



A Note on Constructing Quasi Modules for Quantum Vertex Algebras from Twisted Yangians

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Abstract

In this note, we consider the twisted Yangians $Y(\mathfrak{g}_N)$ associated with the orthogonal and symplectic Lie algebras $\mathfrak{g}_N = \mathfrak{o}_N, \mathfrak{sp}_N$. First, we introduce a certain subalgebra $A_c(\mathfrak{g}_N)$ of the double Yangian for \mathfrak{gl}_N at the level $c \in \mathbb{C}$, which contains the centrally extended $Y(\mathfrak{g}_N)$ at the level c as well as its vacuum module $\mathcal{M}_c(\mathfrak{g}_N)$. Next, we employ its structure to construct examples of quasi modules for the quantum affine vertex algebra $\mathcal{V}_c(\mathfrak{gl}_N)$ associated with the Yang R -matrix. Finally, we use the description of the center of $\mathcal{V}_c(\mathfrak{gl}_N)$ to obtain explicit formulae for families of central elements for a certain completion of $A_c(\mathfrak{g}_N)$ and invariants of $\mathcal{M}_c(\mathfrak{g}_N)$.

Keywords Quantum vertex algebras · Quasi modules · Twisted yangians

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1 Introduction

The twisted Yangians are certain subalgebras of the Yangian for \mathfrak{gl}_N which are associated with the orthogonal and symplectic Lie algebras. They were introduced and studied by G. Olshanski in [17]. Later on, their properties were further investigated by A. Molev, M. Nazarov and G. Olshanski [15]. In particular, the Sklyanin determinant, which is a twisted analogue of the quantum determinant for the Yangian for \mathfrak{gl}_N , was defined and studied in [15, 17]; see also the papers [7, 11–13]. It was named after E. K. Sklyanin in recognition of his work [19], where the new type of determinant for a certain class of reflection algebras was introduced.

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For more information on twisted Yangians and the Sklyanin determinant see the book by A. Molev [14].

The goal of this paper is to establish a connection between the quantum vertex algebra theory and the twisted Yangians for $\mathfrak{g}_N = \mathfrak{o}_N, \mathfrak{sp}_N$. In contrast with the Etingof–Kazhdan construction, which can be used to associate quantum vertex algebras with the double Yangians of classical types [1, 2], the vertex operators coming from the twisted Yangians no longer possess the \mathcal{S} -locality property, a quantum analogue of the locality of vertex operators. Thus, we were only able to employ the structure of twisted Yangians to construct families of quasi modules for Etingof–Kazhdan’s quantum vertex algebra $\mathcal{V}_c(\mathfrak{gl}_N)$ [2] associated with the Yang R -matrix. We should say that the notion of quasi module, which was introduced by H.-S. Li [10], presents a generalization of the vertex algebra module.

Our construction goes in parallel with the construction of quasi $\mathcal{V}_c(\mathfrak{gl}_N)$ -modules associated with reflection algebras [9], which is to be expected given the numerous similarities between the properties of these two classes of algebras. We start by introducing a certain subalgebra $A_c(\mathfrak{g}_N)$ of the double Yangian for \mathfrak{gl}_N at the level $c \in \mathbb{C}$, which contains both the suitably defined central extension of the twisted Yangian $Y(\mathfrak{g}_N)$ at the level c as well as its vacuum module $\mathcal{M}_c(\mathfrak{g}_N)$. The action of $A_c(\mathfrak{g}_N)$ is then used to equip the $\mathbb{C}[[h]]$ -module of $\mathcal{M}_c(\mathfrak{g}_N)$ with the structure of quasi $\mathcal{V}_{2c}(\mathfrak{gl}_N)$ -module so that, in particular, the action of the twisted Yangian resembles the annihilation operators. In addition, we show that a certain wide class of $A_c(\mathfrak{g}_N)$ -modules, the so-called restricted modules, is naturally equipped with the structure of quasi $\mathcal{V}_{2c}(\mathfrak{gl}_N)$ -module.

Finally, as an application, we combine the corresponding quasi module maps with the description of the center of $\mathcal{V}_{2c}(\mathfrak{gl}_N)$, as given by N. Jing, A. Molev, F. Yang and the first author [5], to obtain explicit formulae for families of central elements of the completed algebra $\tilde{A}_c(\mathfrak{g}_N)$ and invariants of the quasi module $\mathcal{M}_c(\mathfrak{g}_N)$. At the critical level $2c = -N$, we recover large families of central elements and invariants parametrized by certain Young diagrams, while at the noncritical level $2c \neq -N$, we obtain a vertex operator-theoretic interpretation of the Sklyanin determinant.

2 Preliminaries

In this section, we recall the double Yangian for \mathfrak{gl}_N and Etingof–Kazhdan’s construction of the quantum affine vertex algebra associated with the Yang R -matrix.

2.1 Double Yangian for \mathfrak{gl}_N

Let $N \geq 2$ be an integer and h a formal parameter. The Yang R -matrix over the commutative ring $\mathbb{C}[[h]]$ is defined by

$$R(u) = 1 - \frac{h}{u} P. \tag{2.1}$$

Here 1 denotes the identity and P the permutation operator on $\mathbb{C}^N \otimes \mathbb{C}^N$, i.e. we have

$$1 = \sum_{i,j=1}^N e_{ii} \otimes e_{jj} \quad \text{and} \quad P = \sum_{i,j=1}^N e_{ij} \otimes e_{ji}, \tag{2.2}$$

where e_{ij} are matrix units. Note that $P^{t_1} = P^{t_2}$, where t_n is the transposition $e_{rs} \mapsto e_{sr}$ applied on the n -th tensor factor. The R -matrix (2.1) satisfies the Yang–Baxter equation

$$R_{12}(u) R_{13}(u + v) R_{23}(v) = R_{23}(v) R_{13}(u + v) R_{12}(u). \tag{2.3}$$

Let $g(u)$ be the unique series in $1 + u^{-1}\mathbb{C}[[u^{-1}]]$ such that $g(u + N) = g(u)(1 - u^{-2})$. Then the R -matrix $\bar{R}(u) = g(u/h)R(u)$ satisfies the *crossing symmetry relation*,

$$(\bar{R}(u)^{-1})^t \bar{R}(u + hN)^t = 1, \tag{2.4}$$

where $\bar{R}(u)^t$ denotes the transposed R -matrix $\bar{R}(u)^t = \bar{R}(u)^{t2}$, and the *unitarity relation*,

$$\bar{R}(u) \bar{R}(-u) = 1, \tag{2.5}$$

see, e.g., [5, Sect. 2] for more details.

The *double Yangian* $DY(\mathfrak{gl}_N)$ for \mathfrak{gl}_N is defined as the associative algebra over $\mathbb{C}[[h]]$ generated by the central element C and the elements $t_{ij}^{(\pm r)}$, where $i, j = 1, \dots, N$ and $r = 1, 2, \dots$; see, e.g., [4]. Its defining relations are given by

$$R(u - v) T_1(u) T_2(v) = T_2(v) T_1(u) R(u - v), \tag{2.6}$$

$$R(u - v) T_1^+(u) T_2^+(v) = T_2^+(v) T_1^+(u) R(u - v), \tag{2.7}$$

$$\bar{R}(u - v + hC/2) T_1(u) T_2^+(v) = T_2^+(v) T_1(u) \bar{R}(u - v - hC/2). \tag{2.8}$$

The matrices $T(u)$ and $T^+(u)$ are defined by

$$T(u) = \sum_{i,j=1}^N e_{ij} \otimes t_{ij}(u) \quad \text{and} \quad T^+(u) = \sum_{i,j=1}^N e_{ij} \otimes t_{ij}^+(u),$$

while its entries, the series $t_{ij}(u)$ and $t_{ij}^+(u)$, are given by

$$t_{ij}(u) = \delta_{ij} + h \sum_{r=1}^{\infty} t_{ij}^{(r)} u^{-r} \quad \text{and} \quad t_{ij}^+(u) = \delta_{ij} - h \sum_{r=1}^{\infty} t_{ij}^{(-r)} u^{r-1}.$$

Throughout the paper we use the subscript to indicate a copy of the matrix in the tensor product algebra $(\text{End } \mathbb{C}^N)^{\otimes m} \otimes DY(\mathfrak{gl}_N)$, so that, for example, we have

$$T_k(u) = \sum_{i,j=1}^N 1^{\otimes(k-1)} \otimes e_{ij} \otimes 1^{\otimes(m-k)} \otimes t_{ij}(u).$$

In particular, we have $k = 1, 2$ and $m = 2$ in the defining relations (2.6)–(2.8).

Recall that the *h -adic topology* on an arbitrary $\mathbb{C}[[h]]$ -module V is the topology generated by the basis $v + h^n V, v \in V, n \in \mathbb{Z}_{\geq 0}$. We shall often write V_h to indicate that the $\mathbb{C}[[h]]$ -module V is h -adically completed. For example, if W is a $\mathbb{C}[[h]]$ -module, then $V = W[z^{-1}]_h$ (resp. $V = W((z))_h$) denotes the $\mathbb{C}[[h]]$ -module of all power series $\sum_{r \in \mathbb{Z}} a_r z^r$ in V such that $a_r \rightarrow 0$ when $r \rightarrow -\infty$ with respect to the h -adic topology.

The *Yangian* $Y(\mathfrak{gl}_N)$ (resp. the *dual Yangian* $Y^+(\mathfrak{gl}_N)$) is the subalgebra of $DY(\mathfrak{gl}_N)$ generated by the elements $t_{ij}^{(r)}$ (resp. $t_{ij}^{(-r)}$), where $i, j = 1, \dots, N, r = 1, 2, \dots$. For any $c \in \mathbb{C}$ we denote by $DY_c(\mathfrak{gl}_N)$ the *double Yangian at the level c* , i.e. the quotient of the algebra $DY(\mathfrak{gl}_N)$ by the ideal generated by $C - c$. The *vacuum module* $\mathcal{V}_c(\mathfrak{gl}_N)$ at the level c over the double Yangian is defined as the h -adic completion of the quotient of the algebra $DY_c(\mathfrak{gl}_N)$ by its left ideal generated by the elements $t_{ij}^{(r)}, i, j = 1, \dots, N, r = 1, 2, \dots$. The Poincaré–Birkhoff–Witt theorem for the double Yangian, see [5, Thm. 2.2] or [16, Thm. 15.3], implies that the vacuum module is isomorphic, as a $\mathbb{C}[[h]]$ -module, with the h -adically completed dual Yangian.

2.2 Quantum Affine Vertex Algebra

Let n and m be positive integers. For the families of variables $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_m)$ and the variable z define the formal power series with coefficients in $(\text{End } \mathbb{C}^N)^{\otimes n} \otimes (\text{End } \mathbb{C}^N)^{\otimes m}$ by

$$\overrightarrow{R}_{nm}^{12}(u|v|z) = \prod_{i=1, \dots, n}^{\rightarrow} \prod_{j=n+1, \dots, n+m}^{\leftarrow} \overline{R}_{ij}(z + u_i - v_{j-n}), \tag{2.9}$$

$$\overline{R}_{nm}^{12}(u|v|z) = \prod_{i=1, \dots, n}^{\rightarrow} \prod_{j=n+1, \dots, n+m}^{\rightarrow} \overline{R}_{ij}(z + u_i + v_{j-n}), \tag{2.10}$$

where the arrows indicate the order of the factors. Moreover, we use the superscript t to denote the product of transposed R -matrices, e.g.,

$$\overrightarrow{R}_{nm}^{t\ 12}(u|v|z) = \prod_{i=1, \dots, n}^{\rightarrow} \prod_{j=n+1, \dots, n+m}^{\rightarrow} \overline{R}_{ij}^t(z + u_i + v_{j-n}). \tag{2.11}$$

The series $R_{nm}^{12}(u|v|z)$ and $\underline{R}_{nm}^{12}(u|v|z)$ which employ the original (non-normalized) Yang R -matrix (2.1) are introduced analogously. In (2.9)–(2.11), as well as in the rest of the paper, we use the common expansion convention, where the expressions of the form $(a_1 z_1 + \dots + a_n z_n)^k$, with $a_i \in \mathbb{C}, a_i \neq 0$ and $k < 0$, are expanded in negative powers of the variable which appears on the left. Hence, for example, we have

$$(z_1 - z_2)^{-1} = \frac{1}{z_1} \sum_{l \geq 0} \left(\frac{z_2}{z_1}\right)^l \neq \frac{1}{z_2} \sum_{l \geq 0} \left(\frac{z_1}{z_2}\right)^l = (-z_2 + z_1)^{-1}.$$

Consider the following operators on $(\text{End } \mathbb{C}^N)^{\otimes n} \otimes \mathcal{V}_c(\mathfrak{gl}_N)$:

$$T_{[n]}^+(u|z) = T_1^+(z + u_1) \dots T_n^+(z + u_n) \text{ and } T_{[n]}(u|z) = T_1(z + u_1) \dots T_n(z + u_n). \tag{2.12}$$

Using defining relations (2.6)–(2.8), one can verify the equalities from [2, Subsect. 2.1.2],

$$R_{nm}^{12}(u|v|z - w) T_{[n]}^{+13}(u|z) T_{[m]}^{+23}(v|w) = T_{[m]}^{+23}(v|w) T_{[n]}^{+13}(u|z) R_{nm}^{12}(u|v|z - w), \tag{2.13}$$

$$R_{nm}^{12}(u|v|z - w) T_{[n]}^{13}(u|z) T_{[m]}^{23}(v|w) = T_{[m]}^{23}(v|w) T_{[n]}^{13}(u|z) R_{nm}^{12}(u|v|z - w), \tag{2.14}$$

$$\overline{R}_{nm}^{12}(u|v|z - w + hc/2) T_{[n]}^{13}(u|z) T_{[m]}^{+23}(v|w) = T_{[m]}^{+23}(v|w) T_{[n]}^{13}(u|z) \overline{R}_{nm}^{12}(u|v|z - w - hc/2), \tag{2.15}$$

where the superscripts 1, 2, 3 indicate the tensor product factors as follows:

$$\overbrace{(\text{End } \mathbb{C}^N)^{\otimes n}}^1 \otimes \overbrace{(\text{End } \mathbb{C}^N)^{\otimes m}}^2 \otimes \overbrace{\mathcal{V}_c(\mathfrak{gl}_N)}^3. \tag{2.16}$$

In (2.9)–(2.11) and (2.12), we sometimes omit the variable z and then write, e.g.,

$$T_{[n]}^+(u) = T_1^+(u_1) \dots T_n^+(u_n) \text{ and } R_{nm}^{12}(u|v) = \prod_{i=1, \dots, n}^{\rightarrow} \prod_{j=n+1, \dots, n+m}^{\leftarrow} R_{ij}(u_i - v_{j-n}). \tag{2.17}$$

For example, the identity in (2.13) with omitted variables z and w takes the form

$$R_{nm}^{12}(u|v)T_{[n]}^{+13}(u)T_{[m]}^{+23}(v) = T_{[m]}^{+23}(v)T_{[n]}^{+13}(u)R_{nm}^{12}(u|v).$$

Finally, we recall Etingof–Kazhdan’s construction [2, Thm. 2.3]:

Theorem 2.1 *For any $c \in \mathbb{C}$ there exists a unique structure of quantum vertex algebra on the $\mathbb{C}[[\hbar]]$ -module $\mathcal{V}_c(\mathfrak{gl}_N)$ such that the vacuum vector is the unit $\mathbf{1} \in \mathcal{V}_c(\mathfrak{gl}_N)$ and the vertex operator map is given by*

$$Y(T_{[n]}^+(u) \mathbf{1}, z) = T_{[n]}^+(u|z) T_{[n]}(u|z + \hbar c/2)^{-1}. \tag{2.18}$$

3 Quasi Modules for the Quantum Affine Vertex Algebra

In this section, we employ the twisted Yangians associated with the orthogonal and symplectic Lie algebras $\mathfrak{g}_N = \mathfrak{o}_N, \mathfrak{sp}_N$ to introduce certain subalgebras $A_c(\mathfrak{g}_N)$ of the double Yangian $DY_c(\mathfrak{gl}_N)$. Using these subalgebras we construct a family of quasi $\mathcal{V}_c(\mathfrak{gl}_N)$ -modules and investigate their connection with the representation theory of $A_c(\mathfrak{g}_N)$.

3.1 Twisted Yangians

We follow [14, Ch. 2] to define (slightly modified) twisted Yangians. In addition, we introduce another family of related algebras which resemble dual Yangians. Let $G = (g_{ij})_{i,j=1}^N$ be a nonsingular complex matrix satisfying $G^t = \pm G$. If G is symmetric (resp. antisymmetric), denote by \mathfrak{g}_N the orthogonal Lie algebra \mathfrak{o}_N (resp. the symplectic Lie algebra \mathfrak{sp}_N), where in the antisymmetric case N is even. Note that \mathfrak{g}_N coincides with the fixed point subalgebra of the automorphism σ of \mathfrak{gl}_N given by

$$\sigma(A) = -G^{-1}A^t G \quad \text{for all } A \in \mathfrak{gl}_N. \tag{3.1}$$

Consider the series

$$S^+(u) = T^+(u) G T^{+t}(-u) \quad \text{and} \quad S(u) = T(u + \hbar c) G T^t(-u). \tag{3.2}$$

They can be written in the form

$$S^+(u) = \sum_{i,j=1}^N e_{ij} \otimes s_{ij}^+(u) \quad \text{and} \quad S(u) = \sum_{i,j=1}^N e_{ij} \otimes s_{ij}(u),$$

where the matrix entries are given by

$$s_{ij}^+(u) = g_{ij} - \hbar \sum_{r=1}^{\infty} s_{ij}^{(-r)} u^{r-1} \quad \text{and} \quad s_{ij}(u) = g_{ij} + \hbar \sum_{r=1}^{\infty} s_{ij}^{(r)} u^{-r}$$

for some elements $s_{ij}^{(-r)} \in Y^+(\mathfrak{gl}_N)$ and $s_{ij}^{(r)} \in Y(\mathfrak{gl}_N)$. The series (3.2) satisfy

$$S^{+t}(-u) = \pm S^+(u) + \frac{\hbar}{2u} (S^+(u) - S^+(-u)), \tag{3.3}$$

$$S^t(-u - \hbar c) = \pm S(u) + \frac{\hbar}{2u + \hbar c} (S(u) - S(-u - \hbar c)), \tag{3.4}$$

where the plus (resp. minus) sign corresponds to the symmetric case $\mathfrak{g}_N = \mathfrak{o}_N$ (resp. anti-symmetric case $\mathfrak{g}_N = \mathfrak{sp}_N$). Indeed, both identities are easily verified using (2.6) and (2.7). Furthermore, the relations (2.6)–(2.8), along with the equality

$$R(u - v) G_1 R^t(-u - v) G_2 = G_2 R^t(-u - v) G_1 R(u - v) \tag{3.5}$$

from [14, Lemma 2.4.1], imply the identities

$$R(u - v) S_1^+(u) R^t(-u - v) S_2^+(v) = S_2^+(v) R^t(-u - v) S_1^+(u) R(u - v), \tag{3.6}$$

$$R(u - v) S_1(u) R^t(-u - v - hc) S_2(v) = S_2(v) R^t(-u - v - hc) S_1(u) R(u - v), \tag{3.7}$$

$$\begin{aligned} \overline{R}(u - v + 3hc/2) S_1(u) \overline{R}^t(-u - v + hc/2) S_2^+(v) \\ = S_2^+(v) \overline{R}^t(-u - v - 3hc/2) S_1(u) \overline{R}(u - v - hc/2). \end{aligned} \tag{3.8}$$

Let $A_c(\mathfrak{g}_N)$ be the subalgebra of the double Yangian $DY_c(\mathfrak{gl}_N)$ at the level c generated by the elements $s_{ij}^{(\pm r)}$ with $i, j = 1, \dots, N$ and $r = 1, 2, \dots$. Next, let $Y^+(\mathfrak{g}_N)$ (resp. $Y_c(\mathfrak{g}_N)$) be the subalgebra of the dual Yangian $Y^+(\mathfrak{gl}_N)$ (resp. the Yangian $Y(\mathfrak{gl}_N)$) generated by the elements $s_{ij}^{(-r)}$ (resp. $s_{ij}^{(r)}$) with $r \geq 1$. Finally, let $\mathcal{M}'_c(\mathfrak{g}_N)$ be the $A_c(\mathfrak{g}_N)$ -submodule of $\mathcal{V}_c(\mathfrak{gl}_N)$ generated by the vacuum vector $\mathbf{1}$. Note that $\mathcal{M}'_c(\mathfrak{g}_N)$ coincides with $Y^+(\mathfrak{g}_N)$ as a $\mathbb{C}[[h]]$ -module. We define the *vacuum module* $\mathcal{M}_c(\mathfrak{g}_N)$ for the algebra $A_c(\mathfrak{g}_N)$ as the h -adic completion of $\{v \in \mathcal{V}_c(\mathfrak{gl}_N) : h^n v \in \mathcal{M}'_c(\mathfrak{g}_N) \text{ for some } n \geq 0\}$. By the definition, the induced topology on $\mathcal{M}_c(\mathfrak{g}_N)$ from $\mathcal{V}_c(\mathfrak{gl}_N)$ coincides with the h -adic topology of $\mathcal{M}_c(\mathfrak{g}_N)$; cf. [10, Lemma 3.5]. Moreover, $\mathcal{M}_c(\mathfrak{g}_N)$ is topologically free.

Remark 3.1 The algebra $Y_0(\mathfrak{g}_N)$ at $h = 1$ becomes the (ordinary) twisted Yangian associated with the Lie algebra \mathfrak{g}_N over the complex field; cf. [15, Sect. 3] or [14, Ch. 2].

Remark 3.2 The definition of algebra $A_c(\mathfrak{g}_N)$ gives rise to the natural question whether (3.3), (3.4) and (3.6)–(3.8) are precisely its defining relations, which we plan to investigate in the continuation of this study. Motivated by such a result for the twisted Yangian [15, Thm. 3.8], one might expect a positive answer.

Consider the degree operator on $Y^+(\mathfrak{gl}_N)$ given by $\text{deg } t_{ij}^{(-r)} = -r$. It defines the ascending filtration over the dual Yangian such that the corresponding graded algebra $\text{gr } Y^+(\mathfrak{gl}_N)$ is isomorphic to the universal enveloping algebra $U(\mathfrak{gl}_N \otimes t^{-1}\mathbb{C}[t^{-1}]) \otimes_{\mathbb{C}} \mathbb{C}[[h]]$. The corresponding isomorphism is defined by the assignments

$$\overline{t}_{ij}^{(-r)} \mapsto e_{ij} \otimes t^{-r}, \tag{3.9}$$

where $\overline{t}_{ij}^{(-r)}$ denote the images of the elements $t_{ij}^{(-r)}$ in the corresponding component of $\text{gr } Y^+(\mathfrak{gl}_N)$; see [5, Sect. 2.2] for more details.

Relations (3.3) imply that the algebra $Y^+(\mathfrak{g}_N)$ is generated by the elements

$$s_{ij}^{(-2r)}, \quad i > j \quad \text{and} \quad s_{ij}^{(-2r+1)}, \quad i \geq j \quad \text{for} \quad r = 1, 2, \dots \tag{3.10}$$

in the symmetric case $\mathfrak{g}_N = \mathfrak{o}_N$, and by the elements

$$s_{ij}^{(-2r)}, \quad i \geq j \quad \text{and} \quad s_{ij}^{(-2r+1)}, \quad i > j \quad \text{for} \quad r = 1, 2, \dots \tag{3.11}$$

in the antisymmetric case $\mathfrak{g}_N = \mathfrak{sp}_N$. Let us write $\bar{s}_{ij}^{(-r)}$ for the images of the elements $s_{ij}^{(-r)}$ in the corresponding graded algebra $\text{gr } Y^+(\mathfrak{g}_N)$. Using (3.2) we can compute the images of the generators (3.10) and (3.11) under the isomorphism (3.9), thus getting

$$\bar{s}_{ij}^{(-r)} \mapsto \sum_{k=1}^N g_{kj} e_{ik} \otimes t^{-r} + (-1)^{r-1} \sum_{k=1}^N g_{ik} e_{jk} \otimes t^{-r}. \tag{3.12}$$

Consider the twisted polynomial current Lie algebra

$$(\mathfrak{gl}_N \otimes t^{-1}\mathbb{C}[t^{-1}])^\sigma = \{A(t) \in \mathfrak{gl}_N \otimes t^{-1}\mathbb{C}[t^{-1}] : \sigma(A(t)) = A(-t)\},$$

where the involutive automorphism σ is given by (3.1). Its elements are polynomials in t^{-1} of the form $\sum_{k < 0} A_k \otimes t^k$ such that their even (resp. odd) coefficients A_{2i} (resp. A_{2i+1}) belong to the subalgebra \mathfrak{g}_N (resp. (-1) -eigenspace of σ). Thus, the Lie algebra $(\mathfrak{gl}_N \otimes t^{-1}\mathbb{C}[t^{-1}])^\sigma$ is spanned by the images of $\bar{s}_{ij}^{(-r)}$ given by (3.12), so that we have

Proposition 3.3 *The restriction of (3.9) gives the isomorphism between the graded algebra $\text{gr } Y^+(\mathfrak{g}_N)$ and the universal enveloping algebra $U((\mathfrak{gl}_N \otimes t^{-1}\mathbb{C}[t^{-1}])^\sigma) \otimes_{\mathbb{C}} \mathbb{C}[[h]]$.*

The analogous results can be also obtained for the generators of the twisted Yangian; see [15, Sect. 3] or [14, Ch. 2]. In fact, Proposition 3.3 was established by closely following the approach from these references.

3.2 Vacuum Module $\mathcal{M}_c(\mathfrak{g}_N)$ as a Quasi $\mathcal{V}_{2c}(\mathfrak{gl}_N)$ -Module

We now present the main result of this section, the construction of the structure of quasi $\mathcal{V}_{2c}(\mathfrak{gl}_N)$ -module over $\mathcal{M}_c := \mathcal{M}_c(\mathfrak{g}_N)$. More precisely, following the definition of *quasi module* for h -adic nonlocal vertex algebra [10, Def. 2.23], we construct the $\mathbb{C}[[h]]$ -module map

$$Y_{\mathcal{M}_c}(z) : \mathcal{V}_{2c}(\mathfrak{gl}_N) \otimes \mathcal{M}_c(\mathfrak{g}_N) \rightarrow \mathcal{M}_c(\mathfrak{g}_N)((z))_h$$

$$v \otimes w \mapsto Y_{\mathcal{M}_c}(z)(v \otimes w) = Y_{\mathcal{M}_c}(v, z)w = \sum_{r \in \mathbb{Z}} v_r w z^{-r-1}$$

which satisfies $Y_{\mathcal{M}_c}(\mathbf{1}, z)w = w$ for all $w \in \mathcal{M}_c(\mathfrak{g}_N)$ and possesses the *quasi weak associativity* property: For any positive integer k and elements $u, v \in \mathcal{V}_{2c}(\mathfrak{gl}_N)$, $w \in \mathcal{M}_c(\mathfrak{g}_N)$ there exists a nonzero polynomial $p(x_1, x_2) \in \mathbb{C}[x_1, x_2]$ such that

$$p(z_0 + z_2, z_2)Y_{\mathcal{M}_c}(u, z_0 + z_2)Y_{\mathcal{M}_c}(v, z_2)w - p(z_0 + z_2, z_2)Y_{\mathcal{M}_c}(Y(u, z_0)v, z_2)w \in h^k \mathcal{M}_c(\mathfrak{g}_N)[[z_0^{\pm 1}, z_2^{\pm 1}]]. \tag{3.13}$$

In order to present the aforementioned result, we need the following notation for the operators on $(\text{End } \mathbb{C}^N)^{\otimes n} \otimes \mathcal{M}_c(\mathfrak{gl}_N)$, where $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_m)$,

$$S_{[n]}^+(u|z) = \prod_{i=1, \dots, n}^{\rightarrow} \left(S_i^+(z + u_i) \bar{R}_{i+1}^t(-2z - u_i - u_{i+1}) \dots \bar{R}_{in}^t(-2z - u_i - u_n) \right), \tag{3.14}$$

$$S_{[n]}(u|z) = \prod_{i=1, \dots, n}^{\rightarrow} \left(S_i(z + u_i) \bar{R}_{i+1}^t(-2z - u_i - u_{i+1} - hc) \dots \bar{R}_{in}^t(-2z - u_i - u_n - hc) \right).$$

As before, $S_{[n]}^+(u)$ and $S_{[n]}(u)$ stand for the analogous expressions with the variable z omitted; recall (2.17). Using Yang–Baxter equation (2.3) and relations (3.5)–(3.8) one easily verifies the following equalities for operators on $(\text{End } \mathbb{C}^N)^{\otimes n} \otimes (\text{End } \mathbb{C}^N)^{\otimes m} \otimes \mathcal{M}_c(\mathfrak{g}_N)$:

$$R_{nm}^{12}(u|v|z-w) S_{[n]}^{+13}(u|z) \underline{R}_{nm}^{t12}(-u|-v|-z-w) S_{[m]}^{+23}(v|w) \\ = S_{[m]}^{+23}(v|w) \underline{R}_{nm}^{t12}(-u|-v|-z-w) S_{[n]}^{+13}(u|z) R_{nm}^{12}(u|v|z-w), \tag{3.15}$$

$$R_{nm}^{12}(u|v|z-w) S_{[n]}^{13}(u|z) \underline{R}_{nm}^{t12}(-u|-v|-z-w-hc) S_{[m]}^{23}(v|w) \\ = S_{[m]}^{23}(v|w) \underline{R}_{nm}^{t12}(-u|-v|-z-w-hc) S_{[n]}^{13}(u|z) R_{nm}^{12}(u|v|z-w), \tag{3.16}$$

$$\overline{R}_{nm}^{12}(u|v|z-w+3hc/2) S_{[n]}^{13}(u|z) \overline{\underline{R}}_{nm}^{t12}(-u|-v|-z-w+hc/2) S_{[m]}^{+23}(v|w) \\ = S_{[m]}^{+23}(v|w) \overline{\underline{R}}_{nm}^{t12}(-u|-v|-z-w-3hc/2) S_{[n]}^{13}(u|z) \overline{R}_{nm}^{12}(u|v|z-w-hc/2), \tag{3.17}$$

where the meaning of superscripts 1, 2, 3 is the same as in (2.16). The next theorem is the main result in this section. It is proved in Subsection 3.3 below.

Theorem 3.4 *For any $c \in \mathbb{C}$ there exists a unique structure of quasi $\mathcal{V}_{2c}(\mathfrak{gl}_N)$ -module on the vacuum module $\mathcal{M}_c(\mathfrak{g}_N)$ such that*

$$Y_{\mathcal{M}_c}(T_{[n]}^+(u) \mathbf{1}, z) = S_{[n]}^+(u|z) S_{[n]}(u|z + hc/2)^{-1}. \tag{3.18}$$

Remark 3.5 The expression for the module map in (3.18) is motivated by the *quantum current commutation relation* from [18]. In this particular case it takes the following form: For any integer $n \geq 1$ there exists an integer $r \geq 0$ such that the equality

$$(u^2 - v^2)^r \overline{R}(u-v) \mathcal{L}_1(u) \overline{R}(u-v+2hc)^{-1} \mathcal{L}_2(v) \\ = (u^2 - v^2)^r \mathcal{L}_2(v) \overline{R}(v-u+2hc)^{-1} \mathcal{L}_1(u) \overline{R}(v-u)$$

holds modulo h^n with $\mathcal{L}(u) = Y_{\mathcal{M}_c}(T_{[n]}^+(0) \mathbf{1}, u)$.

We now derive a simple consequence of the proof of Theorem 3.4. An $A_c(\mathfrak{g}_N)$ -module M is said to be *restricted* if it is topologically free as a $\mathbb{C}[[h]]$ -module and the action of $S(z)$ on M belongs to $\text{End } \mathbb{C}^N \otimes \text{Hom}(M, M[[z^{-1}]_h])$. The argument from the proof of Lemma 3.8 shows that $\mathcal{M}_c(\mathfrak{g}_N)$ is a restricted module. One can easily prove that on any restricted $A_c(\mathfrak{g}_N)$ -module M for any $m \geq 1$ we have

$$S_{[m]}(u|z) \in (\text{End } \mathbb{C}^N)^{\otimes m} \otimes \text{Hom}(M, M[[z^{-1}]][[u_1, \dots, u_m]]_h).$$

This is due to the fact that for any $n \geq 1$ the coefficients of the powers of u in $\overline{R}^t(-2z-u)$ and $S(z+u)w$, where $w \in M$, are polynomials in z^{-1} when regarded modulo h^n . Thus the proof of the next corollary goes in parallel with the proof of Theorem 3.4.

Corollary 3.6 *Let M be a restricted $A_c(\mathfrak{g}_N)$ -module. There exists a unique structure of quasi $\mathcal{V}_{2c}(\mathfrak{gl}_N)$ -module on M such that*

$$Y_M(T_{[n]}^+(u) \mathbf{1}, z) = S_{[n]}^+(u|z) S_{[n]}(u|z + hc/2)^{-1}.$$

3.3 Proof of Theorem 3.4

The proof of Theorem 3.4 is divided in three lemmas which verify the requirements imposed by the definition of quasi module, as given in Subsection 3.2. Even though their proofs go

in parallel with the proof of [9, Thm. 2.7], we provide most of the details in order to take care of the variations which occur in this setting. Throughout the proof, we use the ordered product notation, where the subscript of the product symbol determines the order of the tensor factors. More specifically, for any elements $a = \sum_i a_1^{(i)} \otimes a_2^{(i)}$ and $b = \sum_j b_1^{(j)} \otimes b_2^{(j)}$ of $\text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N$ we write

$$a \cdot_{\text{LR}} b = \sum_{i,j} a_1^{(i)} b_1^{(j)} \otimes b_2^{(j)} a_2^{(i)} \quad \text{and} \quad a \cdot_{\text{RL}} b = \sum_{i,j} b_1^{(j)} a_1^{(i)} \otimes a_2^{(i)} b_2^{(j)}.$$

Using this notation we can express the crossing symmetry property (2.4) as

$$\overline{R}(u)^{-1} \cdot_{\text{LR}} \overline{R}(u + hN) = \overline{R}(u)^{-1} \cdot_{\text{RL}} \overline{R}(u + hN) = 1. \tag{3.19}$$

Lemma 3.7 *Formula (3.18), together with $Y_{\mathcal{M}_c}(\mathbf{1}, z) = 1_{\mathcal{M}_c(\mathfrak{g}_N)}$, defines a unique $\mathbb{C}[[h]]$ -module map $\mathcal{V}_{2c}(\mathfrak{gl}_N) \otimes \mathcal{M}_c(\mathfrak{g}_N) \rightarrow \mathcal{M}_c(\mathfrak{g}_N)[[z^{\pm 1}]]$.*

Proof The coefficients of the matrix entries of $T_{[n]}^+(u) \mathbf{1}$ with $n \geq 0$ span an h -adically dense $\mathbb{C}[[h]]$ -submodule of $\mathcal{V}_{2c}(\mathfrak{gl}_N)$, so (3.18) uniquely determines the quasi module map. Hence it remains to verify that the quasi module map is well-defined by (3.18). It is sufficient to show that it maps the ideal of relations (2.7) to itself. Indeed, by the Poincaré–Birkhoff–Witt theorem for the double Yangian [5, Thm. 2.2], the dual Yangian coincides with the algebra defined by the generators $t_{ij}^{(-r)}$, where $r \geq 1$ and $i, j = 1, \dots, N$, subject to the defining relations given by (2.7).

First, we introduce some notation and establish some Yang–Baxter-like identities which we shall use in the later stage of the proof. Let $n \geq 2$ be an arbitrary integer and $u = (u_1, \dots, u_n)$ a family of variables. For all $i, j = 1, \dots, n$ such that $i \neq j$ we write

$$R_{ij} = R_{ij}(u_i - u_j), \quad \overline{R}_{ij} = \overline{R}_{ij}(-2z - u_i - u_j), \quad \widehat{R}_{ij} = \overline{R}_{ij}(-2z - u_i - u_j - 2hc).$$

The Yang–Baxter equation (2.3) and the unitarity property (2.5) imply the following identities for all $1 \leq j < k < k + 1 < l \leq n$:

$$R_{kk+1} \overline{R}_{jk} \overline{R}_{jk+1} = \overline{R}_{jk+1} \overline{R}_{jk} R_{kk+1}, \quad R_{kk+1} \overline{R}_{kl} \overline{R}_{k+1l} = \overline{R}_{k+1l} \overline{R}_{kl} R_{kk+1}, \tag{3.20}$$

$$R_{kk+1} \widehat{R}_{k+1l}^{-1} \widehat{R}_{kl}^{-1} = \widehat{R}_{kl}^{-1} \widehat{R}_{k+1l}^{-1} R_{kk+1}, \quad R_{kk+1} \widehat{R}_{j k+1}^{-1} \widehat{R}_{jk}^{-1} = \widehat{R}_{jk}^{-1} \widehat{R}_{j k+1}^{-1} R_{kk+1}. \tag{3.21}$$

For any $k = 1, \dots, n - 1$ defining relation (2.7) for the dual Yangian implies

$$R_{kk+1} T_{[n]}^+(u) = T_1^+(u_1) \dots T_{k-1}^+(u_{k-1}) T_{k+1}^+(u_{k+1}) T_k^+(u_k) T_{k+2}^+(u_{k+2}) \dots T_n^+(u_n) R_{kk+1}. \tag{3.22}$$

We shall prove that the difference of the left and the right-hand side of (3.22) belongs to the kernel of $Y_{\mathcal{M}_c}(\cdot, z)$, which implies the lemma. It is clear from (3.18) that the image of the left-hand side under $Y_{\mathcal{M}_c}(\cdot, z)$ is equal to

$$R_{kk+1} S_{[n]}^+(u|z) S_{[n]}(u|z + hc/2)^{-1}. \tag{3.23}$$

As for the right-hand side, using (3.18) again, we find that its image is given by

$$P_{kk+1} S_{[n]}^+(u^{(k)}|z) S_{[n]}(u^{(k)}|z + hc/2)^{-1} P_{kk+1} R_{kk+1}, \tag{3.24}$$

where $u^{(k)} = (u_1, \dots, u_{k-1}, u_{k+1}, u_k, u_{k+2}, \dots, u_n)$ and P_{kk+1} stands for the action of the permutation operator P on the tensor factors k and $k + 1$; recall (2.2). By employing the relation (3.6) and the equalities in (3.20) we find

$$R_{kk+1} S_{[n]}^+(u|z) = P_{kk+1} S_{[n]}^+(u^{(k)}|z) P_{kk+1} R_{kk+1}. \tag{3.25}$$

Analogously, by using (3.7) and the equalities in (3.21) we obtain

$$R_{kk+1} S_{[n]}(u|z + hc/2)^{-1} = P_{kk+1} S_{[n]}(u^{(k)}|z + hc/2)^{-1} P_{kk+1} R_{kk+1}. \tag{3.26}$$

Finally, from (3.25) and (3.26) we easily see that the expressions in (3.23) and (3.24) coincide, so that the map $Y_{\mathcal{M}_c}(\cdot, z)$ is well-defined by (3.18), as required. \square

Lemma 3.8 *For any $v \in \mathcal{V}_{2c}(\mathfrak{gl}_N)$ and $w \in \mathcal{M}_c(\mathfrak{gl}_N)$ the series $Y_{\mathcal{M}_c}(v, z)w$ belongs to $\mathcal{M}_c(\mathfrak{gl}_N)((z))_h$, i.e. it possesses finitely many negative powers of z modulo h^k for all $k \geq 0$.*

Proof Note that the coefficients of the matrix entries of $S_1^+(v_1) \dots S_m^+(v_m)$ with $m \geq 0$ span an h -adically dense $\mathbb{C}[[h]]$ -submodule of $\mathcal{M}_c(\mathfrak{gl}_N)$. Furthermore, from

$$S_{[m]}^+(v) = \prod_{i=1, \dots, m}^{\rightarrow} \left(S_i^+(v_i) \overline{R}_{i+1}^t(-v_i - v_{i+1}) \dots \overline{R}_{im}^t(-v_i - v_m) \right) \tag{3.27}$$

one easily shows that the coefficients of the matrix entries of $S_{[m]}^+(v)$ with $m \geq 0$ span an h -adically dense $\mathbb{C}[[h]]$ -submodule of $\mathcal{M}_c(\mathfrak{gl}_N)$. Indeed, we can move all R -matrices to the left-hand side of (3.27) by using the identity $\overline{R}^t(u) \cdot \overline{R}^t(-u) = 1$; recall (2.5). Thus, on the right-hand side we get $S_1^+(v_1) \dots S_m^+(v_m)$, while the left-hand side is expressed in terms of $S_{[m]}^+(v)$. Hence the coefficients of the matrix entries of $S_1^+(v_1) \dots S_m^+(v_m)$ can be expressed in terms of the coefficients of the matrix entries of $S_{[m]}^+(v)$, as required.

By the preceding discussion, it is sufficient to check that for any integers $m, n \geq 1$ the image of (3.27) under (3.18) belongs to $\mathcal{M}_c(\mathfrak{gl}_N)((z))_h$. Clearly, this image equals

$$S_{[n]}^{+13}(u|z) S_{[n]}^{13}(u|z + hc/2)^{-1} S_{[m]}^{+23}(v). \tag{3.28}$$

Using relation (3.17), along with the crossing symmetry property (3.19), we rewrite it as

$$S_{[n]}^{+13}(u|z) \left(\overline{R}_{nm}^{12}(u|v|z + 2hc + hN) \cdot \left(\overline{R}_{nm}^t \right)_{RL}^{12}(-u| -v| -z) S_{[m]}^{+23}(v) \right. \\ \left. \times \overline{R}_{nm}^{12}(u|v|z)^{-1} S_{[n]}^{13}(u|z + hc/2)^{-1} \mathbf{1} \overline{R}_{nm}^t \right) (-u| -v| -z - 2hc)^{-1} \Big). \tag{3.29}$$

Note that by (3.2) we have $S(x)^{\pm 1} \mathbf{1} = G^{\pm} \otimes \mathbf{1}$. In addition, recall that the normalized R -matrix $\overline{R}(x)$ belongs to $(\text{End } \mathbb{C}^N)^{\otimes 2}[x^{-1}]_h$. Therefore, we conclude that for any choice of $a_1, \dots, a_n, b_1, \dots, b_m, k$, by regarding the expression in (3.29) modulo

$$u_1^{a_1} \dots u_n^{a_n} v_1^{b_1} \dots v_m^{b_m} h^k, \tag{3.30}$$

we obtain only finitely many negative powers of the variable z . Indeed, the negative powers of z in (3.29) come from the R -matrices only. However, the expression contains finitely many R -matrices and each of them produces finitely many negative powers of z modulo (3.30). Hence the image of $Y_{\mathcal{M}_c}(\cdot, z)$ belongs to $\mathcal{M}_c(\mathfrak{gl}_N)((z))_h$, as required. \square

Lemma 3.9 *The map $Y_{\mathcal{M}_c}(\cdot, z)$ possesses the quasi weak associativity property (3.13).*

Proof Consider the second summand in (3.13). Let us apply the vertex operator map $Y(\cdot, z_0)$ of $\mathcal{V}_{2c}(\mathfrak{gl}_N)$, as defined in (2.18), on the series

$$T_{[n]}^{+13}(u) T_{[m]}^{+24}(v) (\mathbf{1} \otimes \mathbf{1}) \in (\text{End } \mathbb{C}^N)^{\otimes n} \otimes (\text{End } \mathbb{C}^N)^{\otimes m} \otimes \mathcal{V}_{2c}(\mathfrak{gl}_N) \otimes \mathcal{V}_{2c}(\mathfrak{gl}_N)[[u, v]], \tag{3.31}$$

where $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_m)$. By using the relation (2.15) at the level $2c$, the crossing symmetry (3.19) and the identity $T_{[n]}^{13}(u|z_0 + hc)^{-1} \mathbf{1} = 1^{\otimes n} \otimes \mathbf{1}$, we obtain

$$T_{[n]}^{+13}(u|z_0) T_{[n]}^{13}(u|z_0 + hc)^{-1} T_{[m]}^{+23}(v) \mathbf{1} = \bar{R}_{nm}^{12,+} \underset{\text{RL}}{\cdot} \left(T_{[n]}^{+13}(u|z_0) T_{[m]}^{+23}(v) \bar{R}_{nm}^{12,-} \right), \tag{3.32}$$

where the terms $\bar{R}_{nm}^{12,\pm}$ are defined by

$$\bar{R}_{nm}^{12,+} = \bar{R}_{nm}^{12}(u|v|z_0 + 2hc + hN) \quad \text{and} \quad \bar{R}_{nm}^{12,-} = \bar{R}_{nm}^{12}(u|v|z_0)^{-1}.$$

Next, we apply the map $Y_{\mathcal{M}_c}(\cdot, z_2)$ to the right-hand side of (3.32). By (3.18) we get

$$\begin{aligned} &\bar{R}_{nm}^{12,+} \underset{\text{RL}}{\cdot} \left(S_{[n]}^{+13}(u|z_2 + z_0) \bar{R}_{nm}^{12,+} S_{[m]}^{+23}(v|z_2) \right. \\ &\quad \times S_{[m]}^{23}(v|z_2 + hc/2)^{-1} \bar{R}_{nm}^{12,-} S_{[n]}^{13}(u|z_2 + z_0 + hc/2)^{-1} \bar{R}_{nm}^{12,-} \left. \right), \end{aligned} \tag{3.33}$$

where the terms $\bar{R}_{nm}^{12,\pm}$ are given by

$$\bar{R}_{nm}^{12,+} = \bar{R}_{nm}^{12}(-u| -v| -2z_2 - z_0) \quad \text{and} \quad \bar{R}_{nm}^{12,-} = \bar{R}_{nm}^{12}(-u| -v| -2z_2 - z_0 - 2hc)^{-1}.$$

Using relation (3.16) we rearrange the last four factors in (3.33), thus getting

$$\begin{aligned} &\bar{R}_{nm}^{12,+} \underset{\text{RL}}{\cdot} \left(S_{[n]}^{+13}(u|z_2 + z_0) \bar{R}_{nm}^{12,+} S_{[m]}^{+23}(v|z_2) \right. \\ &\quad \times \bar{R}_{nm}^{12,-} S_{[n]}^{13}(u|z_2 + z_0 + hc/2)^{-1} \bar{R}_{nm}^{12,-} S_{[m]}^{23}(v|z_2 + hc/2)^{-1} \left. \right). \end{aligned}$$

Finally, we employ relation (3.17) to rewrite this as

$$\begin{aligned} &\bar{R}_{nm}^{12,+} \underset{\text{RL}}{\cdot} \left(S_{[n]}^{+13}(u|z_2 + z_0) S_{[n]}^{13}(u|z_2 + z_0 + hc/2)^{-1} \right. \\ &\quad \times \bar{R}_{nm}^{12}(u|v|z_0 + 2hc)^{-1} S_{[m]}^{+23}(v|z_2) (\bar{R}_{nm}^{12,-})^{-1} \bar{R}_{nm}^{12,-} S_{[m]}^{23}(v|z_2 + hc/2)^{-1} \left. \right). \end{aligned}$$

It remains to observe that by the crossing symmetry property (3.19) the R -matrix factors in the given expression cancel, so that it equals

$$S_{[n]}^{+13}(u|z_2 + z_0) S_{[n]}^{13}(u|z_2 + z_0 + hc/2)^{-1} S_{[m]}^{+23}(v|z_2) S_{[m]}^{23}(v|z_2 + hc/2)^{-1}. \tag{3.34}$$

Recall that the expression in (3.34) corresponds to the second summand in (3.13). As for the first summand, one immediately sees from (3.18) that it is equal to

$$S_{[n]}^{+13}(u|z_0 + z_2) S_{[n]}^{13}(u|z_0 + z_2 + hc/2)^{-1} S_{[m]}^{+23}(v|z_2) S_{[m]}^{23}(v|z_2 + hc/2)^{-1}. \tag{3.35}$$

Observe that (3.34) and (3.35) do not coincide as the former should be expanded in negative powers of z_2 and the latter in the negative powers of z_0 . However, let us apply both (3.34) and (3.35) on an arbitrary element $w \in \mathcal{M}_c(\mathfrak{g}_N)$. By arguing as in the proof of Lemma

3.8, one checks that there exists $r \geq 0$ such that the products of the resulting expressions with $p(z_0 + z_2, z_2)$, where $p(x_1, y_1) = x_1^r(x_1 + x_2)^r$, coincide modulo (3.30). Thus the map $Y_{\mathcal{M}_c}(\cdot, z)$ satisfies the quasi weak associativity requirement (3.13).¹ \square

4 Central Elements in $\tilde{\mathcal{A}}_c(\mathfrak{g}_N)$ and Invariants in $\mathcal{M}_c(\mathfrak{g}_N)$

In this section, we construct families of central elements of the completed algebras $\tilde{\mathcal{A}}_c(\mathfrak{g}_N)$ and invariants of the quasi module $\mathcal{M}_c(\mathfrak{g}_N)$. Our constructions employ the quasi module map established by Theorem 3.4. Moreover, they rely on the fusion procedure for the Yang R -matrix and the explicit formulae for certain families of central elements of the quantum vertex algebra $\mathcal{V}_c(\mathfrak{gl}_N)$, so we start by briefly recalling these results.

4.1 Fusion Procedure and the Center of $\mathcal{V}_c(\mathfrak{gl}_N)$

Let us recall the fusion procedure for the Yang R -matrix (2.1) found in [8]; see also [14, Sect. 6.4] for more details. Let ν be a Young diagram with n boxes of length less than or equal to N and let \mathcal{U} be a standard ν -tableau with entries in $\{1, \dots, n\}$. For any $p = 1, \dots, n$ the contents $c_p = c_p(\mathcal{U})$ of \mathcal{U} are defined by $c_p = j - i$ if p occupies the box (i, j) of \mathcal{U} . Let $e_{\mathcal{U}}$ be the primitive idempotent in the group algebra $\mathbb{C}[\mathfrak{S}_n]$ of the symmetric group \mathfrak{S}_n associated with \mathcal{U} through the use of the orthonormal Young bases for the irreducible representations of \mathfrak{S}_n . The group \mathfrak{S}_n acts on $(\mathbb{C}^N)^{\otimes n}$ by permuting the tensor factors. Let $\mathcal{E}_{\mathcal{U}}$ be the image of $e_{\mathcal{U}}$ with respect to this action. By [8], the consecutive evaluations $u_k = hc_k$ of

$$R(u_1, \dots, u_n) = \overrightarrow{\prod}_{1 \leq i < j \leq n} R_{ij}(u_i - u_j)$$

are well-defined. Furthermore, the result is proportional to $\mathcal{E}_{\mathcal{U}}$, i.e. we have

$$R(u_1, \dots, u_n) \Big|_{u_1=hc_1} \Big|_{u_2=hc_2} \dots \Big|_{u_n=hc_n} = \pi_{\nu} \mathcal{E}_{\mathcal{U}}, \tag{4.1}$$

where π_{ν} stands for the product of all hook lengths of the boxes of ν .

Consider the quantum affine vertex algebra at the critical level $\mathcal{V}_{-N}(\mathfrak{gl}_N)$. Let

$$u_{\nu} = (u_1, \dots, u_n), \quad \text{where} \quad u_k = u + hc_k \quad \text{for} \quad k = 1, \dots, n. \tag{4.2}$$

Due to [5, Thm. 2.4], all coefficients of the series

$$\mathbb{T}_{\nu}^+(u) = \text{tr}_{1, \dots, n} \mathcal{E}_{\mathcal{U}} T_{[n]}^+(u_{\nu}) \mathbf{1} = \text{tr}_{1, \dots, n} \mathcal{E}_{\mathcal{U}} T_1^+(u_1) \dots T_n^+(u_n) \mathbf{1} \in \mathcal{V}_{-N}(\mathfrak{gl}_N)[[u]],$$

where the trace is taken over all n copies of $\text{End } \mathbb{C}^N$, belong to the center $\mathfrak{z}(\mathcal{V}_{-N}(\mathfrak{gl}_N))$ of the quantum vertex algebra $\mathcal{V}_{-N}(\mathfrak{gl}_N)$. As for the noncritical level $c \neq -N$, all coefficients of the *quantum determinant*

$$\text{qdet} T^+(u) = \sum_{\sigma \in \mathfrak{S}_N} \text{sgn } \sigma \cdot t_{\sigma(1)1}^+(u) \dots t_{\sigma(N)N}^+(u - (N - 1)h) \mathbf{1} \in \mathcal{V}_c(\mathfrak{gl}_N)[[u]] \tag{4.3}$$

belong to the center $\mathfrak{z}(\mathcal{V}_c(\mathfrak{gl}_N))$ of the quantum vertex algebra $\mathcal{V}_c(\mathfrak{gl}_N)$; see [5, Prop. 4.10].

¹ Note that the polynomial $p(x_1, y_1) = x_1^r(x_1 + x_2)^r$ can not be replaced by the simpler polynomial $q(x_1, y_1) = x_1^r$. In other words, the quasi module map $Y_{\mathcal{M}_c}(\cdot, z)$ does not need to satisfy the usual *weak associativity* which is obtained from (3.13) by replacing $p(z_0 + z_2, z_2)$ with $(z_0 + z_2)^r$.

4.2 Central Elements of the Completed Algebra $\tilde{A}_c(\mathfrak{g}_N)$

Let I_p for $p \geq 1$ be the left ideal of the double Yangian $DY_c(\mathfrak{gl}_N)$ at the level $c \in \mathbb{C}$ generated by all elements $t_{ij}^{(r)}$ such that $r \geq p$. Following [5], we define the completed double Yangian $\tilde{DY}_c(\mathfrak{gl}_N)$ at the level c as the h -adic completion of the inverse limit $\lim_{\leftarrow} DY_c(\mathfrak{gl}_N)/I_p$. Finally, we introduce the algebra $\tilde{A}_c(\mathfrak{g}_N)$ as the h -adic completion of the inverse limit

$$\lim_{\leftarrow} A_c(\mathfrak{g}_N)/(A_c(\mathfrak{g}_N) \cap I_p).$$

4.2.1 Critical Level

In this subsection, we consider the algebra $\tilde{A}_{-N/2}(\mathfrak{g}_N)$. For any integer $n = 1, \dots, N$ introduce the Laurent series with coefficients in $\tilde{A}_{-N/2}(\mathfrak{g}_N)$ by

$$\mathbb{A}_v(u) = \text{tr}_{1, \dots, n} \mathcal{E}_U S_{[n]}^+(u_v) S_{[n]}(u_v - hN/4)^{-1}, \tag{4.4}$$

where the family of variables u_v is given by (4.2). It is worth noting that the action of the series $\mathbb{A}_v(u)$ on the vacuum module $\mathcal{M}_{-N/2}(\mathfrak{g}_N)$ is given by

$$\mathbb{A}_v(u) = Y_{\mathcal{M}_{-N/2}}(\mathbb{T}_v^+(0), u). \tag{4.5}$$

In what follows, we shall often use the arrow at the top of the symbol to indicate that the factors are arranged in the opposite order, e.g., for $\overline{R}_{i\ n+1} = \overline{R}_{i\ n+1}(u_i - v)$ we have

$$\overline{R}_{n1}(u|v) = \overline{R}_{1n+1} \cdots \overline{R}_{nn+1} \quad \text{and} \quad \overleftarrow{R}_{n1}(u|v) = \overleftarrow{R}_{nn+1} \cdots \overleftarrow{R}_{1n+1}.$$

In the next lemma u_0 denotes a single variable while u_v again stands for the family (4.2).

Lemma 4.1 *The following equalities hold on $\text{End } \mathbb{C}^N \otimes (\text{End } \mathbb{C}^N)^{\otimes n} \otimes \tilde{A}_{-N/2}(\mathfrak{g}_N)$:*

$$\mathcal{E}_u \overline{R}_{1n}^{01}(u_0|u_v) = \overleftarrow{R}_{1n}^{01}(u_0|u_v) \mathcal{E}_u, \quad \mathcal{E}_u \overline{R}_{1n}^{01}(u_0|u_v)^{-1} = \overleftarrow{R}_{1n}^{01}(u_0|u_v)^{-1} \mathcal{E}_u, \tag{4.6}$$

$$\mathcal{E}_u \overleftarrow{R}_{1n}^{\leftarrow r01}(u_0|u_v) = \overleftarrow{R}_{1n}^{r01}(u_0|u_v) \mathcal{E}_u, \quad \mathcal{E}_u \overleftarrow{R}_{1n}^{\leftarrow r01}(-u_0| -u_v) = \overleftarrow{R}_{1n}^{\leftarrow r01}(-u_0| -u_v) \mathcal{E}_u, \tag{4.7}$$

$$\mathcal{E}_u S_{[n]}^{+12}(u_v) = \overleftarrow{S}_{[n]}^{+12}(u_v) \mathcal{E}_u, \quad \mathcal{E}_u S_{[n]}^{12}(u_v - hN/4)^{-1} = \overleftarrow{S}_{[n]}^{12}(u_v - hN/4)^{-1} \mathcal{E}_u, \tag{4.8}$$

where the $n + 1$ copies of $\text{End } \mathbb{C}^N$ are labeled by $0, \dots, n$, \mathcal{E}_U is applied on the tensor factors $1, \dots, n$ and the superscripts indicate the tensor factors as follows:

$$\overbrace{\text{End } \mathbb{C}^N}^0 \otimes \overbrace{(\text{End } \mathbb{C}^N)^{\otimes n}}^1 \otimes \overbrace{\tilde{A}_{-N/2}(\mathfrak{g}_N)}^2. \tag{4.9}$$

Proof The identities in (4.6) are proved in [9, Lemma 3.1]. The remaining equalities can be verified analogously, by using the Yang–Baxter equation (2.3), the fusion procedure (4.1) and the relations (3.15) and (3.16). \square

The next theorem is the main result of this section. As with the similar constructions of the families of central elements in the completions of the double Yangian and the reflection algebra, such as [5, Thm. 4.4], [6, Thm. 4.4] and [9, Thm. 3.2], its proof relies on the usual R -matrix techniques and the fusion procedure for the Yang R -matrix.

Theorem 4.2 *All coefficients of $\mathbb{A}_v(u)$ belong to the center of the algebra $\tilde{A}_{-N/2}(\mathfrak{g}_N)$.*

Proof Let us prove

$$S(u_0) \mathbb{A}_v(u) = \mathbb{A}_v(u) S(u_0). \tag{4.10}$$

By applying $S(u_0)$ to the right-hand side of (4.4) we get

$$\text{tr}_{1,\dots,n} \mathcal{E}_U S_0(u_0) S_{[n]}^{+12}(u_v) S_{[n]}^{12}(u_v - hN/4)^{-1}. \tag{4.11}$$

Observe that our notation coincides with (4.9) and, in particular, that the trace is taken over the tensor factors $1, \dots, n$ while $S(u_0)$ is applied on the 0-th factor. By employing the unitarity (2.5) and the relation (3.17) we express (4.11) as

$$\text{tr}_{1,\dots,n} \mathcal{E}_U \left(\overleftarrow{\underline{R}}_n^{t01}(u_0 + hN/4|u_v) \cdot \left(\overline{\underline{R}}_{1n}^{01}(u_0 - 3hN/4|u_v)^{-1} S_{[n]}^{12}(u_v) \overline{\underline{R}}_{1n}^{01}(-u_0 - 3hN/4|-u_v) S_0(u_0) \overline{\underline{R}}_{1n}^{01}(u_0 + hN/4|u_v) S_{[n]}^{12}(u_v - hN/4)^{-1} \right) \right). \tag{4.12}$$

Since \mathcal{E}_U is idempotent, the first equality in (4.7) implies

$$\mathcal{E}_u \overleftarrow{K} = \mathcal{E}_u^2 \overleftarrow{K} = \mathcal{E}_u K \mathcal{E}_u = \mathcal{E}_u K \mathcal{E}_u^2 \text{ for } \overleftarrow{K} = \overleftarrow{\underline{R}}_{1n}^{t01}(u_0 + hN/4|u_v) \tag{4.13}$$

By using (4.13) we can write (4.12) as

$$\text{tr}_{1,\dots,n} \mathcal{E}_U \left(\overline{\underline{R}}_{1n}^{t01}(u_0 + hN/4|u_v) \cdot \left(\mathcal{E}_U^2 \overline{\underline{R}}_{1n}^{01}(u_0 - 3hN/4|u_v)^{-1} S_{[n]}^{+12}(u_v) \overline{\underline{R}}_{1n}^{01}(-u_0 + 3hN/4|-u_v) S_0(u_0) \overline{\underline{R}}_{1n}^{01}(u_0 + hN/4|u_v) S_{[n]}^{12}(u_v - hN/4)^{-1} \right) \right).$$

Due to the cyclic property of the trace, this equals to

$$\text{tr}_{1,\dots,n} \mathcal{E}_U \overline{\underline{R}}_{1n}^{01}(u_0 - 3hN/4|u_v)^{-1} S_{[n]}^{+12}(u_v) \overline{\underline{R}}_{1n}^{t01}(-u_0 + 3hN/4|-u_v) S_0(u_0) \overline{\underline{R}}_{1n}^{01}(u_0 + hN/4|u_v) S_{[n]}^{12}(u_v - hN/4)^{-1} \mathcal{E}_U \overline{\underline{R}}_{1n}^{t01}(u_0 + hN/4|u_v) \mathcal{E}_U. \tag{4.14}$$

By the second equality in (4.6) and $\mathcal{E}_U^2 = \mathcal{E}_U$, we have

$$\mathcal{E}_u L = \mathcal{E}_u^2 L = \mathcal{E}_u \overleftarrow{L} \mathcal{E}_u = \mathcal{E}_u \overleftarrow{L} \mathcal{E}_u^2 = \mathcal{E}_u^2 L \mathcal{E}_u = \mathcal{E}_u L \mathcal{E}_u \text{ for } L = \overline{\underline{R}}_{1n}^{01}(u_0 - 3hN/4|u_v)^{-1}.$$

Therefore, using the cyclic property of the trace, we can write (4.14) as

$$\text{tr}_{1,\dots,n} \overline{\underline{R}}_{1n}^{01}(u_0 - 3hN/4|u_v)^{-1} \mathcal{E}_U S_{[n]}^{+12}(u_v) \overline{\underline{R}}_{1n}^{t01}(-u_0 + 3hN/4|-u_v) S_0(u_0) \overline{\underline{R}}_{1n}^{01}(u_0 + hN/4|u_v) S_{[n]}^{12}(u_v - hN/4)^{-1} \mathcal{E}_U \overline{\underline{R}}_{1n}^{t01}(u_0 + hN/4|u_v) \mathcal{E}_U.$$

We now employ the first equality in (4.6), the second equality in (4.7) and both equalities in (4.8) to move the leftmost copy of \mathcal{E}_U to the right, which gives us

$$\text{tr}_{1,\dots,n} \overline{\underline{R}}_{1n}^{01}(u_0 - 3hN/4|u_v)^{-1} \overleftarrow{S}_{[n]}^{+12}(u_v) \overleftarrow{\underline{R}}_{1n}^{t01}(-u_0 + 3hN/4|-u_v) S_0(u_0) \overleftarrow{\underline{R}}_{1n}^{t01}(u_0 + hN/4|u_v) \overleftarrow{S}_{[n]}^{12}(u_v - hN/4)^{-1} \mathcal{E}_U^2 \overline{\underline{R}}_{1n}^{01}(u_0 + hN/4|u_v) \mathcal{E}_U. \tag{4.15}$$

Using (4.13) and $\mathcal{E}_U^2 = \mathcal{E}_U$ we replace $\mathcal{E}_U^2 K \mathcal{E}_U$ by $\mathcal{E}_U \overleftarrow{K}$ in (4.15). Next, we use the corresponding equalities from Lemma 4.1 to move the remaining copy of \mathcal{E}_U to the left:

$$\text{tr}_{1,\dots,n} \overline{\underline{R}}_{1n}^{01}(u_0 - 3hN/4|u_v)^{-1} \mathcal{E}_U S_{[n]}^{+12}(u_v) \overline{\underline{R}}_{1n}^{t01}(-u_0 + 3hN/4|-u_v) S_0(u_0) \overline{\underline{R}}_{1n}^{01}(u_0 + hN/4|u_v) S_{[n]}^{12}(u_v - hN/4)^{-1} \overleftarrow{\underline{R}}_{1n}^{t01}(u_0 + hN/4|u_v).$$

By the relation (3.16) this coincides with

$$\text{tr}_{1, \dots, n} \overline{R}_{1n}^{01}(u_0 - 3hN/4|u_v)^{-1} \mathcal{E}_{\mathcal{U}} S_{[n]}^{+12}(u_v) S_{[n]}^{12}(u_v - hN/4)^{-1} \overline{R}_{1n}^{01}(u_0 + hN/4|u_v) S_0(u_0).$$

Indeed, the last two terms which appear on the right, $\overline{R}_{1n}^{01}(-u_0 + 3hN/4| - u_v)$ and $\overleftarrow{R}_{1n}^{01}(u_0 + hN/4|u_v)$ cancel because of the crossing symmetry property (2.4). Finally, we use the cyclic property of the trace and (3.19) to rewrite this as

$$\begin{aligned} &\text{tr}_{1, \dots, n} \mathcal{E}_{\mathcal{U}} S_{[n]}^{+12}(u_v) S_{[n]}^{12}(u_v - hN/4)^{-1} \left(\overline{R}_{1n}^{01}(u_0 - 3hN/4|u_v)^{-1} \cdot \overline{R}_{1n}^{01}(u_0 + hN/4|u_v) \right) S_0(u_0) \\ &= \text{tr}_{1, \dots, n} \mathcal{E}_{\mathcal{U}} S_{[n]}^{+12}(u_v) S_{[n]}^{12}(u_v - hN/4)^{-1} S_0(u_0) = \mathbb{A}_v(u) S(u_0), \end{aligned}$$

so that the commutation relation (4.10) follows. A similar calculation verifies the equality

$$S^+(u_0) \mathbb{A}_v(u) = \mathbb{A}_v(u) S^+(u_0),$$

which, together with (4.10), implies the statement of the theorem. □

4.2.2 Noncritical Level

Consider the algebra $\tilde{\mathbb{A}}_c(\mathfrak{g}_N)$ at the noncritical level $c \in \mathbb{C}, c \neq -N/2$. Introduce the family of variables $u_- = (u, u - h, \dots, u - (N - 1)h)$ and denote by $A^{(N)}$ the action of the anti-symmetrizer from the group algebra $\mathbb{C}[\mathfrak{S}_N]$ on the tensor product space $(\mathbb{C}^N)^{\otimes N}$. Let $\mathbb{A}(u)$ be the Laurent series with coefficients in $\tilde{\mathbb{A}}_c(\mathfrak{g}_N)$,

$$\mathbb{A}(u) = \text{tr}_{1, \dots, N} A^{(N)} S_{[N]}^+(u_-) S_{[N]}(u_- + hc/2)^{-1}. \tag{4.16}$$

Note that $u_- = u_v$ if v is the column diagram with N boxes; recall (4.2). Moreover, we have $A^{(N)} = \mathcal{E}_{\mathcal{U}}$ if \mathcal{U} is the unique standard column tableaux with N boxes. In parallel with the critical level case, we observe that the action of the series $\mathbb{A}(u)$ on the vacuum module $\mathcal{M}_c(\mathfrak{g}_N)$ is given by

$$\mathbb{A}(u) = Y_{\mathcal{M}_c}(\text{qdet} T^+(0), u). \tag{4.17}$$

We have the following simple construction of the family of central elements in $\tilde{\mathbb{A}}_c(\mathfrak{g}_N)$.

Proposition 4.3 *All coefficients of $\mathbb{A}(u)$ belong to the center of the algebra $\tilde{\mathbb{A}}_c(\mathfrak{g}_N)$.*

Proof A direct calculation which relies on the properties of the anti-symmetrizer and goes in parallel with the proof of [14, Thm. 2.5.3] shows that there exists a power series $\gamma(u) \in \mathbb{C}[u^{-1}][[h]]$ such that we have

$$\begin{aligned} \mathbb{A}(u) &= \gamma(u) \text{qdet} T^+(u) \text{qdet} T^+(-u + (N - 1)h) \\ &\quad \times (\text{qdet} T(-u - hc/2 + (N - 1)h))^{-1} (\text{qdet} T(u + 3hc/2))^{-1}. \end{aligned}$$

However, it is well-known that the coefficients of these quantum determinants belong to the center of the double Yangian; see, e.g., [5, Prop. 2.8]. Thus, we conclude that all coefficients of $\mathbb{A}(u)$ belong to the center of the algebra $\tilde{\mathbb{A}}_c(\mathfrak{g}_N)$, as required. □

4.3 Invariants of the Vacuum Module $\mathcal{M}_c(\mathfrak{g}_N)$

In this section, we present some simple applications of Theorem 4.2 and Proposition 4.3, in particular, to the submodule of *invariants* of the vacuum module $\mathcal{M}_c(\mathfrak{g}_N)$, which we define as

$$\mathfrak{z}(\mathcal{M}_c(\mathfrak{g}_N)) = \{w \in \mathcal{M}_c(\mathfrak{g}_N) : S(u)w = G \otimes w\}.$$

Our definition is motivated by the usual notion of a subspace of invariants of the vacuum module for the universal enveloping algebra of affine Lie algebra; see, e.g., the book by E. Frenkel [3]. As before, we shall consider the critical and the noncritical level separately. It is worth noting that for any $c \in \mathbb{C}$ we have $1 \in \mathfrak{z}(\mathcal{M}_c(\mathfrak{g}_N))$ as $S(u)1 = G \otimes 1$.

4.3.1 Critical Level

Consider the vacuum module $\mathcal{M}_{-N/2}(\mathfrak{g}_N)$ at the critical level $c = -N/2$. By applying the series (4.4), whose coefficients belong to the center of the algebra $\tilde{\mathbb{A}}_{-N/2}(\mathfrak{g}_N)$, on $1 \in \mathcal{M}_{-N/2}(\mathfrak{g}_N)$ we obtain

$$\mathbb{M}_v(u) = \mathbb{A}_v(u)1 \in \mathcal{M}_{-N/2}(\mathfrak{g}_N)[[u^{\pm 1}]].$$

Corollary 4.4 *All coefficients of the series $\mathbb{M}_v(u)$ belong to $\mathfrak{z}(\mathcal{M}_{-N/2}(\mathfrak{g}_N))$.*

Proof The corollary is a simple consequence of Theorem 4.2. We have

$$S(v)\mathbb{M}_v(u) = S(v)\mathbb{A}_v(u)1 = \mathbb{A}_v(u)S(v)1 = \mathbb{A}_v(u)(G \otimes 1) = G \otimes \mathbb{A}_v(u)1 = G \otimes \mathbb{M}_v(u),$$

so the coefficients of $\mathbb{M}_v(u)$ belong to the submodule of invariants $\mathfrak{z}(\mathcal{M}_{-N/2}(\mathfrak{g}_N))$. □

Note that the coefficients of the series $\mathbb{M}_v(u)$ can be naturally regarded as elements of the h -adically completed algebra $Y^+(\mathfrak{g}_N)_h$. This leads to another simple consequence of Theorem 4.2, which produces commutative subalgebras in $Y^+(\mathfrak{g}_N)_h$.

Corollary 4.5 *The coefficients of all series $\mathbb{M}_v(u) \in Y^+(\mathfrak{g}_N)_h[[u^{\pm 1}]]$ pairwise commute.*

Proof Let v_i with $i = 1, 2$ be partitions with at most N parts. By Theorem 4.2 we have

$$\mathbb{M}_{v_i}(u)\mathbb{M}_{v_j}(v) = \mathbb{M}_{v_i}(u)\mathbb{A}_{v_j}(v)1 = \mathbb{A}_{v_j}(v)\mathbb{M}_{v_i}(u)1 = \mathbb{A}_{v_j}(v)\mathbb{A}_{v_i}(u)1.$$

for all $i, j = 1, 2$. Since $\mathbb{A}_{v_i}(u)$ and $\mathbb{A}_{v_j}(v)$ commute, this implies the corollary. □

Let μ and ν be partitions which have at most N parts. Theorem 4.2 implies the identity $\mathbb{A}_\nu(u)\mathbb{A}_\mu(v) = \mathbb{A}_\mu(v)\mathbb{A}_\nu(u)$ for operators on $\mathcal{M}_{-N/2}(\mathfrak{g}_N)$. Hence, by using (4.5) we find

$$Y_{\mathcal{M}_{-N/2}}(\mathbb{T}_\nu^+(u), z_1)Y_{\mathcal{M}_{-N/2}}(\mathbb{T}_\mu^+(v), z_2) = Y_{\mathcal{M}_{-N/2}}(\mathbb{T}_\mu^+(v), z_2)Y_{\mathcal{M}_{-N/2}}(\mathbb{T}_\nu^+(u), z_1). \tag{4.18}$$

This commutation relation can be generalized as follows.

Corollary 4.6 *For any two elements a and b of the center $\mathfrak{z}(\mathcal{V}_{-N}(\mathfrak{gl}_N))$ we have*

$$Y_{\mathcal{M}_{-N/2}}(a, z_1)Y_{\mathcal{M}_{-N/2}}(b, z_2) = Y_{\mathcal{M}_{-N/2}}(b, z_2)Y_{\mathcal{M}_{-N/2}}(a, z_1). \tag{4.19}$$

Proof By [5, Thm. 4.9] and (4.18), there exists a family of topological generators of the center $\mathfrak{z}(\mathcal{V}_{-N}(\mathfrak{gl}_N))$ such that their images under the quasi module map (3.18) commute. However, if the commutativity relation (4.19) holds for all elements a and b of the family of topological generators of the center, then it holds for all elements of the center as well; see [9, Prop. 3.4]. Hence the identity (4.19) holds for any $a, b \in \mathfrak{z}(\mathcal{V}_{-N}(\mathfrak{gl}_N))$. \square

We conclude this section by the method for constructing invariants of $\mathcal{M}_{-N/2}(\mathfrak{g}_N)$.

Corollary 4.7 *For any $a \in \mathfrak{z}(\mathcal{V}_{-N}(\mathfrak{gl}_N))$ and $w \in \mathfrak{z}(\mathcal{M}_{-N/2}(\mathfrak{g}_N))$ all coefficients of the series $Y_{\mathcal{M}_{-N/2}}(a, z)w$ belong to the submodule of invariants $\mathfrak{z}(\mathcal{M}_{-N/2}(\mathfrak{g}_N))$.*

Proof As with the proof of Corollary 4.6, we can combine [5, Thm. 4.9], [9, Prop. 3.4] and (4.18) to conclude that for operators on $\mathcal{M}_{-N/2}(\mathfrak{g}_N)$ we have

$$Y_{\mathcal{M}_{-N/2}}(a, z) S(u) = S(u) Y_{\mathcal{M}_{-N/2}}(a, z) \quad \text{for all } a \in \mathfrak{z}(\mathcal{V}_{-N}(\mathfrak{gl}_N)).$$

Therefore, for any $a \in \mathfrak{z}(\mathcal{V}_{-N}(\mathfrak{gl}_N))$ and $w \in \mathfrak{z}(\mathcal{M}_{-N/2}(\mathfrak{gl}_N))$ we have

$$S(u) Y_{\mathcal{M}_{-N/2}}(a, z)w = Y_{\mathcal{M}_{-N/2}}(a, z) S(u)w = Y_{\mathcal{M}_{-N/2}}(a, z)(G \otimes w) = G \otimes Y_{\mathcal{M}_{-N/2}}(a, z)w.$$

Hence the coefficients of $Y_{\mathcal{M}_{-N/2}}(a, z)w$ belong to $\mathfrak{z}(\mathcal{M}_{-N/2}(\mathfrak{g}_N))$, as required. \square

4.3.2 Noncritical Level

Consider the vacuum module $\mathcal{M}_c(\mathfrak{g}_N)$ at the noncritical level $c \in \mathbb{C}$, $c \neq -N/2$. As with the critical level case, applying the series (4.16), whose coefficients belong to the center of $\tilde{\mathbb{A}}_c(\mathfrak{g}_N)$, on $1 \in \mathcal{M}_c(\mathfrak{g}_N)$ we obtain a power series

$$\mathbb{M}(u) = \mathbb{A}(u)1 \in \mathcal{M}_c(\mathfrak{g}_N)[[u^{\pm 1}]].$$

By arguing as in the proof of Corollary 4.4 and using Proposition 4.3 one obtains

Corollary 4.8 *All coefficients of the series $\mathbb{M}(u)$ belong to $\mathfrak{z}(\mathcal{M}_c(\mathfrak{g}_N))$.*

The analogues of Corollaries 4.5, 4.6 and 4.7 for the series $\mathbb{M}(u)$ can be easily established as well. However, their proofs now rely on the Proposition 4.3 and the explicit description of the center $\mathfrak{z}(\mathcal{V}_c(\mathfrak{gl}_N))$ at the noncritical level $c \neq -N$, as given by [5, Prop. 4.10].

Remark 4.9 It is worth to single out the following identity for operators on $\mathcal{M}_c(\mathfrak{g}_N)$:

$$Y_{\mathcal{M}_c}(\text{qdet}T^+(0), z) = \text{sdet}S^+(z) \text{sdet}S(z + hc/2)^{-1}.$$

It connects the quantum determinant (4.3) and the Sklyanin determinants,

$$\text{sdet}S^+(u) = \text{tr}_{1, \dots, N} A^{(N)} S_{[N]}^+(u_-) \quad \text{and} \quad \text{sdet}S(u) = \text{tr}_{1, \dots, N} A^{(N)} S_{[N]}(u_-).$$

The Sklyanin determinant $\text{sdet}S^+(u)$ exhibits similar properties as its more intensively studied Yangian counterpart $\text{sdet}S(u)$; cf. [14, 15, 17].

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