Algebras and Representation Theory (2023) 26:1713–1729
https://doi.org/10.1007/s10468-022-10152-3

On Maximal Green Sequence for Quivers Arising from Weighted Projective Lines

Changjian Fu[1] · **Shengfei Geng**[1]

Received: 2 November 2021 / Accepted: 3 July 2022 / Published online: 19 July 2022

Abstract
We investigate the existence and non-existence of maximal green sequences for quivers arising from weighted projective lines. Let Q be the Gabriel quiver of the endomorphism algebra of a basic cluster-tilting object in the cluster category $\mathcal{C}_{\mathbb{X}}$ of a weighted projective line $\mathbb{X}$. It is proved that there exists a quiver Q' in the mutation equivalence class $\mathrm{Mut}(Q)$ of Q such that Q' admits a maximal green sequence. Furthermore, there is a quiver in $\mathrm{Mut}(Q)$ which does not admit a maximal green sequence if and only if $\mathbb{X}$ is of wild type.

Keywords Quiver mutation · Maximal green sequence · Weighted projective line

Mathematics Subject Classification (2010) 16G10 · 16E10 · 18E30

1 Introduction

Maximal green sequences were introduced by Keller [19] in the study of refined Donaldson-Thomas invariants for quivers and implicitly by Gaiotto et al. in [14]. They are certain sequences of quiver mutations satisfying a certain combinatorial condition. It is known that not all quivers have maximal green sequences, but they do exist for important classes of quivers. We refer to the survey [21] for examples and recent progress. Although there are a lot of results on the existence of maximal green sequences, it is still a challenging problem to characterize the class of quivers which have maximal green sequences.

The existence of maximal green sequences yields quantum dilogarithm identities in the associated quantum torus and provides explicit formulas for Kontsevich-Soibelman's

Presented by: Christof Geiss

Partially supported by the National Natural Science Foundation of China (Grant No. 11971326, 12071315).

✉ Shengfei Geng
genshengfei@scu.edu.cn

Changjian Fu
changjianfu@scu.edu.cn

[1] Department of Mathematics, SiChuan University, 610064 Chengdu, People's Republic of China

refined Donaldson-Thomas invariants (cf. [19]). It also has important applications in the theory of cluster algebras. In particular, Gross et al. [16] proved that the Fock-Goncharov conjecture about the existence of a canonical basis for cluster algebras holds when a cluster algebra has a quiver with a maximal green sequence and the cluster algebra is equal to its upper cluster algebra. It is also a sufficient condition for the existence of a generic basis in certain upper cluster algebras [27].

Cluster-tilting theory of hereditary abelian categories produces a large class of important quivers, which we denote it by $\mathcal{Q}_{\mathrm{ct}}$. Let K be an algebraically closed field and $\mathcal{H}$ a hereditary abelian category over K with tilting objects. The *cluster category* $\mathcal{C}(\mathcal{H})$ [4] is defined as the orbit category of the bounded derived category $\mathcal{D}^b(\mathcal{H})$ with respect to the auto-equivalence $\tau^{-1} \circ \Sigma$, where τ is the Auslander-Reiten translation and Σ is the suspension functor of $\mathcal{D}^b(\mathcal{H})$, respectively. The cluster category $\mathcal{C}(\mathcal{H})$ is a 2-Calabi-Yau triangulated category with cluster-tilting objects (cf. [18]). For each basic cluster-tilting object $T \in \mathcal{C}(\mathcal{H})$, we denote by Q_T the Gabriel quiver of the endomorphism algebra $\mathsf{End}_{\mathcal{C}(\mathcal{H})}(T)$. Then $\mathcal{Q}_{ct}$ consists of quivers which are isomorphic to Q_T for some basic cluster-tilting object T and hereditary abelian category $\mathcal{H}$.

According to Happel's classification theorem [17], each connected hereditary abelian K-category with tilting objects is either derived equivalent to the path algebra KQ of a finite acyclic quiver Q or to the category $\mathsf{coh}\,\mathbb{X}$ of coherent sheaves over a weighted projective line in the sense of Geigle-Lenzing [15]. Therefore, $\mathcal{Q}_{\mathrm{ct}}$ can be written as the union of two subclasses: $\mathcal{Q}_{\mathrm{pa}}$ consists of quivers arising from path algebras and $\mathcal{Q}_{\mathrm{wpl}}$ consists of quivers arising from weighted projective lines. Cluster categories associated to path algebras were extensively studied in their connection to cluster algebras (cf. [3–7] for instance), but those associated to weighted projective lines were not well studied [1, 13]. The aim of this note is to study the existence and non-existence of maximal green sequences for quivers in $\mathcal{Q}_{\mathrm{wpl}}$. Our main result is an existence and non-existence theorem (cf. Theorem 4.3) for quivers arising from weighted projective lines. Surprisingly, the existence and non-existence theorem is compatible with the classification of weighted projective lines.

The paper is structured as follows. In Section 2, we recall the definitions of quiver mutation and maximal green sequence. Quivers of finite mutation type are also discussed. In Section 3, we collect basic properties for weighted projective lines. It is proved that a quiver arising from a weighted projective line $\mathbb{X}$ is of finite mutation type if and only if $\mathbb{X}$ is not of wild type (Proposition 3.7). In Section 4, we present the proof of the main result (Theorem 4.3).

Conventions Let $m \geq n$ be positive integers. For an integer matrix $B \in \mathrm{M}_{m\times n}(\mathbb{Z})$, we refer to the submatrix formed by the first n rows of B the *principal part* of B and the submatrix formed by the last $m-n$ rows the *coefficient part*.

For any integer vectors $\alpha = [a_1, \cdots, a_n]^T, \beta = [b_1, \cdots, b_n]^T \in \mathbb{Z}^n$, we denote by $\alpha \leq \beta$ if $a_i \leq b_i$ for $1 \leq i \leq n$. This endows a partial order on $\mathbb{Z}^n$. For $b \in \mathbb{Z}$, let $\mathsf{sgn}(b)$ be $1, 0$, or -1, depending on whether b is positive, zero, or negative.

2 Preliminaries

2.1 Quivers and Mutation

A quiver is an oriented graph, i.e., a quadruple $Q = (Q_0, Q_1, s, t)$ formed by a set of vertices Q_0, a set of arrows Q_1 and two maps s and t from Q_1 to Q_0 which send an arrow

α respectively to its source $s(\alpha)$ and its target $t(\alpha)$. An arrow whose source and target coincide is a *loop*; a *2-cycle* is a pair of distinct arrows α and β such that $s(\alpha) = t(\beta)$ and $t(\alpha) = s(\beta)$. By convention, in the sequel, by a quiver we always mean a finite quiver without loops nor 2-cycles. An *ice quiver* is a pair (Q, F), where Q is a quiver and F is a subset of Q_0 called *frozen vertices*, such that there are no arrows between frozen vertices. The non-frozen vertices of (Q, F) are *mutable vertices*. The *mutable part* of (Q, F) is the full subquiver of (Q, F) consisting of mutable vertices.

Definition 2.1 Let (Q, F) be an ice quiver and k a mutable vertex. The *mutation* $\mu_k(Q, F)$ of (Q, F) at vertex k is the ice quiver obtained from (Q, F) as follows:

- for each subquiver $i \overset{\beta}{\to} k \overset{\alpha}{\to} j$, we add a new arrow $[\alpha\beta] : i \to j$;
- we reverse all arrows with source or target k;
- we remove the arrows in a maximal set of pairwise disjoint 2-cycles and any arrows that created between frozen vertices.

When $F = \varnothing$, we also write $\mu_k(Q)$ for $\mu_k(Q, \varnothing)$.

Let (Q, F) be an ice quiver with non frozen vertices $\{1, \ldots, n\}$ and frozen vertices $\{n + 1, \ldots, m\}$. Up to an isomorphism fixing the vertices, such an ice quiver is given by an $m \times n$ integer matrix $B(Q, F)$ whose coefficient b_{ij} is the difference between the number of arrows from j to i and the number of arrows from i to j. In particular, the principal part of $B(Q, F)$ is skew-symmetric. Conversely, each $m \times n$ integer matrix B with skew-symmetric principal part comes from an ice quiver. Let $B(Q, F) = (b_{ij})$ be the associated matrix of (Q, F). For any mutable vertex k, we denote by $\mu_k(B(Q, F)) = (b'_{ij})$ the matrix associated to the ice quiver $\mu_k(Q, F)$, then

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \mathsf{sgn}(b_{ik}) \max(0, b_{ik}b_{kj}) & \text{else.} \end{cases}$$

This is the *matrix mutation rule* introduced by Fomin and Zelevinsky [11]. It is clear that $\mu_k(B(Q, F)) = B(\mu_k(Q, F))$.

Mutation at a fixed vertex is an involution. Two ice quivers are *mutation-equivalent* if they are linked by a finite sequence of mutations. We will denote by $\mathrm{Mut}(Q, F)$ the set of all quivers that can be obtained from (Q, F) by a finite sequence of mutations. We write $\mathrm{Mut}(Q) := \mathrm{Mut}(Q, \varnothing)$.

2.2 Maximal Green Sequence

Definition 2.2 Let Q be a quiver. The *framed quiver* $\hat{Q}$ of Q is the ice quiver $(\hat{Q}, Q_0^*)$ such that:

$$Q_0^* = \{i^* \mid i \in Q_0\},\ \hat{Q}_0 = Q_0 \sqcup Q_0^*,\ \hat{Q}_1 = Q_1 \sqcup \{i \to i^* \mid i \in Q_0\}.$$

The *coframed quiver* $\check{Q}$ is the ice quiver $(\check{Q}, Q_0^*)$ such that:

$$Q_0^* = \{i^* \mid i \in Q_0\},\ \check{Q}_0 = Q_0 \sqcup Q_0^*,\ \check{Q}_1 = Q_1 \sqcup \{i \leftarrow i^* \mid i \in Q_0\}.$$

Definition 2.3 Let $R \in \mathrm{Mut}(\hat{Q}, Q_0^*)$. A mutable vertex $k \in R_0$ is *green* if $\{j^* \in Q_0^* \mid \exists\ j^* \to k \in R_1\} = \varnothing$. It is *red* if $\{j^* \in Q_0^* \mid \exists\ j^* \leftarrow k \in R_1\} = \varnothing$.

We have the following *sign-coherence* property.

Theorem 2.4 [9, Theorem 1.7] *Every mutable vertex of* $R \in \mathrm{Mut}(\hat{Q}, Q_0^*)$ *is either green or red.*

Remark 2.5 A non-zero integer vector $c \in \mathbb{Z}^n$ is *sign-coherent* if $c \leq 0$ or $0 \leq c$. Let Q be a quiver with vertex set $\{1, \dots, n\}$. For $R \in \mathrm{Mut}(\hat{Q}, Q_0^*)$, recall that $B(R, Q_0^*)$ is the associated $2n \times n$ integer matrix. Theorem 2.4 can be restated as follows: each column vector of the coefficient part of $B(R, Q_0^*)$ is sign-coherent.

Definition 2.6 A *green sequence* for a quiver Q is a sequence $\mathbf{i} = (i_1, \dots, i_l)$ of vertices of Q such that for any $1 \leq k \leq l$, the vertex i_k is green in $\mu_{i_{k-1}} \circ \cdots \circ \mu_{i_1}(\hat{Q}, Q_0^*)$. The green sequence $\mathbf{i}$ is *maximal* if every mutable vertex in $\mu_{i_l} \circ \cdots \circ \mu_{i_1}(\hat{Q}, Q_0^*)$ is red. We will simply denote the composition $\mu_{i_l} \circ \cdots \circ \mu_{i_1}$ by $\mu_{\mathbf{i}}$. A *green-to-red sequence* is a sequence $\mathbf{i}$ of vertices of Q such that every mutable vertex in $\mu_{\mathbf{i}}(\hat{Q}, Q_0^*)$ is red.

Proposition 2.7 [2, Proposition 2.10] *Suppose that Q admits a green-to-red sequence $\mathbf{i}$. Then there is a unique isomorphism $\mu_{\mathbf{i}}(\hat{Q}, Q_0^*) \xrightarrow{\sim} \check{Q}$ fixing the frozen vertices and sending a non frozen vertex i to $\sigma(i)$ for a unique permutation σ of the vertices of Q.*

Remark 2.8 By definition and Proposition 2.7, it is known that a sequence $\mathbf{i}$ is a green-to-red sequence of Q if and only if the coefficient part of the matrix $B(\mu_{\mathbf{i}}(\hat{Q}, Q_0^*)) = \mu_{\mathbf{i}}(B(\hat{Q}, Q_0^*))$ is a permutation of $-I_n$. A sequence $\mathbf{i} = (i_1, \dots, i_l)$ is a maximal green sequence if and only if

- the i_k-th column vector of the coefficient part of $B(\mu_{i_{k-1}} \circ \cdots \circ \mu_{i_1}(\hat{Q}, Q_0^*))$ is positive for $1 \leq k \leq l$;
- the coefficient part of the matrix $B(\mu_{\mathbf{i}}(\hat{Q}, Q_0^*)) = \mu_{\mathbf{i}}(B(\hat{Q}, Q_0^*))$ is a permutation of $-I_n$.

By definition, all maximal green sequences are green-to-red sequences. There are quivers for which a maximal green sequence does not exist, but a green-to-red sequence does. Furthermore, there are quivers for which no green-to-red sequence exists.

Example 2.9 Let a, b, c be non negative integers, denote by $Q_{a,b,c}$ the quiver with three vertices 1, 2, 3 and a arrows from 1 to 2, b arrows from 2 to 3 and c arrows from 3 to 1. It is known that $Q_{2,2,2}$ does not admit a green-to-red sequence. Furthermore, Muller [24, Theorem 12] proved that $Q_{a,b,c}$ does not admit maximal green sequences whenever $a, b, c \geq 2$.

Lemma 2.10 [24, Corollary 19] *If a quiver Q admits a green-to-red sequence, then any quiver mutation-equivalent to Q also admits a green-to-red sequence.*

Muller [24] also proved that the property of having a maximal green sequence is not invariant under mutation. The following is useful to show the non-existence of maximal green sequence for a given quiver.

Lemma 2.11 [24, Theorem 9 and 17] *If a quiver Q admits a green-to-red sequence (resp. maximal green sequence), then any full subquiver of Q also admits a green-to-red sequence (resp. maximal green sequence). In particular, if Q has a full subquiver $Q_{a,b,c}$ with $a, b, c \geq 2$, then Q does not admit a maximal green sequence.*

Definition 2.12 Let Q be a quiver and Q', Q'' full subquivers. We say that Q is a *triangular extension* of Q' by Q'' if the set of vertices of Q is the disjoint union of the sets of vertices of Q' and Q'' and there are no arrows from vertices of Q'' to vertices of Q'.

The following result was proved in [8, Theorem 4.5] using Lemma 2.11.

Lemma 2.13 [8, Theorem 4.5] *If Q is a triangular extension of Q' by Q'', then Q has a maximal green sequence if and only if Q' and Q'' have maximal green sequences.*

2.3 Tropical Dualities Between *c*-vectors and *g*-vectors

Let Q be a quiver with vertex set $\{1, 2, \ldots, n\}$. Denote by $\mathbb{T}_n$ the *n-regular tree* whose edges are labeled by the numbers $1, \ldots, n$ such that the n edges emanating from each vertex have different labels. We write $t \overset{k}{\text{——}} t'$ to indicate that vertices t and t' are linked by an edge labeled by k.

A *quiver pattern* of $(\hat{Q}, Q_0^*)$ is an assignment of an ice quiver $R_t \in \mathrm{Mut}(\hat{Q}, Q_0^*)$ to each vertex $t \in \mathbb{T}_n$ such that

(1) there is a vertex $t_0 \in \mathbb{T}_n$ such that $R_{t_0} = (\hat{Q}, Q_0^*)$;

(2) if $t \overset{k}{\text{——}} t'$, then $R_{t'} = \mu_k(R_t)$.

Clearly, a quiver pattern of $(\hat{Q}, Q_0^*)$ is uniquely determined by assigning $(\hat{Q}, Q_0^*)$ to the vertex $t_0 \in \mathbb{T}_n$ and t_0 is called the *root* vertex of the quiver pattern.

We relabel the vertex i^* as $n + i$ for each $i \in Q_0$ and fix a quiver pattern of $(\hat{Q}, Q_0^*)$. In particular, for each vertex $t \in \mathbb{T}_n$, we have a $2n \times n$ integer matrix $B(R_t) := (b_{ij;t})$. The coefficient part C_t of $B(R_t)$ is the *C-matrix* at t. Its column vectors are *c-vectors*.

Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{Z}^n$. For $1 \leq j \leq n$, denote by β_j the jth column of the principal part of $B(R_{t_0})$. For each vertex $t \in \mathbb{T}_n$, we also assign an integer matrix $G_t := (g_{1;t}, \ldots, g_{n;t})$ by the following recursion:

(1) for any $1 \leq i \leq n$, $g_{i;t_0} = e_i$;

(2) suppose that G_t is defined and let $t \overset{k}{\text{——}} t'$ be an edge of $\mathbb{T}_n$, then

$$g_{i;t'} = \begin{cases} g_{i;t} & i \neq k; \\ -g_{k;t} + \sum_{j=1}^{n} [b_{jk;t}]_+ g_{j;t} - \sum_{j=1}^{n} [b_{(n+j)k;t}]_+ \beta_j & i = k. \end{cases}$$

We call G_t the *G-matrix* at t and its column vectors are *g-vectors*.

Proposition 2.14 [9, Theorem 1.7] *For each vertex $t \in \mathbb{T}_n$, every row vector of G_t is sign-coherent.*

The following is known as the tropical duality between c-vectors and g-vectors (cf. [20, 25, 26]).

Theorem 2.15 [25, Theorem 4.1] *For each vertex $t \in \mathbb{T}_n$, we have*

$$G_t^T C_t = I_n.$$

2.4 Finite Mutation Type

A quiver Q is of *finite mutation type* if $\mathrm{Mut}(Q)$ is a finite set. Quivers of finite mutation type have been classified in [10]. Here, we only recall the following.

Lemma 2.16 (1) *Every quiver with two vertices is of finite mutation type.*
(2) *If Q is acyclic with at least three vertices, then Q is of finite mutation type if and only if Q is of Dynkin type or extended Dynkin type.*
(3) *Each quiver in* Fig. 1 *is of finite mutation type.*

Proof The statement (1) is obvious, (2) is proved by [3, Theorem 3.6]. For (3), one can verify the finiteness by the MutationApp of Keller [22] directly (cf. also [10, Theorem 6.1]). □

Lemma 2.17 *Let $2 \leq c \leq b \leq a$ be integers. If $a \geq 3$, then the quiver $Q_{a,b,c}$ is not of finite mutation type.*

Proof For a non negative integer t, we set $Q^t := (\mu_2\mu_1)^t(Q_{(a,b,c)})$ and a pair of integers (b_t, c_t) by the following recursion:

$$b_0 = b, c_0 = c, b_t = ac_{t-1} - b_{t-1}, c_t = ab_t - c_{t-1}.$$

We claim that $0 < b_1 < c_1 \cdots < b_t < c_t < \cdots$. Indeed, it is straightforward to see that $0 < b_1 < c_1$. For $t \geq 2$, we have

$$b_t - c_{t-1} = ac_{t-1} - b_{t-1} - c_{t-1} = (a-1)c_{t-1} - b_{t-1} \geq 2c_{t-1} - b_{t-1} > 0$$

and

$$c_t - b_t = (a-1)b_t - c_{t-1} > 0$$

by induction. As a consequence, $Q^t = Q_{(a,b_t,c_t)}$ and $Q_{(a,b,c)}$ is not of finite mutation type. □

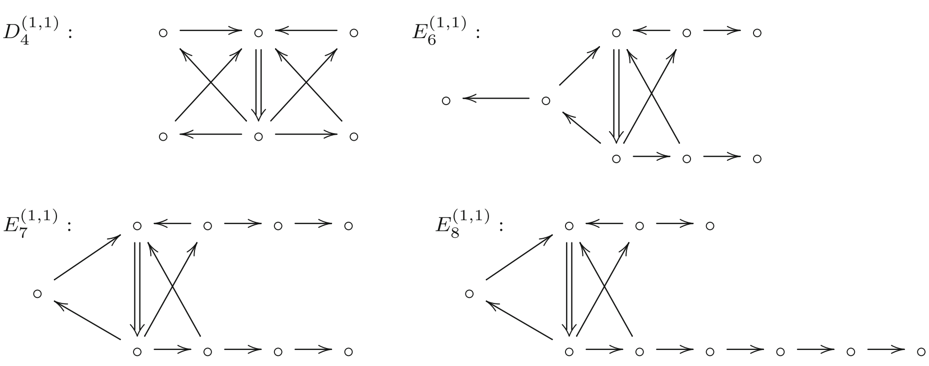


Fig. 1 Quivers of tubular type $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$

3 Quivers Arising from Weighted Projective Lines

3.1 Weighted Projective Lines

Fix a positive integer $t \geq 2$. A *weighted projective line* $\mathbb{X} = \mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$ over K is given by a weight sequence $\mathbf{p} = (p_1, \ldots, p_t)$ of positive integers, and a parameter sequence $\boldsymbol{\lambda} = (\lambda_1 \ldots, \lambda_t)$ of pairwise distinct points of the projective line $\mathbb{P}_1(K)$. Let $\mathbb{L}$ be the rank one abelian group generated by $\vec{x}_1, \ldots, \vec{x}_t$ with the relations

$$p_1\vec{x}_1 = p_2\vec{x}_2 = \cdots = p_t\vec{x}_t =: \vec{c},$$

where the element $\vec{c}$ is called the *canonical element* of $\mathbb{L}$. Denote by

$$\vec{\omega} := (t-2)\vec{c} - \sum_{i=1}^{t} \vec{x}_i \in \mathbb{L},$$

which is called the *dualizing element* of $\mathbb{L}$. Each element $\vec{x} \in \mathbb{L}$ can be uniquely written into the *normal form*

$$\vec{x} = \sum_{i=1}^{t} l_i\vec{x}_i + l\vec{c}, \text{ where } 0 \leq l_i < p_i \text{ and } l \in \mathbb{Z}.$$

Let $\vec{x} = \sum_{i=1}^{t} l_i\vec{x}_i + l\vec{c}$ and $\vec{y} = \sum_{i=1}^{t} m_i\vec{x}_i + m\vec{c} \in \mathbb{L}$ be in normal form, denote by $\vec{x} \leq \vec{y}$ if $l_i \leq m_i$ for $i = 1, \ldots, t$ and $l \leq m$. This defines a partial order on $\mathbb{L}$. It is known that each $\vec{x} \in \mathbb{L}$ satisfies exactly one of the two possibilities:

$$0 \leq \vec{x} \quad \text{or} \quad \vec{x} \leq \vec{c} + \vec{\omega}.$$

3.2 The Category coh $\mathbb{X}$ of Coherent Sheaves

Let

$$S := S(\mathbf{p}, \boldsymbol{\lambda}) = K[X_1, \cdots, X_t]/I$$

be the quotient of the polynomial ring $K[X_1, \cdots, X_t]$ by the ideal I generated by $f_i = X_i^{p_i} - X_2^{p_2} + \lambda_i X_1^{p_1}$ for $3 \leq i \leq t$. The algebra S is $\mathbb{L}$-graded by setting $\deg X_i = \vec{x}_i$ for $i = 1, \ldots, t$ and we have the decomposition of S into K-subspace

$$S = \bigoplus_{\vec{x} \in \mathbb{L}} S_{\vec{x}}.$$

The category $\mathsf{coh}\,\mathbb{X}$ of coherent sheaves over $\mathbb{X}$ is defined to be the quotient category

$$\mathsf{coh}\,\mathbb{X} := \mod^{\mathbb{L}} S / \mod_0^{\mathbb{L}} S,$$

where $\mod^{\mathbb{L}} S$ is the category of finitely generated $\mathbb{L}$-graded S-modules, while $\mod_0^{\mathbb{L}} S$ is the Serre subcategory of $\mathbb{L}$-graded S-modules of finite length. For each sheaf E and $\vec{x} \in \mathbb{L}$, denote by $E(\vec{x})$ the grading shift of E with $\vec{x}$. The free module S gives the structure sheaf $\mathcal{O}$, and each line bundle is given by the grading shift $\mathcal{O}(\vec{x})$ for a unique element $\vec{x} \in \mathbb{L}$. Moreover, we have

$$\mathsf{Hom}_{\mathbb{X}}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{y})) = S_{\vec{y}-\vec{x}} \text{ for any } \vec{x}, \vec{y} \in \mathbb{L}. \tag{1}$$

In [15], Geigle and Lenzing proved that $\mathsf{coh}\,\mathbb{X}$ is a connected hereditary abelian category with tilting objects and has Serre duality of the form

$$\mathbb{D}\,\mathsf{Ext}_{\mathbb{X}}^1(E, F) = \mathsf{Hom}_{\mathbb{X}}(F, E(\vec{\omega})) \tag{2}$$

for all $E, F \in \operatorname{coh}\mathbb{X}$, where $\mathbb{D} = \operatorname{Hom}_K(-, K)$ is the standard duality. In particular, $\operatorname{coh}\mathbb{X}$ admits almost split sequences with the Auslander-Reiten translation τ given by the grading shift with $\vec{\omega}$. Recall that an object $T \in \operatorname{coh}\mathbb{X}$ is a *tilting object* if $\operatorname{Ext}^1_{\mathbb{X}}(T, T) = 0$ and for $X \in \operatorname{coh}\mathbb{X}$ with $\operatorname{Hom}_{\mathbb{X}}(T, X) = 0 = \operatorname{Ext}^1_{\mathbb{X}}(T, X)$, we have that $X = 0$.

Denote by $\operatorname{vect}\mathbb{X}$ the full subcategory of $\operatorname{coh}\mathbb{X}$ consisting of vector bundles, i.e., torsion-free sheaves, and by $\operatorname{coh}_0\mathbb{X}$ the full subcategory consisting of sheaves of finite length, i.e., torsion sheaves. Each coherent sheaf is the direct sum of a vector bundle and a finite length sheaf. Each vector bundle has a finite filtration by line bundles and there is no nonzero morphism from $\operatorname{coh}_0\mathbb{X}$ to $\operatorname{vect}\mathbb{X}$. We remark that $\operatorname{coh}\mathbb{X}$ does not contain nonzero projective objects. Denote by

$$\mathbf{p}_{\boldsymbol{\lambda}} : \mathbb{P}_1(k) \to \mathbb{N},\ \mathbf{p}_{\boldsymbol{\lambda}}(\mu) = \begin{cases} p_i & \text{if } \mu = \lambda_i \text{ for some } i, \\ 1 & \text{else.} \end{cases}$$

the weight function associated with $\mathbb{X}$.

Proposition 3.1 [15, Proposition 2.5] *The category* $\operatorname{coh}_0\mathbb{X}$ *is an exact abelian, uniserial subcategory of* $\operatorname{coh}\mathbb{X}$ *which is stable under Auslander-Reiten translation. The components of the Auslander-Reiten quiver of* $\operatorname{coh}_0\mathbb{X}$ *form a family of pairwise orthogonal standard tubes* $(\mathcal{T}_\mu)_{\mu\in\mathbb{P}_1(k)}$*, where each tube* $\mathcal{T}_\mu$ *has rank* $\mathbf{p}_{\boldsymbol{\lambda}}(\mu)$.

For λ_i with weight $p_i \geq 2$, there is exactly one simple object S_i in $\mathcal{T}_{\lambda_i}$ satisfying $\operatorname{Hom}_{\mathbb{X}}(\mathcal{O}, S_i) \neq 0$. Moreover, there exists a sequence of exceptional objects and epimorphims

$$S_i^{[p_i-1]} \twoheadrightarrow S_i^{[p_i-2]} \twoheadrightarrow \cdots \twoheadrightarrow S_i^{[1]} = S_i,$$

where $S_i^{[j]}$ has length j and top S_i.

The following is well-known (cf. [15]).

Proposition 3.2 *Both*

$$T_{can}(\mathbb{X}) := \bigoplus_{0\leq\vec{x}\leq\vec{c}} \mathcal{O}(\vec{x}) \text{ and } T_{sq}(\mathbb{X}) := \mathcal{O} \oplus \mathcal{O}(\vec{c}) \oplus \bigoplus_{i=1}^{t} \left(\bigoplus_{k=1}^{p_i-1} S_i^{[p_i-k]} \right)$$

are tilting objects of $\operatorname{coh}\mathbb{X}$.

3.3 Quivers Associated with $T_{\mathrm{can}}(\mathbb{X})$ and $T_{\mathrm{sq}}(\mathbb{X})$

Denote by $\mathcal{D}^b(\operatorname{coh}\mathbb{X})$ the bounded derived category of $\operatorname{coh}\mathbb{X}$ with suspension functor Σ. Let $\tau : \mathcal{D}^b(\operatorname{coh}\mathbb{X}) \to \mathcal{D}^b(\operatorname{coh}\mathbb{X})$ be the Auslander-Reiten (AR) translation functor, which restricts to the AR translation of $\operatorname{coh}\mathbb{X}$.

Definition 3.3 The *cluster category* $\mathcal{C}_{\mathbb{X}}$ associated with $\mathbb{X}$ is defined as the orbit category $\mathcal{D}^b(\operatorname{coh}\mathbb{X})/\langle\tau^{-1}\circ\Sigma\rangle$; it has the same objects as $\mathcal{D}^b(\operatorname{coh}\mathbb{X})$, morphism spaces are given by $\bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{D}^b(\operatorname{coh}\mathbb{X})}(X, (\tau^{-1}\circ\Sigma)^i Y)$ with obvious composition.

The cluster category $\mathcal{C}_{\mathbb{X}}$ admits a canonical triangle structure such that the projection $\pi_{\mathbb{X}} : \mathcal{D}^b(\operatorname{coh}\mathbb{X}) \to \mathcal{C}_{\mathbb{X}}$ is a triangle functor (cf. [18]). The suspension functor Σ (resp. the AR translation τ) of $\mathcal{D}^b(\operatorname{coh}\mathbb{X})$ induces the suspension functor (resp. the AR translation) of $\mathcal{C}_{\mathbb{X}}$,

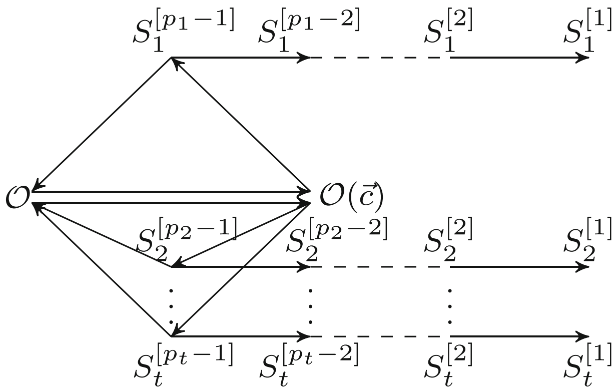


Fig. 2 Quiver $Q_{T_{\text{sq}}(\mathbb{X})}$ with weight sequence $(p_1, \ldots, p_t)$

which will be denoted by Σ (resp. τ) as well. It was shown in [18] that $\mathcal{C}_{\mathbb{X}}$ is a 2-Calabi-Yau triangulated category, i.e., for any $X, Y \in \mathcal{C}_{\mathbb{X}}$, we have bifunctorially isomorphisms

$$\mathsf{Hom}_{\mathcal{C}_{\mathbb{X}}}(X, \Sigma^2 Y) \cong \mathbb{D}\,\mathsf{Hom}_{\mathcal{C}_{\mathbb{X}}}(Y, X).$$

By the 2-Calabi-Yau property, we clearly have $\tau = \Sigma$ in $\mathcal{C}_{\mathbb{X}}$.

Definition 3.4 An object $T \in \mathcal{C}_{\mathbb{X}}$ is a *cluster-tilting object* if $\mathsf{Ext}^1_{\mathcal{C}_{\mathbb{X}}}(T, T) = 0$ and $\mathsf{Ext}^1_{\mathcal{C}_{\mathbb{X}}}(T, X) = 0$ implies that $X \in \mathsf{add}\, T$, where $\mathsf{add}\, T$ is the full subcategory of $\mathcal{C}_{\mathbb{X}}$ consisting of direct summands of direct sum of finite copies of T.

Since $\mathsf{coh}\,\mathbb{X}$ has no nonzero projective objects, the composition of the embedding of $\mathsf{coh}\,\mathbb{X}$ into $\mathcal{D}^b(\mathsf{coh}\,\mathbb{X})$ with the projection functor $\pi_{\mathbb{X}}$ yields a bijection between the set of isomorphism classes of indecomposable objects of $\mathsf{coh}\,\mathbb{X}$ and the set of isomorphism classes of indecomposable objects of $\mathcal{C}_{\mathbb{X}}$. We may identify the objects of $\mathsf{coh}\,\mathbb{X}$ with the ones of $\mathcal{C}_{\mathbb{X}}$ by the bijection.

Lemma 3.5 [4, Section 3] *An object $T \in \mathsf{coh}\,\mathbb{X}$ is a tilting object if and only if T is a cluster-tilting object of $\mathcal{C}_{\mathbb{X}}$.*

In particular, $T_{\text{can}}(\mathbb{X})$ and $T_{\text{sq}}(\mathbb{X})$ are basic cluster-tilting objects of $\mathcal{C}_{\mathbb{X}}$. We denote by $Q_{T_{\text{can}}(\mathbb{X})}$ (resp. $Q_{T_{\text{sq}}(\mathbb{X})}$) the Gabriel quiver of the endomorphism algebra $\mathsf{End}_{\mathcal{C}_{\mathbb{X}}}(T_{\text{can}}(\mathbb{X}))$ (resp. $\mathsf{End}_{\mathcal{C}_{\mathbb{X}}}(T_{\text{sq}}(\mathbb{X}))$). The quivers have been listed in Figs. 2 and 3 respectively. We remark that the relation of the corresponding algebra is quite complicated in general and we do not need in the sequel.

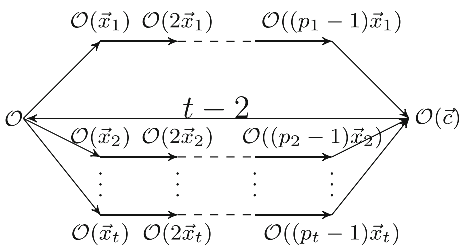


Fig. 3 Quiver $Q_{T_{\text{can}}(\mathbb{X})}$ with weight sequence $(p_1, \ldots, p_t)$, where the label $t-2$ means that there are $t-2$ arrows from $\mathcal{O}(\vec{c})$ to $\mathcal{O}$

The following is a direct consequence of [12, Theorem 1.2].

Proposition 3.6 *Let T be a basic cluster-tilting object of $\mathcal{C}_{\mathbb{X}}$ and Q_T the Gabriel quiver of the endomorphism algebra of T. Then*

(1) *Q_T is mutation-equivalent to $Q_{T_{sq}(\mathbb{X})}$. In particular, the quiver $Q_{T_{can}(\mathbb{X})}$ is mutation-equivalent to $Q_{T_{sq}(\mathbb{X})}$.*
(2) *Q_T admits a green-to-red sequence.*

3.4 The Classification

Denote by $p = \mathrm{lcm}(p_1, \ldots, p_t)$ the least common multiple of $p_1, \ldots, p_t$. The *genus* $g_{\mathbb{X}}$ of $\mathbb{X}$ is defined as

$$g_{\mathbb{X}} = 1 + \frac{1}{2}\left((t-2)p - \sum_{i=1}^{t} \frac{p}{p_i}\right).$$

A weighted projective line of genus $g_{\mathbb{X}} < 1 (g_{\mathbb{X}} = 1$, resp. $g_{\mathbb{X}} > 1)$ is of *domestic* (*tubular*, resp. *wild*) type. The domestic types are, up to permutation, $(1, p)$ with $p \geq 1$, (p, q) with $p, q \geq 2$, $(2, 2, n)$ with $n \geq 2$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$, whereas the tubular types are, up to permutation, $(2, 2, 2, 2)$, $(3, 3, 3)$, $(2, 4, 4)$ and $(2, 3, 6)$. It is worth pointing out that a weighted projective line of domestic type is derived equivalent to a finite dimensional hereditary algebra of tame type.

Proposition 3.7 *Let $\mathbb{X}$ be a weighted projective line. The quiver $Q_{T_{sq}(\mathbb{X})}$ is of finite mutation type if and only if $\mathbb{X}$ is of domestic type or of tubular type.*

Proof The "if" part follows from Lemma 2.16. More precisely, if $\mathbb{X}$ is of domestic type, then $\mathsf{coh}\,\mathbb{X}$ is derived equivalent to a finite dimensional hereditary algebra of tame type. As a consequence, the quiver $Q_{T_{\mathrm{sq}}(\mathbb{X})}$ is mutation-equivalent to an acyclic quiver of extended Dynkin type. If $\mathbb{X}$ is of tubular type, then the quiver $Q_{T_{\mathrm{sq}}(\mathbb{X})}$ is as in Fig. 1.

For the "only if" part, it suffices to prove that $Q_{T_{\mathrm{sq}}(\mathbb{X})}$ is not of finite mutation type provided that $\mathbb{X}$ is of wild type. According to Lemma 2.17, it suffices to show that there is a quiver Q in $\mathrm{Mut}(Q_{T_{\mathrm{sq}}(\mathbb{X})})$ such that Q admits a full subquiver $Q_{a,b,c}$ for some $2 \leq c \leq b \leq a$ and $a \geq 3$.

Let $\mathbb{X}$ be a wild weighted projective line. According to the classification of weighted projective lines, the quiver $Q_{T_{\mathrm{sq}}(\mathbb{X})}$ admits one of the following quivers as a subquiver

(1) $Q_{T_{\mathrm{sq}}(\mathbb{X}')}$ with weight sequence $(2, 3, 7)$;
(2) $Q_{T_{\mathrm{sq}}(\mathbb{X}')}$ with weight sequence $(2, 4, 5)$;
(3) $Q_{T_{\mathrm{sq}}(\mathbb{X}')}$ with weight sequence $(3, 3, 4)$;
(4) $Q_{T_{\mathrm{sq}}(\mathbb{X}')}$ with weight sequence $(2, 2, 2, 3)$;
(5) $Q_{T_{\mathrm{sq}}(\mathbb{X}')}$ with weight sequence $(2, 2, 2, 2, 2)$.

Let $\mathbf{p}$ be one of the weight sequences in (1)–(5). According to Proposition 3.6, there is a quiver $Q_{\mathbf{p}}$ in $\mathrm{Mut}(Q_{T_{\mathrm{sq}}(\mathbb{X})})$ such that $Q_{\mathbf{p}}$ admits $Q_{T_{\mathrm{can}}(\mathbb{X}')}$ as a full subquiver, where $\mathbb{X}'$ has the weight sequence $\mathbf{p}$. It suffices to show that there is a quiver in $\mathrm{Mut}(Q_{T_{\mathrm{can}}(\mathbb{X}')})$ which admits a subquiver $Q_{a,b,c}$ for $2 \leq c \leq b \leq a$ and $3 \leq a$. Let us label the vertices of $Q_{T_{\mathrm{can}}(\mathbb{X}')}$ as in Fig. 3. For $\mathbf{p} = (2, 3, 7)$, let

$$\begin{aligned}\mathrm{i} = (&\mathcal{O}, \mathcal{O}(6\vec{x}_3), \mathcal{O}(\vec{c}), \mathcal{O}(2\vec{x}_3), \mathcal{O}(\vec{x}_3), \mathcal{O}(2\vec{x}_2), \mathcal{O}(6\vec{x}_3), \mathcal{O}(5\vec{x}_3),\\ &\mathcal{O}(\vec{x}_2), \mathcal{O}(2\vec{x}_3), \mathcal{O}(3\vec{x}_3), \mathcal{O}(2\vec{x}_2), \mathcal{O}(\vec{x}_3), \mathcal{O}(\vec{c})).\end{aligned}$$

For $\mathbf{p} = (2, 4, 5)$, let

$$\mathfrak{i} = (\mathcal{O}, \mathcal{O}(\vec{c}), \mathcal{O}(\vec{x}_3), \mathcal{O}(2\vec{x}_3), \mathcal{O}(3\vec{x}_3), \mathcal{O}(3\vec{x}_2), \mathcal{O}(\vec{c}), \mathcal{O}(3\vec{x}_3), \mathcal{O}(4\vec{x}_3), \mathcal{O}(\vec{x}_2), \mathcal{O}(2\vec{x}_3)).$$

For $\mathbf{p} = (3, 3, 4)$, let $\mathfrak{i} = (\mathcal{O}, \mathcal{O}(\vec{x}_1), \mathcal{O}(\vec{x}_2), \mathcal{O}(\vec{x}_3), \mathcal{O}(\vec{c}), \mathcal{O})$. For $\mathbf{p} = (2, 2, 2, 3)$, let $\mathfrak{i} = (\mathcal{O}(\vec{c}), \mathcal{O})$. It is straightforward to check that $Q_{2,2,3}$ is a subquiver of $\mu_{\mathfrak{i}}(Q_{T_{\text{can}}(\mathbb{X}')})$ in each case. Finally, for $\mathbf{p} = (2, 2, 2, 2, 2)$, denote by $\mathfrak{i} = (\mathcal{O}(\vec{c}), \mathcal{O})$. We find that $Q_{2,3,5}$ is a subquiver of $\mu_{\mathcal{O}}\mu_{\mathcal{O}(\vec{c})}(Q_{T_{\text{can}}(\mathbb{X}')})$ in this case. This completes the proof. □

4 The Existence and Non-existence of Maximal Green Sequence

This section is devoted to proving the main result of this note. We begin with the hyperbolic case. Recall that a weighted projective line $\mathbb{X}$ with weight sequence $(p_1, \ldots, p_t)$ is of *hyperbolic type* if $p_1 = p_2 = \cdots = p_t = 2$.

Let $\mathbb{X}$ be of hyperbolic type. We denote by Q_t the quiver $Q_{T_{\text{sq}}(\mathbb{X})}$ in this case and relabel the vertices of Q_t as in Fig. 4. We will always identify Q_t with a full subquiver of Q_{t+1} such that the vertex $t+1$ is the unique vertex which does not belong to Q_t.

Lemma 4.1 *Let* $\mathbf{i_t}$ *be a maximal green sequence of* Q_t*. Denote by* $\bullet \rightrightarrows \circ$ *the unique multiple arrows in* $\mu_{\mathfrak{i}_t}(Q_t)$*. If* $\mathbf{i}_{t+1} := (\mathbf{i}_t, t+1, \circ, \bullet)$ *is a maximal green sequence of* Q_{t+1}*, then* $\circ \rightrightarrows t+1$ *is the unique multiple arrows in* $\mu_{\mathfrak{i}_{t+1}}(Q_{t+1})$ *and* $\mathbf{i}_{t+2} := (\mathfrak{i}_{t+1}, t+2, t+1, \circ)$ *is a maximal green sequence of* Q_{t+2}.

Proof We apply $\mu_{\mathfrak{i}_t}$ to the quiver Q_{t+2}. Since $\mathbf{i_t}$ is a sequence of vertices of Q_t, $\mu_{\mathbf{i_t}}(Q_t)$ is a full subquiver of $\mu_{\mathbf{i_t}}(Q_{t+2})$. In particular, the vertex set of $\mu_{\mathbf{i_t}}(Q_t)$ is a subset of the vertex set of $\mu_{\mathbf{i_t}}(Q_{t+2})$. Since $\mu_{\mathbf{i_t}}(Q_t) \cong Q_t$, we will denote the vertex set of $\mu_{\mathbf{i_t}}(Q_t)$ by $\{\bullet, \circ, 1, \ldots, t\}$ and the vertex set of $\mu_{\mathbf{i_t}}(Q_{t+2})$ by $\{\bullet, \circ, 1, \ldots, t, t+1, t+2\}$.

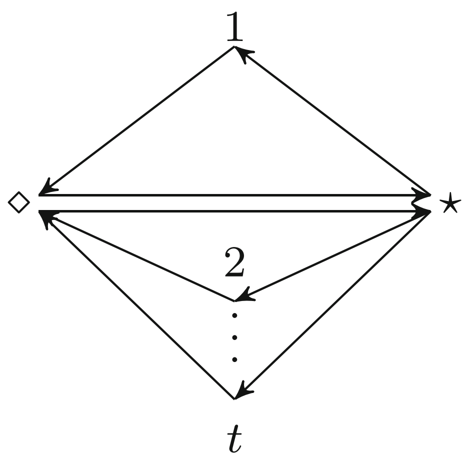


Fig. 4 Quiver $Q_t = Q_{T_{\text{sq}}(\mathbb{X})}$ with weight sequence $(2, 2, \ldots, 2)$

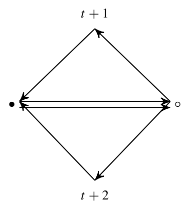
Let $\hat{B} = (b_{ij}) \in M_{2(t+4)}(\mathbb{Z})$ be the skew-symmetric matrix associated to the framed quiver $\hat{Q}_{t+2}$ and $\hat{B}^\circ$ the submatrix of $\hat{B}$ consisting of the first $t+4$ columns. We index the columns of $\hat{B}^\circ$ by $\bullet, \circ, 1, \ldots, t+2$.

Claim 1: The principal part of $\mu_{\mathrm{i_t}}(\hat{B}^\circ)$ is

$$\begin{bmatrix} 0 & -2 & 1 & \cdots & 1 & 1 & 1 \\ 2 & 0 & -1 & \cdots & -1 & -1 & -1 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ & \vdots & & \vdots & & \vdots & \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \in M_{t+4}(\mathbb{Z}).$$

The proof of this claim will be separated into three steps. Here we work on the quivers.

Step 1: *The full subquiver of* $\mu_{\mathrm{i_t}}(Q_{t+2})$ *consisting of vertices* $\bullet, \circ, t+1$ *and* $t+2$ *has the following form:*



Suppose that there are a arrows from vertex $t+1$ to vertex $\bullet$ and b arrows from vertex $\circ$ to vertex $t+1$ and denote by $Q(a,b)$ the full subquiver consisting of vertices $\bullet, \circ$ and $t+1$. Since there is a symmetry between $t+1$ and $t+2$ in Q_{t+2}, it suffices to prove that $a = b = 1$.

Denote by $B = \begin{bmatrix} 0 & -2 & a \\ 2 & 0 & -b \\ -a & b & 0 \end{bmatrix}$ the associated skew-symmetric matrix of $Q(a,b)$. Denote by

$$c_{12} = -2 + \mathrm{sgn}(a)[ab]_+, c_{13} = a - \mathrm{sgn}(c_{12})[bc_{12}]_+, c_{23} = -b + \mathrm{sgn}(c_{12})[-c_{12}c_{13}]_+.$$

By Fomin-Zelevinsky's matrix mutation formula, we obtain

$$C := \mu_\bullet(\mu_\circ(\mu_{t+1}(B))) = \begin{bmatrix} 0 & c_{12} & c_{13} \\ -c_{12} & 0 & c_{23} \\ -c_{13} & -c_{23} & 0 \end{bmatrix}.$$

Note that the associated skew-symmetric matrix of $\mu_\bullet(\mu_\circ(\mu_{t+1}(Q(a,b))))$ is the matrix C. By the assumption that $\mathrm{i_{t+1}}$ is a maximal green sequence of Q_{t+1}, we have $\mu_{\mathrm{i}_{t+1}}(Q_{t+1}) \cong Q_{t+1}$. In particular, the quiver $\mu_\bullet(\mu_\circ(\mu_{t+1}(Q(a,b))))$ is a full subquiver of Q_{t+1} via the isomorphism $\mu_{\mathrm{i}_{t+1}}(Q_{t+1}) \cong Q_{t+1}$. The remaining proof is a discussion of the values of a and b, from which we can deduce that $a = 1 = b$. We will denote by $Q(C)$ the associated quiver of the skew-symmetric matrix C.

Case 1*:* $a < 0,\ b > 0$. A direct computation shows that $c_{12} = -2$, $c_{13} = a$ and $c_{23} = -b$. Consequently, the associate quiver $Q(C)$ is not a full subquiver of Q_{t+1}.

Case 2*:* $a > 0, b < 0$. We have $c_{12} = -2$, $c_{13} = a - 2b \geq 3$, which implies that the associated quiver $Q(C)$ is not a full subquiver of Q_{t+1}.

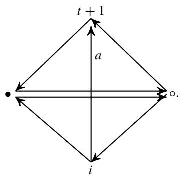
Case 3*:* $a \leq 0, b \leq 0$. We have $c_{12} = -2 - ab \leq -2$. Since $Q(C)$ is a full subquiver of Q_{t+1}, we have $c_{12} = -2$. Hence $ab = 0$, i.e., $a = 0$ or $b = 0$. In each case, one can show that $Q(C)$ is not a full subquiver of Q_{t+1}.

Case 4*:* $a \geq 0,\ b \geq 0$. Similar to the Case 3, we obtain $-2 \leq c_{12} = -2 + ab \leq 2$. In particular, $0 \leq ab \leq 4$. A direct computation shows that $a = 1 = b$ is the unique value such that $Q(C)$ is a full subquiver of Q_{t+1}. This completes the proof for the statement in **Step** 1.

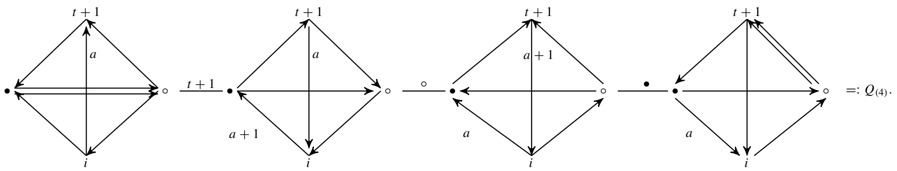
As a direct consequence of the statement of **Step** 1, the quiver $Q(C)$ has the form as in Fig. 5. Consequently, $\circ \rightrightarrows t+1$ is the unique multiple arrows in $\mu_{\mathrm{i}_{t+1}}(Q_{t+1})$.

Step 2: *There re no arrows between vertex* $k \in \{1, \ldots, t\}$ *and vertex* $t+1$ *in the quiver* $\mu_{\mathrm{i}_t}(Q_{t+2})$.

Without loss of generality, we may assume that there are a arrows from vertex i to vertex $t+1$ and we consider the full subquiver consisting of vertices $\bullet, \circ, t+1, i$:



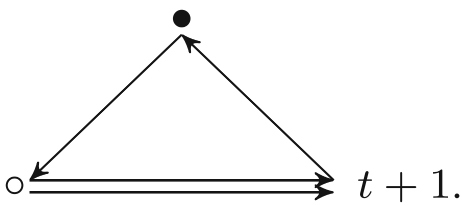
By applying the mutation sequence $t+1, \circ, \bullet$ to the above quiver, we obtain



By the assumption that $\mathbf{i_{t+1}}$ is a maximal green sequence of Q_{t+1}, we know that the quiver $Q_{(4)}$ is a full subquiver of $\mu_{\mathbf{i_{t+1}}}(Q_{t+1})$ and we conclude that $a = 0$. This completes the proof of the statement in **Step** 2.

Step 3: *There are no arrows between vertex* $k \in \{1, \ldots, t+1\}$ *and vertex* $t+2$ *in the quiver* $\mu_{\mathrm{i}_t}(Q_{t+2})$.

Fig. 5 Quiver $Q(C)$



Note that there is a symmetry between vertex $t+1$ and $t+2$ in the quiver Q_{t+2} and the mutation sequence $\mathbf{i_t}$ does not involve the vertices $t+1$ and $t+2$. Then the statement of **Step** 3 follows the statement of **Step** 2 directly.

Now **Claim** 1 is a direct consequence of the statements in **Step** 1, 2, 3**.**

Claim 2**:** *Up to a permutation of the rows associated to* $\bullet^*, \circ^*, 1^*, \ldots, t^*$*, the coefficient part* $C_{\mathbf{i_t}}$ *of* $\mu_{\mathbf{i_t}}(\hat{B}^\circ)$ *has the following form:*

$$\begin{bmatrix} -I_{t+2} & X \\ 0 & I_2 \end{bmatrix}$$

where $X \in M_{(t+2)\times 2}(\mathbb{Z})$ *with non-negative entries.*

We fix a quiver pattern of the framed quiver $\hat{Q}_{t+2}$ of Q_{t+2} by assigning $\hat{Q}_{t+2}$ to the root vertex $t_0 \in \mathbb{T}_{t+4}$. Each sequence $\mathfrak{i}$ of vertices of Q_{t+2} induces a path of $\mathbb{T}_{t+4}$ with starting point t_0. We denote by the ending point $s_{\mathfrak{i}}$ and denote by $G_{\mathfrak{i}} := G_{s_{\mathfrak{i}}}$ (resp. $C_{\mathfrak{i}}$) the G-matrix (resp. C-matrix) at $s_{\mathfrak{i}}$.

Since the sequence $\mathbf{i_t}$ does not involves the vertices $t+1$ and $t+2$. It follows that $G_{\mathbf{i_t}} = \begin{bmatrix} A & 0 \\ Y & I_2 \end{bmatrix}$, where $A \in M_{t+2}(\mathbb{Z})$ is invertible and $Y \in M_{2\times(t+2)}(\mathbb{Z})$ with non negative entries. By the tropical dualities between G-matrices and C-matrices (2.15), we have

$$C_{\mathbf{i_t}} = G_{\mathbf{i_t}}^{-T} = \begin{bmatrix} A^{-T} & -A^{-T}Y^T \\ 0 & I_2 \end{bmatrix}.$$

Since $\mathbf{i_t}$ is a maximal green sequence, it follows that A^{-T} is a permutation of $-I_{t+2}$. On the other hand, the entries of $-A^{-T}Y^T$ are non negative by the sign-coherence of c-vectors. This finishes the proof of **Claim** 2.

According to **Claims** 1 and 2, up to permutation of indices, we may assume that

$$\mu_{\mathbf{i_t}}(\hat{B}^\circ) = \begin{array}{c} \begin{matrix} \bullet & \circ & 1 & \cdots & t & t+1 & t+2 \end{matrix} \\ \left[\begin{matrix} 0 & -2 & 1 & \cdots & 1 & 1 & 1 \\ 2 & 0 & -1 & \cdots & -1 & -1 & -1 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ & \vdots & & \vdots & & \vdots & \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 0 & 0 & \cdots & 0 & a_\bullet & a_\bullet \\ 0 & -1 & 0 & \cdots & 0 & a_\circ & a_\circ \\ 0 & 0 & -1 & \cdots & 0 & a_1 & a_1 \\ & \vdots & & \vdots & & \vdots & \\ 0 & 0 & 0 & \cdots & -1 & a_t & a_t \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{matrix}\right] \end{array} \begin{matrix} \bullet \\ \circ \\ 1 \\ \vdots \\ t \\ t+1 \\ t+2 \\ \\ \\ \\ \\ \\ \\ \\ \end{matrix}$$

where $\bullet, \circ, 1, \cdots, t+2$ are (relabelled) vertices of $\mu_{\mathbf{i_t}}(Q_{t+2})$, $a_\bullet, a_\circ, a_1, \ldots, a_t$ are non negative integers.

By Fomin-Zelevinsky's mutation rule, we obtain $\mu_{\mathbf{i_{t+1}}}(\hat{B}^\circ)$ as in Fig. 6. Note that $\mathbf{i_{t+1}} = (\mathbf{i_t}, t+1, \circ, \bullet)$ is a maximal green sequence of Q_{t+1}. It follows that the submatirx formed

$$\mu_{\mathbf{i_{t+1}}}(\widehat{B}^{\circ}) = \mu_{\bullet} \circ \mu_{\circ} \circ \mu_{t+1}(\mu_{\mathbf{i_t}}(\widehat{B}^{\circ}))$$

$$= \begin{array}{c} \begin{array}{ccccccc} \bullet & \circ & 1 & \cdots & t & t+1 & t+2 \end{array} \\ \left[\begin{array}{ccccccc} 0 & -1 & 0 & \cdots & 0 & 1 & 0 \\ 1 & 0 & 1 & \cdots & 1 & -2 & 1 \\ 0 & -1 & 0 & \cdots & 0 & 1 & 0 \\ & \vdots & & \vdots & & \vdots & \\ 0 & -1 & 0 & \cdots & 0 & 1 & 0 \\ -1 & 2 & -1 & \cdots & -1 & 0 & -1 \\ 0 & -1 & 0 & \cdots & 0 & 1 & 0 \\ 1-a_{\bullet} & -1 & 0 & \cdots & 0 & 0 & a_{\bullet} \\ 1-a_{\circ} & 0 & 0 & \cdots & 0 & -1 & a_{\circ} \\ -a_1 & 0 & -1 & \cdots & 0 & 0 & a_1 \\ & \vdots & & \vdots & & \vdots & \\ -a_t & 0 & 0 & \cdots & -1 & 0 & a_t \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \bullet \\ \circ \\ 1 \\ \vdots \\ t \\ t+1 \\ t+2 \\ \\ \\ \\ \\ \\ t+1^* \\ t+2^* \end{array} \end{array}$$

Fig. 6 The matrix $\mu_{\mathbf{i_{t+1}}}(\hat{B}^{\circ})$

by the first $t+3$ row indices and the first $t+3$ column indices of the coefficient part of $\mu_{\mathbf{i_{t+1}}}(\hat{B}^{\circ})$ is a permutation of $-I_{t+1}$. Hence we have

$$a_{\bullet} = 1, a_{\circ} = 1, a_1 = 0, \ldots, a_t = 0.$$

Finally, we apply the mutation sequence $\circ, t+1, t+2$ to the matrix $\mu_{\mathbf{i_{t+1}}}(\hat{B}^{\circ})$, we compute the matrix $\mu_{\mathbf{i_{t+2}}}(\hat{B}^{\circ})$ as in Fig. 7. Note that the coefficient part of the matrix $\mu_{\mathbf{i_{t+2}}}(\hat{B}^{\circ})$ is a permutation of $-I_{t+4}$. According to Remark 2.8, we conclude that $\mathbf{i_{t+2}}$ is a maximal green sequence of Q_{t+2}. This completes the proof of the lemma. □

Proposition 4.2 *Assume that $t \geq 3$. The quiver $Q_{T_{sq}(\mathbb{X})}$ admits a maximal green sequence.*

Proof Note that the quiver $Q_{T_{\rm sq}(\mathbb{X})}$ for any weight sequence is a triangular extension of Q_t by disjoint union of quivers of type $\mathbb{A}$. Every acyclic quiver admits a maximal green sequence (cf. [2]). According to Lemma 2.13, it suffices to show that the quiver Q_t admits a maximal green sequence for $t \geq 3$. We label the vertices of Q_t as in Fig. 4. It is straightforward to check that $\mathbb{i}_3 = (\diamond, 1, 2, \diamond, \star, 3, 2, 1, \star, \diamond)$ is a maximal green sequence for Q_3 and $\star \rightrightarrows 3$ is the unique multiple arrows of $\mu_{\mathbb{i}_3}(Q_3)$. Furthermore, $\mathbb{i}_4 := (\mathbb{i}_3, 4, 3, \star)$ is a maximal green sequence of Q_4. Now the result follows from Lemma 4.1. □

Theorem 4.3 *Let $\mathbb{X}$ be a weighted projective line.*

(1) *There is a quiver Q' in* $\mathrm{Mut}(Q_{T_{can}(\mathbb{X})})$ *such that Q' admits a maximal green sequence.*
(2) *There is a quiver Q'' in* $\mathrm{Mut}(Q_{T_{can}(\mathbb{X})})$ *such that Q'' does not admit a maximal green sequence if and only if $\mathbb{X}$ is of wild type.*

Proof If $t = 2$, then $Q_{T_{\rm can}(\mathbb{X})}$ is an acyclic quiver. Hence $Q_{T_{\rm can}(\mathbb{X})}$ admits a maximal green sequce (cf. [2] for instance). Now assume that $t \geq 3$. According to Proposition 3.6, we

$$
\mu_{\mathbf{i_{t+1}}}(\widehat{B}^{\circ}) = \begin{array}{ccccccc|c}
\bullet & \circ & 1 & \cdots & t & t+1 & t+2 & \\
\hline
0 & -1 & 0 & \cdots & 0 & 1 & 0 & \bullet \\
1 & 0 & 1 & \cdots & 1 & -2 & 1 & \circ \\
0 & -1 & 0 & \cdots & 0 & 1 & 0 & 1 \\
 & \vdots & & \vdots & & \vdots & & \vdots \\
0 & -1 & 0 & \cdots & 0 & 1 & 0 & t \\
-1 & 2 & -1 & \cdots & -1 & 0 & -1 & t+1 \\
0 & -1 & 0 & \cdots & 0 & 1 & 0 & t+2 \\
0 & -1 & 0 & \cdots & 0 & 0 & 1 & \\
0 & 0 & 0 & \cdots & 0 & -1 & 1 & \\
0 & 0 & -1 & \cdots & 0 & 0 & 0 & \\
 & \vdots & & \vdots & & \vdots & & \\
0 & 0 & 0 & \cdots & -1 & 0 & 0 & \\
-1 & 0 & 0 & \cdots & 0 & 0 & 0 & t+1^{*} \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & t+2^{*}
\end{array}
\xrightarrow{t+2}
\begin{array}{ccccccc|c}
\bullet & \circ & 1 & \cdots & t & t+1 & t+2 & \\
\hline
0 & -1 & 0 & \cdots & 0 & 1 & 0 & \bullet \\
1 & 0 & 1 & \cdots & 1 & -1 & -1 & \circ \\
0 & -1 & 0 & \cdots & 0 & 1 & 0 & 1 \\
 & \vdots & & \vdots & & \vdots & & \vdots \\
0 & -1 & 0 & \cdots & 0 & 1 & 0 & t \\
-1 & 1 & -1 & \cdots & -1 & 0 & 1 & t+1 \\
0 & 1 & 0 & \cdots & 0 & -1 & 0 & t+2 \\
0 & -1 & 0 & \cdots & 0 & 1 & -1 & \\
0 & 0 & 0 & \cdots & 0 & 0 & -1 & \\
0 & 0 & -1 & \cdots & 0 & 0 & 0 & \\
 & \vdots & & \vdots & & \vdots & & \\
0 & 0 & 0 & \cdots & -1 & 0 & 0 & \\
-1 & 0 & 0 & \cdots & 0 & 0 & 0 & t+1^{*} \\
0 & 0 & 0 & \cdots & 0 & 1 & -1 & t+2^{*}
\end{array}
$$

$$
\xrightarrow{t+1}
\begin{array}{ccccccc|c}
\bullet & \circ & 1 & \cdots & t & t+1 & t+2 & \\
\hline
0 & 0 & 0 & \cdots & 0 & -1 & 1 & \bullet \\
0 & 0 & 0 & \cdots & 0 & 1 & -1 & \circ \\
0 & 0 & 0 & \cdots & 0 & -1 & 1 & 1 \\
 & \vdots & & \vdots & & \vdots & & \vdots \\
0 & 0 & 0 & \cdots & 0 & -1 & 1 & t \\
1 & -1 & 1 & \cdots & 1 & 0 & -1 & t+1 \\
-1 & 1 & -1 & \cdots & -1 & 1 & 0 & t+2 \\
0 & 0 & 0 & \cdots & 0 & -1 & 0 & \\
0 & 0 & 0 & \cdots & 0 & 0 & -1 & \\
0 & 0 & -1 & \cdots & 0 & 0 & 0 & \\
 & \vdots & & \vdots & & \vdots & & \\
0 & 0 & 0 & \cdots & -1 & 0 & 0 & \\
-1 & 0 & 0 & \cdots & 0 & 0 & 0 & t+1^{*} \\
0 & 1 & 0 & \cdots & 0 & -1 & 0 & t+2^{*}
\end{array}
\xrightarrow{\circ}
\mu_{\mathbf{i_{t+2}}}(\widehat{B}^{\circ}) = \begin{array}{ccccccc|c}
\bullet & \circ & 1 & \cdots & t & t+1 & t+2 & \\
\hline
0 & 0 & 0 & \cdots & 0 & -1 & 1 & \bullet \\
0 & 0 & 0 & \cdots & 0 & -1 & 1 & \circ \\
0 & 0 & 0 & \cdots & 0 & -1 & 1 & 1 \\
 & \vdots & & \vdots & & \vdots & & \vdots \\
0 & 0 & 0 & \cdots & 0 & -1 & 1 & t \\
1 & 1 & 1 & \cdots & 1 & 0 & -2 & t+1 \\
-1 & -1 & -1 & \cdots & -1 & 2 & 0 & t+2 \\
0 & 0 & 0 & \cdots & 0 & -1 & 0 & \\
0 & 0 & 0 & \cdots & 0 & 0 & -1 & \\
0 & 0 & -1 & \cdots & 0 & 0 & 0 & \\
 & \vdots & & \vdots & & \vdots & & \\
0 & 0 & 0 & \cdots & -1 & 0 & 0 & \\
-1 & 0 & 0 & \cdots & 0 & 0 & 0 & t+1^{*} \\
0 & -1 & 0 & \cdots & 0 & 0 & 0 & t+2^{*}
\end{array}
$$

Fig. 7 The matrix $\mu_{\mathbf{i_{t+2}}}(\hat{B}^{\circ})$

know that $Q_{T_{\mathrm{sq}}(\mathbb{X})}$ belongs to $\mathrm{Mut}(Q_{T_{\mathrm{can}}(\mathbb{X})})$. Consequently, the first statement follows from Proposition 4.2 directly.

The "only if" part of (2) follows from the main result of [23]. Namely, let us assume that $\mathbb{X}$ is not of wild type, then $Q_{T_{\mathrm{sq}}(\mathbb{X})}$ is of finite mutation type by Proposition 3.7. According to the main result of [23], each quiver in $\mathrm{Mut}(Q_{T_{\mathrm{sq}}(\mathbb{X})})$ admits a maximal green sequence. To prove the "if" part, we use Lemma 2.11. Let $\mathbb{X}$ be a weighted projective line of wild type. Similar to the proof of the "only if" of Proposition 3.7, we conclude that there is a quiver Q in $\mathrm{Mut}(Q_{T_{\mathrm{can}}(\mathbb{X})})$ such that $Q_{a,b,c}$ is a full subquiver of Q, where $2 \leq c \leq b \leq a$ and $3 \leq a$. Consequently, Q does not admit a maximal green sequence by Lemma 2.11. □

Acknowledgments We are very grateful to the anonymous referee for valuable comments.

Data Availability All data generated or analysed during this study are included in this article.

Declarations

Conflict of Interest The authors declared that they have no conflicts of interest to this work.

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