}}D}}(R_1, R_2)$. Let x be an element of D such that ${}^xR_1 \le R_2 \le D$. For any $u \in R_1$ the following equality holds

$$u^{-1}c_x(u) = u^{-1}xux^{-1} = [u^{-1}, x] \in D' \le \tilde{D},$$

hence $c_x \in \mathrm{Hom}_{\mathscr{F}_{D\times^{\bar{D}}D}}(R_1, R_2)$.

(iii) Clearly $\mathscr{F}_{D\times^{\bar{D}}D} \subseteq \mathscr{F}$. By Alperin's Fusion Theorem [8, Theorem 8.2.8] it is enough to show that $\mathrm{Aut}_{\mathscr{F}}(R) \subseteq \mathrm{Aut}_{\mathscr{F}_{D\times^{\bar{D}}D}}(R)$, where $R \le D$; for this let $\phi \in \mathrm{Aut}_{\mathscr{F}}(R)$ and $u \in R$. It is clear that

$$u^{-1}\phi(u) \in \mathrm{foc}(\mathscr{F}) \le \tilde{D},$$

hence the conclusion.

(iv) Since $\tilde{D} = 1$ we know that $D \times^{\bar{D}} D = \Delta(D)$.

□

3.2 Normalizer Subcategories

Let R be a subgroup of D. Statement (i) of Propostion 3.2 assures us that the following definition makes sense.

Definition 3.2 The *normalizer subcategory* $N_{\mathscr{F}_{D\times^{\bar{D}}D}}(R)$ is the subcategory of $\mathscr{F}_{D\times^{\bar{D}}D}$ defined by:

(i) $\mathrm{Ob}(N_{\mathscr{F}_{D\times^{\bar{D}}D}}(R)) := \{R_1 | R_1 \le N_D(R)\}$;

(ii) for any $R_1, R_2 \le N_D(R)$, a morphism $\phi : R_1 \to R_2$ in $N_{\mathscr{F}_{D\times^{\bar{D}}D}}(R)$ is a morphism $\phi \in \mathrm{Hom}_{\mathscr{F}_{D\times^{\bar{D}}D}}(R_1, R_2)$ which extends to some $\phi' \in \mathrm{Hom}_{\mathscr{F}_{D\times^{\bar{D}}D}}(RR_1, RR_2)$ such that $\phi'|_R(R) = R$.

As a consequence of Definitions 3.1, 3.2 and Proposition 3.1 we obtain the next proposition.

Proposition 3.2 *(i)* $N_{\mathscr{F}_{D\times^{\tilde{D}}D}}(R)$ *is a subcategory of* $\mathscr{F}_{D\times^{\tilde{D}}D}$ *on the p-group* $N_D(R)$*;*

(ii) If If the commutator subgroup D' *of* D *is included in* $\tilde{D}$ *then* $N_{\mathscr{F}_{D\times^{\tilde{D}}D}}(R)$ *is a fusion subsystem of* $\mathscr{F}$*;*

(iii) Assume $\mathscr{F}$ *is saturated. If* $\mathrm{foc}(\mathscr{F})\leq\tilde{D}$ *then* $N_{\mathscr{F}_{D\times^{\tilde{D}}D}}(R)=N_{\mathscr{F}}(R)$*, the usual normalizer fusion system of* R *in* $\mathscr{F}$*, see [2, Definition 4.26 (ii)];*

(iv) If $\tilde{D}=1$ *and* $R=D$ *then* $N_{\mathscr{F}_{\Delta(D)}}(D)=\mathscr{F}_{\Delta(D)}$*.*

4 Proof of Theorem 1.5

For the proof of Theorem 1.5 we need to recall in the following remark what is Puig's notion of "isofusion" between pointed groups.

Remark 4.1 Let $\mathbf{A}$ be an N-interior G-algebra, where N is a normal subgroup of G. Let U_μ, V_ν be pointed groups on $\mathbf{A}$, where $U, V\leq G$ and let $i\in\mu$, $j\in\nu$. A map $\phi\in\mathfrak{I}_{G\times^{\bar{G}}G}(G)$ (with $\mathrm{dom}(\phi)=V$, $\mathrm{cod}(\phi)=U$) for which there is $r\in\mathbf{A}^\times$ such that $(\phi(v)v^{-1})\cdot{}^{v}(ir)=rj$, for any $v\in V$, is called and $\mathbf{A}$-*isofusion*. The set of $\mathbf{A}$-isofusions is denoted by $F_{\mathbf{A}}(V_\nu,U_\mu)$, see [11, 2.2.1]. When $U_\mu=V_\nu$, the notation $F_{\mathbf{A}}(V_\nu):=F_{\mathbf{A}}(V_\nu,U_\mu)$ is used. In the particular case in which $N=G$ and $\mathbf{A}$ is a G-interior algebra we have that $\phi\in F_{\mathbf{A}}(V_\nu)$ if and only if $\phi:V\to V$ is a group isomorphism for which there is $r\in\mathbf{A}^\times$ such that $\phi(v)j={}^{r}(vj)$, for any $v\in V$.

Recall that in the statement of Theorem 1.5 we considered: b to be block idempotent $\mathscr{O}G$ with defect group D; the idempotent $l\in(\mathscr{O}Gb)^D$ to be a primitive idempotent such that $\mathrm{Br}_D^{\mathscr{O}G}(l)\neq 0$; $A=l\mathscr{O}Gl$ denotes the source algebra of b and $\mathscr{F}$ the saturated fusion system of A on D. Let $\tilde{D}$ be a normal subgroup of D such that $\mathrm{hyp}(\mathscr{F})\leq\tilde{D}$ and $\tilde{A}$ be the hyperfocal subalgebra with respect to $\tilde{D}$. Let λ_D be the point $\{ala^{-1}|a\in((\mathscr{O}Gb)^D)^\times\}$, which is the unique point of D on $\mathscr{O}Gb$ associated with (D,e_D). Applying Remark 4.1 to the D-interior algebra $\mathscr{O}Gb$ we obtain the set of isofusions $F_{\mathscr{O}Gb}(D_{\lambda_D})$.

Proof We will show that if $\mathscr{F}_{D\times^{\bar{D}}D}=N_{\mathscr{F}_{D\times^{\bar{D}}D}}(D)$ then $\tilde{A}$ has a $D\times^{\bar{D}}D$-stable basis. We already know that, since $\tilde{A}$ is a direct summand of $\mathscr{O}G$ as an $\mathscr{O}[D\times^{\bar{D}}D]$-module, $\tilde{A}$ has a $D\times^{\bar{D}}D$-stable basis which is $\tilde{D}$-bifree, see 2. By Theorem 2.5 (with $\mathbf{A}$, N, G replaced by $\tilde{A}$, $\tilde{D}$, D, respectively) we need to verify that, given $\phi\in\mathfrak{I}_{D\times^{\bar{D}}D}(D)$ such that $\tilde{A}(\phi)\neq 0$, then $\tilde{A}^\times\cap\tilde{A}^{\Delta(\phi)}\neq\emptyset$.

Let $\phi:R_1\to R_2$, $\phi\in\mathfrak{I}_{D\times^{\bar{D}}D}(D)$ such that $\tilde{A}(\phi)\neq 0$. Now $\tilde{A}$ is a direct summand of A as $\mathscr{O}[D\times^{\bar{D}}D]$-modules, hence $\tilde{A}(\phi)$ is a direct summand of $A(\phi)$. It follows $A(\phi)\neq 0$, hence $\Omega^{\Delta(\phi)}\neq\emptyset$, for some $D\times D$-stable basis of A. Applying [3, Theorem 7.2] we obtain $\phi\in\mathrm{Hom}_{\mathscr{F}_{D\times^{\bar{D}}D}}(R_1,R_2)$. Next, since $\mathscr{F}_{D\times^{\bar{D}}D}=N_{\mathscr{F}_{D\times^{\bar{D}}D}}(D)$, it follows that there is $\psi:R_1D\to R_2D$ in $\mathscr{F}_{D\times^{\bar{D}}D}$ such that

$$\psi(D)=D,\quad \psi_{|R_1}=\phi.$$

So there is $\psi\in\mathrm{Aut}_{\mathscr{F}}(D)$ such that

$$\psi_{|R_1}=\phi,\quad u^{-1}\psi(u)\in\tilde{D},$$

for any $u \in D$.

Since D is a defect group of b, we know that $N_G(D, e_D) = N_G(D_{\lambda_D})$, hence there is $g \in N_G(D_{\lambda_D})$ such that $\psi = c_g$. Then $\psi \in F_{\mathcal{O}Gb}(D_{\lambda_D})$ (see [9, Theorem 3.1]); equivalently, we say that $\psi : D_{\lambda_D} \to D_{\lambda_D}$ is an $\mathcal{O}Gb$-isofusion, which also verifies $\Delta(\psi) \leq D \times^{\bar{D}} D$. Since $l = 1_{\tilde{A}}$ is a primitive idempotent of $\tilde{A}$ then $\tilde{\lambda}_D = \{l\}$ is a point of D on $\tilde{A}$ such that

$$\tilde{\lambda}_D = \lambda'_D \subseteq \lambda_D,$$

where $\lambda'_D = \{l\}$ is considered as a point of D on A. We now apply [9, Proposition 2.14] and [11, Proposition 4.2] to obtain

$$F_{\tilde{A}}(D_{\tilde{\lambda}_D}) = F_A(D_{\lambda'_D}) = F_{\mathcal{O}Gb}(D_{\lambda_D}),$$

where $F_{\tilde{A}}(D_{\tilde{\lambda}_D})$ and $F_A(D_{\lambda'_D})$ are obtained using Remark 4.1; since $\tilde{A}$ is $\tilde{D}$-interior D-algebra and A is a D-interior algebra. It follows that $\psi : D_{\tilde{\lambda}_D} \to D_{\tilde{\lambda}_D}$ is an $\tilde{A}$-isofusion.

This means that ψ is an isomorphism lying in $\mathfrak{I}_{D \times^{\bar{D}} D}(D)$ and satisfying a statement similar to statement (b) of Proposition 2.6. That is, there are elements

$$s \in l(\tilde{A}^{\Delta(\psi)})l = \tilde{A}^{\Delta(\psi)}, \quad s' \in l(\tilde{A}^{\Delta(\psi^{-1})})l = \tilde{A}^{\Delta(\psi^{-1})},$$

such that

$$l = ss' = s's,$$

thus $s \in \tilde{A}^{\times}$. Since $\Delta(\phi) \subseteq \Delta(\psi)$, we conclude that $s \in \tilde{A}^{\times} \cap \tilde{A}^{\Delta(\phi)}$. □

5 Stable Unital Bases Under Basic Morita Equivalences

Recall that we denoted by b a block idempotent of $\mathcal{O}G$ with defect group D and by $l \in (\mathcal{O}Gb)^D$ a primitive idempotent such that $\mathrm{Br}_D^{\mathcal{O}G}(l) \neq 0$. We shall keep the notation $A := l\mathcal{O}Gl$. For any subgroup R of D there is a unique block e_R of $kC_G(R)$ such that $\mathrm{Br}_R^{\mathcal{O}G}(l)e_R \neq 0$. Fixing (D, e_D), a maximal $(\mathcal{O}G, b, G)$-Brauer pair, we denoted by $\mathcal{F} := \mathcal{F}_{(D,e_D)}(\mathcal{O}G, b, G)$, the saturated fusion system of A on D, associated with b. Also recall that we considered in the Introduction another finite group H and a block idempotent b' of $\mathcal{O}H$ with defect group E. Let $l' \in (\mathcal{O}Hb')^E$ be a primitive idempotent such that $\mathrm{Br}_E^{\mathcal{O}H}(l') \neq 0$. Similar to the case of the block idempotent b, we shall use the notations: $A' := l'\mathcal{O}Hl'$ is the source algebra of b', for any subgroup Q in E the block e'_Q is the unique block of $kC_H(Q)$ such that $\mathrm{Br}_Q^{\mathcal{O}H}(l')e'_Q \neq 0$, $\mathcal{F}'$ is the saturated fusion of A' on E, etc. Let $\gamma := \{ala^{-1} | a \in ((\mathcal{O}Gb)^D)^{\times}\}$ be the point of D on $\mathcal{O}Gb$ which contains l, similarly we introduce γ'. It follows that D_γ is a defect pointed group of $G_{\{b\}}$ and $E_{\gamma'}$ is a defect pointed group of $H_{\{b'\}}$. We denoted by π, π' the first and second projection of a direct product of groups, see the paragraph before Lemma 2.3.

5.1 *Basic Morita equivalences.* We recall the characterization of Morita equivalences between Brauer blocks, see [14, 3.2.1, Theorem 3.2]. If $\mathcal{O}Gb$ is Morita equivalent to $\mathcal{O}Hb'$ through an $\mathcal{O}Gb - \mathcal{O}Hb'$-bimodule M (which can be viewed as an indecomposable $\mathcal{O}[G \times H]$-module, such that $bMb' = M$) then there is a suitable p-subgroup Δ in $G \times H$ such that

$$\pi(\Delta) = D, \pi'(\Delta) = E$$

and, there is a suitable indecomposable $\mathscr{O}\Delta$-module N such that the restriction modules $\mathrm{Res}^{\Delta}_{\Delta\cap(G\times 1)} N$ and $\mathrm{Res}^{\Delta}_{\Delta\cap(1\times H)} N$ are projective. We shall use the following notations for the surjective homomorphisms of p-groups

$$\sigma : \Delta \to D, \sigma := \pi_{|\Delta}, \quad \sigma' : \Delta \to E, \sigma' := \pi'_{|\Delta}.$$

We say that $\mathscr{O}Gb$ is *basic Morita equivalent* to $\mathscr{O}Hb'$, see [14, Corollary 3.6], if $\mathscr{O}Gb$ is Morita equivalent to $\mathscr{O}Hb'$ through a $\mathscr{O}Gb - \mathscr{O}Hb'$-bimodule M (with the above notations) and σ (or σ') is bijective.

We collect, from various references, the following properties, which will be useful for the next proofs.

5.2 *Properties of basic Morita equivalences.* We assume in this Subsection that $\mathscr{O}Gb$ is basic Morita equivalent to $\mathscr{O}Hb'$, through the $\mathscr{O}Gb - \mathscr{O}Hb'$-bimodule M, as in 5.

1) The map $\sigma : \Delta \to D$ is an isomorphism if and only if $\sigma' : \Delta \to E$ is an isomorphism. We denote by

$$\lambda : E \to D, \lambda := \sigma \circ (\sigma')^{-1}, \quad \lambda' : D \to E, \lambda' := \sigma' \circ \sigma^{-1}$$

the induced isomorphisms between the defect groups.

2) There is an embedding of D-interior algebras

$$f : A \to S \otimes_{\mathscr{O}} A' \tag{3}$$

and an embedding of E-interior algebras

$$f' : A' \to S^{op} \otimes_{\mathscr{O}} A, \tag{4}$$

where S is the $\mathscr{O}$-simple, permutation D-interior algebra $\mathrm{End}_{\mathscr{O}}(N)$ with the property $S(D) \neq 0$.

3) For any R, a subgroup of D, we denote the isomorphic subgroups of E by $Q := \lambda'(R)$. We denote by $\mathscr{LP}_A(R)$ the set of local points of R on A and similarly for $\mathscr{LP}_{A'}(Q)$. By [13, 7.6.2] there is a bijection between these two sets of local points

$$\mathscr{LP}_A(R) \rightarrowtail\!\!\!\!\!\rightarrow \mathscr{LP}_{A'}(Q) \tag{5}$$

Moreover, for any $R_1, R_2 \leq D$, if $\delta' \in \mathscr{LP}_{A'}(Q_1)$ corresponds uniquely to $\delta \in \mathscr{LP}_A(R_1)$ (respectively, $\epsilon' \in \mathscr{LP}_{A'}(Q_2)$ corresponds uniquely to $\epsilon \in \mathscr{LP}_A(R_2)$) through the bijection in (5) then, there is also a bijection induced by (3) and (4) between the set of isofusions

$$F_A((R_1)_\delta, (R_2)_\epsilon) \rightarrowtail\!\!\!\!\!\rightarrow F_{A'}((Q_1)_{\delta'}, (Q_2)_{\epsilon'}), \tag{6}$$

see [9, Proposition 2.14] and [5, Lemma 1.17]

4) By [13, Theorem 19.7] we know that if b is basic Morita (hence Rickard) equivalent to b' then λ' induces an equivalence between the Brauer categories of b and b'. In our case $\lambda' : D \to E$ induces an isomorphism of saturated fusion systems

$$\mathscr{F} \cong \mathscr{F}' \quad (R, e_R) \mapsto (Q, e'_Q), Q = \lambda'(R) \tag{7}$$

Given a pointed group U_μ on any G-algebra B, we define the *multiplicity* of U_μ, denoted $m_B(U_\mu)$, to be the number of elements of μ appearing in a primitive idempotent decomposition of the identity element of B in the subalgebra B^U of U-fixed elements. Given a second pointed group T_τ on B with $T_\tau \leq U_\mu$, we denote the *relative multiplicity* of T_τ in U_μ, denoted $m_B(T_\tau, U_\mu)$, to be the number of elements of τ that appear in a primitive idempotent decomposition of i in the algebra B^T, where i is any element of μ.

Proposition 5.1 *Let b, b' be basic Morita equivalent blocks admitting defect pointed groups groups D_γ and $E_{\gamma'}$, respectively. Let $R_\delta \leq D_\gamma$, $Q_{\delta'} \leq E_{\gamma'}$ be local pointed groups determined by the bijection (5). If $m_\delta := m_{\mathcal{O}G}(R_\delta, D_\gamma) = m_A(R_\delta)$ and $m_{\delta'} := m_{\mathcal{O}H}(Q_{\delta'}, E_{\gamma'}) = m_{A'}(Q_{\delta'})$ then $m_\delta = m_{\delta'}$.*

Proof For any subgroup $R \leq D$, a decomposition of the idempotent l in $(\mathcal{O}Gb)^R$ is a decomposition in A^R. We fix

$$l = i_1 + \ldots + i_{m_\delta} + \ldots + i_r$$

a primitive idempotent decomposition of l in A^R (of length r). Denote by l_1 the sum $i_1 + \ldots + i_{m_\delta}$, which is an idempotent in A^R; with the terms of l_1 belonging to δ. Using the embedding of D-interior algebras given in (3), it follows $f(\delta) \subseteq \bar{\delta}$, where $\bar{\delta}$ is a point of Q on $S \otimes_{\mathcal{O}} A'$ and $Q_{\delta'}$ is the unique pointed group on $(A')^Q$ which corresponds to $Q_{\bar{\delta}}$, see [10, Theorem 5.3].

We also fix

$$l' = i'_1 + \ldots + i'_{m_{\delta'}} + \ldots + i'_{r'}$$

a primitive idempotent decomposition of l' in $(A')^Q$ (of length r'). Denote by l'_1 the sum $i'_1 + \ldots + i'_{m_{\delta'}}$, which is an idempotent in $(A')^Q$; with the terms of l'_1 belonging to δ'.

The map

$$g : A' \to S \otimes_{\mathcal{O}} A', \quad g(a') = 1_S \otimes a'$$

is an injective homomorphism (of Q-algebras), verifying $g_{|_{(A')^Q}}((A')^Q) \subseteq (S \otimes_{\mathcal{O}} A')^Q$. It follows that we can identify A' with a unital subalgebra of $S \otimes_{\mathcal{O}} A'$, $(A')^Q$ with a unital subalgebra of $(S \otimes_{\mathcal{O}} A')^Q$, hence $g(l') = 1 \otimes l'$. This means that the idempotent l' can be identified with $1 \otimes l'$ through g. In $(S \otimes_{\mathcal{O}} A')^Q$ we have

$$1_{S \otimes_{\mathcal{O}} A'} = 1 \otimes l' = 1 \otimes l'_1 + \ldots + 1 \otimes i'_{r'} = g(l'_1) + \ldots + g(i'_{r'})$$

and

$$f(l_1)(1 \otimes l') = f(l_1) = (1 \otimes l')f(l_1) \tag{8}$$

But $f(l_1) = f(i_1) + \ldots + f(i_{m_\delta})$ is a primitive idempotent decomposition in $(S \otimes_{\mathcal{O}} A')^Q$, with $f(i_1) \in \bar{\delta}, \ldots, f(i_{m_\delta}) \in \bar{\delta}$. It follows that the correspondence between $Q_{\bar{\delta}}$ and $Q_{\delta'}$ forces, using (8), to obtain

$$f(l_1)g(l'_1) = g(l'_1)f(l_1) = f(l_1).$$

Then $f(l_1)$ is an idempotent which appears in the primitive idempotent decomposition of $g(l'_1)$ in $(S \otimes_{\mathcal{O}} A')^Q$, which forces $m_\delta \leq m_{\delta'}$.

Analogously, using the embedding (4), see [13, 7.3.3], we obtain $m_{\delta'} \leq m_\delta$. □

Finally, we prove that stable unital bases are preserved by basic Morita equivalences.

Proof **(of Theorem 1.7)** We will show that statement (a) of [3, Theorem 1.2] is true for A'.

Let $\phi' : Q_1 \to Q_2$ be an $\mathcal{F}'$-isomorphism. Since $\mathcal{O}Gb$ is basic Morita equivalent to $\mathcal{O}Hb'$ then there is $\phi : R_1 \to R_2$ an $\mathcal{F}$-isomorphism, which corresponds to ϕ' through (7), where

$$R_1 = \lambda(Q_1), R_2 = \lambda(Q_2).$$

Applying [3, Theorem 1.2, (a)] to ϕ we obtain a bijective correspondence between the local points ϵ of R_2 on $\mathscr{O}G$ satisfying $(R_2)_\epsilon \leq D_\gamma$ and the local points δ of R_1 on $\mathscr{O}G$ satisfying $(R_1)_\delta \leq D_\gamma$, hence a bijection

$$\mathscr{L}\mathscr{P}_A(R_2) \rightarrowtail\!\!\!\!\rightarrow \mathscr{L}\mathscr{P}_A(R_1)$$

The correspondence is such that $\epsilon \leftrightarrow \delta$ if and only if $\phi : (R_1)_\delta \to (R_2)_\epsilon$ is an A-isofusion. Furthermore, in the above case

$$m_{\mathscr{O}G}((R_2)_\epsilon, D_\gamma) = m_{\mathscr{O}G}((R_1)_\delta, D_\gamma),$$

equality that is equivalent to

$$m_A((R_2)_\epsilon) = m_A((R_1)_\delta) \tag{9}$$

Applying (5) to R_1 and R_2 we get

$$\mathscr{L}\mathscr{P}_A(R_1) \rightarrowtail\!\!\!\!\rightarrow \mathscr{L}\mathscr{P}_{A'}(Q_1), \quad \mathscr{L}\mathscr{P}_A(R_2) \rightarrowtail\!\!\!\!\rightarrow \mathscr{L}\mathscr{P}_{A'}(Q_2),$$

hence there is a bijective correspondence between $\mathscr{L}\mathscr{P}_{A'}(Q_1)$ and $\mathscr{L}\mathscr{P}_{A'}(Q_2)$, which is what we need: a bijective correspondence between the local points ϵ' of Q_2 on $\mathscr{O}H$, satisfying $(Q_2)_{\epsilon'} \leq E_{\gamma'}$ and the local points δ' of Q_1 on $\mathscr{O}H$, satisfying $(Q_1)_{\delta'} \leq E_{\gamma'}$. Recall that, by (6), we have a bijection between the sets of isofusions

$$F_A((R_1)_\delta, (R_2)_\epsilon) \rightarrowtail\!\!\!\!\rightarrow F_{A'}((Q_1)_{\delta'}, (Q_2)_{\epsilon'}),$$

hence $\phi' : (Q_1)_{\delta'} \to (Q_2)_{\epsilon'}$ is an isofusion.

Finally, applying Proposition 5.1, we obtain

$$m_A((R_1)_\delta) = m_{A'}((Q_1)_{\delta'}) \text{ and } m_A((R_2)_\epsilon) = m_{A'}((Q_2)_{\epsilon'}),$$

which, using (9), give

$$m_{A'}((Q_2)_{\epsilon'}) = m_{A'}((Q_1)_{\delta'}).$$

□

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