Algebras and Representation Theory (2023) 26:553–607
https://doi.org/10.1007/s10468-021-10103-4

Locally Finite Central Simple Algebras

Tamar Bar-On[1] · Shira Gilat[1] · Eliyahu Matzri[1] · Uzi Vishne[1]

Received: 9 February 2020 / Accepted: 10 October 2021 / Published online: 19 November 2021

Abstract

We develop a comprehensive theory of algebras over a field which are locally both finite dimensional and central simple. We generalize fundamental concepts of the theory of finite dimensional central simple algebras, and introduce supernatural matrix algebras, the supernatural degree and matrix degree, and so on. We define a Brauer monoid, whose unique maximal subgroup is the classical Brauer group, and show that once infinite dimensional division algebras exist over the field, they are abundant.

Mathematics Subject Classification (2010) Primary 16D30 · Secondary 16K20, 12E15

1 Introduction

Finite dimensional central simple algebras over a field are a classical topic, which interacts with several current research trends, and is being actively studied by many authors. Numerous textbooks are available ([11, 16, 19, 29, 31]; also see the list of open problems [4]). However, with very few exceptions, the class of algebras which only locally belong to this class has been ignored. In this paper, a property holds locally if every finitely generated subalgebra is contained in a subalgebra with this property.

Let F be a field. An F-algebra is central if its center is equal to F. In this paper we develop the theory of algebras over a field, which locally are central simple of finite

Presented by: Iain Gordon

The authors are partially supported by Israeli Science Foundation grants no. 1623/16 and 630/17.

✉ Uzi Vishne
vishne@math.biu.ac.il

Tamar Bar-On
tamarnachshoni@gmail.com

Shira Gilat
shira.gilat@live.biu.ac.il

Eliyahu Matzri
elimatzri@gmail.com

1 Bar Ilan University, Ramat Gan, Israel

dimension. These are “locally Azumaya F-algebras”: every finite dimensional subspace is contained in an Azumaya F-subalgebra.

Let $\mathcal{C}_F$ denote the class of such algebras. We observe that every algebra $\mathbf{A} \in \mathcal{C}_F$ can be presented as an injective direct limit of a directed system of finite dimensional central simple algebras. We find it useful to keep track of the order type of the directed system, which invites some notions from order theory, most prominently cofinality of subsets.

Let $\mathcal{C}_F^\Gamma$ denote the class of F-algebras which are direct limits of a system of finite dimensional and central simple algebras indexed by the directed set Γ. This class is a monoid with respect to the tensor product, and is closed under countable direct limits if Γ is infinite and every initial segment is finite. Infinite tensor products serve as an interesting example.

We assume familiarity with the basics of the finite dimensional theory, including the degree, splitting fields and the double centralizer, as well as standard properties of the Brauer group, whose elements are finite dimensional central simple algebras modulo matrices, with the tensor product operation.

One of the main themes of the paper is that while the basic invariants of finite dimensional algebras are natural numbers, such as the degree and index, for locally finite algebras the appropriate notions require supernatural numbers (such as $2 \cdot 5^\infty$ or the product of all primes). We develop supernatural matrices, a supernatural degree, a supernatural matrix degree and a supernatural dimension of fields (supernatural matrices over $\mathbb{C}$ appear in the theory of C^*-algebras as “hyperfinite algebras”, see [12]). All these combine well with primary decomposition, which is fundamental in the finite dimensional theory.

Countably generated algebras which are locally finite dimensional and central simple compose the class $\mathcal{C}_F^\omega$. The p-primary component of $\mathcal{C}_F^\omega$ has a particularly nice structure, of a semigroup with a zero element, which is the supernatural matrix algebra $\mathrm{M}_{p^\infty}(F)$. Absorption by this algebra gives the monoid its distinct flavor. Using the primary decomposition (which was proven by Koethe [20] for division algebras in $\mathcal{C}_F^\omega$), we obtain a precise description of algebras in $\mathcal{C}_F^\omega$.

Finite dimensional central simple algebras are Brauer equivalent if they are isomorphic, give or take finite matrices. Similarly, countably generated locally finite central simple algebras are Brauer equivalent if they are isomorphic, give or take locally finite supernatural matrices. This leads to the “countable Brauer monoid” $\mathrm{Br}^\omega(F)$ of the equivalence classes, which decomposes as an infinite product of p-primary monoids, each containing the respective p-part of the Brauer group as its unique maximal subgroup.

The quotient of the countable Brauer monoid by the action of the Brauer group is the semigroup of countably generated locally finite central simple algebras, up to finite factors. This is clearly the right setup in which to study the interactions of countably many finite dimensional central simple algebras. We show that this semigroup is nil of index 2, and has no irreducible elements (according to [17], such semigroups “are rarely seen”). We also show that once this quotient is nonzero, it actually supports an uncountable chain of infinite dimensional division algebras.

The paper has four parts. **Part I.** In Section 2 we present the class $\mathcal{C}_F$ of locally finite central simple algebras, contrasted with some of its close relatives. The local properties invite a presentation of the algebras as direct limits over directed sets. The directed sets filter $\mathcal{C}_F$ as a union of the sub-classes $\mathcal{C}_F^\Gamma$, ranging over all directed sets Γ. In Section 4 we observe that every $\mathcal{C}_F^\Gamma$ is closed under the tensor product. Interactions between the directed sets, such as completeness and cofinality, are discussed with their corresponding monoids $\mathcal{C}_F^\Gamma$ in Section 5.

Since all algebras in $\mathcal{C}_F$ are simple, Artinian algebras of $\mathcal{C}_F$ are isomorphic to matrices over a unique division algebras. In Section 6 we observe that this underlying division algebra is in $\mathcal{C}_F$, and has the same order type as the original algebra. The degree, which is a fundamental invariant in the finite case, is defined with supernatural values in Section 7.

Part II. A significant portion of the paper is devoted to the class $\mathcal{C}_F^\omega$ of locally finite central simple algebras which are countably generated. In Section 9 we discuss their presentation as direct limits. In particular we show that such an algebra is uniquely determined by its finite dimensional central simple subalgebras. Other diverse classes of examples, that of infinite and countable tensor products, are presented in Section 10. Indeed, infinite tensor products are the most accessible example. In Section 17 we construct an infinite crossed product which is also a locally matrix algebra, but is not an infinite tensor product.

The double centralizer theorem provides ample decompositions of of finite dimensional as tensor products. After establishing a somewhat technical description of morphisms between algebras in $\mathcal{C}_F^\omega$ in Section 11, we can show in Section 12 that the double centralizer theorem fails quite often; in fact, every factor subalgebra is also present as a pathological subalgebra. In this context, it is interesting to note that Barsotti [5] constructed a proper endomorphism for every division algebra $D \in \mathcal{C}_F^\omega$, and used this to build an algebra in $\mathcal{C}_F$ containing D, which is not countably generated.

In Section 13 we define Sylow subalgebras, which lead to existence and uniqueness of primary decomposition for the algebras in $\mathcal{C}_F^\omega$.

In Section 14 we study Clifford algebras of infinite dimensional nonsingular quadratic spaces. Such algebras are always in $\mathcal{C}_F$. We compute the centralizer of a Clifford subalgebra, which provides an easy example where the centralizer of a central subalgebra is not necessarily central.

Part III. In order to obtain an underlying division algebra for non-Artinian algebras, we develop supernatural matrices in Section 15. In Section 16 we introduce the supernatural matrix degree. This leads to a classification of countably-generated algebras in $\mathcal{C}_F$, as a unique supernatural matrix algebra over a unique division algebra of minimal degree. It is then easy to show in Section 19 that the only idempotents in $\mathcal{C}_F^\omega$ are supernatural matrix algebras, with certain infinite supernatural matrices as zero elements.

Part IV. In the final part of the paper we study the countable Brauer monoid, defined by declaring algebras in $\mathcal{C}_F^\omega$ to be equivalent if they are isomorphic up to locally finite supernatural matrix algebras. For comparison, Morita equivalence is shown in Section 20 to be isomorphism up to finite matrix algebras. We show in Section 21 that in each primary component, the Brauer group is the subgroup of invertible elements in the countable Brauer monoid. In Section 22 we focus on the infinite part of the countable Brauer monoid, which is obtained by taking the quotient of the countable Brauer monoid with respect to the Brauer group action. Each primary component is a nil semigroup of index 2, with no irreducible elements. In particular, if there are (infinite) countably dimensional division algebra which are locally finite, then there is a continuum of division algebras which are inequivalent under tensoring with finite dimensional algebras (22.6). Section 23 is devoted to countable tensor products over special fields: each primary component is finite over a local or global field, and some interesting observations can be made when F has finite Brauer dimension.

Finally, in Section 24, we study restriction and splitting fields, and show that every infinite dimensional subfield of an algebra of infinite prime-power degree splits the algebra.

We are indebted to an anonymous referee, who carefully read the paper and suggested numerous improvements to its presentation.

Part I. Local Properties and Direct Limits

2 Local Properties and Direct Limits

Our algebras and morphisms are always over a fixed field F, and unital. We begin the discussion with some general comments on algebras defined by local conditions. Let $\mathcal{P}$ be any class of algebras. We say that an algebra A is **locally**-$\mathcal{P}$ if every finitely generated subalgebra of A is contained in some $P \subseteq A$ such that $P \in \mathcal{P}$. We do not assume a-priori that P is finitely generated, although this will be the case most often. For example, one can stipulate that all algebras in $\mathcal{P}$ are finitely generated. A finitely generated locally-$\mathcal{P}$ algebra is of course $\mathcal{P}$.

Recall that a **directed set** is a partially ordered set in which every finite subset is bounded. Our direct limits ("colimits" in the categorical sense) are always taken over a directed set, which may vary. We say that a direct limit $\varinjlim A_\lambda$ is **injective** if all the morphisms $A_\lambda \to A_{\lambda'}$ are injective. More details on direct limits are recalled in Section 4.

Proposition 2.1 *Let $\mathcal{P}$ be any class of algebras. Any injective direct limit of a system of algebras from $\mathcal{P}$, is locally-$\mathcal{P}$.*

Proof Consider an injective direct limit $A = \varinjlim_\Lambda A_\lambda$ where the $A_\lambda \in \mathcal{P}$. Let $S \subseteq A$ be a finite set. Every element of $s \in S$ is contained in some A_{λ_s}, and by definition of the directed set, there is $\bar{\lambda}$ such that $\lambda_s \leq \bar{\lambda}$ for all s; but then $F[S] \subseteq A_{\bar{\lambda}}$, since the maps $A_{\lambda_s} \to A_{\bar{\lambda}}$ are injective. □

Proposition 2.2 *Let $\mathcal{P}$ be a class of finitely generated algebras. An algebra is locally-$\mathcal{P}$ if and only if it is an injective direct limit of a system of algebras from $\mathcal{P}$.*

Proof Assume A is locally-$\mathcal{P}$. Every algebra is an (injective) direct limit of its finitely generated (unital) subalgebras. Since every such subalgebra is contained in an algebra from $\mathcal{P}$, which is finitely generated, the subsystem of the subalgebras $P \subseteq A$ such that $P \in \mathcal{P}$ is cofinal. Therefore, A is a direct limit of its subalgebras from $\mathcal{P}$. The other direction holds by Proposition 2.1. □

Proposition 2.3 *Let $\mathcal{P}$ be a class of algebras. The class of locally-$\mathcal{P}$ algebras is closed under injective direct limits.*

Proof Let $\mathbf{A} = \varinjlim A_\lambda$ be an injective direct limit of locally-$\mathcal{P}$ algebras A_λ. By Proposition 2.1, A is locally-(locally-$\mathcal{P}$). But this just means every finitely generated subalgebra is contained in a locally-$\mathcal{P}$ subalgebra, and is therefore contained in a subalgebra from $\mathcal{P}$; so A is locally-$\mathcal{P}$. □

3 Locally Finite and Central Simple Algebras

An algebra A (over F) is **central** if its center is equal to F. Throughout this paper, we are concerned with the following class of algebras:

Definition 3.1 An algebra over F is **locally central simple and finite dimensional** if every finitely generated subalgebra is contained in a central simple subalgebra which is finite dimensional over F. The class of such algebras is denoted by $\mathcal{C}_F$.

Notice that we require finite dimension over F. This should be compared to another interesting class, of locally simple-and-PI-algebras (not to be confused with simple algebras which are locally PI). Kaplansky proved that simple PI algebras are finite over their center [18], and thus simple Artinian. Therefore, a locally simple-PI algebra is locally simple-Artinian. There is a famous example due to J.C. McConnell (unpublished) of a non-algebraic locally PI division ring, which is thus locally simple-PI, but is not PI. Our assumption is stronger, that the dimension of finitely generated subalgebras is finite over F, so in particular the algebra must be algebraic. Locally PI division rings are called "weakly locally finite" in [9], also see [8]. Division algebras in which every subalgebra generated by two elements is finite dimensional are studied in [10].

Every direct limit of simple algebras is injective, so by Proposition 2.2, we have:

Corollary 3.2 *An algebra is in $\mathcal{C}_F$ if and only if it is a direct limit of finite dimensional.*

To get started, we state the obvious

Remark 3.3 A finite dimensional algebra over F is in $\mathcal{C}_F$ if and only if it is simple and central.

Remark 3.4 A direct limit of division algebras is a division algebra.

We also have by Proposition 2.3 that

Corollary 3.5 *The class $\mathcal{C}_F$ is closed under direct limits.*

Proposition 3.6 *A locally central algebra is central; and a locally simple algebra is simple. In particular, any algebra $\mathbf{A} \in \mathcal{C}_F$ is central and simple.*

Proof Let $\mathbf{A}$ be a locally central algebra. Let $z \in \mathrm{Cent}(\mathbf{A})$. By definition $z \in A_0$ for some central $A_0 \subseteq \mathbf{A}$, so obviously $z \in \mathrm{Cent}(A_0) = F$. Now let $\mathbf{A}$ be a locally simple algebra. Let $0 \neq I \lhd \mathbf{A}$. Let $0 \neq a \in I$. Then $a \in A_0$ for some simple subalgebra A_0, and $a \in I \cap A_0 \lhd A_0$, showing that $1 \in I \cap A_0 \subseteq I$. □

Remark 3.7 Algebras in $\mathcal{C}_F$ are von Neumann regular (since the class of von Neumann regular algebras is closed under direct limits).

4 Direct Limits and Tensor Products

On several occasions below it will become necessary to follow explicit morphisms. Towards this end we recall an elaborate notation for direct limits, as follows. A **directed set** is an ordered set $(\Lambda, \leq)$ such that for every $\lambda, \lambda' \in \Lambda$ there is $\lambda'' \in \Lambda$ such that $\lambda, \lambda' \leq \lambda''$. A **directed system** of algebras and morphisms is a set $\{A_\gamma, \varphi_{\gamma\gamma'}\}$, indexed by a directed set Γ, such that $\varphi_{\gamma\gamma'} : A_\gamma \to A_{\gamma'}$ satisfy the standard compatibility condition. We further assume that the maps are unital, and since the algebras are simple, all morphisms are injective.

The direct limit $\{\mathbf{A}, \varphi_\gamma\} = \varinjlim_\Gamma \{A_\gamma, \varphi_{\gamma\gamma'}\}$ is an algebra $\mathbf{A}$ with morphisms $\varphi_\gamma : A_\gamma \to \mathbf{A}$ which happen to be embeddings. It is customary to omit some or all the maps and write $\mathbf{A} = \varinjlim_\Gamma \{A_\gamma, \varphi_{\gamma\gamma'}\}$ or $\mathbf{A} = \varinjlim_\Gamma A_\gamma$.

4.1 Morphisms

Let $\{\mathbf{A}, \varphi_\gamma\} = \varinjlim_\Gamma \{A_\gamma, \varphi_{\gamma\gamma'}\}$ be a direct limit of algebras. We occasionally apply the categorical definition of the direct limit, as follows.

Remark 4.1 (1) Let R be any algebra, and Γ a directed set. A system of morphisms $\{f_\gamma : A_\gamma \to R\}$ is *compatible* if it satisfies the condition $f_{\gamma'} \circ \varphi_{\gamma\gamma'} = f_\gamma$ for every $\gamma < \gamma'$. A compatible system defines a morphism $f : \varinjlim A_\gamma \to R$ such that $f \circ \varphi_\gamma = f_\gamma$; since every element of $\mathbf{A}$ is of the form $\varphi_\gamma(x)$ for some $\gamma \in \Gamma$ and $x \in A_\gamma$, this condition in fact defines f. Moreover, if all the f_γ are injective, then so is f.

(2) More generally, let $\alpha : \Gamma \to \Delta$ be a map of directed sets, and let $\{\mathbf{A}, \varphi_\gamma\} = \varinjlim_\Gamma \{A_\gamma, \varphi_{\gamma\gamma'}\}$ and $\{\mathbf{B}, \psi_\delta\} = \varinjlim_\Delta \{B_\delta, \psi_{\delta\delta'}\}$ be direct limits over Γ and Δ, respectively. A system of morphisms $\{f_\gamma : A_\gamma \to B_{\alpha(\gamma)}\}$ is *compatible* if it satisfies the condition $f_{\gamma'} \circ \varphi_{\gamma\gamma'} = \psi_{\alpha(\gamma)\alpha(\gamma')} \circ f_\gamma$ for every $\gamma < \gamma'$ in Γ. A compatible system defines a morphism $f : \varinjlim_\Gamma A_\gamma \longrightarrow \varinjlim_\Delta B_\delta$ such that $f \circ \varphi_\gamma = \psi_{\alpha(\gamma)} \circ f_\gamma$. Again, if all the f_γ are injective, then so is f.

(3) When $\Delta = \Gamma$ and α is the identity in (2), we denote $f = \varinjlim f_\gamma$.

4.2 Tensor Products

The tensor product is of fundamental importance in the theory of finite dimensional central simple algebras. We show in this subsection that $\mathcal{C}_F$ is closed under tensor products.

Remark 4.2 [3, Exer. 2.20] The tensor product commutes with direct limits.

More precisely, given a direct limit $\{\mathbf{A}, \varphi_\gamma\} = \varinjlim A_\gamma$ and an algebra $\mathbf{B}$, the morphisms $1 \otimes \varphi_\gamma : \mathbf{B} \otimes A_\gamma \longrightarrow \mathbf{B} \otimes \varinjlim_\Gamma A_\gamma$ are compatible, and the map $\varinjlim (1 \otimes \varphi_\gamma) : \varinjlim_\Gamma (\mathbf{B} \otimes A_\gamma) \longrightarrow \mathbf{B} \otimes \varinjlim_\Gamma A_\gamma$ defined as in Remark 4.1. (1), is an isomorphism.

Let Γ, Δ be directed sets. The direct product $\Gamma \times \Delta$ is ordered by setting $(\gamma, \delta) \leq (\gamma', \delta')$ if both $\gamma \leq \gamma'$ and $\delta \leq \delta'$, and becomes a directed set. Let $\{C_{\gamma\delta}, \varphi_{\gamma\delta,\gamma'\delta'}\}$ be a system of algebras over $\Gamma \times \Delta$. If $\gamma < \gamma'$, there is a morphism $\varphi_{\gamma\gamma'} = \varinjlim_\Delta \varphi_{\gamma\delta,\gamma'\delta} : \varinjlim_\Delta C_{\gamma\delta} \to \varinjlim_\Delta C_{\gamma'\delta}$ defined by Remark 4.1. (3). The system $\{\varinjlim_\Delta C_{\gamma\delta}, \varphi_{\gamma\gamma'}\}$, indexed by Γ, is compatible, and has its own direct limit $\varinjlim_\Gamma \varinjlim_\Delta C_{\gamma\delta}$.

Proposition 4.3 (e.g. [30, Cor 13.3]) $\varinjlim_{\Gamma\times\Delta} C_{\gamma\delta} \cong \varinjlim_\Gamma \varinjlim_\Delta C_{\gamma\delta}$.

Corollary 4.4 $\varinjlim_\Gamma \varinjlim_\Delta C_{\gamma\delta} = \varinjlim_\Delta \varinjlim_\Gamma C_{\gamma\delta}$.

The double limit applies to tensor product, as follows. Let Γ and Δ be directed sets, and let $\mathbf{A} = \varinjlim_\Gamma \{A_\gamma, \varphi_{\gamma\gamma'}\}$ and $\mathbf{B} = \varinjlim_\Delta \{B_\delta, \varphi_{\delta\delta'}\}$ be direct limits over them. By $\varinjlim_{\Gamma\times\Delta}(A_\gamma \otimes B_\delta)$ we mean the direct limit of the system $\varinjlim_{\Gamma\times\Delta} \{A_\gamma \otimes B_\delta, \varphi_{\gamma\gamma'} \otimes \varphi_{\delta\delta'}\}$.

Proposition 4.5 *The class $\mathcal{C}_F$ is closed under tensor products.*

Proof Write $\mathbf{A} = \varinjlim_\Gamma A_\gamma$ and $\mathbf{B} = \varinjlim_\Delta B_\delta$ where A_γ, B_δ are finite dimensional and central simple. By Remarks 4.3 and 4.2,

$$\begin{aligned}\mathbf{A} \otimes_F \mathbf{B} &= (\varinjlim_{\Gamma} A_\gamma) \otimes \mathbf{B} \\ &\cong \varinjlim_{\Gamma}(A_\gamma \otimes \mathbf{B}) \\ &= \varinjlim_{\Gamma}(A_\gamma \otimes \varinjlim_{\Delta} B_\delta) \\ &\cong \varinjlim_{\Gamma}(\varinjlim_{\Delta}(A_\gamma \otimes B_\delta)) \\ &\cong \varinjlim_{\Gamma \times \Delta} (A_\gamma \otimes B_\delta),\end{aligned}$$

which is a direct limit of finite dimensional central simple algebras. □

Furthermore, F is the identity element with respect to the tensor product operation. Notice that $\mathcal{C}_F$ is not a monoid, being a proper class in the set-theoretic sense.

4.3 Cofinality

Given a system $\{A_\gamma, \varphi_{\gamma\gamma'}\}$ of algebras and morphisms over Γ, its restriction to a subset $\Gamma_0 \subseteq \Gamma$ is the obvious subsystem with $\gamma, \gamma' \in \Gamma_0$.

Lemma 4.6 *Let Γ be a directed set and $\Gamma_0 \subseteq \Gamma$ a directed subset. For any injective directed system $\{A_\gamma, \varphi_{\gamma\gamma'}\}$ over Γ, we have that $\varinjlim_{\Gamma_0} A_\gamma \subseteq \varinjlim_\Gamma A_\gamma$.*

Proof Let $\left\{\mathbf{A}', \varphi'_\gamma\right\} = \varinjlim_{\Gamma_0} A_\gamma$ and $\{\mathbf{A}, \varphi_\gamma\} = \varinjlim_\Gamma A_\gamma$. The system of maps $\varphi_\gamma : A_\gamma \to \mathbf{A}$ is compatible, and its limit over Γ_0 induces a map $f : \mathbf{A}' \to \mathbf{A}$, which is injective. □

Recall that a subset $\Gamma_0 \subseteq \Gamma$ is **cofinal** if for every $\gamma \in \Gamma$ there is $\gamma' \in \Gamma_0$ such that $\gamma < \gamma'$. (A cofinal subset of a directed set is itself directed).

Lemma 4.7 *Let Γ be a directed set and $\Gamma_0 \subseteq \Gamma$ a cofinal subset. Let $\{A_\gamma, \varphi_{\gamma\gamma'}\}$ be an injective directed system of algebras over Γ. Then $\varinjlim_{\Gamma_0} A_\gamma = \varinjlim_\Gamma A_\gamma$.*

Proof Continuing the proof of Lemma 4.6, the map f is onto because for every γ there is some $\gamma_0 \in \Gamma_0$ such that $\varphi_\gamma(A_\gamma) \subseteq \varphi_{\gamma_0}(A_{\gamma_0}) = (f \circ \varphi'_{\gamma_0})(A_{\gamma_0}) \subseteq f(\mathbf{A}')$, and so $\mathbf{A} \subseteq \mathbf{A}'$. □

5 The Classes $\mathcal{C}_F^\Gamma$

For a directed set Γ, let $\mathcal{C}_F^\Gamma$ denote the class of direct limits of the form $\mathbf{A} = \varinjlim_\Gamma A_\gamma$, over the given set Γ, where the A_γ are finite dimensional and central simple algebras, as always. For example, letting $1 = \{0\}$ denote the singleton, we have that $\mathcal{C}_F^1$ is the class of finite dimensional central simple algebras. If $\Gamma \cong \Gamma'$ as partially ordered sets, then clearly $\mathcal{C}_F^\Gamma = \mathcal{C}_F^{\Gamma'}$.

5.1 Tensor Product in $\mathcal{C}_F^\Gamma$

We can now refine Proposition 4.5.

Proposition 5.1 *For every directed system* Γ*,*

$$\varinjlim_{\Gamma} A_\gamma \otimes \varinjlim_{\Gamma} B_\gamma = \varinjlim_{\Gamma} (A_\gamma \otimes B_\gamma). \tag{1}$$

In particular, $\mathcal{C}_F^\Gamma$ *is closed under the tensor product.*

Proof As we have seen in Proposition 4.5,

$$\varinjlim_{\Gamma} A_\gamma \otimes \varinjlim_{\Gamma} B_\gamma = \varinjlim_{\Gamma\times\Gamma} (A_\gamma \otimes B_{\gamma'}),$$

but the diagonal subsystem $\{(\gamma,\gamma) : \gamma \in \Gamma\}$ is cofinal in $\Gamma \times \Gamma$, so by Lemma 4.7, $\varinjlim_{\Gamma\times\Gamma}(A_\gamma \otimes B_{\gamma'}) = \varinjlim_{\Gamma}(A_\gamma \otimes B_\gamma)$. □

Corollary 5.2 $\mathcal{C}_F^\Gamma$ *are monoids contained in* $\mathcal{C}_F$*.*

5.2 Changing the Directed Set

Let Γ be a directed set. Recall the right order topology on Γ, whose basis is composed of the sets we denote as $[\gamma, \Gamma) = \left\{\gamma' \in \Gamma : \gamma \leq \gamma'\right\}$. A subset $\Delta \subseteq \Gamma$ is open if whenever $\delta \leq \gamma \in \Gamma$ and $\delta \in \Delta$, we have that $\gamma \in \Delta$. Similarly, a subset $\Delta \subseteq \Gamma$ is cofinal if and only if it is dense in this topology.

We say that a subdirected set Δ is **complete** in Γ if for every $\gamma \in \Gamma$, the set $(-\Delta, \gamma] = \{\delta \in \Delta : \delta \leq \gamma\}$ is either empty or has a maximum.

Example 5.3 (1) Every open subset $\Delta \subseteq \Gamma$ is complete, because if $(-\Delta, \gamma]$ is not empty then necessarily $\gamma \in \Delta$ is the maximum.
(2) Let Γ be a directed set, and let $\gamma \in \Gamma$. Then the open subset $[\gamma, \Gamma)$ is cofinal in Γ.
(3) For directed sets Γ, Δ and fixed $\delta_0 \in \Delta$, $\Gamma \times \{\delta_0\}$ is complete in $\Gamma \times \Delta$. Indeed, we require that every non-empty $(-(\Gamma \times \{\delta_0\}), (\gamma, \delta)]$ will have a maximum. If $\delta_0 \not\leq \delta$ this set is empty, and if $\delta_0 \leq \delta$ then $(-(\Gamma \times \{\delta_0\}), (\gamma, \delta)] = (-\Gamma, \gamma] \times \{\delta_0\}$ has a maximum (γ, δ_0).

We need the following generalization of Proposition 4.3.

Proposition 5.4 *Let* Γ *be a directed set. Let* Δ *be a directed set, with an order-preserving map* $\Gamma \to P(\Delta)$ *taking each* $\gamma \in \Gamma$ *to a directed subset* $\Delta_\gamma \subseteq \Delta$*, such that* $\Delta = \bigcup_{\gamma\in\Gamma} \Delta_\gamma$*. Let* $\{A_\delta, \varphi_{\delta\delta'}\}$ *be a system of algebras and morphisms indexed by* Δ*. For* $\gamma < \gamma'$ *there is a natural morphism* $\varinjlim_{\Delta_\gamma} A_\delta \to \varinjlim_{\Delta_{\gamma'}} A_{\delta'}$*. Taking the limit over* Γ*, we get an isomorphism* $\varinjlim_{\gamma\in\Gamma}(\varinjlim_{\Delta_\gamma} A_\delta) \cong \varinjlim_{\delta\in\Delta} A_\delta$*.*

Proposition 5.5 *Let* Δ *and* Γ *be directed sets.*

(1) *If* $\Delta \subseteq \Gamma$ *is complete in* Γ*, then* $\mathcal{C}_F^\Delta \subseteq \mathcal{C}_F^\Gamma$*.*
(2) *If* $\Delta \subseteq \Gamma$ *is cofinal in* Γ*, then* $\mathcal{C}_F^\Delta \supseteq \mathcal{C}_F^\Gamma$*.*

(3) *If* $\Delta \subseteq \Gamma$ *is cofinal and complete in* Γ, *then* $\mathcal{C}_F^{\Delta} = \mathcal{C}_F^{\Gamma}$.
(4) *If there is an order-preserving surjection* $\Delta \to \Gamma$, *then* $\mathcal{C}_F^{\Delta} \supseteq \mathcal{C}_F^{\Gamma}$.

Proof (1) If $\Delta \subseteq \Gamma$ is open, the proof is straightforward: given a direct limit $\varinjlim_{\Delta} A_\delta$, we can set $A_\gamma = F$ for every $\gamma \in \Gamma$ which is not in Δ, with the obvious morphisms $A_\gamma \to A_{\gamma'}$ for every γ, γ' which are not both in Δ. Then $\varinjlim_{\Gamma} A_\gamma = \varinjlim_{\Delta} A_\gamma$.

In the general case we apply Proposition 5.4 with the same Γ and Δ, and with $\Delta_\gamma = (-\Delta, \gamma]$, which are directed by assumption. Clearly $\bigcup_{\gamma \in \Gamma} \Delta_\gamma \supseteq \bigcup_{\delta \in \Delta} \Delta_\delta = \Delta$. Given a direct limit $\varinjlim_{\Delta} A_\delta$ with finite dimensional central simple algebras A_δ, for every $\gamma \in \Gamma$, either Δ_γ is empty, in which case $\varinjlim_{\delta \leq \gamma} A_\delta = \varinjlim_{\Delta_\gamma} A_\delta$ is equal to F; or Δ_γ has a maximum δ_0, in which case the limit is $\varinjlim_{\delta \leq \gamma} A_\delta = \varinjlim_{\Delta_\gamma} A_\delta = A_{\delta_0}$, which is finite dimensional. Therefore, $\varinjlim_{\delta \in \Delta} A_\delta = \varinjlim_{\gamma \in \Gamma} (\varinjlim_{\delta \leq \gamma} A_\delta)$ is a limit over Γ of finite dimensional central simple algebras.
(2) Follows from Lemma 4.7.
(3) Mutual inclusion.
(4) Let $\pi : \Delta \to \Gamma$ be the given projection. Given a system of finite dimensional central simple algebras $\{A_\gamma, \varphi_{\gamma\gamma'}\}$ indexed by Γ such that $\mathbf{A} = \varinjlim A_\gamma$, the system $\{A_{\pi(\delta)}, \varphi_{\pi(\delta)\pi(\delta')}\}$, indexed by Δ, has the same limit.

□

Corollary 5.6 *For every two directed sets* Γ, Δ *we have that* $\mathcal{C}_F^{\Gamma}, \mathcal{C}_F^{\Delta} \subseteq \mathcal{C}_F^{\Gamma \times \Delta}$.

Proof Standard copies of Γ, Δ are complete in $\Gamma \times \Delta$ by Example 5.3. (3). □

Recall that $\mathcal{C}_F^1$ is the monoid of finite dimensional central simple algebras.

Corollary 5.7 *If a directed set* Γ *is bounded then it has a maximum and* $\mathcal{C}_F^{\Gamma} = \mathcal{C}_F^1$.

Proof The embedding of the singleton to the maximum point is both cofinal and complete.
□

5.3 Being Properly in $\mathcal{C}_F^{\Gamma}$

We say that an algebra $\mathbf{A}$ is **properly** in $\mathcal{C}_F^{\Gamma}$ if it can be presented as a direct limit $\mathbf{A} = \varinjlim_{\gamma \in \Gamma} A_\gamma$ of finite dimensional central simple algebras, such that $A_\gamma \subset A_{\gamma'}$ (proper inclusions) whenever $\gamma < \gamma'$.

For any algebra $\mathbf{A} \in \mathcal{C}_F$ we define $\Sigma_{\mathbf{A}}$ to be the set of finite dimensional central simple subalgebras of $\mathbf{A}$, ordered by inclusion, so this is a directed set precisely because $\mathbf{A}$ is locally both finite dimensional and central simple. Clearly $\mathbf{A} = \varinjlim_{A_0 \in \Sigma_{\mathbf{A}}} A_0$, and $\Sigma_{\mathbf{A}}$ is the largest directed set Γ for which $\mathbf{A}$ is properly in $\mathcal{C}_F^{\Gamma}$.

Proposition 5.8 *Let* Γ *be a directed set and* $\mathbf{A} \in \mathcal{C}_F^{\Gamma}$. *Then there is a cofinal subset* $\Gamma^0 \subseteq \Sigma_{\mathbf{A}}$, *with an order-preserving surjection* $\Gamma \to \Gamma^0$, *such that* $\mathbf{A}$ *is properly in* $\mathcal{C}_F^{\Gamma^0}$.

Proof A presentation $\{\mathbf{A}, \varphi_\gamma\} = \varinjlim_{\Gamma} A_\gamma$ over Γ, where A_γ are finite dimensional central simple algebras, induces an order-preserving map $\pi : \Gamma \to \Sigma_{\mathbf{A}}$ defined by $\gamma \mapsto \varphi_\gamma(A_\gamma)$.

Let Γ^0 be the image of this map. Since $\mathbf{A}$ is the limit over Γ of the A_γ, every $A_0 \in \Sigma_{\mathbf{A}}$ is contained in some $\varphi_\gamma(A_\gamma)$ ($\gamma \in \Gamma$), so $A_0 \leq A_\gamma$ in $\Sigma_{\mathbf{A}}$ and Γ^0 is cofinal. □

More precisely, the proof yields:

Proposition 5.9 *An algebra $\mathbf{A} \in \mathcal{C}_F$ is properly in $\mathcal{C}_F^\Gamma$ if and only if Γ is isomorphic to a cofinal subset of $\Sigma_{\mathbf{A}}$.*

5.4 The Height of Γ

We propose a trichotomy of directed sets according to height, as follows. The **height** of a directed set Γ, denoted $\mathrm{ht}(\Gamma)$, is finite if there is no order-preserving injection $\omega \to \Gamma$; the height is ω if there is an injection $\omega \to \Gamma$ but there is no injection $\omega + 1 \to \Gamma$; and the height is $> \omega$ if there is an injection $\omega + 1 \to \Gamma$.

Proposition 5.10 *Let Γ be a directed set. If there is any algebra $\mathbf{A}$ which is properly in $\mathcal{C}_F^\Gamma$, then $\mathrm{ht}(\Gamma) \leq \omega$.*

Proof Let $\mathbf{A} = \varinjlim_\Gamma A_\gamma$ be a proper presentation. Then the function $\Gamma \to \mathbb{N}$ defined by $\gamma \mapsto \dim A_\gamma$ is strictly increasing, so $\omega + 1$ cannot embed in Γ because it does not embed in $\mathbb{N}$. □

Remark 5.11 If a directed set Γ has finite height, then it is bounded and therefore has a maximal element; thus $\mathcal{C}_F^\Gamma = \mathcal{C}_F^1$.

The bounded case is uninteresting by Corollary 5.7. Everything interesting happens in the Goldilocks zone of $\mathrm{ht}(\Gamma) = \omega$.

Remark 5.12 If $\mathrm{ht}(\Gamma) = \omega$ then $\mathcal{C}_F^\omega \subseteq \mathcal{C}_F^\Gamma$. Indeed, in this case there are embeddings $\omega \hookrightarrow \Gamma$, and the image of every such map is complete because of the height.

6 Artinian Algebras in $\mathcal{C}_F$

After defining factors, to become more useful later on, we derive an analog of Artin-Wedderburn structure theorem for the classes $\mathcal{C}_F^\Gamma$.

Definition 6.1 Let $\mathbf{A} \in \mathcal{C}_F$. We call a subalgebra $\mathbf{B} \in \mathcal{C}_F$ a **factor** of $\mathbf{A}$ if there is a subalgebra $\mathbf{B}' \in \mathcal{C}_F$ of $\mathbf{A}$, such that the natural homomorphism $\mathbf{B} \otimes \mathbf{B}' \xrightarrow{\sim} \mathbf{A}$, defined by $b \otimes b' \mapsto bb'$, is an isomorphism.

Necessarily in this case, $\mathbf{B}'$ is contained in the centralizer $\mathrm{C}_{\mathbf{A}}(\mathbf{B})$. Since $\mathbf{A} \cong \mathbf{B} \otimes \mathbf{B}'$, and since $\mathbf{B}$ is central, it follows immediately that $\mathbf{B}' = \mathrm{C}_{\mathbf{A}}(\mathbf{B})$, and $\mathbf{B}'$ itself is a factor of $\mathbf{A}$. In particular $\mathrm{C}_{\mathbf{A}}(\mathrm{C}_{\mathbf{A}}(\mathbf{B})) = \mathbf{B}$.

By the double centralizer theorem, if $\mathbf{A}$ is finite dimensional, every central simple subalgebra is a factor. However, as we will see in Section 12 below, this is far from being the case in general.

Remark 6.2 If $\mathbf{B}, \mathbf{B}' \in \mathcal{C}_F$ then both are factors of $\mathbf{B} \otimes \mathbf{B}'$.

Finite dimensional central simple algebras B_0 are characterized by the fact that whenever $B_0 \hookrightarrow R$, and R is any algebra, then $R \cong B_0 \otimes \mathrm{C}_R(B_0)$ [15, Theorem V.11.2]. We refine this characterization in Proposition 6.3. Pathological behavior of infinite central simple algebras in this context is exhibited in Section 12.

Proposition 6.3 *Every finite dimensional central simple subalgebra of an algebra* $\mathbf{A} \in \mathcal{C}_F$ *is a factor.*

More precisely, if $\mathbf{A} \in \mathcal{C}_F^\Gamma$ *and* $B_0 \subseteq \mathbf{A}$ *is a finite dimensional central simple subalgebra, then* $\mathrm{C}_{\mathbf{A}}(B_0) \in \mathcal{C}_F^\Gamma$ *and* $\mathbf{A} \cong B_0 \otimes \mathrm{C}_{\mathbf{A}}(B_0)$.

Proof Write $\mathbf{A} = \varinjlim_\Gamma A_\gamma$ where A_γ are finite dimensional central simple subalgebras. There is some $\gamma_0 \in \Gamma$ such that $B_0 \subseteq A_{\gamma_0}$. By Lemma 4.7 applied to Example 5.3, we may assume γ_0 is the minimal element of Γ (preserving the assumption that $\mathbf{A} \in \mathcal{C}_F^\Gamma$ by Proposition 5.5(3), because $[\gamma_0, \Gamma)$ is confinal and complete). In particular, for every $\gamma \in \Gamma$, $B_0 \subseteq A_\gamma$. Therefore, for every $\gamma \in \Gamma$, $A_\gamma = B_0 \otimes \mathrm{C}_{A_\gamma}(B_0)$, and if $\gamma < \gamma'$, the morphism $A_\gamma \to A_{\gamma'}$ restricts to a morphism $\mathrm{C}_{A_\gamma}(B_0) \to \mathrm{C}_{A_{\gamma'}}(B_0)$ in a compatible manner. By Remark 4.2,

$$\mathbf{A} = \varinjlim_{\Gamma} A_\gamma \cong \varinjlim_{\Gamma}(B_0 \otimes \mathrm{C}_{A_\gamma}(B_0)) \cong B_0 \otimes \varinjlim_{\Gamma} \mathrm{C}_{A_\gamma}(B_0).$$

It remains to show that $\mathrm{C}_{\mathbf{A}}(B_0) = \varinjlim_\Gamma \mathrm{C}_{A_\gamma}(B_0)$, which is obvious. In particular $\mathrm{C}_{\mathbf{A}}(B_0) \in \mathcal{C}_F^\Gamma$. □

Proposition 6.4 (Skolem-Noether for $\mathcal{C}$) *Let* $\mathbf{A} \in \mathcal{C}_F$. *Every isomorphism between finite dimensional central simple subalgebras of* $\mathbf{A}$ *is induced by conjugation.*

Proof Let $B_1, B_2 \subseteq \mathbf{A}$ be isomorphic subalgebras. There is a finite dimensional central simple algebra $B \subseteq \mathbf{A}$ containing both B_1 and B_2. By Skolem-Noether for B, there is $b \in B^\times$ inducing the given isomorphism, and clearly $b \in \mathbf{A}^\times$. □

We thus have matrix cancellation in $\mathcal{C}_F$ (see [22]):

Corollary 6.5 *Let* B, B' *be arbitrary algebras. If* $\mathrm{M}_n(B) \cong \mathrm{M}_n(B') \in \mathcal{C}_F^\Gamma$, *then* $B \cong B'$ *and this algebra is in* $\mathcal{C}_F^\Gamma$.

Proof Apply Propositions 6.4 and 6.3. □

Theorem 6.6 *Every Artinian algebra in* $\mathcal{C}_F^\Gamma$ *is a (finite) matrix algebra over a division algebra from* $\mathcal{C}_F^\Gamma$.

Proof Since $\mathbf{A}$ is simple Artinian, we can write $\mathbf{A} = \mathrm{M}_n(\mathbf{D})$ for a unique division ring $\mathbf{D}$ and unique n. By Corollary 6.5, $\mathbf{D} \in \mathcal{C}_F^\Gamma$. □

7 Supernatural Numbers and the Degree

Recall that the degree of a central simple algebra is the square root of the dimension. This is a fundamental invariant, but natural numbers cannot serve as degrees when the dimension is infinite. Moreover, the lattice of natural numbers (without zero), with respect to the divisibility relation, is not complete. Both issues are resolved by moving to supernatural numbers.

Definition 7.1 A supernatural number is a formal product $\mathbf{n} = \Pi p^{\alpha_p}$, where the product ranges over all the prime numbers p, and each α_p is a finite natural number or infinity. (If all α_p are finite and almost all $\alpha_p = 0$ we recover the natural numbers.)

Supernatural numbers can be multiplied, by setting $p^\infty p^\alpha = p^\infty$ for every α. Multiplication then defines the divisibility relation, which is a weak order relation (namely transitive, reflexive and antisymmetric). As with natural numbers, the divisibility relation defines the greatest common divisor (gcd) and the least common multiple (lcm). Unlike the lattice of natural numbers, the lattice of supernatural numbers is complete: every set of supernatural numbers has gcd and lcm.

Definition 7.2 Let $\mathbf{A}$ be an algebra in $\mathcal{C}_F$. The **degree** of $\mathbf{A}$ is defined as the lcm of the degrees of its finite dimensional central simple subalgebras $A_0 \subseteq \mathbf{A}$.

In particular, the degree of a finite dimensional central simple algebra,, as defined here, coincides with the standard degree of the algebra.

Lemma 7.3 *For algebras $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{C}_F$, if there is an embedding $\boldsymbol{A} \hookrightarrow \boldsymbol{B}$, then* $\deg \boldsymbol{A}$ *divides* $\deg \boldsymbol{B}$.

Proof Every finite dimensional central simple subalgebra of $\mathbf{A}$ is also a subalgebra of $\mathbf{B}$. □

Proposition 7.4 *Let $\boldsymbol{A} = \varinjlim \boldsymbol{A}_\gamma$ where $\boldsymbol{A}_\gamma \in \mathcal{C}_F$. Then* $\deg(\boldsymbol{A}) = \mathrm{lcm}\{\deg(\boldsymbol{A}_\gamma)\}$.

Proof On one hand every $\mathbf{A}_\gamma$ is a subalgebra of $\mathbf{A}$, so $\mathrm{lcm}\{\deg(\mathbf{A}_\gamma)\}$ divides $\deg(\mathbf{A})$. On the other hand, every finite dimensional (central simple) subalgebra of $\mathbf{A}$ is a subalgebra of some $\mathbf{A}_\gamma$, so its degree divides $\deg(\mathbf{A}_\gamma)$. □

In particular,

Corollary 7.5 *Let $A = \varinjlim A_\gamma$ where A_γ are finite dimensional central simple algebras. Then* $\deg(A) = \mathrm{lcm}\{\sqrt{[A_\gamma : F]}\}$.

If we express $\mathbf{A} \in \mathcal{C}_F$ as a direct limit of its finite dimensional subalgebras, Corollary 7.5 repeats the definition of the degree; at the same time this proposition gives an explicit formula for the degree, applicable for any presentation of $\mathbf{A}$ as a direct limit.

Proposition 7.6 *For every $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{C}_F$,* $\deg(\boldsymbol{A} \otimes \boldsymbol{B}) = \deg(\boldsymbol{A}) \deg(\boldsymbol{B})$

Proof Writing $\mathbf{A} = \varinjlim_{\Gamma} A_\gamma$ and $\mathbf{B} = \varinjlim_{\Delta} B_\delta$ for finite dimensional central simple algebras A_γ, B_δ, and following the proof of Proposition 4.5, we have that

$$\begin{aligned}
\deg(\mathbf{A} \otimes \mathbf{B}) &= \mathrm{lcm}_{\gamma,\delta}\big\{\deg(A_\gamma \otimes B_\delta)\big\} \\
&= \mathrm{lcm}_{\gamma,\delta}\big\{\deg(A_\gamma)\deg(B_\delta)\big\} \\
&= \mathrm{lcm}_{\gamma}\big\{\mathrm{lcm}_{\delta}\big\{\deg(A_\gamma)\deg(B_\delta)\big\}\big\} \\
&= \mathrm{lcm}_{\gamma}\big\{\deg(A_\gamma) \cdot \mathrm{lcm}_{\delta}\{\deg(B_\delta)\}\big\} \\
&= \mathrm{lcm}_{\gamma}\big\{\deg(A_\gamma)\big\} \cdot \mathrm{lcm}_{\delta}\{\deg(B_\delta)\} \\
&= \deg(\mathbf{A}) \cdot \deg(\mathbf{B}).
\end{aligned}$$
□

Remark 7.7 If $\mathbf{A} = \varinjlim A_\lambda$ is of infinite dimension, then the dimensions $\dim A_\lambda$ are unbounded.

Otherwise let A_{λ_0} be of maximal dimension among the components; for every λ there is an algebra $A_{\lambda'}$ containing copies of both A_λ and A_{λ_0}, but $A_{\lambda'} = A_{\lambda_0}$ by maximality, so $A_\lambda \subseteq A_{\lambda_0}$ and $\mathbf{A} \subseteq A_{\lambda_0}$, contrary to assumption.

Corollary 7.8 *If* $\deg(\boldsymbol{A})$ *is finite then* $\dim(\boldsymbol{A})$ *is finite as well.*

Part II. Countably Generated Locally Central Simple Algebras

8 Countably Generated Algerbas

Recall that the first infinite ordinal, ω, is a directed set. In the next several sections we develop the theory of $\mathcal{C}_F^\omega$, namely algebras which are the direct limits over ω of finite dimensional central simple algebras. We have shown in Section 5.3 that $\mathcal{C}_F^\omega$ is the minimal class which properly hosts infinite dimensional algebras from $\mathcal{C}_F$.

Proposition 8.1 *An algebra* $\boldsymbol{A} \in \mathcal{C}_F$ *is in* $\mathcal{C}_F^\omega$ *if and only if it is countably generated.*

Proof Let $\mathbf{A}$ be countably generated algebra which locally is finite dimensional and central simple over F. Write $\mathbf{A} = F[x_1, x_2, \dots]$. By assumption, for every n there is a finite dimensional central simple subalgebra $A_n \subseteq \mathbf{A}$ such that $A_{n-1} \subseteq A_n$ and $x_n \in A_n$. The embeddings $A_{n-1} \hookrightarrow A_n$ define a directed system whose limit is $\mathbf{A}$. On the other hand, if $\mathbf{A} = \varinjlim_{\omega} A_n$, then $\mathbf{A}$ is countably generated since each A_n is finitely generated. □

As a refinement of Corollary 3.5, we have:

Proposition 8.2 *The class* $\mathcal{C}_F^\omega$ *is closed under countable direct limits (namely direct limits over a countable directed set).*

Proof A countable direct limit of countably generated algebras is countably generated. □

An algebra over F is countably generated if and only if it has countable dimension. This property passes down to subalgebras, so we have:

Corollary 8.3 *Suppose* $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{C}_F$ *where* $\boldsymbol{A} \hookrightarrow \boldsymbol{B}$ *and* $\boldsymbol{B} \in \mathcal{C}_F^\omega$. *Then* $\boldsymbol{A} \in \mathcal{C}_F^\omega$ *as well.*

9 Filtrations of Countably Generated Algebras

9.1 Filtered Algebras

The main technical feature of direct limits over ω is that all arrows into A_n factor through a single object, namely A_{n-1}. This basic observation allows us to give concrete presentations of countably generated algebras and their subalgebras. Let $\mathbf{A} \in \mathcal{C}_F^{\omega}$. A sequence of finite dimensional central simple algebras $A_1 \to A_2 \to \cdots$ such that $\mathbf{A} = \varinjlim A_n$ is called a **filtration** of $\mathbf{A}$. By definition, every algebra if $\mathcal{C}_F^{\omega}$ can be filtered.

Let $\{A_n\}$ and $\{B_n\}$ be filtrations of $\mathbf{A}$ and $\mathbf{B}$, respectively. A homomorphism of algebras $\mu : \mathbf{A} \to \mathbf{B}$ (necessarily an embedding) is a **homomorphism of filtered algebras** if $\mu(A_n) \subseteq B_n$.

Proposition 9.1 *Let $\mu : \boldsymbol{B} \to \boldsymbol{A}$ be an embedding of algebras in $\mathcal{C}_F^{\omega}$. For any filtration of $\boldsymbol{B}$ there is a filtration of $\boldsymbol{A}$ with respect to which μ is an embedding of filtered algebras.*

Proof Let $\{B_n\}$ be a filtration of $\mathbf{B}$. Let $\{A_n\}$ be arbitrary filtration of $\mathbf{A}$. For every n, choose $m_n > m_{n-1}$ such that $\mu(B_n) \subseteq A_{m_n}$. Replace the chain A_n by the cofinal chain $A_{m_1} \hookrightarrow A_{m_2} \hookrightarrow \cdots$. □

Let $\alpha = \{A_1, A_2, \dots\}$ be a filtration of $\mathbf{A}$. Then $\alpha^+ = \{A_2, A_3, \dots\}$ is also a filtration of $\mathbf{A}$.

Proposition 9.2 *Let $\boldsymbol{A} \in \mathcal{C}_F^{\omega}$. For any endomorphism $\mu : \boldsymbol{A} \to \boldsymbol{A}$ there is filtration α of $\boldsymbol{A}$ such that $\mu : (\boldsymbol{A}, \alpha) \to (\boldsymbol{A}, \alpha^+)$ is an embedding of filtered algebras.*

Proof Let $\{A'_n\}$ be any filtration of $\mathbf{A}$. Take $A_1 = A'_1$. For every $n \geq 0$, let A_{n+1} be a finite dimensional central simple algebra containing both $\mu(A_n)$ and A'_{n+1}. So $\mu(A_n) \subseteq A_{n+1}$ by construction, and at the same time $\mathbf{A} = \varinjlim A'_n \subseteq \varinjlim A_n \subseteq \mathbf{A}$. □

The advantage of an endomorphism $\mu : (\mathbf{A}, \alpha) \to (\mathbf{A}, \alpha^+)$ of filtered algebras is that the restriction of μ to each component A_n is by Skolem-Noether a conjugation by an element from the next component A_{n+1}. This was used by Barsotti [5] to prove that every division algebra $\mathbf{A} \in \mathcal{C}_F^{\omega}$ of infinite dimension has endomorphisms which are not onto.

An important property of algebras $\mathbf{A} \in \mathcal{C}_F^{\omega}$ is that if such an algebra embeds in some $\mathbf{B} \in \mathcal{C}_F$ locally, then it embeds there globally:

Proposition 9.3 *Let $\boldsymbol{A} \in \mathcal{C}_F^{\omega}$ and $\boldsymbol{B} \in \mathcal{C}_F$. The following are equivalent:*

(1) *There is an embedding $\boldsymbol{A} \hookrightarrow \boldsymbol{B}$.*
(2) *Every finite dimensional central simple subalgebra of $\boldsymbol{A}$ embeds in $\boldsymbol{B}$.*
(3) *There is a filtration $\{A_n\}$ of $\boldsymbol{A}$ such that every $A_n \hookrightarrow \boldsymbol{B}$ (where we do not assume the maps are compatible).*

Proof (1) $\implies$ (2) $\implies$ (3) are obvious. We prove that (3) $\implies$ (1). Consider a filtration $\{A_n\}$ of $\mathbf{A}$, so there is a chain of maps, $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots$, defining a directed system $\{A_n, f_{nn'}\}$ by composition.

By assumption there is an embedding $i_1 : A_1 \to B_1$ where $B_1 \subseteq \mathbf{B}$ is a finite dimensional central simple subalgebra. Fix an automorphism $\pi_1 : B_1 \to B_1$. We define automorphisms π_n of B_n by induction. Suppose π_n is defined. Let B_{n+1} be a finite dimensional central simple subalgebra of $\mathbf{B}$, which is large enough so that $B_n \subseteq B_{n+1}$ and there is an embedding $i_{n+1}: A_{n+1} \to B_{n+1}$. Let $j_n : B_n \to B_{n+1}$ denote the inclusion map. Consider the diagram

$$\begin{array}{ccccc} A_{n+1} & \xhookrightarrow{\;i_{n+1}\;} & B_{n+1} & \xrightarrow{\;\pi_{n+1}\;} & B_{n+1} \\ {\scriptstyle f_n}\uparrow & & & & \uparrow{\scriptstyle j_n} \\ A_n & \xhookrightarrow{\;i_n\;} & B_n & \xrightarrow{\;\pi_n\;} & B_n \end{array}$$

The subalgebras $i_{n+1} f_n(A_n)$ and $j_n \pi_n i_n(A_n)$ are isomorphic simple subalgebras of B_{n+1}, so by the Skolem-Neother theorem there is an (inner) automorphism $\pi_{n+1}: B_{n+1} \to B_{n+1}$ such that $\pi_{n+1} \circ i_{n+1} \circ f_n = j_n \circ \pi_n \circ i_n$. The system of maps $\pi_n \circ i_n : A_n \to B_n$ is thus compatible, and the map $\varinjlim (\pi_n \circ i_n) : \varinjlim \{A_n, f_{nn'}\} \to \varinjlim B_n$ defined by Remark 4.1. (3) is injective. By Lemma 4.6, $\varinjlim B_n \subseteq \mathbf{B}$, so we are done. □

Corollary 9.4 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be algebras in* $\mathcal{C}_F^{\omega}$ *with filtrations* $\{A_n\}$ *and* $\{B_n\}$*, respectively. If there are embeddings* $A_n \hookrightarrow B_n$ *for every* n*, then there is an embedding* $\mathbf{A} \hookrightarrow \mathbf{B}$*.*

In particular, the objects participating in the direct limit $\varinjlim A_n$ determine the limit:

Proposition 9.5 *Let* $A_1, A_2, \cdots$ *be finite dimensional central simple algebras such that for every* n*,* A_n *can be embedded into* A_{n+1}*. All direct limits* $\varinjlim_{\omega} A_n$ *(regardless of the morphisms* $A_n \to A_{n+1}$*) are isomorphic to each other.*

Proof This is a special case of Corollary 9.4: in the proof of Proposition 9.3 we take $B_n = A_n$, and since the $i_n : A_n \to A_n$ are now isomorphisms, so are the compositions $\pi_n \circ i_n$, and the limit $\varinjlim (\pi_n \circ i_n)$ is an isomorphism. □

We can generalize this to a direct limit of algebras in $\mathcal{C}_F^{\omega}$.

Proposition 9.6 *The direct limit of a chain* $\mathbf{A}_1 \to \mathbf{A}_2 \to \cdots$ *of algebras in* $\mathcal{C}_F^{\omega}$ *is independent of the morphisms of the chain.*

Proof Let $\mu_n, \mu'_n : \mathbf{A}_n \to \mathbf{A}_{n+1}$ be two given chains. Given a presentation $\mathbf{A}_n = \varinjlim_m A_{nm}$ where A_{nm} are finite dimensional central simple algebras (and where the presentation for $n = 1$ is arbitrary), the proof of Proposition 9.1 provides a presentation $\mathbf{A}_{n+1} = \varinjlim_m A_{n+1,m}$ such that both $\mu(A_{nm}) \subseteq A_{n+1,m}$ and $\mu'(A_{nm}) \subseteq A_{n+1,m}$.

Now $\mathbf{A} = \varinjlim \{\mathbf{A}_n, \mu_{nn'}\} = \varinjlim_{(n,m)\in\omega\times\omega} \{A_{nm}, \mu_{nn'}\}$ by Proposition 4.3, which is equal to $\varinjlim_{n\in\omega} \{A_{nn}, \mu_{nn'}\}$ because the diagonal is cofinal. Likewise $\mathbf{A}' = \varinjlim \{\mathbf{A}_n, \mu'_{nn'}\} = \varinjlim_{n\in\omega} \{A_{nn}, \mu'_{nn'}\}$, but the diagonal limits are isomorphic by Proposition 9.5. □

However, in Example 16.11 below we show that over uncountable directed sets, direct limits of finite dimensional central simple algebras strictly depend on the morphism.

9.2 Isomorphisms in $\mathcal{C}_F^{\omega}$

Let $A_1 \to A_2 \to \cdots$ be a filtration of some $\mathbf{A} \in \mathcal{C}_F^{\omega}$. For an increasing sequence $n_1 < n_2 < \cdots$, we say that $A_{n_1} \to A_{n_2} \to \cdots$ (with the induced arrows) is a **subfiltration**. The natural map $\varinjlim A_{n_i} \to \varinjlim A_i$, defined as the identity on the components in the left-hand side, induces by Lemma 4.7 an isomorphism of the algebras.

We say that two filtered algebra **A** and **B** are **quasi-isomorphic** if the filtrations have infinitely many common components. As we have seen, both algebras are isomorphic (as algebras) to the direct limit over the intersection of the filtrations.

Proposition 9.7 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be filtered algebras in* $\mathcal{C}_F^{\omega}$ *which are isomorphic as algebras, then there is a filtered algebra* $\mathbf{C}$ *which is quasi-isomorphic to both.*

Proof Let $(\mathbf{A}, \{A_n\})$ and $(\mathbf{B}, \{B_n\})$ be two filtered algebra, with an isomorphism $f : \mathbf{A} \to \mathbf{B}$. Define a chain of algebras C_n by taking $C_1 = A_1$; for odd k, C_k is one of the algebras A_n, and we take C_{k+1} to be an algebra B_m containing $f(C_k)$; for even k, C_k is one of the algebras B_m, and we take C_{k+1} to be an algebra A_n containing $f^{-1}(C_k)$. In this manner, the chain $C_1 \hookrightarrow C_2 \hookrightarrow \cdots$ has infinitely many common components both with $A_1 \hookrightarrow A_2 \hookrightarrow \cdots$ and $B_1 \hookrightarrow B_2 \hookrightarrow \cdots$ (with the same maps $C_n \to C_{n+2}$ as in the original series), $\mathbf{C} = \varinjlim C_k$ is quasi-isomorphic to both $(\mathbf{A}, \{A_n\})$ and $(\mathbf{B}, \{B_n\})$. □

Proposition 9.7 holds for any countable direct limit of finite dimensional algebras. However in the following "Kantor-Schröder-Bernstein type" theorem, we need to apply Proposition 9.5, which does rely on the Skolem-Noether property:

Proposition 9.8 *Let* $\mathbf{A}, \mathbf{B} \in \mathcal{C}_F^{\omega}$*. If there are embeddings* $\mathbf{A} \hookrightarrow \mathbf{B}$ *and* $\mathbf{B} \hookrightarrow \mathbf{A}$ *then* $\mathbf{A} \cong \mathbf{B}$*.*

Proof Write $\mathbf{A} = \varinjlim A_n$ and $\mathbf{B} = \varinjlim B_n$ where A_n, B_m are finite dimensional central simple algebras. Let $f : \mathbf{A} \to \mathbf{B}$ and $f' : \mathbf{B} \to \mathbf{A}$ be the given embeddings. As in Proposition 9.7, we define a series of algebras C_n by taking $C_1 = A_1$, then C_{k+1} is one of the algebras B_n containing $f(C_k)$ when k is odd, and C_{k+1} is one of the algebras A_n containing $f'(C_k)$ when k is even. Let $C = \varinjlim C_k$. All the components C_{2t+1} participate in the series defining $\mathbf{A}$ (albeit with different morphisms). By Proposition 9.5, $\mathbf{A} \cong \varinjlim C_{2t+1} = C$. For the same reason, $\mathbf{B} \cong \varinjlim C_{2t} = C$. □

Combining this fact with Proposition 9.3, we conclude:

Corollary 9.9 *An algebra* $\mathbf{A} \in \mathcal{C}_F^{\omega}$ *is completely determined by its finite dimensional central simple subalgebras.*

10 Infinite Tensor Products

We define a general tensor product suitable for our needs, and then specialize to the case of a countable set. In Section 17 we construct locally finite dimensional central simple algebras which are not infinite tensor products.

10.1 Infinite Tensor Products

Let I be any (unordered) set of indices, and let A_i be arbitrary algebras, $i \in I$. For a finite subset $I_0 \subseteq I$, $\otimes_{i \in I_0} A_i$ denotes the tensor product, with the obvious embeddings $\otimes_{i \in I_0} A_i \to \otimes_{i \in I_1} A_i$ for finite $I_0 \subseteq I_1$. We then define the **infinite tensor product** (over I)

$$\bigotimes_I A_i = \varinjlim (\otimes_{i \in I_0} A_i),$$

where the direct limit is over the directed set $P^{<\omega}(I)$ of finite subsets of I, ordered by inclusion.

Proposition 10.1 *The class $\mathcal{C}_F$ is closed under infinite tensor products.*

Proof Indeed, $\mathcal{C}_F$ is closed under finite tensor products by Proposition 4.5, and under direct limits by Corollary 3.5. □

Example 10.2 (Koethe, [20]) Let I be an arbitrary set. Fix a natural number n. Let α_i, β_i be indeterminates over a field k with a primitive nth root of unity ρ of order n, and suppose $F = k(\{\alpha_i, \beta_i : i \in I\})$ is the transcendental field extension. Let $(\alpha, \beta) = F[x, y : x^n = \alpha,\ y^n = \beta,\ yx = \rho xy]$ denote be the symbol algebra of degree n over F, which is central simple. Then $\mathbf{D} = \bigotimes_I (\alpha_i, \beta_i)_n$ is a division algebra, which is (properly) in $\mathcal{C}_F^{P^{<\omega}(I)}$.

Remark 10.3 Let I, I' be disjoint sets. Let A_i $(i \in I)$ and $A_{i'}$ $(i' \in I')$ be algebras. Then

$$\bigotimes_{I\cup I'} A_i \cong \bigotimes_I A_i \otimes \bigotimes_{I'} A_{i'}.$$

Indeed, by Proposition 5.1 for the directed set $P^{<\omega}(I \cup I')$,

$$\begin{aligned}\bigotimes_{I\cup I'} A_i &= \varinjlim_{I_0 \in P^{<\omega}(I\cup I')} (A_i \otimes B_i)\\ &\cong \left(\varinjlim_{I_0 \in P^{<\omega}(I)} \otimes_{i\in I} A_i\right) \otimes \left(\varinjlim_{I_0' \in P^{<\omega}(I')} \otimes_{i'\in I_0'} A_{i'}\right)\\ &\cong \bigotimes_I A_i \otimes \bigotimes_{I'} A_i.\end{aligned}$$

Similarly, we have:

Remark 10.4 If A_i, B_i are algebras indexed by a set I, then

$$\bigotimes (A_i \otimes B_i) \cong \bigotimes A_i \otimes \bigotimes B_i.$$

Indeed, by Proposition 5.1 for the directed set $P^{<\omega}(I)$,

$$\begin{aligned}\bigotimes_I (A_i \otimes B_i) &= \varinjlim (\otimes_{i\in I_0} (A_i \otimes B_i))\\ &\cong \varinjlim ((\otimes_{i\in I_0} A_i) \otimes (\otimes_{i\in I_0} B_i))\\ &\cong \varinjlim (\otimes_{i\in I_0} A_i) \otimes \varinjlim (\otimes_{i\in I_0} B_i) = \bigotimes A_i \otimes \bigotimes B_i.\end{aligned}$$

For any subset $J_0 \subseteq J$ we use the shorthand notation $\bigcup J_0 = \bigcup_{I\in J_0} I$.

Proposition 10.5 *Let J be a set of disjoint sets $I \in J$; and suppose an algebra A_i is given for any $i \in \bigcup J = \bigcup_{I\in J} I$. Then*

$$\bigotimes_{I\in J} (\bigotimes_I A_i) \cong \bigotimes_{\bigcup J} A_i.$$

Proof

$$\begin{aligned}\bigotimes_{I\in J}(\bigotimes_I A_i) &= \bigotimes_{I\in J} \varinjlim_{I_0\in P^{<\omega}(I)} A_{I_0}\\ &= \varinjlim_{J_0\in P^{<\omega}(J)} \bigotimes_{I\in J_0} \varinjlim_{I_0\in P^{<\omega}(I)} A_{I_0}\end{aligned}$$

In Proposition 5.4 we take $\Gamma = P^{<\omega}(J)$ and $\Delta = P^{<\omega}(\bigcup J)$; and for each finite $J_0 \subseteq J$, we take $\Delta_{J_0} = P^{<\omega}(\bigcup J_0)$, so that $\bigcup_{J_0\subseteq J} P^{<\omega}(\bigcup J_0) = P^{<\omega}(\bigcup J) = \Delta$. For each $I \in J$ we let $B_I = \bigotimes_{i\in I} A_i$. For every finite $I_0 \subseteq \bigcup J$ we set $A_{I_0} = \bigotimes_{i\in I_0} A_i$ with the obvious morphisms $A_{I_0} \to A_{I'_0}$ when $I_0 \subseteq I'_0$.

By Remark 10.3, $\varinjlim_{I_0\in P^{<\omega}(\bigcup J_0)} A_{I_0} = \bigotimes_{i\in\bigcup J_0} A_i = \otimes_{I\in J_0} B_I$, so by Proposition 5.4,

$$\begin{aligned}\bigotimes_{I\in J}(\bigotimes_I A_i) &= \bigotimes_{I\in J} B_I\\ &= \varinjlim_{J_0\in P^{<\omega}(J)} (\otimes_{I\in J_0} B_I)\\ &= \varinjlim_{J_0\in P^{<\omega}(J)} (\varinjlim_{I_0\in P^{<\omega}(\bigcup J_0)} A_{I_0})\\ &\cong \varinjlim_{I_0\in P^{<\omega}(\bigcup J)} A_{I_0}\\ &\cong \bigotimes_{\bigcup J} A_i\end{aligned}$$

□

Corollary 10.6 *Let $A_i \in \mathcal{C}_F$ be algebras with factors $B_i \subseteq A_i$ (see Definition 6.1), where i ranges over an index set I. Then $\bigotimes B_i$ is a factor of $\bigotimes A_i$. Indeed, write $A_i = B_i \otimes C_i$. Then $\bigotimes_I A_i \cong (\bigotimes_I B_i) \otimes (\bigotimes_I C_i)$.*

Corollary 10.7 *Let $A_i \in \mathcal{C}_F$ be algebras, where i ranges over an index set I. For every subset $I' \subseteq I$, $\bigotimes_{I'} A_i$ is a factor of $\bigotimes_I A_i$. Indeed, $\bigotimes_I A_i \cong (\bigotimes_{I'} A_i) \otimes (\bigotimes_{I\setminus I'} A_i)$.*

This leads to the following (compare to Proposition 8.3):

Proposition 10.8 *If $\mathbf{A} \subseteq \mathbf{B} \in \mathcal{C}_F^\Gamma$ where $\mathbf{A}$ is an infinite tensor product of finite dimensional central simple algebras, then $\mathbf{A} \in \mathcal{C}_F^\Gamma$.*

Proof Write $\mathbf{A} = \bigotimes_I A_i$ and $\mathbf{B} = \varinjlim_\Gamma B_\gamma$, where the A_i and B_γ are finite dimensional central simple algebras. We define $\mu : \Gamma \to P^{<\omega}(I)$ by $\mu(\gamma) = \{i \in I : A_i \subseteq B_\gamma\}$; this is well defined because $\bigotimes_{i\in\mu(\gamma)} A_i \subseteq B_\gamma$, and B_γ is finite dimensional. Moreover, for every finite $I_0 \subseteq I$, $\bigotimes_{I_0} A_i$ is finitely generated and thus contained in some B_γ, whence $I_0 \subseteq \mu(\gamma)$. This shows that $\mathrm{Im}(\mu)$ is cofinal in $P^{<\omega}(I)$, so $\mathbf{A} \in \mathcal{C}_F^{P^{<\omega}(I)} \subseteq \mathcal{C}_F^{\mathrm{Im}(\mu)} \subseteq \mathcal{C}_F^\Gamma$ by Proposition 5.5, items (2) and (4). □

Let us describe the infinite tensor product of arbitrary algebras from $\mathcal{C}_F$. We say that a directed set Γ is **pointed** if it has an initial element. When considering a class $\mathcal{C}_F^\Gamma$ we may always assume that Γ is pointed, because $\{-\infty\} \cup \Gamma$, with the obvious order, is a directed set in which Γ is cofinal and complete. By Proposition 5.5. (3), $\mathcal{C}_F^\Gamma = \mathcal{C}_F^{\{-\infty\}\cup\Gamma}$.

Let I be an arbitrary set and let Γ_i be pointed directed sets for every $i \in I$.

Definition 10.9 Let Γ_i be pointed directed sets. For $f \in \prod_{i \in I}$, let

$$\mathrm{supp}(f) = \{i \in I : \ f(i) \neq -\infty\}$$

be the **support** of f. The **direct sum** $\sum_{i \in I} \Gamma_i$ is the set of functions $f \in \prod_{i \in I} \Gamma_i$ with finite support. The direct sum is ordered by the product order. This is clearly a directed set. If I is finite then $\sum_{i \in I} \Gamma_i = \prod_{i \in I} \Gamma_i$.

Proposition 10.10 *Let I be an arbitrary set and Γ_i directed sets ($i \in I$). Let $\mathbf{A}_i \in \mathcal{C}_F^{\Gamma_i}$. We may assume each Γ_i is pointed. Then $\bigotimes_I \mathbf{A}_i \in \mathcal{C}_F^{\sum \Gamma_i}$.*

In fact, $\bigotimes_{i \in I} \mathbf{A}_i = \varinjlim_{f \in \sum \Gamma_i} A_f$ where $A_f = \otimes_{i \in \mathrm{supp}(f)} A_{i, f(i)}$ and $\mathbf{A}_i = \varinjlim_{\Gamma_i} A_{i\gamma}$.

Proof For each i write $\mathbf{A}_i = \varinjlim_{\Gamma_i} A_{i\gamma}$, where $A_{i\gamma}$ are finite dimensional central simple algebras. For $f \in \sum \Gamma_i$ we set $A_f = \otimes_{i \in \mathrm{supp}(f)} A_{i\, f(i)}$, with the obvious inclusions $A_f \hookrightarrow A_{f'}$ if $f \leq f'$. Now $\bigotimes_I \mathbf{A}_i = \bigotimes_{i \in I} (\varinjlim_{\gamma_i \in \Gamma_i} A_{i\gamma_i}) = \varinjlim_{I_0 \in P^{<\omega}(I)} \varinjlim_{f \in \sum_{i \in I_0} \Gamma_i} \bigotimes_{i \in I_0} A_{i\, f(i)} = \varinjlim_{f \in \sum \Gamma_i} A_f$ by Proposition 5.4. □

Example 10.11 Let I be an arbitrary set. For each $i \in I$ let $\Gamma_i = 1$, the one-point directed set. Then $\sum_{i \in I} 1 \cong P^{<\omega}(I)$. If A_i ($i \in I$) are finite dimensional central simple algebras, then $\bigotimes A_i \in \mathcal{C}_F^{\sum 1} = \mathcal{C}_F^{P^{<\omega}(I)}$ by definition, in accordance with Proposition 10.10.

We also know that limits and tensor products commute:

Proposition 10.12 *Let Γ be a directed system, and I an index set. Let $A_{\gamma,i}$ be algebras, such that for every $i \in I$, the system $\{A_{\gamma,i}\}$ is endowed with compatible morphisms $A_{\gamma,i} \to A_{\gamma',i}$, so that $A^{(i)} = \varinjlim_{\gamma \in \Gamma} A_{\gamma,i}$ are defined. Then*

$$\bigotimes_I (\varinjlim_{\Gamma} A_{\gamma,i}) = \varinjlim_{\Gamma} \bigotimes_I A_{\gamma,i}.$$

Proof By Corollary 4.4, and since direct limit commutes with finite tensor products, we have that

$$\begin{aligned}
\varinjlim_{\Gamma} \bigotimes_I A_{\gamma,i} &= \varinjlim_{\Gamma} \varinjlim_{I_0 \in P^{<\omega}(I)} \bigotimes_{i \in I_0} A_{\gamma,i} \\
&= \varinjlim_{I_0 \in P^{<\omega}(I)} \varinjlim_{\Gamma} \bigotimes_{i \in I_0} A_{\gamma,i} \\
&= \varinjlim_{I_0 \in P^{<\omega}(I)} \bigotimes_{i \in I_0} \varinjlim_{\Gamma} A_{\gamma,i} \\
&= \bigotimes_I \varinjlim_{\Gamma} A_{\gamma,i}
\end{aligned}$$

□

10.2 Countable Tensor Products

Let us specialize to the case where the index set I is ω. The set $\{\{1,\ldots,n\}: n\in\omega\}$ is cofinal in $P^{<\omega}(\omega)$, so by Proposition 4.7, the **countable tensor product**

$$\varinjlim_{i\in\omega}(A_1\otimes\cdots\otimes A_i) \tag{2}$$

is equal to $\bigotimes_\omega A_i$. Being more convenient, we will use the characterization (2) for infinite tensor products over ω. Since the infinite tensor product is independent of the order of the components, so is the countable tensor product.

Proposition 10.13 (Koethe, [20]) *Every algebra $\mathbf{A}\in\mathcal{C}_F^\omega$ has the form $\mathbf{A}\cong\bigotimes A_n$ where A_n are finite dimensional central simple algebras.*

Proof Write $\mathbf{A}=\varinjlim A'_n$ where $A'_1\subseteq A'_2\subseteq\cdots$ are finite dimensional central simple algebras. Since $A'_n\subseteq A'_{n+1}$ is central simple, the double centralizer theorem gives a decomposition $A'_{n+1}=A'_n\otimes A_{n+1}$ for a finite dimensional central simple algebra A_{n+1}, so by induction $A'_n=A_1\otimes\cdots\otimes A_n$ and by definition $\varinjlim A'_n=\bigotimes_\omega A_n$. □

Proposition 10.14 *Let Λ be a directed set of height $\mathrm{ht}(\Lambda)=\omega$. Then there is a cofinal isomorphic copy of Λ in $\omega\times\Lambda$.*

Proof Fix an unbounded countable chain $\lambda_1<\lambda_2<\cdots$ in Λ. Define a map $t:\Lambda\to\omega\times\Lambda$ by $\lambda\mapsto(\sup\{i:\lambda_i\leq\lambda\},\lambda)$, where $\sup\emptyset=0$, which is well defined because the chain is unbounded. Let $(n,\lambda)\in\omega\times\Lambda$. There is $\lambda'\in\Lambda$ such that $\lambda,\lambda_n<\lambda'$, and then $(n,\lambda)<t(\lambda')$. □

Recall that by Proposition 5.10, a directed set Γ has height ω once there is even one infinite-dimensional algebra which is properly in $\mathcal{C}_F^\Gamma$.

Proposition 10.15 *Let Γ be a directed set with height $\mathrm{ht}(\Gamma)=\omega$. Consider a system $\{\mathbf{A}_n,f_{nn'}\}$ indexed by ω, such that each $\mathbf{A}_n\in\mathcal{C}_F^\Gamma$, and assume there are presentations $\mathbf{A}_n=\varinjlim_\Gamma A_{n\gamma}$ such that $f_{nn'}(A_{n\gamma})\subseteq A_{n'\gamma}$ for every $\gamma\in\Gamma$ and $n<n'$. Then $\varinjlim_\omega\mathbf{A}_n\in\mathcal{C}_F^\Gamma$.*

Proof By Remark 4.3, $\varinjlim_\omega\mathbf{A}_n=\varinjlim_\omega\varinjlim_\Gamma A_{n\gamma}=\varinjlim_{\omega\times\Gamma}A_{n\gamma}$, which can be presented as a limit over Γ by Proposition 10.14. □

Theorem 10.16 *The class $\mathcal{C}_F^\Gamma$ is closed under countable tensor products, provided that* $\mathrm{ht}(\Gamma)=\omega$.

Proof Let $\mathbf{A}_1,\mathbf{A}_2,\ldots\in\mathcal{C}_F^\Gamma$, and write $\mathbf{A}_i=\varinjlim_\Gamma A_{i\gamma}$ with finite components. The countable tensor product $\bigotimes_\omega\mathbf{A}_n$ is a direct limit of the finite tensor products $\mathbf{A}_1\otimes\cdots\otimes\mathbf{A}_n$, each of which can be expressed as $\varinjlim_\Gamma(A_{1\gamma}\otimes\cdots\otimes A_{n\gamma})$ by Proposition 5.1. The components of this limit are preserved by the maps $\mathbf{A}_1\otimes\cdots\otimes\mathbf{A}_n\to\mathbf{A}_1\otimes\cdots\otimes\mathbf{A}_{n'}$, so Proposition 10.15 applies and the limit is in $\mathcal{C}_F^\Gamma$. □

Of particular interest in Theorem 10.16 is the case $\Gamma = \omega$, for which we provide an explicit formula. Suppose $\mathbf{A}^{(i)} = \varinjlim A_{in}$ are direct limits in $\mathcal{C}_F^{\omega}$. Then

$$\bigotimes_{\omega} \mathbf{A}^{(i)} = \varinjlim_{n} (A_{1n} \otimes \cdots \otimes A_{nm_n}) \tag{3}$$

for every increasing series $m_1 < m_2 < \cdots$.

Proposition 10.17 *Let $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots \in \mathcal{C}_F$. The degree of the countable tensor product is*

$$\deg(\bigotimes_{\omega} \boldsymbol{A}_i) = \prod_{i=1}^{\infty} \deg(\boldsymbol{A}_i)$$

(where an infinite product of supernatural numbers is defined in the obvious manner).

Proof By Proposition 7.4,

$$\begin{aligned}\deg(\bigotimes_{\omega} \mathbf{A}_i) &= \deg(\varinjlim_{i\in\omega} (\mathbf{A}_1 \otimes \cdots \otimes \mathbf{A}_i)) \\ &= \operatorname{lcm}\{\deg(\mathbf{A}_1 \otimes \cdots \otimes \mathbf{A}_i)\} = \operatorname{lcm}_i \prod_{j=1}^{i} \deg \mathbf{A}_j = \prod_{j=1}^{\infty} \deg(\mathbf{A}_j).\end{aligned}$$

□

11 Subalgebras in $\mathcal{C}_F^{\omega}$

By definition and by Proposition 9.1, every morphism $\mathbf{B} \hookrightarrow \mathbf{A}$ between algebras in $\mathcal{C}_F^{\omega}$ is the limit of an infinite commuting diagram of the form

$$\begin{array}{ccccccc} \cdots \longrightarrow & A_n & \longrightarrow & A_{n+1} & \longrightarrow \cdots \\ & \uparrow & & \uparrow & \\ \cdots \longrightarrow & B_n & \longrightarrow & B_{n+1} & \longrightarrow \cdots \end{array} \tag{4}$$

where the entries are finite dimensional central simple algebras. We wish to refine this description to take into account Proposition 10.13, which presents an algebra in $\mathcal{C}_F^{\omega}$ as a countable tensor product.

11.1 Explicit Presentation for Subalgebras

Example 11.1 Let C_n and D_n be finite dimensional central simple algebras ($n = 1, 2, \ldots$), and let

$$\psi_n : C_n \longrightarrow D_{n+1} \otimes C_{n+1} \tag{5}$$

be embeddings (we call this our **data**). We set:

(1) $\mathbf{B} = \varinjlim B_n$, with
$$B_n = D_1 \otimes \cdots \otimes D_n$$
with the natural embeddings $B_n \hookrightarrow B_n \otimes D_{n+1} = B_{n+1}$;

(2) $\mathbf{A} = \varinjlim \{A_n, \alpha_n\}$, where
$$A_n = B_n \otimes C_n = D_1 \otimes \cdots \otimes D_n \otimes C_n$$

$$\begin{array}{ccc} \to A_n = D_1 \otimes \cdots \otimes D_n \otimes C_n & \xrightarrow{1^{\otimes n} \otimes \psi_n} & A_{n+1} = D_1 \otimes \cdots \otimes D_n \otimes (D_{n+1} \otimes C_{n+1}) \to \\ \uparrow & & \uparrow \\ \longrightarrow B_n = D_1 \otimes \cdots \otimes D_n & \longrightarrow & B_{n+1} = D_1 \otimes \cdots \otimes D_n \otimes D_{n+1} \longrightarrow \end{array}$$

Fig. 1 The algebras $\bigotimes D_n \hookrightarrow (C_n, D_n; \psi_n)$ defined in Example 11.1

and the maps $\alpha_n : A_n \to A_{n+1}$ are defined by composing

$$\alpha_n : A_n = B_n \otimes C_n \xrightarrow{1 \otimes \psi_n} B_n \otimes D_{n+1} \otimes C_{n+1} = A_{n+1} \ .$$

The arrows defining **A**, **B** are presented in the commutative diagram of Fig. 1.

Notation 11.2 *Given the data of Example 11.1 we denote*

$$(C_n, D_n; \psi_n) = \varinjlim \{(D_1 \otimes \cdots \otimes D_n) \otimes C_n, 1 \otimes \psi_n\} = \varinjlim \{A_n, \alpha_n\}.$$

Moreover, an embedding $\pi_\psi : \bigotimes D_n \hookrightarrow (C_n, D_n; \psi_n)$ *is defined by the inclusions* $B_n = (D_1 \otimes \cdots \otimes D_n) \hookrightarrow (D_1 \otimes \cdots \otimes D_n) \otimes C_n = A_n$.

We can now describe any pair of algebras $\mathbf{B} \subseteq \mathbf{A}$ in $\mathcal{C}^\omega_F$.

Proposition 11.3 *Let* $\boldsymbol{B} \subseteq \boldsymbol{A}$ *be algebras in* $\mathcal{C}^\omega_F$. *Then there is data for Example 11.1 such that* $\boldsymbol{A} = (C_n, D_n; \psi_n)$, $\boldsymbol{B} = \bigotimes D_n$, *and the inclusion is given by the left inclusions* $B_n \hookrightarrow B_n \otimes C_n = A_n$. *Moreover in this case* $\boldsymbol{A} \cong \bigotimes E_n$, *where* $E_1 = A_1$ *and* E_{n+1} *is the centralizer of* $\psi_n(C_n)$ *in* $D_{n+1} \otimes C_{n+1}$.

Proof By Proposition 9.1 we can write $\mathbf{A} = \varinjlim A_n$ and $\mathbf{B} = \varinjlim B_n$ where $B_n \subseteq A_n$ are finite dimensional central simple algebras, and there is a chain of embeddings $A_n \to A_{n+1}$ inducing the embeddings $B_n \to B_{n+1}$ (throughout, $n \geq 1$). Let $C_n = \mathrm{C}_{A_n}(B_n)$ be the centralizers. Also let $D_{n+1} = \mathrm{C}_{B_{n+1}}(B_n)$, so we can decompose $B_{n+1} = B_n \otimes D_{n+1}$. We formally set $A_0 = B_0 = C_0 = F$ and $D_1 = B_1$. The formula for B_n then follows, and $\mathbf{B} = \bigotimes D_n$. Now the embedding $\alpha_n : A_n = B_n \otimes C_n \to B_n \otimes (D_{n+1} \otimes C_{n+1})$ restricts to the identity on B_n, so it is determined by the embedding of the centralizer $C_n = \mathrm{C}_{A_n}(B_n)$ in the centralizer $\mathrm{C}_{A_{n+1}}(B_n) = D_{n+1} \otimes C_{n+1}$. The isomorphism $C_n \otimes E_{n+1} \to D_{n+1} \otimes C_{n+1}$ extends $\psi : C_n \to D_{n+1} \otimes C_{n+1}$.

Finally, $\mathrm{C}_{A_{n+1}}(A_n) = \mathrm{C}_{D_{n+1} \otimes C_{n+1}}(C_n) \cong E_{n+1}$, so $A_{n+1} \cong A_n \otimes E_{n+1}$, and setting $E_1 = A_1$ we have by induction $A_n = E_1 \otimes \cdots \otimes E_n$. Thus, applying Proposition 9.5, $\mathbf{A} \cong \bigotimes E_i$. $\square$

Remark 11.4 The presentation of $\mathbf{B} \subseteq \mathbf{A}$ by the data of Example 11.1 has two advantages: **B** has a natural description, and the embedding $\pi_\psi : \mathbf{B} \hookrightarrow \mathbf{A}$ is natural; however, the construction of **A** is 'twisted' by the maps ψ_n. We can straighten this out by considering $A'_n = E_1 \otimes \cdots \otimes E_n$ with the natural inclusions, and taking $\mathbf{A}' = \bigotimes E_i = \varinjlim A'_n$, which is isomorphic to **A**, at the expense of twisting the embedding of $\mathbf{B} = \bigotimes D_i$ into $\mathbf{A}'$. This embedding is obtained by choosing embeddings $B_n \to A'_n$ so that (4) commutes, which are best described by an example:

$$\begin{aligned} B_4 = D_1 \otimes D_2 \otimes D_3 \otimes D_4 & \xhookrightarrow{1 \otimes 1 \otimes 1 \otimes \iota} D_1 \otimes D_2 \otimes D_3 \otimes D_4 \otimes C_4 \\ & \xrightarrow{1 \otimes 1 \otimes 1 \otimes \psi_3^{-1}} D_1 \otimes D_2 \otimes D_3 \otimes C_3 \otimes E_4 \\ & \xrightarrow{1 \otimes 1 \otimes \psi_2^{-1} \otimes 1} D_1 \otimes D_2 \otimes C_2 \otimes E_3 \otimes E_4 \\ & \xrightarrow{1 \otimes \psi_1^{-1} \otimes 1 \otimes 1} D_1 \otimes C_1 \otimes E_2 \otimes E_3 \otimes E_4 \\ & \xrightarrow{=} E_1 \otimes E_2 \otimes E_3 \otimes E_4 \,. \end{aligned}$$

11.2 Decomposing $(C_n, D_n; \psi_n)$ as a Tensor Product

As noted above, $(C_n, D_n; \psi_n) \cong \bigotimes E_n$. It will be more convenient to obtain decompositions in terms of the C_n and D_n, as we now do in two special cases.

Remark 11.5 Let C_n, D_n and ψ_n be data for Example 11.1. For any series of embeddings $\psi'_n : C_n \to D_{n+1} \otimes C_{n+1}$,

$$(C_n, D_n; \psi'_n) \cong (C_n, D_n; \psi_n);$$

this follows from Proposition 9.5. Notice that in general, the subalgebra $\mathbf{B} = \bigotimes D_n$ is not preserved by the isomorphism.

If it is possible to form the limit of the centralizers C_n, we obtain the following useful decompositon:

Proposition 11.6 *If there are embeddings $C_n \hookrightarrow C_{n+1}$, then we have the decomposition $(C_n, D_n; \psi_n) \cong (\bigotimes D_n) \otimes \varinjlim C_n$.*

Proof We may write $C_{n+1} = C_n \otimes P_{n+1}$, and taking centralizes in (5) we may also identify $E_{n+1} = D_{n+1} \otimes P_{n+1}$, being the centralizer of C_n in $D_{n+1} \otimes C_{n+1}$. We can then define

$$\omega_n : C_n \otimes E_{n+1} = C_n \otimes D_{n+1} \otimes P_{n+1} \longrightarrow D_{n+1} \otimes C_n \otimes P_{n+1} = D_{n+1} \otimes C_{n+1}$$

to be the switching map $x \otimes y \otimes 1 \mapsto y \otimes x \otimes 1$ on the intermediate step; thus $\omega_n(C_n \otimes F) \subseteq F \otimes C_{n+1}$. Recall that $\iota_n : C_n \to C_n \otimes E_{n+1}$ is the left inclusion. Recall that in Proposition 11.3, $\mathbf{B} = \varinjlim B_n = \bigotimes D_n$ where $B_n = D_1 \otimes \cdots \otimes D_n$. It follows that the map

$$B_n \otimes C_n \longrightarrow B_n \otimes C_n \otimes E_{n+1} \xrightarrow{1 \otimes \omega_n} B_n \otimes D_{n+1} \otimes C_{n+1} = B_{n+1} \otimes C_{n+1}$$

decomposes as a tensor product of the natural embedding $B_n \hookrightarrow B_{n+1}$ and an embedding $\omega_n|_{C_n} : C_n \to C_{n+1}$. Therefore,

$$(C_n, D_n; \omega_n) = \varinjlim (B_n \otimes C_n, 1 \otimes \omega_n \iota_n) = \varinjlim B_n \otimes \varinjlim C_n = \mathbf{B} \otimes \varinjlim C_n.$$

But by Remark 11.5, $(C_n, D_n; \psi_n) \cong (C_n, D_n; \omega_n)$. □

Let us consider the other extreme case:

Proposition 11.7 *If there are embeddings $C_n \hookrightarrow D_{n+1}$ then $(C_n, D_n; \psi_n) \cong \bigotimes D_n$.*

Proof By assumption there are isomorphisms $\theta_n : C_n \otimes E_{n+1} \to D_{n+1} \otimes C_{n+1}$ such that $\theta_n(C_n \otimes F) \subseteq D_{n+1} \otimes F$. Consider the algebra $\mathbf{A} = (C_n, D_n; \theta_n)$. Then

$$\begin{aligned}(1 \otimes \theta_n \iota_n) A_n &= (1 \otimes \theta_n \iota_n)(B_n \otimes C_n) \\ &= (1 \otimes \theta_n)(B_n \otimes C_n \otimes F) = B_n \otimes \theta_n (C_n \otimes F) \\ &\subseteq B_n \otimes D_{n+1} \otimes F = B_{n+1} \otimes F.\end{aligned}$$

So the series interlace, and in this situation $\mathbf{B} = \varinjlim B_{n+1} = \varinjlim A_n = \mathbf{A}$; in other words π_θ is onto.

But now given any ψ_n, we have that $(C_n, D_n; \psi_n) \cong (C_n, D_n; \theta_n) = \bigotimes D_n$ by Remark 11.5. ☐

12 Centralizers

The double centralizer theorem and the Skolem-Noether theorem are amongst the most basic tools in the theory of finite dimensional central simple algebras. The double centralizer in fact characterizes finite dimensional central simple algebras [15, Thm V.11.2] (as quoted in Section 6), so it is important to understand to what extent it fails in our class of algebras. We analyze subalgebras in $\mathcal{C}_F^\omega$ to the extent that we can explicitly compute centralizers, which we use to give counterexamples to both theorems. Moreover, in Example 14.9 we present a central subalgebra whose centralizer is not central.

12.1 Computing Centralizers

Let $A_1 \subseteq A_2 \subseteq \cdots$ be arbitrary algebras, and $B_n \subseteq A_n$ subalgebras. Let $\mathbf{A} = \bigcup A_n$ and $\mathbf{B} = \bigcup B_n$. Then

$$\mathrm{C}_{\mathbf{A}}(\mathbf{B}) = \bigcup_n \bigcap_{m \geq n} \mathrm{C}_{A_m}(B_m), \tag{6}$$

easily proved by mutual inclusion.

We need to generalize this formula to cover direct limits.

Proposition 12.1 *Let $B_n \subseteq A_n$ be algebras with maps $f_n : A_n \to A_{n+1}$ such that $f_n(B_n) \subseteq B_{n+1}$. Let $\boldsymbol{A} = \varinjlim A_n$ and $\boldsymbol{B} = \varinjlim B_n$. For $n \leq n'$ set $f_{nn'} = f_{n'-1} \circ \cdots \circ f_n$. Let $C_n = \mathrm{C}_{A_n}(B_n)$, and define*

$$C_n^* = \left\{x \in C_n : (\forall n' \geq n) f_{nn'}(x) \in C_{n'}\right\}. \tag{7}$$

Then $\left\{C_n^*, f_{nn'}|_{C_n^*}\right\}$ is a subsystem of $\{A_n, f_{nn'}\}$, and

$$\mathrm{C}_{\boldsymbol{A}}(\boldsymbol{B}) = \varinjlim C_n^*.$$

Proof First we verify that $f_{nn'}(C_n^*) \subseteq C_{n'}^*$ when $n \leq n'$. Indeed, let $x \in C_n^*$ and let $n'' \geq n'$. Then $f_{n'n''} f_{nn'}(x) = f_{nn''}(x) \in C_{n''}$ by assumption. The limit $\mathbf{C}^* = \varinjlim \left\{C_n^*, f_{nn'}\right\}$ is thus a well defined subalgebra of $\mathbf{A}$. Since $\mathbf{A} = \bigcup \varphi_n(A_n)$ and $\mathbf{B} = \bigcup \varphi_n(B_n)$, we have by Eq. 6 that

$$\begin{aligned} \mathrm{C}_{\mathbf{A}}(\mathbf{B}) &= \bigcup_n \bigcap_{m \geq n} \mathrm{C}_{\varphi_m(A_m)}(\varphi_m(B_m)) \\ &= \bigcup_n \bigcap_{m \geq n} \varphi_m(\mathrm{C}_{A_m}(B_m)) \\ &= \bigcup_n \bigcap_{m \geq n} (\varphi_n \circ f_{nm}^{-1})(C_m) \\ &= \bigcup_n \varphi_n \left(\bigcap_{m \geq n} f_{nm}^{-1}(C_m) \right) = \bigcup_n \varphi_n(C_n^*) = \mathbf{C}^*. \end{aligned}$$

□

12.2 Separating Conjugates

In order to compute C_n^* in our setup, we need a lemma on intersection of conjugate subalgebras.

Lemma 12.2 *Let T, T' be central simple algebras. Let $a_0, a_1, \ldots, a_k \in T$ be elements such that $F[a_1, \ldots, a_k] = T$ and $a_0 = 1$, and let $b_0, b_1, \ldots, b_k \in T'$ be linearly independent in T'. If $t = \sum a_i \otimes b_i$ is invertible, then the subalgebras $t(T \otimes F)t^{-1}$ and $T \otimes F$ intersect trivially in $T \otimes T'$.*

Proof The equation $t(x \otimes 1)t^{-1} = y \otimes 1$ implies $\sum (a_i x - y a_i) \otimes b_i = 0$, so $a_i x = y a_i$ for each i, and in particular $y = x \in \mathrm{C}_T(F[a_1, \ldots, a_k]) = F$. □

For the rest of this section *we assume the base field F is infinite*.

Lemma 12.3 *Let T, T' be finite dimensional central simple algebras over F, such that $T' \neq F$. Then there is an invertible element $t \in T \otimes T'$ such that the conjugate subalgebras $t(T \otimes F)t^{-1}$ and $T \otimes F$ intersect trivially in $T \otimes T'$.*

Proof Take $a_0 = 1$ and let a_1, a_2 be generators of T (such a pair always exists by e.g. [15, Theorem VII.3]). The Zariski-open set

$$U = \left\{ (b_1, b_2, b_3) \in T'^3 : \sum a_i \otimes b_i \text{ is invertible} \right\}$$

is non-empty because $(1, 0, 0) \in U$. The Zariski-open set

$$U' = \left\{ (b_1, b_2, b_3) \in T'^3 : \{b_1, b_2, b_3\} \text{ are linearly independent} \right\}$$

is also non-empty because $\dim T' \geq 4$. So U, U' are dense in T'^3, and therefore have non-empty intersection. Choose $(b_1, b_2, b_3) \in U \cap U'$, then $t = \sum a_i \otimes b_i$ is invertible and $t(T \otimes F)t^{-1} \cap (T \otimes F) = F \otimes F$ by Lemma 12.2. □

12.3 Centralizers in $\mathcal{C}_F^\omega$

We apply Proposition 12.1 to the algebras $\mathbf{B} \subseteq \mathbf{A}$ considered in the previous section. Using Proposition 11.3, fix data C_n, D_n and ψ_n for Example 11.1, and let $\mathbf{A} = (C_n, D_n; \psi_n)$

be the resulting algebra with the subalgebra $\mathbf{B} = \varinjlim B_n = \bigotimes D_n$. In this notation, the formula in Proposition 12.1 presents $\mathrm{C}_{\mathbf{B}}(\mathbf{A})$ as a limit $\varinjlim C_n^*$, where

$$C_n^* = \left\{x \in C_n : (\forall n' \geq n)\, \alpha_{n'-1} \cdots \alpha_n(x) \in C_{n'}\right\},$$

as we replace the f_n of Eq. 7 by $\alpha_n : A_n \to A_{n+1}$. But for $x \in C_n$, $\alpha_n(x) = \psi_n(x)$, so imposing the first out of the infinitely many conditions defining C_n^* (namely the condition for $n' = n + 1$), we have that

$$C_n^* \subseteq \{x \in C_n : \psi_n(x) \in F \otimes C_{n+1}\}.$$

We construct a situation where C_n^* are trivial, and moreover such that Proposition 11.6 applies and provides an explicit presentation for $\mathbf{A}$ as a tensor product.

Proposition 12.4 *Let P_n, D_n be finite dimensional central simple algebras, such that all $D_n \neq F$ (except possibly for D_1). Set $C_n = P_1 \otimes \cdots \otimes P_n$. Then there are maps $\psi_n : C_n \to D_{n+1} \otimes C_{n+1}$ such that the image of $\pi_\psi : \bigotimes D_n \to \mathbf{A} = (C_n, D_n; \psi_n)$ has trivial centralizer in $\mathbf{A}$.*

Proof Denote $E_n = D_n \otimes P_n$. In Corollary 11.6 we defined

$$\omega_n : C_n \otimes E_{n+1} = C_n \otimes D_{n+1} \otimes P_{n+1} \longrightarrow D_{n+1} \otimes C_n \otimes P_{n+1} = D_{n+1} \otimes C_{n+1}$$

by switching the two left components. Thus $\omega_n(C_n \otimes F) = F \otimes (C_n \otimes F) \subseteq F \otimes C_{n+1}$.

Since $D_{n+1} \neq F$, Lemma 12.3 shows that there are invertible elements $t_n \in D_{n+1} \otimes C_{n+1}$ with the property that $t_n(F \otimes C_{n+1})t_n^{-1}$ intersects $F \otimes C_{n+1}$ trivially. We take $\psi_n = \gamma_{t_n} \circ \omega_n$, where $\gamma_t(z) = tzt^{-1}$ is the conjugation map. Then $\psi_n(C_n \otimes F) \subseteq \gamma_{t_n}(F \otimes C_{n+1})$ which intersects $F \otimes C_{n+1}$ trivially, showing that $C_n^* = F$. It follows that $\mathrm{C}_{\mathbf{A}}(\mathbf{B}) = \varinjlim C_n^* = F$. □

Definition 12.5 We say that a proper subalgebra $\mathbf{B}$ of a central algebra $\mathbf{A}$ is **pathological** if $\mathrm{C}_{\mathbf{A}}(\mathbf{B}) = F$ (equivalently, by taking centralizers back and forth, if $\mathrm{C}_{\mathbf{A}}(\mathrm{C}_{\mathbf{A}}(\mathbf{B})) = \mathbf{A}$).

Theorem 12.6 *Let $\mathbf{A} \in \mathcal{C}_F^\omega$. Every infinite dimensional factor $\mathbf{B}$ of $\mathbf{A}$ is isomorphic to either $\mathbf{A}$ or to a pathological subalgebra.*

Proof Let $\mathbf{C} = \mathrm{C}_{\mathbf{A}}(\mathbf{B})$ which is in $\mathcal{C}_F$ by the definition of a factor. Then $\mathbf{B}, \mathbf{C} \in \mathcal{C}_F^\omega$ by Corollary 8.3, and we may write $\mathbf{A} = \mathbf{B} \otimes \mathbf{C}$.

By Proposition 10.13 there are decompositions $\mathbf{B} = \bigotimes D_n$ and $\mathbf{C} = \bigotimes P_n$ as countable tensor products of finite dimensional components. We may assume all $D_n \neq F$ using the following argument. Since $\deg(\mathbf{B})$ is infinite, $D_n \neq F$ infinitely often. Choose an injection f from $Z = \{n : D_n = F\}$ to the complement $\{n : D_n \neq F\}$. For every $n \in Z$, replace the algebras $P_n, P_{f(n)}$ by $F, P_n \otimes P_{f(n)}$, respectively. Then remove Z from the set of indices over which $\bigotimes D_n$ and $\bigotimes P_n$ are computed, retaining the same algebras $\mathbf{B}$ and $\mathbf{C}$, when now we always have $D_n \neq F$, as asserted.

Let $C_n = P_1 \otimes \cdots \otimes P_n$, so that there are embeddings $C_n \hookrightarrow C_{n+1}$. By Proposition 12.4, ψ_n can be chosen so that the image of the map $\pi_\psi : \mathbf{B} = \bigotimes D_n \to (C_n, D_n; \psi_n)$ has trivial centralizer. But we already established in Proposition 11.6 that there is an isomorphism $\mu : (C_n, D_n; \psi_n) \to \bigotimes D_n \otimes \bigotimes P_n$, which carries $\pi_\psi(\mathbf{B})$ into a subalgebra $\mu\pi_\psi(\mathbf{B}) \subseteq \mathbf{B} \otimes \mathbf{C} = \mathbf{A}$, with trivial centralizer. If $\mathbf{B} \not\cong \mathbf{A}$, then necessarily $\mu\pi_\psi(\mathbf{B})$ is a proper subalgebra, isomorphic to $\mathbf{B}$, which is pathological by construction. □

In order for Theorem 12.6 to produce a true pathological subalgebra, we need to ensure that $\mathbf{B}$ is not isomorphic to $\mathbf{A}$, for example by imposing the condition that $\deg(\mathbf{B}) \neq \deg(\mathbf{A})$, or that $\mathbf{B}$ is a division algebra and $\mathbf{A}$ is not. We classify factors which are isomorphic to the whole algebra in Proposition 19.3 below, and show in Theorem 19.7 that for every infinite dimensional $\mathbf{A} \in \mathcal{C}_F^\omega$ there is an isomorphism between proper subalgebras which cannot be extended to $\mathbf{A}$. Example 14.11 is an explicit pathological subalgebra, in the context of Clifford algebras.

Remark 12.7 If $\mathbf{B} \subseteq \mathbf{A}$ is a factor, then $\mathrm{C}_{\mathbf{A}}(\mathrm{C}_{\mathbf{A}}(\mathbf{B})) = \mathbf{B}$.

Let $\mathbf{A} \in \mathcal{C}_F^\omega$. For any subalgebra $\mathbf{B} \subseteq \mathbf{A}$, $\mathbf{B} \subseteq \mathrm{C}_{\mathbf{A}}(\mathrm{C}_{\mathbf{A}}(\mathbf{B})) \subseteq \mathbf{A}$. As we have seen, every infinite dimensional factor $\mathbf{B}$, for which $\mathrm{C}_{\mathbf{A}}(\mathrm{C}_{\mathbf{A}}(\mathbf{B})) = \mathbf{B}$, is isomorphic to a pathological subalgebra $\mathbf{B}'$, for which $\mathrm{C}_{\mathbf{A}}(\mathrm{C}_{\mathbf{A}}(\mathbf{B}')) = \mathbf{A}$. We show that many other cases appear between these two extremities.

Proposition 12.8 *Let $\boldsymbol{A} \in \mathcal{C}_F^\omega$ and $\boldsymbol{A}'$ a factor. Every infinite dimensional factor $\boldsymbol{B}$ of $\boldsymbol{A}'$ is isomorphic to a subalgebra $\boldsymbol{B}' \subseteq \boldsymbol{A}$ such that $\mathrm{C}_{\boldsymbol{A}}(\mathrm{C}_{\boldsymbol{A}}(\boldsymbol{B}')) = \boldsymbol{A}'$.*

Proof We may decompose $\mathbf{A} = \mathbf{A}' \otimes \mathbf{A}''$ and $\mathbf{A}' = \mathbf{B} \otimes \mathbf{C}$. By Theorem 12.6 $\mathbf{B}$ is isomorphic to a subalgebra $\mathbf{B}' \subseteq \mathbf{A}'$ with $\mathrm{C}_{\mathbf{A}'}(\mathbf{B}') = F$. But then $\mathrm{C}_{\mathbf{A}}(\mathbf{B}') = \mathrm{C}_{\mathbf{A}' \otimes \mathbf{A}''}(\mathbf{B}' \otimes F) = \mathrm{C}_{\mathbf{A}'}(\mathbf{B}') \otimes \mathbf{A}'' = F \otimes \mathbf{A}'' = \mathbf{A}''$, and $\mathrm{C}_{\mathbf{A}}(\mathrm{C}_{\mathbf{A}}(\mathbf{B}')) = \mathrm{C}_{\mathbf{A}}(\mathbf{A}'') = \mathbf{A}'$. □

Dimensions of pathological subalgebras are used later on to construct a direct limit of matrix algebras which is not an infinite tensor product (see Remark 17.9).

13 Sylow Subalgebras and Primary Decomposition

In this section we define a notion of Sylow subalgebras for algebras in $\mathcal{C}_F$. Every finite dimensional central simple algebra decomposes uniquely as a tensor product of components of prime-power degree. We show that this fact generalizes to $\mathcal{C}_F^\omega$, where we allow degree p^∞, and countable tensor products. This was proved by Koethe [20] for division algebras.

13.1 *p*-Algebras

Fix a prime p. We say that an algebra $\mathbf{A} \in \mathcal{C}_F$ (finite or infinite) is a **p-algebra** if its degree is a p-power. Notice that by definition every p-algebra is locally finite and central simple. (Traditionally the term p-algebra refers to finite dimensional central simple algebras over a field of characteristic p, whose degree [2] or index [16] are a power of p; we are indifferent to the characteristic throughout the paper).

Proposition 13.1 *A direct limit of p-algebras is a p-algebra.*

Proof This is a direct consequence of Proposition 7.4. □

Corollary 13.2 *Let $\boldsymbol{A} \in \mathcal{C}_F$. Every p-subalgebra of $\boldsymbol{A}$ is contained in a maximal p-subalgebra.*

Indeed, Proposition 13.1 allows us to apply Zorn's lemma, since the union over a chain of subalgebras is its direct limit. By Propositions 7.6 and 13.1 we also have:

Corollary 13.3 *An infinite tensor product of p-algebras is a p-algebra.*

We move to study p-subalgebras which are "locally maximal".

13.2 Sylow Subalgebras

Let $\mathbf{A} = \varinjlim \{A_\gamma, \varphi_{\gamma\gamma'}\}$ be a direct limit of the system $\{A_\gamma, \varphi_{\gamma\gamma'}\}$ index by a directed set Γ. A **system of subalgebras** is composed of subalgebras $B_\gamma \subseteq A_\gamma$, for each $\gamma \in \Gamma$, such that the maps $\varphi_{\gamma\gamma'} : A_\gamma \to A_{\gamma'}$ restrict to maps $B_\gamma \to B_{\gamma'}$ for every $\gamma < \gamma'$. (Not to be confused with the notion of a subsystem of algebras, which refers to a system of algebras defined on a directed subset of Γ). The direct limit $\varinjlim B_\gamma$ of a system of subalgebras is defined, and there is a natural embedding $\varinjlim B_\gamma \hookrightarrow \varinjlim A_\gamma$.

Definition 13.4 Let $\mathbf{A} \in \mathcal{C}_F$. A subalgebra $\mathbf{A}^\circ \subseteq \mathbf{A}$ is a p-**Sylow subalgebra** of $\mathbf{A}$ if there is a presentation of $\mathbf{A}$ as a direct limit $\mathbf{A} = \varinjlim \{A_\gamma, \varphi_{\gamma\gamma'}\}$ with A_γ finite dimensional and central simple, and a system of subalgebras $\{A_\gamma^\circ\}$ of $\{A_\gamma, \varphi_{\gamma\gamma'}\}$ whose components A_γ° are maximal p-subalgebras of the respective algebras A_γ, such that $\mathbf{A}^\circ = \varinjlim A_\gamma^\circ$.

Notice that if $\mathbf{A}$ is a finite dimensional central simple algebra, then its Sylow subalgebras are precisely the subalgebras of maximal p-power degree, and (by Skolem-Noether) they are all conjugate to each other.

Remark 13.5 By Proposition 13.1, every p-Sylow subalgebra is a p-algebra.

We do not know if Sylow subalgebras exist in general. We do prove below the existence of Sylow subalgebras for some classes of algebras.

Remark 13.6 The unique Sylow subalgebra of a p-algebra $\mathbf{A} \in \mathcal{C}_F$ is $\mathbf{A}$ itself. Indeed, in any presentation $\mathbf{A} = \varinjlim A_\gamma$, the only choice of maximal p-subalgebras $A_\gamma^\circ \subseteq A_\gamma$ is $A_\gamma^\circ = A_\gamma$.

More generally,

Proposition 13.7 *Let $\boldsymbol{P} \in \mathcal{C}_F$ be a p-algebra, and $\boldsymbol{C} \in \mathcal{C}_F$ an algebra of degree prime to p. Then $\boldsymbol{P}$ is a Sylow subalgebra of $\boldsymbol{P} \otimes \boldsymbol{C}$.*

Proof Write $\mathbf{P} = \varinjlim_\Gamma P_\gamma$ and $\mathbf{C} = \varinjlim_\Delta B_\delta$, where P_γ, B_δ are finite dimensional and central simple. Then $\mathbf{P} \otimes \mathbf{C} = \varinjlim_{\Gamma\times\Delta} (P_\gamma \otimes B_\delta)$, as discussed in Section 4.2. Now $P_\gamma \otimes F \subseteq P_\gamma \otimes B_\delta$ is a directed system of subalgebras, composed of maximal p-subalgebras, so the limit $\varinjlim_{\Gamma\times\Delta} (P_\gamma \otimes F) = \varinjlim_\Delta (\varinjlim_\Gamma P_\gamma) = \varinjlim_\Delta P = \mathbf{P}$ is a Sylow subalgebra. □

Proposition 13.8 *Let $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{C}_F$ be algebras. If $\boldsymbol{A}^\circ \subseteq \boldsymbol{B}$ and $\boldsymbol{B}^\circ \subseteq \boldsymbol{B}$ are p-Sylow subalgebras, then $\boldsymbol{A}^\circ \otimes \boldsymbol{B}^\circ$ is a p-Sylow subalgebra of $\boldsymbol{A} \otimes \boldsymbol{B}$.*

Proof Write $\mathbf{A} = \varinjlim_{\Gamma} A_\gamma$ and $\mathbf{A}^\circ = \varinjlim_{\Gamma} A_\gamma^\circ$ where A_γ are finite dimensional central simple algebras and A_γ° are maximal p-subalgebras; and likewise $\mathbf{B} = \varinjlim_{\Delta} B_\delta$ and $\mathbf{B}^\circ = \varinjlim_{\Delta} B_\delta^\circ$. Then $\mathbf{A}^\circ \otimes \mathbf{B}^\circ = \varinjlim_{\Gamma\times\Delta} A_\gamma^\circ \otimes B_\delta^\circ$ is a p-Sylow subalgebra of $\mathbf{A} \otimes \mathbf{B} = \varinjlim_{\Gamma\times\Delta} A_\gamma \otimes B_\delta$ by definition. □

More generally,

Proposition 13.9 *Let $\mathbf{A}_i \in \mathcal{C}_F$ be algebras, where i runs over an arbitrary index set I. If $\mathbf{A}_i^\circ \subseteq \mathbf{A}_i$ are p-Sylow subalgebras then $\bigotimes \mathbf{A}_i^\circ$ is a p-Sylow subalgebra of $\bigotimes \mathbf{A}_i$.*

Proof For each i write $\mathbf{A}_i \in \mathcal{C}_F^{\Gamma_i}$ as a direct limit $\mathbf{A}_i = \varinjlim_{\Gamma_i} A_{i\gamma}$ where $\mathbf{A}_i^\circ = \varinjlim_{\Gamma_i} A_{i\gamma}^\circ$, where each $A_{i\gamma}^\circ$ is a maximal p-subalgebra of $A_{i\gamma}$. We assume each Γ_i is pointed (see Definition 10.9). For $f \in \sum \Gamma_i$, we let $A_f = \otimes_{i \in \mathrm{supp}(f)} A_{i,f(i)}$ and $A_f^\circ = \otimes_{i \in \mathrm{supp}(f)} A_{i,f(i)}^\circ$, so clearly A_f° is a maximal p-subalgebra of A_f. Now, by Proposition 10.10, $\bigotimes \mathbf{A}_i^\circ = \varinjlim_{f\in\sum\Gamma_i} A_f^\circ \subseteq \varinjlim_{f\in\sum\Gamma_i} A_f = \bigotimes \mathbf{A}_i$ is a p-Sylow subalgebra by definition. □

Recall that by Corollary 10.6, if all the $\mathbf{A}_i^\circ$ are factors, then $\bigotimes \mathbf{A}_i^\circ$ is a factor of $\bigotimes \mathbf{A}_i$.

Proposition 13.10 *Let $\mathbf{A} \in \mathcal{C}_F$. Every division p-Sylow subalgebra $\mathbf{P} \subseteq \mathbf{A}$ is maximal as a division p-subalgebra.*

Proof Write $\mathbf{A} = \varinjlim_{\Gamma} A_\gamma$ where $\mathbf{P} = \varinjlim_{\Gamma} P_\gamma$ and $P_\gamma \subseteq A_\gamma$ are maximal p-subalgebras. Assume $\mathbf{P} \subseteq \mathbf{Q} \subseteq \mathbf{A}$ where $\mathbf{Q}$ is a division p-subalgebra. Let $x \in \mathbf{Q}$. There is $\gamma \in \Gamma$ such that $x \in A_\gamma$. Consider the subalgebra $P_\gamma[x] \subseteq A_\gamma$ generated by P_γ and x. There is a finite dimensional central simple subalgebra $Q_0 \subseteq \mathbf{Q}$ containing $P_\gamma[x]$, and since $P_\gamma[x]$ is a division algebra, being a subalgebra of $\mathbf{Q}$, the dimension of $P_\gamma[x]$ divides $\dim Q_0$ which is a p-power. But on the other hand, the maximal p-power dividing $\dim A_\gamma$ is the dimension of P_γ, so $P_\gamma[x] \subseteq P_\gamma$ and $x \in P_\gamma \subseteq \mathbf{P}$. □

It immediately follows that:

Theorem 13.11 *Let $\mathbf{A} \in \mathcal{C}_F$ be a division algebra. Then every p-Sylow subalgebra $\mathbf{P} \subseteq \mathbf{A}$ is maximal as a p-subalgebra.*

In Proposition 18.7 we extend this result to any $\mathbf{A} \subseteq \mathcal{C}_F$ which does not contain a copy of $\mathrm{M}_{p^\infty}(F)$.

13.3 Sylow Subalgebras in $\mathcal{C}_F^\omega$

Proposition 13.12 *Every $\mathbf{A} \in \mathcal{C}_F^\omega$ has Sylow subalgebras.*

Proof For every presentation $\mathbf{A} = \varinjlim_{\omega} \{A_n, f_{nn'}\}$, we can choose maximal p-subalgebras $A_n^\circ \subseteq A_n$ by induction such that $f_{n,n+1}(A_n^\circ) \subseteq A_{n+1}^\circ$ for every n. (This also follows from Proposition Section 10.13SylowTen.) □

Theorem 13.13 *Let $\mathbf{A} \in \mathcal{C}_F^\omega$. The Sylow subalgebras of $\mathbf{A}$ are isomorphic to each other.*

Proof First of all, when we present $\mathbf{A} = \varinjlim A_n$, each A_n° is uniquely determined up to isomorphism, being a maximal p-subalgebra of A_n. By Lemma 9.5, $\varinjlim A_n^\circ$ is uniquely determined up to isomorphism.

We need to compare the Sylow subalgebras obtained from presentations of the same algebra $\mathbf{A} \in \mathcal{C}_F^\omega$ as different direct limits. By Proposition 9.7, we may assume the two presentations have the form $\mathbf{A} = \varinjlim A_n = \varinjlim A_{n_k}$, where $n_1 < n_2 < \cdots$ is a series of natural numbers. But then we may further assume that $\{A_{n_k}^\circ\}$ is a subseries of $\{A_n^\circ\}$. □

Lemma 13.14 *Let* $\boldsymbol{A} \in \mathcal{C}_F^\omega$ *and let* $\boldsymbol{P} \subseteq \boldsymbol{A}$ *be a p-subalgebra. Then there is an embedding of* $\boldsymbol{P}$ *into some (and thus every) Sylow subalgebra of* $\boldsymbol{A}$.

Proof Apply Proposition 9.1 to write $\mathbf{A} = \varinjlim A_n$ and $\mathbf{P} = \varinjlim P_n$ where $P_n \subseteq A_n$ for every n. Applying the proof of Proposition 13.12 choose a Sylow subalgebra $\varinjlim A_n^\circ$ of $\mathbf{A}$ where $A_n^\circ \subseteq A_n$ are maximal p-subalgebras.

Since $P_n \subseteq A_n$, there are embeddings $P_n \hookrightarrow A_n^\circ$ (not necessarily the ones induced from the embedding $P_n \subseteq A_n$). By Proposition 9.4, $\mathbf{P} \hookrightarrow \varinjlim A_n^\circ$. □

(However, it is not clear if any p-subalgebra is *contained* in a p-Sylow subalgebra).

Lemma 13.15 *Let* $\boldsymbol{A}, \boldsymbol{B}$ *be algebras in* $\mathcal{C}_F^\omega$ *with an embedding* $\boldsymbol{A} \hookrightarrow \boldsymbol{B}$. *Let* $\boldsymbol{A}^\circ \subseteq \boldsymbol{A}$ *and* $\boldsymbol{B}^\circ \subseteq \boldsymbol{B}$ *be p-Sylow subalgebras. Then there is an embedding* $\boldsymbol{A}^\circ \hookrightarrow \boldsymbol{B}^\circ$.

Proof By Remark 13.5, $\mathbf{A}^\circ$ is a p-algebra, so the embedding follows from Lemma 13.14. □

Proposition 13.16 *The degree of a p-Sylow subalgebra of* $\boldsymbol{A} \in \mathcal{C}_F^\omega$ *is the maximal p-power dividing* $\deg(\boldsymbol{A})$.

Proof Write $\mathbf{A} = \varinjlim_\omega A_n$. Let $\mathbf{A}^\circ$ be a p-Sylow subalgebra of $\mathbf{A}$. By Remark 13.5, the degree of $\mathbf{A}^\circ$ is a p-power, which divides $\deg(\mathbf{A})$ by Proposition 7.3. Suppose $\deg(\mathbf{A})$ is divisible by p^α. If α is finite, then for some n, $\deg(A_n)$ is divisible by p^α, and the same holds for the p-component of A_n. If α is infinite, then for every k, when n is large enough, $\deg(A_n)$ is divisible by p^k, and the same holds for the p-component of A_n. □

Corollary 13.17 *The base field F is a p-Sylow subalgebra of* $\boldsymbol{A} \in \mathcal{C}_F^\omega$ *iff* $\deg(\boldsymbol{A})$ *is prime to p.*

13.4 Primary Decomposition

If an algebra $\mathbf{A} \in \mathcal{C}_F$ can be presented as a (countable) tensor product $\bigotimes \mathbf{A}_p$, where each $\mathbf{A}_p$ is a p-algebra in $\mathcal{C}_F$ (for distinct primes p), we say that **A has a primary decomposition**.

Proposition 13.18 *If* $\boldsymbol{A} = \bigotimes \boldsymbol{A}_p$ *is a primary decomposition, then each* $\boldsymbol{A}_p$ *is a p-Sylow subalgebra.*

Proof This is a special case of Proposition 13.7, because for every prime p, the degree of $\bigotimes_{p' \neq p} \mathbf{A}^{(p')}$ is prime to p by Proposition 10.17. □

Theorem 13.19 *Let I be an arbitrary set and let $\mathbf{A}_i \in \mathcal{C}_F$ for $i \in I$. If each $\mathbf{A}_i$ has a primary decomposition, then so does $\bigotimes_{i\in I}\mathbf{A}_i$.*

Proof By assumption, for each i there are p-algebras $\mathbf{A}_{ip}$ (for the primes p) such that $\mathbf{A}_i = \bigotimes_p \mathbf{A}_{ip}$. By Proposition 10.5, $\bigotimes_I \mathbf{A}_i = \bigotimes_I \bigotimes_p \mathbf{A}_{ip} = \bigotimes_p (\bigotimes_I \mathbf{A}_{ip})$, and each $\bigotimes_I \mathbf{A}_{ip}$ is a p-algebra by Proposition 13.3. □

In particular, we have:

Corollary 13.20 *An infinite tensor product of finite dimensional central simple algebras has primary decomposition.*

Applying Proposition 10.13 we get:

Corollary 13.21 *Every $\mathbf{A} \in \mathcal{C}_F^\omega$ has a primary decomposition.*

The primary decomposition of algebras in $\mathcal{C}_F^\omega$ is unique:

Theorem 13.22 (Primary Decomposition Theorem) *Every algebra $\mathbf{A} \in \mathcal{C}_F^\omega$ has a unique primary decomposition.*

Proof Existence is Corollary 13.21. Assume $\mathbf{A} = \bigotimes \mathbf{A}^{(p)} \cong \bigotimes \mathbf{B}^{(p)}$ where $\mathbf{A}^{(p)}$ and $\mathbf{B}^{(p)}$ are p-algebras in $\mathcal{C}_F^\omega$.

For any p, by Proposition 13.18 both $\mathbf{A}^{(p)}$ and $\mathbf{B}^{(p)}$ are p-Sylow subalgebras of $\mathbf{A}$, so they are isomorphic by Theorem 13.13. □

Theorem 13.22 was proved by Koethe for division algebras in [20]. More generally, we have:

Corollary 13.23 *Let $\mathbf{A}, \mathbf{B} \in \mathcal{C}_F^\omega$, with primary decompositions $\mathbf{A} = \bigotimes \mathbf{A}^{(p)}$ and $\mathbf{B} = \bigotimes \mathbf{B}^{(p)}$. Then $\mathbf{A} \hookrightarrow \mathbf{B}$ if and only if $\mathbf{A}^{(p)} \hookrightarrow \mathbf{B}^{(p)}$ for every prime p.*

Proof By Proposition 13.18, $\mathbf{A}^{(p)}$ and $\mathbf{B}^{(p)}$ are p-Sylow subalgebras of $\mathbf{A}$ and $\mathbf{B}$, respectively. If $\mathbf{A} \hookrightarrow \mathbf{B}$ then $\mathbf{A}^{(p)} \hookrightarrow \mathbf{B}^{(p)}$ for every p by Lemma 13.15. In the other direction, given $f_p : \mathbf{A}^{(p)} \hookrightarrow \mathbf{B}^{(p)}$, $\bigotimes f_p = \varinjlim (f_2 \otimes f_3 \otimes \cdots \otimes f_p)$ is an embedding of $\mathbf{A}$ in $\mathbf{B}$. □

13.5 Pathological Sylow Subalgebras

The Sylow subalgebra of Proposition 13.7 is a factor. This is far from being true in general.

Theorem 13.24 *Let $\mathbf{B} \in \mathcal{C}_F^\omega$ be an infinite dimensional p-algebra and $F \neq \mathbf{C} \in \mathcal{C}_F^\omega$, such that* $\deg(\mathbf{C})$ *is prime to p. Then $\mathbf{B}$ is isomorphic to a proper pathological Sylow subalgebra of $\mathbf{B} \otimes \mathbf{C}$.*

Proof We follow the notation of Theorem 12.6, and decompose $\mathbf{C} = \bigotimes C_n$. Write $\tilde{D}_n = D_1 \otimes \cdots \otimes D_n$. In Theorem 12.6, we construct an embedding $\mathbf{B} = \varinjlim \tilde{D}_n \hookrightarrow \mathbf{A} = (C_n, D_n; \psi_n) = \varinjlim \left\{ \tilde{D}_n \otimes C_n, 1 \otimes \psi_n \iota_n \right\}$ as a pathological subalgebra, but since $\deg(C_n)$ are all prime to p, each component $\tilde{D}_n$ is a maximal p-subalgebra of the respective component $\tilde{D}_n \otimes C_n$, so **B** embeds as a Sylow subalgebra, which is proper because $\deg(\mathbf{A}) = \deg(\mathbf{B})\deg(\mathbf{C}) \neq \deg(\mathbf{B})$. Now, the isomorphism $\mu: \mathbf{A} \to \mathbf{B} \otimes \mathbf{C}$ carries **B** to a proper pathological Sylow subalgebra. □

Corollary 13.25 *Every* $\mathbf{A} \in \mathcal{C}_F^{\omega}$ *with degree divisible by but not equal to* p^{∞} *has pathological* p*-Sylow subalgebras.*

Indeed, **A** has an infinite dimensional factor which is a p-algebra by Primary Decomposition, so we can apply Theorem 13.24.

14 Clifford Algebras

Clifford algebras serve as an important cohomological invariant of finite dimensional nonsingular quadratic forms, with a wide range of applications. In this section we study Clifford algebras of infinite dimensional nonsingular spaces. We show that these are in $\mathcal{C}_F$, and demonstrate once more pathological behavior of the centralizers. In particular we provide an example of a centralizer of a locally finite central simple subalgebra, which is not central. For simplicity we assume in this section that $\operatorname{char} F \neq 2$, although this is not essential, see [19, Section 8].

A **quadratic space** is a pair (V, q) in which V is a vector space and $q : V \to F$ a quadratic form. One infinite dimensional example is the direct limit of finite dimensional quadratic spaces, $(V, q) = \varinjlim (V_n, q_n)$. Another example would be a Hilbert space H, with the norm form $q(x) = (x, x)_H$, stripped of all the topological data; in particular a basis is in the algebraic sense, and a subspace is not necessarily closed.

Definition 14.1 The **Clifford algebra** of a quadratic space (V, q) is by definition the tensor algebra $\bigoplus_{n \geq 0} V^{\otimes n}$ of V, modulo the ideal generated by the elements $v \otimes v - q(v)$, $v \in V$.

Remark 14.2 The natural grading of the tensor algebra induces a $\mathbb{Z}/2\mathbb{Z}$-grading of the Clifford algebra, $\operatorname{Cl}(V, q) = \operatorname{Cl}_0(V, q) \oplus \operatorname{Cl}_1(V, q)$.

Remark 14.3 Assume V is finite dimensional and nonsingular. Then:

(1) The Clifford algebra $\operatorname{Cl}(V, q)$ is simple, of degree $2^{\dim V}$.
(2) The product $z_V = v_1 \cdots v_n$ of the vectors in an arbitrary orthogonal basis of V is independent of the choice of the basis.
(3) The center of $\operatorname{Cl}(V, q)$ is equal to F if $\dim V$ is even, and to the étale quadratic extension $F[z_V]$ if $\dim V$ is odd.
(4) The center of $\operatorname{Cl}_0(V, q)$ is equal to $F[z_V]$ if $\dim V$ is even, and to F if $\dim V$ is odd.

From the definition it follows that if (V, q) is a quadratic space and $W \subseteq V$ is a subspace, then there is a natural embedding $\operatorname{Cl}(W) \hookrightarrow \operatorname{Cl}(V)$, whose image is the subalgebra of $\operatorname{Cl}(V)$ generated by W.

Recall that a quadratic space (V, q) is **nonsingular** if the radical $\operatorname{rad} V = \{x : (x, V) = 0\}$ equals zero, where $(\cdot, \cdot)$ is the bilinear form associated to q.

Lemma 14.4 *Let* V *be any quadratic space, and* $U \subseteq V$ *a finite dimensional nonsingular subspace. Then* $V = U \oplus U^{\perp}$.

Proof Indeed, $U \cap U^{\perp} = \text{rad}U = 0$ by assumption. Let $\iota : V \to U^* = \text{Hom}(U, F)$ be defined by $x \mapsto (-, x)$. Then $V/(U + U^{\perp}) \cong \iota V/\iota U = 0$ because $\iota U = U^*$. □

(The finiteness assumption is essential here: if U is a dense subspace of a Hilbert space V, then $U + U^{\perp} = U < V$.)

Proposition 14.5 *The following are equivalent for an infinite dimensional quadratic space* (V, q)*:*

1. V *is nonsingular.*
2. *Every finite dimensional subspace* U *is contained in a finite dimensional nonsingular subspace* $U' \supseteq U$.
3. (V, q) *is a direct limit of finite dimensional nonsingular subspaces.*
4. *Every finite dimensional subspace* U *is contained in an even dimensional nonsingular subspace* $U' \supseteq U$.
5. (V, q) *is a direct limit of even dimensional nonsingular subspaces.*

Proof (1) $\Longrightarrow$ (2): let U be a finite dimensional subspace. If U is nonsingular we are done, so let $0 \neq u_0 \in \text{rad}\, U$. Since $u_0 \notin \text{rad}\, V$, there is $v_0 \in V$ such that $(u_0, v_0) \neq 0$. Let $U' = U + Fv_0$. It is easy to see that $\text{rad}\, U' \subsetneq \text{rad}\, U$, so we are done by induction on the dimension of the radical. (2) $\Longrightarrow$ (1): by assumption any vector in the radical is contained in a nonsingular space, and thus must be zero. (2)$\Longleftrightarrow$(3) and (4)$\Longleftrightarrow$(5) similarly to Proposition 2.2. (4) $\Longrightarrow$ (2) is obvious. To prove that (2) $\Longrightarrow$ (4), let $U \subseteq V$ be an odd dimensional nonsingular subspace. Write $V = U \oplus U^{\perp}$ using Lemma 14.4. The form q cannot reduce to zero on the whole subspace $U^{\perp}$ because then subspaces of dimension larger than $\dim U$ cannot be embedded in a finite dimensional nonsingular space, so we can choose an anisotropic $x \in U^{\perp}$, and then $U + Fx$ is nonsingular. □

Proposition 14.6 *Let* (W_λ, q_λ) *be a system of nonsingular quadratic spaces. Then* $\text{Cl}(\varinjlim (W_\lambda, q_\lambda)) = \varinjlim \text{Cl}(W_\lambda, q_\lambda)$.

Proof Indeed, let $(V, q) = \varinjlim (W_\lambda, q_\lambda)$. For each λ the map $W_\lambda \to V$ induces a map $\text{Cl}(W_\lambda, q_\lambda) \to \text{Cl}(V, q)$, and these maps are compatible. On the other hand every element of V is in the image of some W_λ, so the same holds for the Clifford algebras. □

Proposition 14.7 *Let* (V, q) *be a nonsingular quadratic space. Then* $\text{Cl}(V, q)$ *is in* $\mathcal{C}_F$. *Moreover,* $\deg \text{Cl}(V, q)$ *is a (possibly infinite) power of 2.*

Proof By Proposition 14.5 we may write V as a direct limit of even dimensional nonsingular subspaces W_λ. Now $\text{Cl}(V, q) = \varinjlim \text{Cl}(W_\lambda, q_\lambda)$ by Proposition 14.6, which is in $\mathcal{C}_F$ because each $\text{Cl}(W_\lambda, q_\lambda)$ is central simple by Remark 14.3. □

14.1 Centralizers of Clifford Subalgebras

Let U be a nonsingular subspace of an arbitrary quadratic space V. We compute the centralizer of $\text{Cl}(U)$ in $\text{Cl}(V)$. We first settle the case where U has an orthogonal complement.

Lemma 14.8 *Let $V = U \perp W$ be an orthogonal sum of quadratic spaces. Then the centralizer of* $\mathrm{Cl}(U)$ *in* $\mathrm{Cl}(V)$ *is*

$$\mathrm{C}_{\mathrm{Cl}(V)}(\mathrm{Cl}(U)) = \begin{cases} \mathrm{Cl}_0(W) & \textit{if } \dim U \textit{ is infinite} \\ \mathrm{Cl}_0(W) + z_U \mathrm{Cl}_0(W) & \textit{if } \dim U \textit{ is even} \\ \mathrm{Cl}_0(W) + z_U \mathrm{Cl}_1(W) & \textit{if } \dim U \textit{ is odd} \end{cases}.$$

Proof By decomposing every product of elements from V as a sum of products of elements from U and W, and using the fact that the former anticommute with the latter, we see that $\mathrm{Cl}(V) = \mathrm{Cl}(U)\mathrm{Cl}(W) = \mathrm{Cl}(U)\mathrm{Cl}_0(W) \oplus \mathrm{Cl}(U)\mathrm{Cl}_1(W)$.

Our goal is to compute the centralizer of $\mathrm{Cl}(U)$ in $\mathrm{Cl}(V)$. By definition, every element from W anticommutes with every element from U. It follows that $\mathrm{Cl}_0(W)$ commutes with $\mathrm{Cl}(U)$. Fix a basis $B = \{u_t : t \in T\}$ of U (finite or infinite); thus, $B^* = \left\{u_{t_1} \cdots u_{t_m} : t_1 < t_2 < \cdots < t_m\right\}$ is a basis of $\mathrm{Cl}(U)$. A general element $r \in \mathrm{Cl}(V)$ can be presented, uniquely, as $r = \sum f_i(g_i + h_i)$, where $f_i \in B^*$, $g_i \in \mathrm{Cl}_0(W)$ and $h_i \in \mathrm{Cl}_1(W)$, where for each i either $g_i \neq 0$ or $h_i \neq 0$. We assume this element centralizes $\mathrm{Cl}(U)$.

For every $u \in B$, the commutator is $0 = [r, u] = [\sum f_i(g_i + h_i), u] = \sum [f_i, u]g_i - (f_i u + u f_i)h_i$. Because of the direct decomposition, $0 = \sum [f_i, u]g_i = \sum (f_i u + u f_i)h_i$.

Since u is fixed, the expressions $[f_i, u]$ and $(f_i u + u f_i)$ are (possibly zero) multiples of distinct elements from B^*, hence they are independent. Therefore for each i either $[f_i, u] = 0$ or $g_i = 0$; and for each i either $f_i u + u f_i = 0$ or $h_i = 0$.

So, for each i, exactly one of g_i or h_i is zero (because we cannot have $f_i u - u f_i = f_i u + u f_i = 0$). But this partition is independent of u. So for each i, either $f_i u = u f_i$ for all u; which is possible if and only if $f_i = 1$ or, when $\dim U$ is finite and odd, $f_i = z_U$; or $f_i u = -u f_i$ for all u, which is possible if and only if $\dim U$ is finite and even, and $f_i = z_U$.

In particular, r has at most two summands: if $\dim U$ is infinite then $r = g_1$; if $\dim U$ is odd, $r = g_1 + z_U g_2$; and if $\dim U$ is even, then $r = g_1 + z_U h_1$, proving the claim. □

This computation leads to an example of a centralizer of a central simple subalgebra which is not central:

Example 14.9 Let U, U' be nonsingular quadratic spaces such that U has infinite dimension and U' is odd dimensional. Let $V = U \perp U'$, an orthogonal sum. Then $\mathrm{Cl}(V) \in \mathcal{C}_F$, $\mathrm{Cl}(U) \in \mathcal{C}_F$ is a subalgebra, but the centralizer $\mathrm{C}_{\mathrm{Cl}(V)}(\mathrm{Cl}(U)) = \mathrm{Cl}_0(U')$ is non-central by Remark 14.3. (4).

Theorem 14.10 *Let V be a quadratic space, and U an infinite dimensional nonsingular subspace. Then*

$$\mathrm{C}_{\mathrm{Cl}(V)}(\mathrm{Cl}(U)) = \mathrm{Cl}_0(U^{\perp}).$$

Proof Write $U = \varinjlim U_\lambda$, a direct limit of even dimensional nonsingular subspaces. By Lemma 14.4 we have for every λ that $V = U_\lambda \perp U_\lambda^{\perp}$, so we can apply Lemma 14.8 and get

$$\mathrm{C}_{\mathrm{Cl}(V)}(\mathrm{Cl}(U)) = \bigcap_{\lambda} \mathrm{C}_{\mathrm{Cl}(V)}(\mathrm{Cl}(U_\lambda)) = \bigcap_{\lambda} F[z_{U_\lambda}]\mathrm{Cl}_0(U_\lambda^{\perp}).$$

To compute this intersection, notice that for every $\lambda < \lambda'$ we have that $U_\lambda \subseteq U_{\lambda'}$ so $U_\lambda^{\perp} \supseteq U_{\lambda'}^{\perp}$ and $\mathrm{Cl}_0(U_\lambda^{\perp}) \supseteq \mathrm{Cl}_0(U_{\lambda'}^{\perp})$ and now the intersection of $F[z_{U_\lambda}]\mathrm{Cl}_0(U_\lambda^{\perp})$ with

$F[z_{U_{\lambda'}}]\mathrm{Cl}_0(U_{\lambda'}^{\perp})$, as subspaces of $F[z_{U_\lambda}, z_{U_{\lambda'}}]\mathrm{Cl}_0(U_{\lambda'}^{\perp})$, is equal to $\mathrm{Cl}_0(U_{\lambda}^{\perp})$. Therefore $\bigcap_\lambda F[z_{U_\lambda}]\mathrm{Cl}_0(U_\lambda^{\perp}) = \mathrm{Cl}_0(\bigcap U_\lambda^{\perp}) = \mathrm{Cl}(U^{\perp})$ because $\bigcap U_\lambda^{\perp} = U^{\perp}$. □

Example 14.11 (A pathological Clifford subalgebra) Let U be an infinite dimensional nonsingular subspace of a nonsingular space V. Assume further that $U^{\perp} = 0$ (for example, U is a dense subspace of a Hilbert space V). Then $\mathrm{Cl}(U)$ is a pathological subalgebra of $\mathrm{Cl}(V)$, because $\mathrm{C}_{\mathrm{Cl}(V)}(\mathrm{Cl}(U)) = \mathrm{Cl}_0(U^{\perp}) = \mathrm{Cl}_0(0) = F$ by the theorem.

Remark 14.12 Let V be a quadratic space, and U an infinite dimensional nonsingular subspace. Then

$$\mathrm{C}_{\mathrm{Cl}(V)}(\mathrm{Cl}_0(U)) = \mathrm{Cl}(U^{\perp}).$$

The proof is similar to Theorem 14.10, where Lemma 14.8 is replaced by

$$\mathrm{C}_{\mathrm{Cl}(V)}(\mathrm{Cl}_0(U)) = \begin{cases} \mathrm{Cl}(W) & \text{if } \dim U \text{ is infinite} \\ \mathrm{Cl}(W) + z_U\mathrm{Cl}(W) & \text{if } \dim U \text{ is finite} \end{cases},$$

with essentially the same proof.

If $U + U^{\perp}$ is a nonsingular subspace (equivalently U, $U^{\perp}$ are nonsingular), we obtain a double centralizer formula:

$$\mathrm{C}_{\mathrm{Cl}(V)}(\mathrm{C}_{\mathrm{Cl}(V)}(\mathrm{Cl}(U))) = \mathrm{C}_{\mathrm{Cl}(V)}(\mathrm{Cl}_0(U^{\perp})) = \mathrm{Cl}(U^{\perp\perp}).$$

Part III. Supernatural Matrices and the Matrix Degree

15 Supernatural Matrices

We define matrix algebras of arbitrary supernatural degree, and study their properties. A supernatural number is the least common multiple of its (finite) natural divisors, which form a directed set under the divisibility relation. We can thus define:

Definition 15.1 Let $\mathbf{n}$ be a supernatural number. We define the **supernatural matrix algebra** $\mathrm{M}_{\mathbf{n}}(F) = \varinjlim_{n|\mathbf{n}} \mathrm{M}_n(F)$, where the maps $\mathrm{M}_n(F) \to \mathrm{M}_{n'}(F)$ for $n \mid n'$ are the diagonal block embeddings $a \mapsto a \otimes 1 = a \oplus \cdots \oplus a$.

Remark 15.2 By Corollary 7.5, $\deg \mathrm{M}_{\mathbf{n}}(F) = \mathbf{n}$.

Although the divisor lattice of $\mathbf{n}$ is isomorphic to ω only when $\mathbf{n} = p^{\infty}$ for a natural prime p, we always have that $\mathrm{M}_{\mathbf{n}}(F) \in \mathcal{C}_F^{\omega}$ by Proposition 8.1, since being generated by countably many matrix algebras, $\mathrm{M}_{\mathbf{n}}(F)$ has countable dimension.

Notice that up to conjugation, the diagonal maps $\mathrm{M}_n(F) \to \mathrm{M}_m(F)$ are forced by the assumption that the maps are unital: indeed, every embedding $\mathrm{M}_n(F) \to \mathrm{M}_m(F)$ takes $\mathrm{M}_n(F)$ into a central simple subalgebra, and induces a decomposition via the centralizer, so $n \mid m$ and the centralizer is isomorphic to $\mathrm{M}_{m/n}(F)$.

Proposition 15.3 *For any directed system* Γ *of supernatural divisors of* $\mathbf{n}$ *for which* $\mathbf{n} = \mathrm{lcm}\Gamma$*, we have that* $\mathrm{M}_{\mathbf{n}}(F) = \varinjlim_{\mathbf{m}\in\Gamma} \mathrm{M}_{\mathbf{m}}(F)$.

Proof If all the divisors in Γ are finite, the claim is clear by cofinality. In the general case, $\varinjlim_{\mathbf{m}\in\Gamma} \mathrm{M}_{\mathbf{m}}(F) = \varinjlim_{\mathbf{m}\in\Gamma} \varinjlim_{m\,|\,\mathbf{m}} \mathrm{M}_m(F) = \varinjlim_{n\,|\,\mathbf{n}} \mathrm{M}_n(F) = \mathrm{M}_{\mathbf{n}}(F)$ by Corollary 5.4. □

The diagonal embeddings are now seen to be immaterial, because by Proposition 9.5 a limit over ω is determined by the objects, regardless of the morphisms. However the assumption that these maps are unital is essential; otherwise we may obtain the algebra of finite matrices, which is a completely different object.

Example 15.4 The direct limit $\varinjlim \mathrm{M}_n(F)$ of all finite matrix algebras, under the divisibility order, is the supernatural matrix algebra of the largest supernatural degree, namely $\mathrm{M}_{2^\infty 3^\infty 5^\infty \dots}(F)$.

Proposition 15.5 *For every two supernatural numbers* $\boldsymbol{n}$ *and* $\boldsymbol{m}$*,*

$$\mathrm{M}_{\boldsymbol{n}}(F) \otimes \mathrm{M}_{\boldsymbol{m}}(F) \cong \mathrm{M}_{\boldsymbol{nm}}(F).$$

Proof Denote the sets of divisors of $\mathbf{n}$ by $\Gamma_{\mathbf{n}}$. Since there is a confinal copy of ω in $\Gamma_{\mathbf{n}}$, Proposition 5.5. Eq. 5.5 shows that $\mathcal{C}_F^{\Gamma_{\mathbf{n}}} \subseteq \mathcal{C}_F^{\omega}$.

By definition

$$\mathrm{M}_{\mathbf{n}}(F) \otimes \mathrm{M}_{\mathbf{m}}(F) = \varinjlim_{\Gamma_{\mathbf{n}}} \mathrm{M}_n(F) \otimes \varinjlim_{\Gamma_{\mathbf{m}}} \mathrm{M}_m(F) = \varinjlim_{\Gamma_{\mathbf{n}} \times \Gamma_{\mathbf{m}}} (\mathrm{M}_n(F) \otimes \mathrm{M}_m(F))$$

which is isomorphic to $\varinjlim_{\Gamma_{\mathbf{nm}}} \mathrm{M}_{nm}(F) = \mathrm{M}_{\mathbf{nm}}(F)$ by Proposition 9.6. □

In particular, if $m\mathbf{n} = \mathbf{n}$ (which is the case if and only if $m^\infty \mid \mathbf{n}$), then

$$\mathrm{M}_m(\mathrm{M}_{\mathbf{n}}(F)) \cong \mathrm{M}_{\mathbf{n}}(F). \tag{8}$$

Proposition 15.6 *If* $\boldsymbol{n}_1, \boldsymbol{n}_2, \dots$ *are supernatural numbers and* $\boldsymbol{n} = \prod \boldsymbol{n}_i$*, then* $\bigotimes_i \mathrm{M}_{\boldsymbol{n}_i}(F) \cong \mathrm{M}_{\boldsymbol{n}}(F)$.

Proof By Proposition 15.5 and Proposition 15.3, the set of supernatural matrices is closed under countable tensor products. Therefore $\bigotimes_i \mathrm{M}_{\mathbf{n}_i}(F)$ is a supernatural matrix algebra, whose degree is $\mathbf{n}$ by Proposition 10.17. We are done by Remark 15.2. □

Proposition 15.7 *Let* $\boldsymbol{n} = \prod p^{\alpha_p}$ *be the primary decomposition of* $\boldsymbol{n}$*. Then* $\mathrm{M}_{\boldsymbol{n}}(F) \cong \bigotimes \mathrm{M}_{p^{\alpha_p}}(F)$ *is the primary decomposition of the algebra.*

Proof The isomorphism follows from Proposition 15.6, and since $\deg(\mathrm{M}_{p^{\alpha_p}}) = p^{\alpha_p}$, this is a primary decomposition, which is unique by Theorem 13.22.

A future paper [7] discusses the connection of supernatural matrices with Leavitt pat algebras [1] and deep matrices [25]. □

16 The Matrix Degree of an Algebra

The matrix degree of a finite dimensional central simple algebra A_0 is defined as the maximal number n such that $\mathrm{M}_n(F) \hookrightarrow A_0$. In finite dimension, the matrix degree is $\deg(A_0)/\mathrm{ind}(A_0)$, where $\mathrm{ind}(A_0)$ is the degree of the underlying division algebra.

Definition 16.1 For an algebra $\mathbf{A}$ in $\mathcal{C}_F$, we define the **matrix degree** $\deg_{\mathrm{mat}} \mathbf{A}$ as the supernatural number which is the least common multiple of the matrix degrees of its finite dimensional central simple subalgebras.

Remark 16.2 For every $\mathbf{A} \in \mathcal{C}_F$, $\deg_{\mathrm{mat}} \mathbf{A} \mid \deg \mathbf{A}$, because the matrix degree divides the degree for finite dimensional algebras.

Remark 16.3 For algebras $\mathbf{A}, \mathbf{B} \in \mathcal{C}_F$, if there is an embedding $\mathbf{A} \hookrightarrow \mathbf{B}$, then $\deg_{\mathrm{mat}} \mathbf{A}$ divides $\deg_{\mathrm{mat}} \mathbf{B}$.

The argument proving Proposition 7.4 shows the following as well:

Proposition 16.4 *Let $A_\gamma \in \mathcal{C}_F$ be a directed system of algebras. Then* $\deg_{\mathrm{mat}} (\varinjlim A_\gamma) = \mathrm{lcm}\{\deg_{\mathrm{mat}} A_\gamma\}$.

Example 16.5 $\deg_{\mathrm{mat}} \mathrm{M}_{\mathbf{n}}(F) = \mathbf{n}$. Indeed $\deg_{\mathrm{mat}} \mathrm{M}_n(F) = n$ for any natural number n, and since by definition $\mathrm{M}_{\mathbf{n}}(F) = \varinjlim_{n \mid \mathbf{n}} \mathrm{M}_n(F)$, we have that $\deg_{\mathrm{mat}} \mathrm{M}_{\mathbf{n}}(F) = \mathrm{lcm}\{n : n \mid \mathbf{n}\} = \mathbf{n}$.

Proposition 16.6 *Let $\mathbf{A} \in \mathcal{C}_F$ and $\mathbf{n}$ a supernatural number. Then $\mathrm{M}_{\mathbf{n}}(F) \hookrightarrow \mathbf{A}$ if and only if $\mathbf{n} \mid \deg_{\mathrm{mat}} \mathbf{A}$.*

Proof If $\mathrm{M}_{\mathbf{n}}(F) \hookrightarrow \mathbf{A}$ then $\mathbf{n} = \deg_{\mathrm{mat}} \mathrm{M}_{\mathbf{n}}(F) \mid \deg_{\mathrm{mat}} \mathbf{A}$ by Remark 16.3. In the other direction, since $\mathbf{n} \mid \deg_{\mathrm{mat}} \mathbf{A}$, there are embeddings $\mathrm{M}_n(F) \hookrightarrow \mathbf{A}$ for every $n \mid \mathbf{n}$. Since $\mathrm{M}_{\mathbf{n}}(F) \in \mathcal{C}_F^\omega$, Proposition 9.3 applies and gives an embedding $\mathrm{M}_{\mathbf{n}}(F) \hookrightarrow \mathbf{A}$. □

Corollary 16.7 *The matrix degree of $\mathbf{A}$ is the maximal (supernatural) degree of a matrix algebra embedded in the $\mathbf{A}$.*

16.1 The Opposite Algebra

Notice that if $\mathbf{A} \in \mathcal{C}_F^\Gamma$ then the opposite algebra $\mathbf{A}^{\mathrm{op}}$ is in $\mathcal{C}_F^\Gamma$ as well, as $(\varinjlim_\Gamma A_\gamma)^{\mathrm{op}} = \varinjlim_\Gamma A_\gamma^{\mathrm{op}}$.

Proposition 16.8 *Let $\mathbf{A} \in \mathcal{C}_F^\omega$. Then $\mathbf{A} \otimes \mathbf{A}^{\mathrm{op}} \cong \mathrm{M}_{\mathbf{n}}(F)$ where $\mathbf{n} = (\deg \mathbf{A})^2$.*

Proof Write $\mathbf{A} = \varinjlim A_n$ where A_n are finite dimensional central simple subalgebras. Since for each n we have that $A_n \otimes A_n^{\mathrm{op}} \cong \mathrm{M}_{\dim A_n}(F)$, Propositions 5.1 and 9.5 show that $\mathbf{A} \otimes \mathbf{A}^{\mathrm{op}} = \varinjlim A_n \otimes A_n^{\mathrm{op}} \cong \varinjlim \mathrm{M}_{\dim A_n}(F) = \varinjlim \mathrm{M}_{(\deg A_n)^2}(F) = \mathrm{M}_{\mathbf{n}}(F)$. □

Corollary 16.9 *For any $\mathbf{A} \in \mathcal{C}_F^\omega$, $\mathbf{A} \hookrightarrow \mathrm{M}_{\deg(\mathbf{A})^2}(F)$.*

In the spirit of Proposition 16.8, we have:

Example 16.10 For $\mathbf{A} \in \mathcal{C}_F^\omega$, the algebra $\mathbf{A}^\infty = \varinjlim_n \mathbf{A}^{\otimes n}$ is the supernatural matrix algebra of degree $(\deg \mathbf{A})^\infty$.

Indeed, apply Proposition 10.13 to write $\mathbf{A} = \bigotimes A_m$, and take n to range over a divisor series, such as the factorials. By Proposition 10.12 (for $\Gamma = I = \omega$),

$$\begin{aligned}\mathbf{A}^{\infty} &= \varinjlim_n \mathbf{A}^{\otimes n} = \varinjlim_n (\bigotimes A_m)^{\otimes n} \\ &= \varinjlim_n \bigotimes_m (A_m^{\otimes n}) = \bigotimes_m \varinjlim_n (A_m^{\otimes n}) \\ &= \bigotimes_m \mathrm{M}_{(\deg A_m)^{\infty}}(F) = \mathrm{M}_{\prod(\deg A_m)^{\infty}}(F) = \mathrm{M}_{\deg \mathbf{A}^{\infty}}(F)\end{aligned}$$

The following example is a counterpart to Proposition 9.5: while for direct limits over ω only the algebras matter, the morphisms can fundamentally change the limit over uncountable directed sets.

Example 16.11 Let $\mathbf{A} \in \mathcal{C}_F^{\Gamma}$, where Γ is an arbitrary directed set. Write $\mathbf{A} = \varinjlim_{\Gamma} A_\gamma$ where A_γ are finite dimensional central simple algebras. Then $\mathbf{A} \otimes \mathbf{A}^{\mathrm{op}} = \varinjlim_{\Gamma}(A_\gamma \otimes A_\gamma^{\mathrm{op}})$ has the same dimension as $\mathbf{A}$.

We construct an algebra $\mathbf{B}$ which is a limit of the same algebras $A_\gamma \otimes A_\gamma^{\mathrm{op}}$, and is nevertheless countably dimensional.

Take the objects $B_\gamma = A_\gamma \otimes A_\gamma^{\mathrm{op}}$, identified with matrices of the respective dimensions, and whose morphisms are, for each $\gamma < \gamma'$, the diagonal matrix embeddings $\beta_{\gamma\gamma'} : B_\gamma \to B_{\gamma'}$ of Definition 15.1. Let $\mathbf{B} = \varinjlim (B_\gamma, \beta_{\gamma\gamma'})$.

Consider the supernatural matrix algebra $\mathrm{M}_{\mathbf{n}}(F) = \varinjlim_{n \mid \mathbf{n}} \mathrm{M}_n(F)$, where $\mathbf{n} = \deg(\mathbf{A})$. The system of identifications $B_\gamma \to \mathrm{M}_{\dim(A_\gamma)}(F)$ for each $\gamma \in \Gamma$, is compatible, and induces by Remark 4.1. Eq. 4.1 an isomorphism $\mathbf{B} \cong \mathrm{M}_{\mathbf{n}}(F)$.

16.2 Identifying Division Algebras

Proposition 16.12 *Let $\mathbf{A} \in \mathcal{C}_F$. Then:*

1. *$\mathbf{A}$ is a division algebra if and only if* $\deg_{\mathrm{mat}} \mathbf{A} = 1$.
2. *$\mathbf{A}$ is Artinian if and only if* $\deg_{\mathrm{mat}} \mathbf{A}$ *is finite.*

Proof By definition, $\deg_{\mathrm{mat}} \mathbf{A} = 1$ if and only if every finite dimensional central simple subalgebra of $\mathbf{A}$ is a division algebra, if and only if $\mathbf{A}$ itself is a division algebra (of arbitrary dimension).

If $\deg_{\mathrm{mat}} \mathbf{A}$ is infinite then $\mathrm{M}_n(F) \hookrightarrow \mathbf{A}$ for unbounded values of n, so $\mathbf{A}$ cannot be Artinian. If $n = \deg_{\mathrm{mat}} \mathbf{A}$ is finite, then write $\mathbf{A} = \mathrm{M}_n(F) \otimes \mathbf{B}$ by Corollary 6.5. Clearly $\deg_{\mathrm{mat}} \mathbf{A} = n \cdot \deg_{\mathrm{mat}} \mathbf{B}$, so $\deg_{\mathrm{mat}} \mathbf{B} = 1$ and $\mathbf{B}$ is a division algebra, showing that $\mathbf{A}$ is Artinian. □

Let $\mathbf{A}_1, \mathbf{A}_2 \in \mathcal{C}_F$. It is easy to see that

$$\deg_{\mathrm{mat}}(\mathbf{A}_1)\deg_{\mathrm{mat}}(\mathbf{A}_2) \mid \deg_{\mathrm{mat}}(\mathbf{A}_1 \otimes \mathbf{A}_2).$$

Equality does not hold in general, even in the finite dimensional case, because of partial splitting (e.g. when some finite dimensional division algebra B_0 satisfies $B_0 \hookrightarrow \mathbf{A}_1$ and $B_0^{\mathrm{op}} \hookrightarrow \mathbf{A}_2$). However, we do have:

Proposition 16.13 *Let $\mathbf{A}_1, \mathbf{A}_2 \in \mathcal{C}_F$. If* $\deg \mathbf{A}_1, \deg \mathbf{A}_2$ *are co-prime, then* $\deg_{\mathrm{mat}}(\mathbf{A}_1 \otimes \mathbf{A}_2) = \deg_{\mathrm{mat}} \mathbf{A}_1 \cdot \deg_{\mathrm{mat}} \mathbf{A}_2$.

Proof Suppose $\mathrm{M}_n(F) \hookrightarrow \mathbf{A}_1 \otimes \mathbf{A}_2$. There are finite dimensional $B_i \subseteq \mathbf{A}_i$ such that $\mathrm{M}_n(F) \hookrightarrow B_1 \otimes B_2$. Taking $n_i = \gcd(n, \deg \mathbf{A}_i)$ we have that $n = n_1 n_2$. Since $\deg(B_i) \mid \deg(\mathbf{A}_i)$, we have that $\mathrm{M}_{n_i}(F) \hookrightarrow B_i$ and $n_i \mid \deg_{\mathrm{mat}} \mathbf{A}_i$, which are co-prime, showing that $n = n_1 n_2$ divides $\deg_{\mathrm{mat}} \mathbf{A}_1 \cdot \deg_{\mathrm{mat}} \mathbf{A}_2$. □

Proposition 16.14 *Let $\mathbf{A}_i \in \mathcal{C}_F$ be algebras of pairwise co-prime degrees. Then* $\deg_{\mathrm{mat}} \left(\bigotimes_{\omega} \mathbf{A}_i\right) = \prod \deg_{\mathrm{mat}} \mathbf{A}_i$.

Proof For any n, $\deg_{\mathrm{mat}} (\mathbf{A}_1 \otimes \cdots \otimes \mathbf{A}_n) = \prod_{i=1}^{n} \deg_{\mathrm{mat}} \mathbf{A}_i$ by Proposition 16.13, so we are done by Proposition 16.4. □

Therefore, by Proposition 16.12:

Proposition 16.15 *The countable tensor product of division algebras of pairwise co-prime degrees is also a division algebra.*

And in particular:

Proposition 16.16 *Assume $\mathbf{A} \in \mathcal{C}_F$ has a primary decomposition $\mathbf{A} = \bigotimes \mathbf{A}_p$. Then $\mathbf{A}$ is a division algebra if and only if each $\mathbf{A}_p$ is a division algebra.*

We will also need the following easy observation:

Remark 16.17 Let $\mathbf{A} \in \mathcal{C}_F$. If $\mathbf{D} \in \mathcal{C}_F$ is a division algebra, then $\deg_{\mathrm{mat}} (\mathbf{A} \otimes \mathbf{D}) = \deg_{\mathrm{mat}} (\mathbf{A})$.

17 Split Crossed Products

Let $\mathbf{K}/F$ be an algebraic (but possibly infinite) Galois extension of fields. A subgroup $\Gamma \subseteq \mathbf{G} = \mathrm{Gal}(\mathbf{K}/F)$ of the Galois group is dense with respect to the natural (Krull) topology [27, Sec. 2.11] if and only if $\mathbf{K}^{\Gamma} = F$. In our context, a **split crossed product** is an algebra of the form $\mathbf{K}[\Gamma]$, where $\mathbf{K}$ is as above and $\Gamma \subseteq \mathbf{G}$ is dense. The multiplication is defined by the natural action $\sigma \cdot a = \sigma(a) \cdot \sigma$ (for $a \in \mathbf{K}$ and $\sigma \in \Gamma$).

Remark 17.1 A split crossed product $\mathbf{K}[\Gamma]$ is always simple and central over F.

Remark 17.2 The split crossed product $\mathbf{K}[\Gamma]$ is locally finite-dimensional if and only if Γ is locally finite (as a group).

17.1 Split Crossed Products over Tensor Product Fields

In this subsection we consider the following setup. Let I be an infinite index set. Let $\{K_i : i \in I\}$ be a set of linearly independent finite dimensional Galois extensions of F in a fixed algebraic closure. For $i \in I$, let $G_i = \mathrm{Gal}(K_i/F)$ be the Galois group. Let $\mathbf{K}$ be the field generated by all the K_i; so that $\mathbf{K} = \bigotimes_{i \in I} K_i$ is an infinite tensor product (Section 10.1). The full Galois group $\mathrm{Gal}(\mathbf{K}/F)$ is the direct product $\mathbf{G} = \prod G_i$.

Remark 17.3 We assume henceforth that the order of the G_i is bounded.

Notice that by the restricted Burnside problem [32], $\mathbf{G}$ is locally finite if and only if the exponent of the G_i is bounded, which is a weaker condition.

For every finite subset $\lambda \subseteq I$, let $K_\lambda = \otimes_{i\in\lambda} K_i$, let $G_\lambda = \prod_{i\in\lambda} G_i$, and let A_λ denote the split crossed product $K_\lambda[G_\lambda]$, so that $A_\lambda \cong \mathrm{End}(K_\lambda)$. In particular we put $A_i = A_{\{i\}}$, so that $A_\lambda = \otimes_{i\in\lambda} A_i$.

A **locally matrix algebra** is by definition any direct limit of (finite dimensional) matrix algebras over F. For example, a supernatural matrix algebra is a locally matrix algebra, and by Corollary 9.9 the two notions coincide for countably generated algebras. More generally, an infinite tensor product of matrix algebras, of any cardinality, is also a locally matrix algebra. In this section we construct a locally matrix algebra which is not an infinite tensor product.

Proposition 17.4 *The (full) split crossed product* $\mathbf{A} = \boldsymbol{K}[\mathbf{G}]$ *is a locally matrix algebra. In particular* $\mathbf{A} \in \mathcal{C}_F$.

Proof Every finitely generated subalgebra $A_0 \subseteq \mathbf{A}$ is contained in an algebra generated over F by a finite dimensional subfield K_λ and a finitely generated subgroup $H \subseteq \mathbf{G} = \prod_{i\in I} G_i$, which is finite by our assumption that the components G_i have bounded order. We say that $i, i' \in I$ are equivalent if there is an isomorphism $G_i \to G_{i'}$ which for every $h \in H$ maps the ith entry to the i'th entry. Since we assume the orders of the G_i are bounded, there are only finitely many isomorphism types of the G_i, and so there are only finitely many equivalence classes. We say that a subset of I "covers H" if it intersects each equivalence class.

We now take a finite set $\lambda' \supseteq \lambda$ which is large enough both to cover H and for the map $H \to \mathrm{Gal}(K_{\lambda'}/F)$ to be injective (both properties are monotone). Refine the equivalence relation so that no two elements of λ' are equivalent. Under the refined equivalence relation, each $i \notin \lambda'$ is equivalent to a unique $i' \in \lambda'$. Moreover if i, i' are equivalent under the refined relation, the groups G_i and $G_{i'}$ must be isomorphic. Every element of $G_{\lambda'} = \mathrm{Gal}(K_{\lambda'}/F)$ can then be extended to an element of $\mathbf{G}$ via a fixed isomorphisms of equivalent components. Let H' be the subgroup of $\mathbf{G}$ extended in this manner from all of $\mathrm{Gal}(K_{\lambda'}/F)$. Then $H \subseteq H' \cong G_{\lambda'}$, and $K_{\lambda'}[H'] \cong K_{\lambda'}[G_{\lambda'}] \cong \mathrm{End}(K_{\lambda'})$ is a matrix algebra containing A_0. □

Now consider the subgroup $\mathbf{G}_0 = \bigoplus_{i\in I} G_i$, which is a dense subgroup of $\mathbf{G}$. Let $\mathbf{A}_0$ denote the subalgebra $\mathbf{K}[\mathbf{G}_0]$ of $\mathbf{A} = \mathbf{K}[\mathbf{G}]$.

Proposition 17.5 $A_0 = \bigotimes_{i\in I} A_i$.

Proof By definition $\mathbf{A}_0$ is the direct limit of the finite dimensional subalgebras $K_\lambda[G_\lambda] = \otimes_{i\in\lambda} K_i[G_i]$ (for finite $\lambda \subseteq I$), which by definition is the infinite tensor product $\bigotimes_{i\in I} A_i$. □

Remark 17.6 The algebra $\mathbf{A}_0$ has the same matrix degree as $\mathbf{A}$. By controlling the groups G_i, it is easy to obtain matrix degree $\mathbf{p}^\infty$ where $\mathbf{p}$ is any finite product of primes.

Proposition 17.7 *The vector space dimensions are* $\dim(\boldsymbol{K}) = \dim(\boldsymbol{A}_0) = |I|$ *and* $\dim(\boldsymbol{A}) = 2^{|I|}$.

Proof The first claim follows from Proposition 17.5, and the second from $G = \prod_{i \in I} G_i$ being a basis of $\mathbf{A}$ over $\mathbf{K}$. □

Recall the definition of a pathological subalgebra from Definition 12.5.

Proposition 17.8 A_0 *is a pathological subalgebra of* $\boldsymbol{A}$.

Proof An element $a \in \mathbf{A}$ has the form $a = \sum_r k_r g_r$ where $k_r \in \mathbf{K}$ and $g_r \in \mathbf{G}$. For every $k \in \mathbf{K}$, the assumption that $0 = [a, k] = \sum k_r(g_r(k) - k)g_r$ forces $a \in \mathbf{K}$, and the density of $\mathbf{G}_0$ in $\mathbf{G}$ then implies $a \in F$. □

Remark 17.9 If $\mathbf{C} = \bigotimes_J C_j$ is an infinite tensor product of finite dimensional central simple algebras, then every pathological subalgebra has the same dimension as $\mathbf{C}$.

Proof Let $\{b_\alpha\}$ be a basis for a pathological subalgebra $\mathbf{C}'$; for each α let $J_\alpha \subseteq J$ be a finite set of indices such that $b_\alpha \in \bigotimes_{J_\alpha} C_j$. Let $J' = \bigcup J_\alpha$. Then $\left|J'\right| = |\{b_\alpha\}| = \dim(\mathbf{C}')$, so if $\dim(\mathbf{C}') < \dim(\mathbf{C}) = |J|$ there must be some $j_0 \notin J'$, and C_{j_0} centralizes $\mathbf{C}'$. □

Comparing Proposition 17.8 with Remark 17.9, we conclude:

Corollary 17.10 *The locally matrix algebra* $\boldsymbol{A} = \boldsymbol{K}[\boldsymbol{G}]$ *cannot be an infinite tensor product of finite dimensional central simple algebras.*

17.2 Split Abelian Crossed Product

As before, let $\mathbf{K}/F$ be an algebraic Galois extension, and $\mathbf{G} = \mathrm{Gal}(\mathbf{K}/F)$. As mentioned in Remark 17.2, for $\mathbf{K}[\mathbf{G}]$ to be locally finite-dimensional, we must assume that $\mathbf{G}$ is locally finite, and in particular torsion. By [27, Cor. 4.3.9], an abelian profinite group is torsion if and only if it is a direct product of cyclic groups with bounded order.

Corollary 17.11 *Assuming* $\boldsymbol{G} = \mathrm{Gal}(\boldsymbol{K}/F)$ *is abelian, the split crossed product* $\boldsymbol{K}[\boldsymbol{G}]$ *is in* $\mathcal{C}_F$ *if and only if* $\boldsymbol{G}$ *is torsion.*

18 Classification of Algebras in $\mathcal{C}_F^\omega$

A supernatural number $\mathbf{n} = \prod p^{\alpha_p}$ is **locally finite** if all the α_p are finite (so a supernatural number is finite if and only if it is finitely supported and locally finite). We generalize Corollary 6.5 for matrices in $\mathcal{C}_F^\omega$.

Proposition 18.1 *Let* $\boldsymbol{n}$ *be a locally finite supernatural number. Let* $\boldsymbol{A} \in \mathcal{C}_F^\omega$*, and assume* $\mathrm{M}_{\boldsymbol{n}}(F) \hookrightarrow \boldsymbol{A}$. *Then* $\boldsymbol{A} \cong \mathrm{M}_{\boldsymbol{n}}(F) \otimes \boldsymbol{B}$ *for some* $\boldsymbol{B} \in \mathcal{C}_F^\omega$.

Moreover if $\boldsymbol{n} = \deg_{\mathrm{mat}} \boldsymbol{A}$ *then* $\boldsymbol{B}$ *is a division algebra, which in this case is unique.*

Proof Take a primary decomposition $\mathbf{A} = \bigotimes \mathbf{A}_p$, where $\mathbf{A}_p \in \mathcal{C}_F^\omega$ are p-algebras, and a primary decomposition $\mathrm{M}_{\mathbf{n}}(F) = \bigotimes \mathrm{M}_{p^{\alpha_p}}(F)$, where the α_p are finite by assumption. By Corollary 13.23, $\mathrm{M}_{p^{\alpha_p}}(F) \hookrightarrow \mathbf{A}_p$ for each prime p, so by Corollary 6.5, $\mathbf{A}_p = \mathrm{M}_{p^{\alpha_p}}(F) \otimes \mathbf{B}_p$ for some p-algebra $\mathbf{B}_p \in \mathcal{C}_F^\omega$. By Remark 10.4, $\mathbf{A} = \bigotimes \mathbf{A}_p = \bigotimes (\mathrm{M}_{p^{\alpha_p}}(F) \otimes \mathbf{B}_p) = \bigotimes \mathrm{M}_{p^{\alpha_p}}(F) \otimes \bigotimes \mathbf{B}_p = \mathrm{M}_{\mathbf{n}}(F) \otimes \mathbf{B}$, where $\mathbf{B} = \bigotimes \mathbf{B}_p$.

Now, $\mathbf{n} \cdot \deg_{\text{mat}}(\mathbf{B}) \mid \deg_{\text{mat}}(\mathbf{A})$, so if $\deg_{\text{mat}}(\mathbf{A}) = \mathbf{n}$ we are forced to have $\deg_{\text{mat}}(\mathbf{B}) = 1$ because locally finite supernatural numbers cancel, so $\mathbf{B}$ is a division algebra by Remark 16.12. It follows that each $\mathbf{B}_p$ is a division algebra, so $\mathbf{A}_p$ is Artinian and $\mathbf{B}_p$ is unique by Wedderburn-Artin. Finally the $\mathbf{B}_p$ determine $\mathbf{B}$. □

Theorem 18.2 *An algebra* $A \in \mathcal{C}_F^{\omega}$ *is a supernatural matrix algebra if and only if* $\deg_{\text{mat}} A = \deg A$.

Proof If $\mathbf{A} = \mathrm{M}_{\mathbf{n}}(F)$, then $\deg_{\text{mat}} \mathbf{A} = \mathbf{n} = \deg \mathbf{A}$ by Example 16.5.

Let us assume $\mathbf{A} \in \mathcal{C}_F^{\omega}$ has matrix degree equal to $\mathbf{n} = \deg \mathbf{A}$. There is a unique decomposition $\mathbf{n} = \mathbf{n}'\mathbf{m}^{\infty}$ where: $\mathbf{n}'$ is locally finite, $\mathbf{m}$ is a (possibly infinite) product of distinct primes, and $\mathbf{n}'$ and $\mathbf{m}$ are co-prime. Indeed, $\mathbf{n}'$ is the product of the finite prime-powers in the primary decomposition of $\mathbf{n}$.

Since $\mathrm{M}_{\mathbf{n}'}(F) \hookrightarrow \mathrm{M}_{\mathbf{n}}(F) \hookrightarrow \mathbf{A}$, we have that $\mathbf{A} \cong \mathrm{M}_{\mathbf{n}'}(F) \otimes \mathbf{B}$ for some $\mathbf{B} \in \mathcal{C}_F^{\omega}$ by Proposition 18.1. Now $\mathbf{n}'\mathbf{m}^{\infty} = \mathbf{n} = \deg(\mathbf{A}) = \mathbf{n}' \deg(\mathbf{B})$, so $\deg(\mathbf{B}) = \mathbf{m}^{\infty}$ by cancellation. By Proposition 16.8, we have that

$$\mathbf{A} \hookrightarrow \mathbf{A} \otimes \mathbf{B}^{\text{op}} = \mathrm{M}_{\mathbf{n}'}(F) \otimes \mathbf{B} \otimes \mathbf{B}^{\text{op}} = \mathrm{M}_{\mathbf{n}'\mathbf{m}^{\infty}}(F) = \mathrm{M}_{\mathbf{n}}(F),$$

because $(\mathbf{m}^{\infty})^2 = \mathbf{m}^{\infty}$.

Since $\mathbf{n} = \deg_{\text{mat}} \mathbf{A}$, we proved that $\mathrm{M}_{\mathbf{n}}(F) \hookrightarrow \mathbf{A} \hookrightarrow \mathrm{M}_{\mathbf{n}}(F)$. By Proposition 9.8, this shows that $\mathbf{A} \cong \mathrm{M}_{\mathbf{n}}(F)$. □

We immediately obtain:

Proposition 18.3 *Every p-algebra in* $\mathcal{C}_F^{\omega}$ *is either*

(1) *of the form* $\mathrm{M}_n(F) \otimes \boldsymbol{D}$ *for a unique division algebra* $\boldsymbol{D} \in \mathcal{C}_F^{\omega}$ *and a unique finite* p*-power* n *(which is the matrix degree of the algebra);* ***or***
(2) *the matrix algebra* $\mathrm{M}_{p^{\infty}}(F)$.

The only non-Artinian algebra in this class is $\mathrm{M}_{p^{\infty}}(F)$.

Proof Let $\mathbf{A}$ be a p-algebra in $\mathcal{C}_F^{\omega}$. If $n = \deg_{\text{mat}} \mathbf{A}$ is finite, then we can decompose $\mathbf{A} = \mathrm{M}_n(F) \otimes \mathbf{D}$ by Corollary 18.1, and $\mathbf{D}$ is a division algebra; in particular $\mathbf{A}$ is Artinian. On the other hand, if $\deg_{\text{mat}} \mathbf{A} = p^{\infty} = \deg \mathbf{A}$, then $\mathbf{A} = \mathrm{M}_{p^{\infty}}(F)$ by Theorem 18.2. □

Theorem 18.4 *Every* $A \in \mathcal{C}_F^{\omega}$ *can be uniquely presented as* $\mathrm{M}_{\boldsymbol{n}'\boldsymbol{m}^{\infty}}(F) \otimes \boldsymbol{D}$ *where*

- $\boldsymbol{m}$ *is a (possibly infinite) product of primes;*
- $\boldsymbol{n}'$ *is a locally finite supernatural number which is prime to* $\boldsymbol{m}$*;*
- $\boldsymbol{D} \in \mathcal{C}_F^{\omega}$ *is a division algebra whose degree is prime to* $\boldsymbol{m}$*.*

Proof Fix a primary decomposition $\mathbf{A} = \bigotimes \mathbf{A}_p$, where $\mathbf{A}_p \in \mathcal{C}_F^{\omega}$ are p-algebras. For every prime p write $p^{\alpha_p} = \deg_{\text{mat}} \mathbf{A}_p$, where α_p is finite or infinite. Let P be the set of primes p for which α_p is finite. By Proposition 18.3, $\mathbf{A}_p = \mathrm{M}_{p^{\alpha_p}}(F) \otimes \mathbf{D}_p$ for a unique division p-algebra $\mathbf{D}_p$ when $p \in P$, and $\mathbf{A}_p = \mathrm{M}_{p^{\infty}}(F)$ when $p \notin P$. Let $\mathbf{n}' = \prod_{p \in P} p^{\alpha_p}$ and $\mathbf{m} = \prod_{p \notin P} p$, and let $\mathbf{D} = \bigotimes_{p \in P} \mathbf{D}_p$. Now $\bigotimes \mathbf{A}_p = (\bigotimes_{p \in P} \mathbf{A}_p) \otimes (\bigotimes_{p \notin P} \mathbf{A}_p) = (\bigotimes_{p \in P} \mathrm{M}_{p^{\alpha_p}}(F) \otimes \mathbf{D}_p) \otimes (\bigotimes_{p \notin P} \mathrm{M}_{p^{\infty}}(F)) = \mathrm{M}_{\mathbf{n}'}(F) \otimes \mathbf{D} \otimes \mathrm{M}_{\mathbf{m}^{\infty}}(F)$.

The uniqueness follows from the condition that $\deg(\mathbf{D})$ is prime to the infinite matrix part; indeed, any division algebra of degree dividing $\mathbf{m}$ will be absorbed by $\mathrm{M}_{\mathbf{m}}(F)$. □

We say that $\mathbf{D}$ is an **underlying division algebra** for $\mathbf{A} \in \mathcal{C}_F$ if $\mathbf{A} = \mathrm{M}_{\mathbf{n}}(F) \otimes \mathbf{D}$ for some supernatural matrix algebra $\mathrm{M}_{\mathbf{n}}(F)$.

Corollary 18.5 *An algebra $\mathbf{A} \in \mathcal{C}_F^{\omega}$ has a unique underlying division algebra of minimal degree.*

A prime p is **saturated** in $\mathbf{A}$ if $p^{\infty} \mid \deg_{\mathrm{mat}} \mathbf{A}$. The saturated primes in $\mathbf{A}$ are the prime divisors of $\mathbf{m}$ in the presentation of Theorem 18.4.

Corollary 18.6 *Let $\mathbf{D}$ be the underlying division algebra of minimal degree of a given algebra $\mathbf{A} \in \mathcal{C}_F^{\omega}$.*

The underlying division algebras of $\mathbf{A}$ are the algebras of the form $\mathbf{D} \otimes \mathbf{D}'$, where $\mathbf{D}'$ is any division algebra in $\mathcal{C}_F^{\omega}$ whose degree is supported on the saturated primes of $\mathbf{A}$ (Proposition 16.15 shows that $\mathbf{D} \otimes \mathbf{D}'$ is indeed a division algebra).

Proposition 18.7 *Let $\mathbf{A} \in \mathcal{C}_F$, and let p be unsaturated. Every p-Sylow subalgebra of A is a maximal p-subalgebra.*

Proof Let p^n be the maximal power of p dividing $\deg_{\mathrm{mat}} \mathbf{A}$. Let $\mathbf{P} \subseteq \mathbf{A}$ be a Sylow subalgebra. Write $\mathbf{A} = \varinjlim A_{\gamma}$. Starting the net sufficiently high, we may assume every A_{γ} contains $M = \mathrm{M}_{p^n}(F)$, and so does its maximal p-subalgebra. Therefore $M \subseteq \mathbf{P}$. Let $\mathbf{P}'$ and $\mathbf{A}'$ be the centralizers of M; since $\mathbf{A}' = \varinjlim \mathrm{C}_{A_{\gamma}}(M)$, $\mathbf{P}'$ is a p-Sylow subalgebra of $\mathbf{A}'$. But since $\deg_{\mathrm{mat}}(\mathbf{A}')$ is prime to p, every p-subalgebra of $\mathbf{A}'$ is a division algebra, and by Theorem 13.11 $\mathbf{P}'$ is a maximal p-subalgebra of $\mathbf{A}'$. By Proposition 6.3 we have that $\mathbf{P} = M \otimes \mathbf{P}'$, which is a now a maximal p-subalgebra of $\mathbf{A} = M \otimes \mathbf{A}'$. □

19 Elements in the Monoid $\mathcal{C}_F^{\omega}$

In this section we inspect the monoid $\mathcal{C}_F^{\omega}$, and identify zeros and idempotents. Let ${}_p\mathcal{C}_F^{\omega}$ denote the submonoid of $\mathcal{C}_F^{\omega}$ whose elements are p-algebras. The Primary Decomposition Theorem 13.22 provides an isomorphism to the direct product

$$\mathcal{C}_F^{\omega} \cong \prod_p {}_p\mathcal{C}_F^{\omega}. \tag{9}$$

We thus solve a basic equation in ${}_p\mathcal{C}_F^{\omega}$, and then combine the results in $\mathcal{C}_F^{\omega}$.

Proposition 19.1 *Let $\mathbf{A}, \mathbf{B} \in {}_p\mathcal{C}_F^{\omega}$. Then $\mathbf{A} \otimes \mathbf{B} \cong \mathbf{A}$ if and only if $\mathbf{B} = F$ or $\mathbf{A} \cong \mathrm{M}_{p^{\infty}}(F)$.*

Proof If $\mathbf{B} = F$ then clearly $\mathbf{A} \otimes \mathbf{B} \cong \mathbf{A}$, and if $\mathbf{A} \cong \mathrm{M}_{p^{\infty}}(F)$ then $\mathbf{A} \otimes \mathbf{B} \cong \mathbf{A}$ by Proposition 18 3, because the matrix degree cannot drop. On the other hand, assume $\mathbf{A} \otimes \mathbf{B} \cong \mathbf{A}$ where $\mathbf{B} \neq F$. Let $B_0 \subseteq \mathbf{B}$ be a finite dimensional central simple subalgebra, and let $e = \exp(B_0)$ be the order of $[B_0]$ in the Brauer group, so $B_0^{\otimes e}$ is a nontrivial matrix algebra. For every n, $B_0^{\otimes en} \subseteq \mathbf{B}^{\otimes en} \subseteq \mathbf{A} \otimes \mathbf{B}^{\otimes en} \cong \mathbf{A}$, so $\deg_{\mathrm{mat}} \mathbf{A}$ is infinite. □

Corollary 19.2 *The monoid ${}_p\mathcal{C}_F^{\omega}$ has a zero element, namely the algebra $\mathrm{M}_{p^{\infty}}(F)$. There are no nontrivial idempotents. There are no invertible elements other than F (because of the degree).*

We now consider the same questions in $\mathcal{C}_F^\omega$.

Proposition 19.3 *For algebras* $\mathbf{A}, \mathbf{B} \in \mathcal{C}_F^\omega$, $\mathbf{A} \otimes \mathbf{B} \cong \mathbf{A}$ *if and only if* $(\deg \mathbf{B})^\infty \mid \deg_{\mathrm{mat}} \mathbf{A}$.

Proof Decompose $\mathbf{A} = \bigotimes \mathbf{A}_p$ and $\mathbf{B} = \bigotimes \mathbf{B}_p$. Clearly, $\mathbf{A} \otimes \mathbf{B} \cong \mathbf{A}$ if and only if for every prime p we have that $\mathbf{A}_p \otimes \mathbf{B}_p \cong \mathbf{A}_p$, which by Proposition 19.1 happens if and only if whenever $\mathbf{B}_p \neq F$, $\mathbf{A}_p \cong \mathrm{M}_{p^\infty}(F)$. Namely, $\deg(\mathbf{B})^\infty$ divides $\deg_{\mathrm{mat}} \mathbf{A}$. □

Generalizing the isomorphism (8), we have:

Corollary 19.4 *For* $\mathbf{A} \in \mathcal{C}_F^\omega$, $\mathrm{M}_p(\mathbf{A}) \cong \mathbf{A}$ *if and only if* $p^\infty \mid \deg_{\mathrm{mat}}(\mathbf{A})$.

Let $\mathbf{p}$ denote the product of all the natural primes.

Corollary 19.5 *The algebra* $\mathrm{M}_{\mathbf{p}^\infty}(F)$ *is the zero element in the monoid* $\mathcal{C}_F^\omega$. *The idempotents are all of the form* $\mathrm{M}_{\mathbf{m}^\infty}(F)$, *for* $\mathbf{m}$ *a product of distinct primes. There are no invertible elements other than* F.

Proposition 19.3 classifies algebras isomorphic to one of their own factors.

Proposition 19.6 *An algebra* $\mathbf{A} \in \mathcal{C}_F^\omega$ *is isomorphic to one of its own proper factors if and only if* $\mathbf{A}$ *contains* $\mathrm{M}_{p^\infty}(F)$ *for some prime* p.

Proof If $\mathbf{A}$ is isomorphic to a proper factor of itself, then $\mathbf{A} \cong \mathbf{A} \otimes \mathbf{B}$ for some $F \neq \mathbf{B} \in \mathcal{C}_F^\omega$. Take some $p \mid \deg \mathbf{B}$, then $p^\infty \mid \deg_{\mathrm{mat}} \mathbf{A}$ by Proposition 19.3, and $\mathrm{M}_{p^\infty}(F) \hookrightarrow \mathbf{A}$ by Proposition 16.6. On the other hand if $\mathrm{M}_{p^\infty}(F) \subseteq \mathbf{A}$ then $\mathrm{M}_{p^\infty}(F)$ is a factor of $\mathbf{A}$ by primary decomposition, and already $\mathrm{M}_{p^\infty}(F)$ is isomorphic to its own proper factor because $\mathrm{M}_p(F) \otimes \mathrm{M}_{p^\infty}(F) \cong \mathrm{M}_{p^\infty}(F)$. □

As mentioned in the introduction, Barsotti proved in [5] that every infinite dimensional division algebra $\mathbf{A} \in \mathcal{C}_F^\omega$ is isomorphic to a proper subalgebra. By Proposition 19.6, this subalgebra cannot be a factor.

We now show that Skolem-Noether's theorem fails in $\mathcal{C}_F^\omega$.

Theorem 19.7 *For every infinite dimensional* $\mathbf{A} \in \mathcal{C}_F^\omega$ *there is an isomorphism between proper subalgebras in* $\mathcal{C}_F^\omega$ *which cannot be extended to* $\mathbf{A}$.

Proof Let $B_0 \neq F$ be a finite dimensional central simple subalgebra. We can decompose $\mathbf{A} \cong B_0 \otimes \mathbf{B}$ where B is the centralizer. If $\mathbf{B} \cong \mathbf{A}$ then we can further decompose $\mathbf{A} \cong B_0 \otimes \mathbf{A} \cong (B_0 \otimes B_0) \otimes \mathbf{A}$, and the isomorphism between the two infinite dimensional factors cannot be extended because the centralizers B_0 and $B_0 \otimes B_0$ are not isomorphic.

Otherwise, $\mathbf{B} \ncong \mathbf{A}$. By Theorem 12.6, $\mathbf{B}$ can also be embedded as a pathological subalgebra $\mathbf{B}_1$, and the isomorphism $\mathbf{B} \to \mathbf{B}_1$ cannot be extended because an automorphism aught to carry centralizer to centralizer. □

IV. The Brauer Monoid

20 Morita Equivalence

We show that algebras $\mathbf{A}, \mathbf{B} \in \mathcal{C}_F$ are Morita equivalent (see e.g. [28, Chapter 4], [21, Chapter 7]) if and only if they are **matrix-equivalent**, namely if there are natural numbers n, m such that $\mathrm{M}_n(\mathbf{A}) \cong \mathrm{M}_m(\mathbf{B})$.

Remark 20.1 For any algebras $\mathbf{A}$, $\mathbf{A}'$ and $\mathbf{B}$, if $\mathbf{A}$ and $\mathbf{A}'$ are matrix-equivalent, then so are $\mathbf{A} \otimes \mathbf{B}$ and $\mathbf{A}' \otimes \mathbf{B}$.

We say that an algebra $\mathbf{A}$ is a **corner** in an algebra $\mathbf{B}$ if $\mathbf{A} = e\mathbf{B}e$ for some idempotent $e \in \mathbf{B}$. The following useful criterion follows from [21, Proposition 18.33]:

Remark 20.2 Let R be a simple ring. Every ring S which is Morita-equivalent to R is a corner in a ring which is matrix-equivalent to R.

Theorem 20.3 *Let $\mathbf{A} \in \mathcal{C}_F$. An F-algebra $\mathbf{B}$ is Morita-equivalent to $\mathbf{A}$ if and only if they are matrix-equivalent.*

Proof Assume that $\mathbf{A}$ and $\mathbf{B}$ are Morita-equivalent. Replacing $\mathbf{A}$ by a matrix algebra over itself, we may assume by Remark 20.2 that $\mathbf{B}$ is a corner in $\mathbf{A}$. Let $e \in \mathbf{A}$ be an idempotent. There is a FInite dimensional central simple subalgebra $A_0 \subseteq \mathbf{A}$ such that $e \in A_0$. Using Proposition 6.3, write $\mathbf{A} = A_0 \otimes \mathbf{A}'$ where $\mathbf{A}' \in \mathcal{C}_F$. Now $\mathbf{B} = e\mathbf{A}e = (eA_0e) \otimes \mathbf{A}'$, but eA_0e is Morita equivalent to A_0, and thus matrix-equivalent to it; by Remark 20.1, $\mathbf{B}$ is matrix-equivalent to $A_0 \otimes \mathbf{A}' = \mathbf{A}$. The other direction is trivial. □

Recall [22, Section 9] that a class of rings is **Morita-invariant** if, when A is in the class, so is every algebra Morita equivalent to A.

Corollary 20.4 *The class $\mathcal{C}_F$ is Morita invariant.*

Proof The proof of Theorem 20.3 shows that $\mathbf{B} = e_0A_0e_0 \otimes \mathbf{A}'$ where both $e_0A_0e_0$ and $\mathbf{A}'$ are in $\mathcal{C}_F$. □

21 The Brauer Monoid

By definition, the elements of the Brauer group over a field are the finite dimensional central simple algebras up to Morita equivalence, which in this context is referred to as Brauer equivalence. In this section we introduce a similar structure for algebras in $\mathcal{C}_F^{\omega}$. Note that the algebras here are countably generated.

Definition 21.1 Algebras $\mathbf{A}, \mathbf{B} \in \mathcal{C}_F^{\omega}$ are **Brauer equivalent** if there are *locally finite* supernatural numbers $\mathbf{n}', \mathbf{m}'$ such that $\mathbf{A} \otimes \mathrm{M}_{\mathbf{n}'}(F) \cong \mathbf{B} \otimes \mathrm{M}_{\mathbf{m}'}(F)$. As in the finite case, we denote the class of $\mathbf{A}$ by $[\mathbf{A}]$.

This is an equivalence relation because the product of two locally finite supernatural numbers is locally finite, and for the same reason the product of classes is well defined by $[\mathbf{A}] \cdot [\mathbf{B}] = [\mathbf{A} \otimes \mathbf{B}]$. Brauer equivalence is coarser than Morita equivalence, and the two notions coincide for p-algebras, see Section 21.2 below.

Definition 21.2 The (countable) **Brauer monoid** $\mathrm{Br}^{\omega}(F)$ is the monoid of equivalence classes $[\mathbf{A}]$, for $\mathbf{A} \in \mathcal{C}_F^{\omega}$, with the tensor product operation.

The equivalence relation of Definition 21.1 reduces to the standard Brauer equivalence on finite dimensional central simple algebras, so the Brauer group $\mathrm{Br}(F)$ naturally embeds as a submonoid of $\mathrm{Br}^{\omega}(F)$.

Remark 21.3 The term Brauer monoid is used by Haile [13] and others (also see [26] and [14]) for crossed product algebras arising from 2-cocycles which may obtain zero values; the Brauer group is the unique maximal subgroup in this monoid.

In combinatorial topology, the term Brauer monoid refers to a semigroup of partitions of a fixed set of even cardinality, originated in Brauer's paper [6]. This semigroup has a nice diagrammatical representation, and is related to representation theory and knot theory, see for example [23].

In spite of this triple terminological collision, we find the term Brauer monoid, in its new meaning, natural and suitable.

Why do we insist on $\mathbf{n}'$, $\mathbf{m}'$ in Definition 21.1 being locally finite?

Remark 21.4 Declaring two algebras in $\mathbf{A}, \mathbf{B} \in \mathcal{C}_F^{\omega}$ to be equivalent if $\mathbf{A} \otimes \mathrm{M}_{\mathbf{n}}(F) \cong \mathbf{B} \otimes \mathrm{M}_{\mathbf{m}}(F)$ for *arbitrary* supernatural numbers $\mathbf{n}$ and $\mathbf{m}$ will be a step too far, because of the zero element $\mathrm{M}_{\mathbf{p}^{\infty}}(F)$ of Corollary 19.5, which will collapse all the algebras to a single class.

21.1 The Underlying Division Algebra

In the Brauer group of finite dimensional algebras, every class has a unique representative which is the underlying division algebra. For algebras in $\mathcal{C}_F^{\omega}$, infinite matrices need to be taken into account. Indeed, by Theorem 18.4, every algebra $\mathbf{A} \in \mathcal{C}_F^{\omega}$ is equivalent to a unique algebra of the form $\mathrm{M}_{\mathbf{m}^{\infty}}(F) \otimes \mathbf{D}$, where $\mathbf{m}$ is a product of primes and $\mathbf{D} \in \mathcal{C}_F^{\omega}$ is a division algebra whose degree is prime to $\mathbf{m}$.

Corollary 21.5 *For an algebra* $\mathbf{A} \in \mathcal{C}_F^{\omega}$ *the following are equivalent:*

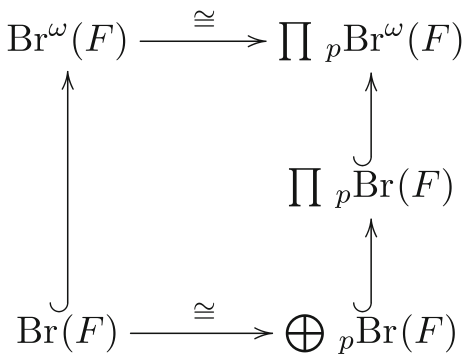
(1) $\mathbf{A}$ *is Brauer equivalent to a division algebra;*
(2) $\deg_{\mathrm{mat}}(\mathbf{A})$ *is locally finite;*
(3) *no proper factor of* $\mathbf{A}$ *is isomorphic to it.*

Proof (1) $\Longleftrightarrow$ (2) is by definition of equivalence, and (2) $\Longleftrightarrow$ (3) is Proposition 19.6. □

Theorem 12.6 now reads:

Corollary 21.6 *If* $\mathbf{A} \in \mathcal{C}_F^{\omega}$ *is Brauer equivalent to a division algebra, then every infinite dimensional proper factor of* $\mathbf{A}$ *is isomorphic to a pathological subalgebra.*

Fig. 2 Primary decomposition for the Brauer group and monoid



21.2 The Monoid ${}_p\mathrm{Br}^\omega(F)$

Let us utilize primary decomposition (Section 13.4). Since a locally finite supernatural p-power is a finite power of p, equivalence of p-algebras in $\mathcal{C}_F^\omega$ is defined by adding and removing finite matrices. Therefore, algebras $\mathbf{A}, \mathbf{B} \in \mathcal{C}^\omega$ are Brauer equivalent if and only if their primary components are matrix-equivalent everywhere.

Let ${}_p\mathrm{Br}^\omega(F) \subseteq \mathrm{Br}^\omega(F)$ denote the submonoid of (matrix-equivalence) classes of p-algebras.

Remark 21.7 Whereas finite dimensional central simple algebras have finitely many non-trivial primary components, the primary decomposition (9) induces an isomorphism of the countable Brauer monoid to the direct product (and not the direct sum) of its primary sub-monids. We thus have a commuting diagram of monoids and subgroups, as depicted in Fig. 2.

Let us now fix a prime p. By Proposition 18.3, every class $[\mathbf{A}] \neq [\mathrm{M}_{p^\infty}(F)]$ has a unique underlying division algebra. The degree and matrix degree are not defined on ${}_p\mathrm{Br}^\omega(F)$, but their finitude *is* well defined. These two invariants partition the countable Brauer monoid ${}_p\mathrm{Br}^\omega(F)$ into three types, as described in the following table:

Degree	Matrix degree	representatives
finite	finite	finite dimensional division algebras
infinite	finite	infinite dimensional division algebras
infinite	infinite	$\mathrm{M}_{p^\infty}(F)$

Since the degree is multiplicative and $1 = [F]$ has finite degree, every invertible element has finite degree:

Corollary 21.8 1. *${}_p\mathrm{Br}(F)$ is the group of invertible elements in ${}_p\mathrm{Br}^\omega(F)$.*
2. *The complement of ${}_p\mathrm{Br}(F)$ is the unique maximal ideal of ${}_p\mathrm{Br}^\omega(F)$.*

Moreover, the Brauer group acts on ${}_p\mathrm{Br}^\omega(F)$ by multiplication, and we have:

Proposition 21.9 *The action of ${}_p\mathrm{Br}(F)$ on ${}_p\mathrm{Br}^\omega(F)$ preserves the partition into types*

$${}_p\mathrm{Br}^\omega(F) = {}_p\mathrm{Br}(F) \;\cup\; ({}_p\mathrm{Br}^\omega(F) \setminus ({}_p\mathrm{Br}(F) \cup \{0\})) \;\cup\; \{0\}.$$

Proof The zero element absorbs by Corollary 19.1. The product of finite classes is clearly finite. It is also clear that if A_0 is a finite dimensional central simple algebra and $\mathbf{B}$ is a division algebra of degree p^∞, then by Remark 16.17, $\deg(A_0 \otimes \mathbf{B}) = p^\infty$. It remains to show that $A_0 \otimes \mathbf{B}$ is not the zero element. But $A_0^{\mathrm{op}} \otimes (A_0 \otimes \mathbf{B}) = \mathrm{M}_{\deg(A_0)}(\mathbf{B})$, which is clearly nonzero. □

21.3 Zero Divisors

Every $[\mathbf{A}] \in {}_p\mathrm{Br}^\omega$ of infinite dimension is a zero divisor. Indeed, by Proposition 16.8, $[\mathbf{A}] \cdot [\mathbf{A}^{\mathrm{op}}] = [\mathbf{A} \otimes \mathbf{A}^{\mathrm{op}}] = [\mathrm{M}_{p^\infty}(F)] = 0$ because $\deg(\mathbf{A}) = p^\infty$.

Even more so, the product of two algebras is zero quite frequently:

Proposition 21.10 *Let $\boldsymbol{A} = \varinjlim A_n$ and $\boldsymbol{B} = \varinjlim B_n$ be algebras of infinite degree in ${}_p\mathcal{C}_F^\omega$. Then $[\boldsymbol{A}] \cdot [\boldsymbol{B}] = 0$ if and only if the matrix degrees $\deg_{\mathrm{mat}}(A_n \otimes B_n)$ are unbounded.*

Proof $\mathbf{A} \otimes \mathbf{B} \sim 0$ if and only if $\mathrm{lcm}\{\deg_{\mathrm{mat}}(A_n \otimes B_n)\} = \deg_{\mathrm{mat}}(\mathbf{A} \otimes \mathbf{B}) = p^\infty$, which is the case if and only if $\deg_{\mathrm{mat}}(A_n \otimes B_n)$ are unbounded. □

Remark 21.11 For a finite dimensional central simple algebra A, we have that $\deg_{\mathrm{mat}}(A \otimes A) \geq \deg(A)$. Indeed $\mathrm{ind}(A \otimes A) \mid \mathrm{ind}(A) \mid \deg(A)$, so $\deg(A) \mid \deg(A \otimes A)/\mathrm{ind}(A \otimes A) = \deg_{\mathrm{mat}}(A \otimes A)$.

Of particular interest is the following:

Corollary 21.12 *For every $\boldsymbol{A} \in {}_p\mathcal{C}_F^\omega$ of infinite degree, $[\boldsymbol{A}]^2 = [\boldsymbol{A}] \cdot [\boldsymbol{A}] = 0$.*

Proof Writing $\mathbf{A} = \varinjlim A_n$, $\deg_{\mathrm{mat}}(A_n \otimes A_n) \geq \deg(A_n) \to \infty$, using Remark 21.11. □

Corollary 21.13 *The idempotents in ${}_p\mathrm{Br}^\omega(F)$ are trivial: $1 = [F]$ and $0 = [\mathrm{M}_{p^\infty}(F)]$.*

22 The Infinite Part of the Brauer Monoid

Proposition 21.9 leads us to consider the countable Brauer monoid ${}_p\mathrm{Br}^\omega(F)$ up to units, and more generally $\mathrm{Br}^\omega(F)$ up to units in each primary component. If D_p are finite dimensional central simple algebras for each prime p, we call the product $\bigotimes D_p$ **primary-finite**. (It should perhaps be called "locally finite" to conform with the terminology for supernatural numbers and matrices, but this would be confusing in the general context of locally finite algebras.)

Up to Brauer equivalence, the primary-finite algebras compose $\prod {}_p\mathrm{Br}(F)$, which is the unique maximal subgroup of $\mathrm{Br}^\omega(F)$.

Definition 22.1 We say that two algebras $\mathbf{A}, \mathbf{B} \in \mathcal{C}_F^\omega$ are **similar**, and write $\mathbf{A} \stackrel{*}{=} \mathbf{B}$, if there are primary-finite algebras $\mathbf{P}, \mathbf{P}' \in \mathcal{C}_F^\omega$ such that $\mathbf{A} \otimes \mathbf{P} \cong \mathbf{B} \otimes \mathbf{P}'$.

Similarity is an equivalence relation. The similarity class of $\mathbf{A} \in \mathcal{C}_F^\omega$ is denoted $[\mathbf{A}]^*$. We let $\mathcal{S} = SS$ denote the semigroup of similarity classes of non-primary-finite algebras $\mathbf{A} \in \mathcal{C}_F^\omega$, with the operation $[\mathbf{A}]^* \cdot [\mathbf{B}]^* = [\mathbf{A} \otimes \mathbf{B}]^*$.

Similarity ignores locally finite supernatural matrix algebras, and is therefore defined on the countable Brauer monoid: if $\mathbf{A} \sim \mathbf{A}'$ and $\mathbf{B} \sim \mathbf{B}'$ are Brauer equivalent, then $\mathbf{A} \stackrel{*}{=} \mathbf{B}$ if and only if $\mathbf{A}' \stackrel{*}{=} \mathbf{B}'$. We can thus write $[\mathbf{A}] \stackrel{*}{=} [\mathbf{B}]$ when $\mathbf{A} \stackrel{*}{=} \mathbf{B}$. We thus have two projections of monoids,

$$\mathcal{C}_F^{\omega} \longrightarrow \mathrm{Br}^{\omega}(F) \longrightarrow \mathcal{S} \cup \{1\},$$

which are defined by gradually forgetting details of the algebra: $\mathbf{A} \mapsto [\mathbf{A}] \mapsto [\mathbf{A}]^*$. All the finite dimensional central simple algebras collapse to a single element $[1]^*$ in the monoid $\mathcal{S} \cup \{1\}$.

22.1 Primary Decomposition

As before, there is a primary decomposition $\mathcal{S} \cup \{1\} = \prod({}_p\mathcal{S} \cup \{1\})$, where ${}_p\mathcal{S}$ is the semigroup of similarity classes of the infinite dimensional $\mathbf{A} \in {}_p\mathcal{C}_F^{\omega}$. For p-algebras $\mathbf{A}, \mathbf{B}$, $\mathbf{A} \stackrel{*}{=} \mathbf{B}$ if and only if there are finite dimensional central simple algebras P, P' of prime power degree, such that $\mathbf{A} \otimes P \cong \mathbf{B} \otimes P'$.

We may identify ${}_p\mathcal{S} \cup \{1\}$ with the quotient space ${}_p\mathrm{Br}^{\omega}(F)/{}_p\mathrm{Br}(F)$, so again there are projections of monoids

$${}_p\mathcal{C}_F^{\omega} \longrightarrow {}_p\mathrm{Br}^{\omega}(F) \longrightarrow {}_p\mathcal{S} \cup \{1\}.$$

As Proposition 21.9 shows, ${}_p\mathcal{S} \cup \{1\}$ has three "parts": a unit $1 = [F]^*$ composed of all the finite dimensional central simple algebras of degree a power of p, a zero element $0 = [\mathrm{M}_{p^\infty}(F)]^*$, and the similarity classes of infinite dimensional division algebras, up to finite algebras. As before, the types are characterized by finitude of the degree and matrix degree.

Since ${}_p\mathcal{S} \cup \{1\}$ is ${}_p\mathrm{Br}^{\omega}(F)$ up to units, notions such as divisibility, ideals or zero divisors in ${}_p\mathrm{Br}^{\omega}(F)$ are best studied in ${}_p\mathcal{S}$, where we work from now on (so a, b, c denote elements of ${}_p\mathcal{S}$). In particular we note that

Remark 22.2 The commutative semigroup ${}_p\mathcal{S}$ is nil of index 2. (Indeed, $a^2 = 0$ for every a by Corollary 21.12.)

We list some basic observations on divisibility. For $a, b \in {}_p\mathcal{S}$, we say that a **divides** b and write $a \,|\, b$ if there is $c \in {}_p\mathcal{S}$ such that $b = ac$. In other words $[\mathbf{A}]^*$ divides $[\mathbf{B}]^*$ when there are a finite dimensional central simple algebra D and an infinite dimensional algebra $\mathbf{C} \in \mathcal{C}_F^{\omega}$, such that $\mathbf{B} \otimes D \cong \mathbf{A} \otimes \mathbf{C}$.

Remark 22.3 Notice that $a \nmid a$ unless $a = 0$, because if $a = ac$ then $a = ac = ac^2 = 0$. It follows that the divisibility relation is anti-symmetric (and "almost" irreflexive).

Proposition 22.4 *There are no irreducible elements in ${}_p\mathcal{S}$ (i.e. ${}_p\mathcal{S}^2 = {}_p\mathcal{S}$).*

Proof Let $0 \neq a \in {}_p\mathcal{S}$. Let $\mathbf{A} \in a$ be a central simple algebra. Write $\mathbf{A} \cong \bigotimes D_i$ where $D_i \neq F$ are finite dimensional central simple algebras. Then $\mathbf{A} \cong (\bigotimes D_{2i}) \otimes (\bigotimes D_{2i+1})$, so taking $b = [\bigotimes D_{2i}]^*$ and $c = [\bigotimes D_{2i+1}]^*$, we have that $a = [\mathbf{A}]^* = bc$. □

Every commutative nil-semigroup of index 2 can be imbedded into such a semigroup without irreducible elements, as proved in [17]. In that paper the authors comment that "examples of these semigroups are rarely seen". By Proposition 22.4, ${}_p\mathcal{S}$ is such an example.

To see how complicated ${}_p\mathcal{S}$ is, consider subsets of ω. We say that $I \subseteq^* I'$ if $I' \setminus I$ is finite, and that $I \stackrel{*}{=} I'$ if both $I \subseteq^* I'$ and $I' \subseteq^* I$. This is an equivalence relation, whose classes are ordered by the strong order relation $\subset^*$, defined by declaring that $I \subset^* I'$ when $I \subseteq^* I'$ but $I' \stackrel{*}{\neq} I$ (a strong order relation is transitive, irreflexive and asymmetric). Let $[I]^*$ denote the equivalence class of a subset I. Let $P^*(\omega)$ denote the set of equivalence classes of infinite sets (so we remove the single class $[\emptyset]^*$, composed of all finite subsets). The order relation $\subset^*$ is defined on $P^*(\omega)$ by setting $[I]^* \subset^* [I']^*$ when $I^* \subset^* I'^*$.

Theorem 22.5 *If ${}_p\mathcal{S} \neq 0$, then there is an order-preserving injection $(P^*(\omega), \subset^*) \to ({}_p\mathcal{S}, |)$.*

Proof Let $0 \neq a \in {}_p\mathcal{S}$, and choose a division algebra $\mathbf{A} \in a$. Decompose $\mathbf{A} = \bigotimes_\omega D_i$ where the D_i are finite dimensional division algebras. Define a map $\mu : P^*(\omega) \to {}_p\mathcal{S}$ by $\mu([I]^*) = [\bigotimes_{i \in I} D_i]^*$. This is well defined, because if $I \stackrel{*}{=} I'$ for $I, I' \subseteq \omega$, then $\bigotimes_{i \in I} D_i \stackrel{*}{=} \bigotimes_{i \in I'} D_i$. If $[I]^* \subset^* [I']^*$ then up to finite subsets we may assume $I \subset I'$, with infinite complement, so $\mu(I') = [\bigotimes_{I'} D_i]^* \,|\, [\bigotimes_I D_i]^* = \mu(I)$ because of the decomposition $\bigotimes_{I'} D_i = (\bigotimes_I D_i) \otimes (\bigotimes_{I' \setminus I} D_i)$. This shows μ is order-preserving.

Suppose $\mu([I]^*) = \mu([I']^*)$. Let $J = I \setminus I'$ and $J' = I' \setminus I$. Let $\mathbf{B} = \bigotimes_{I \cap I'} D_i$, $C = \bigotimes_J D_i$ and $C' = \bigotimes_{J'} D_i$. By assumption $\mathbf{B} \otimes \mathbf{C} \stackrel{*}{=} \mathbf{B} \otimes \mathbf{C}'$, so if C' has infinite dimension, we would have that $\mathbf{B} \otimes \mathbf{C} \otimes \mathbf{C}' \stackrel{*}{=} \mathbf{B} \otimes \mathbf{C}' \otimes \mathbf{C}' \stackrel{*}{=} 0$ by Proposition 21.12, contrary to the fact that $\mathbf{B} \otimes \mathbf{C} \otimes \mathbf{C}' = \bigotimes_{I \cup I'} D_i$ is a factor of the division algebra $\mathbf{A}$, and hence itself a division algebra. This shows J' is finite, and J is finite by symmetry, so $I \stackrel{*}{=} I'$. □

The injection is into the set of divisors of a given division algebra (and depends on its decomposition as a countable tensor product).

Dedekind cuts supply us with an order-preserving embedding of the real line $(\mathbb{R}, <)$ into $(P^*(\mathbb{Q}), \subset^*)$, by $\alpha \mapsto [\{x \in \mathbb{Q} : \alpha < x\}]^*$.

Corollary 22.6 *Assume ${}_p\mathcal{S} \neq 0$. Then there is a continuum of non-isomorphic division algebras $\mathbf{D}_\alpha \in \mathcal{C}_F^\omega$ ($\alpha \in \mathbb{R}$), such that $\mathbf{D}_\alpha$ is a factor of $\mathbf{D}_\beta$ whenever $\alpha < \beta$.*

22.2 Ideals in $\mathcal{S}$

A subset $I \subseteq \mathcal{S}$ is an **ideal** if $\mathcal{S}I \subseteq I$. An ideal of the form $\mathcal{S}a$ is called a **principal ideal**. The subset $\mathcal{S}a \cup \{a\}$ is an ideal as well. From Remark 22.3 it follows that $a \notin \mathcal{S}a$ unless $a = 0$.

Remark 22.7 If $a \,|\, b \neq 0$ then $\mathcal{S}b \cup \{b\} \subsetneq \mathcal{S}a$. In particular $\mathcal{S}b \subsetneq \mathcal{S}a$ and $\mathcal{S}b \cup \{b\} \subsetneq \mathcal{S}a \cup \{a\}$.

It suffices to prove that $\mathcal{S}b \cup \{b\} \neq \mathcal{S}a$. By assumption $b = ac$ for some $c \in \mathcal{S}$ and by Proposition 22.4 we can write $c = c'c''$ for $c', c'' \in \mathcal{S}$. Then $ac' \in \mathcal{S}a$ but $ac' \notin \mathcal{S}b \cup \{b\}$ because $ac' \,|\, b$.

22.3 Annihilators

Let us consider the annihilators

$$\operatorname{ann}(a) = \left\{b \in {}_p\mathcal{S} : ab = 0\right\}$$

for $a \in {}_p\mathcal{S}$. Obviously, if $a \mid b$ then $\operatorname{ann}(a) \subseteq \operatorname{ann}(b)$.

Proposition 22.8 *If* $a \mid b \neq 0$ *then* $\operatorname{ann}(a) \subsetneq \operatorname{ann}(b)$.

Proof By assumption $b = ac$ for some c, and then $bc = ac^2 = 0$ so $c \in \operatorname{ann}(b)$, but $ac = b \neq 0$ so $c \notin \operatorname{ann}(a)$. □

Corollary 22.9 *Assume* ${}_p\mathcal{S} \neq 0$. *The linearly ordered continuum of elements of* ${}_p\mathcal{S}$ *(Corollary 22.6) provides a chain of principal ideals (Remark 22.7) and a chain of annihilators (Proposition 22.8), both of the cardinality of the continuum.*

So we do not have ACC or DCC on principal ideals of annihilators. It should be noted that the annihilators in $\mathcal{S}$ reflect the underlying algebraic structure beyond divisibility:

Proposition 22.10 *For any* $\boldsymbol{A}, \boldsymbol{B} \in {}_p\mathcal{C}_F^{\omega}$, *if* $\boldsymbol{A} \hookrightarrow \boldsymbol{B}$ *then* $\operatorname{ann}([\boldsymbol{A}]^*) \subseteq \operatorname{ann}([\boldsymbol{B}]^*)$.

Proof If $[\mathbf{A}] \cdot [\mathbf{C}] = 0$ then $p^{\infty} = \deg_{\mathrm{mat}}(\mathbf{A} \otimes \mathbf{C}) \mid \deg_{\mathrm{mat}}(\mathbf{B} \otimes \mathbf{C})$ by Proposition 16.3, so $[\mathbf{B}] \cdot [\mathbf{C}] = 0$ as well. □

Since the underlying division algebras can be presented by Proposition 10.13 as countable tensor products of finite dimensional division algebras, we need a criterion for such a product to have infinite matrix degree.

We say that a finite set of division algebras $D_1, \dots, D_k$ is **deficient** if $D_1 \otimes \cdots \otimes D_k$ is not a division algebra.

Proposition 22.11 *Let* $\mathcal{D} = \{D_1, D_2, \dots\}$ *be a countable set of central division algebras of finite p-power degree. Then* $\bigotimes_{D \in \mathcal{D}} D \cong \mathrm{M}_{p^{\infty}}(F)$ *if and only if* $\mathcal{D}$ *contains infinitely many disjoint deficient subsets.*

Proof Call a set of subsets $T \subseteq P(\mathcal{D})$ a "deficient family" if the elements of T are disjoint deficient subsets of $\mathcal{D}$. The collection of deficient families is closed under union of chains (because every element of the union belongs to a member of the chain and is thus a deficient set; and any two elements belong to a member and are thus disjoint), and by Zorn's lemma there is a maximal deficient family T. By maximality, $\mathcal{D} - \bigcup T$ has no deficient set, so any finite product of algebras from $\mathcal{D} - \bigcup T$ is a division algebra, and hence $\bigotimes_{D \notin \bigcup T} D$ is a division algebra.

For every $I \in T$, $\deg_{\mathrm{mat}}(\otimes_{D \in I} D)$ is divisible by p, but finite because it divides $\prod_{D \in I} \deg(D)$. Therefore, the matrix degree of $\bigotimes_{D \in \bigcup T} D = \bigotimes_{I \in T}(\otimes_{D \in I} D)$ is finite or infinite together with T. In light of the decomposition $\bigotimes_{D \in \mathcal{D}} D = (\bigotimes_{D \in \bigcup T} D) \otimes (\bigotimes_{D \notin \bigcup T} D)$, we proved that the matrix degree of $\bigotimes_{D \in \mathcal{D}} D$ is infinite if and only if T is infinite.

This proves a bit more than required: $\bigotimes_{D \in \mathcal{D}} D$ is isomorphic to $\mathrm{M}_{p^{\infty}}(F)$ if and only if every maximal deficient family is infinite, if and only if some maximal deficient family is infinite, if and only if there is an infinite deficient family. □

Corollary 22.12 *Let* $\boldsymbol{A}, \boldsymbol{B} \in {}_p\mathcal{C}_F^{\omega}$ *be division algebras. Then* $\boldsymbol{A} \otimes \boldsymbol{B} \sim 0$ *if and only if there are decompositions* $\boldsymbol{A} = \bigotimes A_i$ *and* $\boldsymbol{B} = \bigotimes B_i$, *with* A_i, B_i *finite dimensional division algebras, such that no* $A_i \otimes B_i$ *is a division algebra.*

Proof First assume $\mathbf{A} \otimes \mathbf{B} \sim 0$. Decompose $\mathbf{A} = \bigotimes A_i$ and $\mathbf{B} = \bigotimes B_i$, where A_i, B_i are finite dimensional division algebra. Write $\mathfrak{a} = \{A_1, A_2, \dots\}$ and $\mathfrak{b} = \{B_1, B_2, \dots\}$. By Proposition 22.11 there are infinitely many disjoint deficient subsets $S_1, S_2, \dots$ of $\mathfrak{a} \cup \mathfrak{b}$. Each deficient subset must intersect both $\mathfrak{a}$ and $\mathfrak{b}$ not trivially. Enumerate the factors A_i in $\mathfrak{a} - \cup S_j$ as $A_{m_1}, A_{m_2}, \dots$; if this set is finite, take $A_{m_k} = F$ for k larger than the size of the set. Likewise enumerate $B_{m'_1}, B_{m'_2}, \dots$. For each n, let $A'_n = A_{m_n} \otimes \bigotimes_{A_0 \in \mathfrak{a} \cap S_n} A_0$; and let $B'_n = B_{m'_n} \otimes \bigotimes_{B_0 \in \mathfrak{b} \cap S_n} B_0$. Then $\mathbf{A} \cong \bigotimes A'_n$, $\mathbf{B} \cong \bigotimes B'_n$, and $A'_n \otimes B'_n$ is never a division algebra, being a product over a set containing S_n which is deficient.

In the other direction, given the decompositions as described, we have that $\deg_{\text{mat}}(\mathbf{A} \otimes \mathbf{B}) = \deg_{\text{mat}}(\bigotimes(A_i \otimes B_i)) = \prod \deg_{\text{mat}}(A_i \otimes B_i) = p^\infty$. □

Corollary 22.13 *For every $a, b \in {}_p\mathcal{S}$, if $ab = 0$ then there are decompositions $a = a'a''$ and $b = b'b''$ such that $a'b' = 0$ and $a''b'' = 0$.*

Proof Choose division algebra representatives $\mathbf{A} \in a$ and $\mathbf{B} \in b$. Decompose $\mathbf{A} = \bigotimes A_i$ and $\mathbf{B} = \bigotimes B_i$ as in Corollary 22.12. Then $[\mathbf{A}]^* = [\bigotimes A_{2i}]^*[\bigotimes A_{2i+1}]^*$, $[\mathbf{B}]^* = [\bigotimes B_{2i}]^*[\bigotimes B_{2i+1}]^*$, and $[\bigotimes A_{2i}]^*[\bigotimes B_{2i}]^* = [\bigotimes A_{2i+1}]^*[\bigotimes B_{2i+1}]^* = 0$. □

With this we can improve Proposition 22.4, taking into account that $\operatorname{ann}(0) = {}_p\mathcal{S}$.

Proposition 22.14 *For any $0 \neq a \in {}_p\mathcal{S}$, there are no irreducible elements in* $\operatorname{ann}(a)$ *(when considered as a semigroup).*

Proof Let $b \in \operatorname{ann}(a)$. By Corollary 22.13 we can write $a = a'a''$ and $b = b'b''$ where $a'b' = 0$ and $a''b'' = 0$. In particular $ab' = ab'' = 0$, so $b', b'' \in \operatorname{ann}(a)$ and b is not irreducible there. □

In contrast, a principal ideal ${}_p\mathcal{S}a$ is the null semigroup, so we conclude with

Corollary 22.15 *An annihilator in ${}_p\mathcal{S}$ cannot be a principal ideal.*

23 The Brauer Monoid over Special Fields

The p-**Brauer dimension** of F is defined to be the minimal number d for which $\operatorname{ind}(D) \mid \exp(D)^d$ for every finite dimensional central division algebra D of p-power degree, if this value is finite. The p-Brauer dimension is 1 when the index equals the exponent, as is the case over local or global fields.

In this section we consider the countable Brauer monoid when F has finite p-Brauer dimension. Fields with this property are abundant by [24]: every C_n field has finite p-Brauer dimension (depending on p).

Proposition 23.1 *Let F be a field over which the exponent of a finite dimensional division algebra of p-power degree is equal to the index. Then the only division algebras of (supernatural) p-power degree in $\mathcal{C}_F$ are finite dimensional.*

Proof Passing to a subalgebra, we may assume that a counterexample $\mathbf{D}$ is countably generated and thus (Proposition 8.1) in ${}_p\mathcal{C}_F^\omega$. But then $\mathbf{D} = \bigotimes D_i$ where D_i are finite dimensional and with $p \mid \mathrm{ind}(D_i)$, so that $\mathrm{ind}(D_1 \otimes D_2) = \exp(D_1 \otimes D_2) < \exp(D_1)\exp(D_2) = \mathrm{ind}(D_1)\mathrm{ind}(D_2)$, contrary to $D_1 \otimes D_2$ being a division algebra. □

By the classification obtained in Proposition 18.3, we get that:

Corollary 23.2 *Over a field with index=exponent,* ${}_p\mathrm{Br}^{\omega} = {}_p\mathrm{Br} \cup \{0\}$.

In light of the fact that $[\mathbf{A}]^2 = 0$ in ${}_p\mathrm{Br}^{\omega}$, actual exponents makes little sense. But similarly to the supernatural degree, we can still define

$$\exp(\mathbf{A}) = \mathrm{lcm}_{A_0 \subseteq \mathbf{A}}\{\exp(A_0)\}$$

ranging over finite dimensional central simple algebras.

Proposition 23.3 *Over a field of finite p-Brauer dimension $d > 1$, every infinite dimensional division algebra in ${}_p\mathcal{C}_F^{\omega}$ has infinite exponent.*

Moreover, if $\mathbf{D} = \bigotimes D_t$ is a division algebra, then the number of D_t with $\exp(D_t) \mid p^N$ *is at most* $\log(\frac{d^2}{d-1}N)/\log(\frac{d}{d-1})$.

Proof Assume $\mathbf{D} = \bigotimes D_i$ is a division algebra, with $\exp(D_i) = p^{n_i}$ where $n_1 \leq n_2 \leq \cdots$. By the assumption on F, $p^{n_i} = \exp(D_i) \mid \mathrm{ind}(D_i) \mid p^{dn_i}$, so $p^{n_1+\cdots+n_t} \mid \prod \mathrm{ind}(D_i) = \mathrm{ind}(D_1 \otimes \cdots \otimes D_t) \mid p^{dn_t}$, so that $\sum_{i=1}^{t} n_i \leq dn_t$. We claim that $n_i \geq \frac{1}{d}(\frac{d}{d-1})^{i-1}$. For $i = 1$ in fact $n_1 \geq 1 \geq \frac{1}{d}$. For $i = 2$ we have that $n_2 \geq \frac{1}{d-1}n_1 \geq \frac{1}{d-1}$ because $n_1 + n_2 \leq dn_2$. For every $t \geq 2$, $n_t \geq \frac{1}{d-1}\sum_{i=1}^{t-1} n_i \geq \frac{1}{d-1}(1 + \sum_{i=2}^{t-1}\frac{1}{d}(\frac{d}{d-1})^{i-1}) \geq \frac{1}{d}(\frac{d}{d-1})^{t-1}$. Now, if $\frac{1}{d}(\frac{d}{d-1})^{t-1} \leq n_t \leq N$ then $t \leq \log(\frac{d^2}{d-1}N)/\log(\frac{d}{d-1})$. □

24 Restriction and Splitting Fields

Having worked so far over a fixed base field, we make some concluding comments on field extensions. Let K/F be an extension of fields. In the finite case, extension of scalars induces the restriction map $\mathrm{Br}(F) \to \mathrm{Br}(K)$, which is a homomorphism (of groups). We can say the same in the infinite case.

Proposition 24.1 *The restriction map* $\mathrm{res}_{K/F} : \mathcal{C}_F \to \mathcal{C}_K$ *is a homomorphism of monoids. It induces, for every directed set* Γ, *a homomorphism* $\mathrm{res}_{K/F} : \mathcal{C}_F^{\Gamma} \to \mathcal{C}_K^{\Gamma}$.

Extension of scalars preserves the degree, and therefore preserves primary decomposition. It carries supernatural matrices to supernatural matrices, and therefore induces a homomorphism

$$\mathrm{res}_{K/F} : \mathrm{Br}^{\omega}(F) \to \mathrm{Br}^{\omega}(K).$$

Let us consider algebras in $\mathcal{C}_F^{\omega}$, where we have primary decomposition. We say that a field extension K/F **splits** an algebra $\mathbf{A} \in \mathcal{C}_F^{\omega}$ if $K \otimes_F \mathbf{A}$ is a supernatural matrix algebra. If an algebra splits, then any algebra Brauer equivalent to it splits as well.

For algebras of finite degree, this notion coincides with the standard notion of splitting. However for an algebra $\mathbf{A}_p$ of infinite p-power degree, splitting means $K \otimes_F \mathbf{A}_p \cong \mathrm{M}_{p^{\infty}}(K)$, so that $\mathrm{res}_{K/F}[\mathbf{A}_p] = 0$ in ${}_p\mathrm{Br}^{\omega}(K)$. Very possibly, if $\mathbf{A}$ is split by K then some of the primary components are finite dimensional algebras split in the usual sense, while others are infinite dimensional algebras that become infinite matrices. Following Corollary 21.13, we can summarize as follows:

Remark 24.2 The idempotents in $\mathrm{Br}^{\omega}(F)$ are the classes of supernatural matrices. A class $[\mathbf{A}] \in \mathrm{Br}^{\omega}(F)$ is split by K if $\mathrm{res}_{K/F}[\mathbf{A}]$ is an idempotent.

A finite dimensional extension can only increase the matrix degree by a finite amount, so a finite dimensional field extension can only split finite dimensional division algebras.

Proposition 24.3 *Let* $[\boldsymbol{A}] \in {}_p\mathrm{Br}^{\omega}$. *Every infinite dimensional subfield* $K \subseteq \boldsymbol{A}$ *splits* $[\boldsymbol{A}]$.

Proof Write $\mathbf{A} = \varinjlim A_n$. We may assume $A_1 \subseteq A_2 \subseteq \cdots$. Let $K_n = K \cap A_n$, which are subfields. We know that $K = \bigcup K_n$, so $\dim K_n \to \infty$. We also know that $K_n \otimes A_n \cong \mathrm{M}_{\dim K_n}(\mathrm{C}_{A_n}(K_n))$. So $\mathrm{M}_{\dim K_n}(F) \subseteq K_n \otimes A_n \subseteq K \otimes A_n$, which proves that $K \otimes \mathbf{A} = \varinjlim K \otimes A_n$ has infinite matrix degree. □

So unlike in the finite theory, any infinite dimensional subfield of $\mathbf{A}$ splits the algebra; we do not need to assume it is maximal in one of its representatives.

Question 24.4 *Let* $\boldsymbol{A} \in \mathcal{C}_F^{\omega}$. *Let* $K \subseteq \boldsymbol{A}$ *be an infinite dimensional subfield. Is* K *necessarily maximal in some algebra* $\boldsymbol{A}' \in [\boldsymbol{A}]$?

Notice that any subfield of a locally finite dimensional algebra is algebraic. For a field K algebraic over F, we define the **supernatural dimension** $\dim(K)$ to be the least common multiple of the finite dimensions of subfields; this is once again a supernatural number.

Proposition 24.5 *Let* $\boldsymbol{D} \in \mathcal{C}_F^{\omega}$ *be a division algebra. If* $K \subseteq \boldsymbol{D}$ *then* $\dim K \mid \deg \boldsymbol{D}$.

Proof As in Proposition 24.3, write $\mathbf{D} = \varinjlim D_n$ and let $K_n = K \cap D_n$. Since $\dim(K_n) \mid \deg(D_n)$ for every n, the relation holds for the least common multiples. □

Similarly, we have:

Proposition 24.6 *Let* K *be a subfield of a division algebra* $\boldsymbol{D} \in \mathcal{C}_F^{\omega}$. *Then* $\deg_{\mathrm{mat}}(K \otimes \boldsymbol{D}) = \dim K$.

Proof Again write $\mathbf{D} = \varinjlim D_n$ and let $K_n = K \cap D_n$. Since $\mathrm{C}_{D_n}(K_n)$ are division algebras, we have that

$$\deg_{\mathrm{mat}}(K \otimes \mathbf{D}) = \mathrm{lcm}\{\deg_{\mathrm{mat}}(K_n \otimes D_n)\} = \mathrm{lcm}\{\dim(K_n)\} = \dim K. \qquad \square$$

We can now supply a criterion for splitting.

Theorem 24.7 *A subfield* K *splits a division algebra* $\boldsymbol{D} \in \mathcal{C}_F^{\omega}$ *if and only if* $\dim K = \deg \boldsymbol{D}$.

Proof We have that $\deg_{\mathrm{mat}}(K \otimes \mathbf{D}) = \dim K$ by Proposition 24.6 while $\deg(K \otimes \mathbf{D}) = \deg(\mathbf{D})$, so we are done by Theorem 18.2. □

References

1. Abrams, G.: Leavitt path algebras: the first decade. Bull. Math. Sci. **5**(1), 59–120 (2015)
2. Albert, A.A.: Structure of algebras. AMS Coll Publ XXIV (1961)
3. Atiyah, M.F., MacDonald, I.G.: Introduction to commutative algebra perseus. Books (1969)
4. Auel, A., Brussel, E., Garibaldi, S., Vishne, U.: Open problems on central simple algebras. Transform. Groups **16**(1), 219–264 (2011)
5. Barsotti, I.: Noncountable normally locally finite division algebras. Proc. Amer. Math. Soc. **8**, 1101–1103 (1957)
6. Brauer, R.: On algebras which are connected with the semisimple continuous groups. Ann. Math. **38**, 857–872 (1937)
7. Bar-On, T., Gilat, S., Matzri, E., Vishne, U.: The algebra of supernatural matrices. Submitted.
8. Deo, T.T., Bien, M.H., Hai, B.X.: On the radicality of maximal subgroups in $GL_n(D)$. J. Algebra **365**, 42–49 (2012)
9. Deo, T.T., Bien, M.H., Hai, B.X.: On weakly locally finite division rings. Acta Math. Vietnam. **44**, 553–569 (2019)
10. Hai, B.X., Deo, T.T., Bien, M.H.: On subgroups in division rings of type 2. Stud. Sci. Math. Hung. **49**(4), 549–557 (2012)
11. Gille, P.H., Szamuely, T.: Central simple algebras and galois cohomology. Cambridge Studies in Advanced Mathematics **101** (2006)
12. Glimm, J.G.: On a certain class of operator algebras. Trans. Amer. Math. Soc. **95**, 318–340 (1960)
13. Haile, D.E.: The Brauer monoid of a field. J. Algebra **81**(2), 521–539 (1983)
14. Haile, D.E., Rowen, L.H.: Weakly Azumaya algebras. J. Algebra **250**(1), 134–177 (2002)
15. Jacobson, N.: Structure of rings, colloquium publications. XXXVII AMS (1956)
16. Jacobson, N.: Finite Dimensional Division Algebras. Springer, Berlin (1996)
17. Ježek, J., Kepka, T., Němec, P.: Commutative semigroups that are nil of index 2 and have no irreducible elements. Math. Bohem. **133**(1), 1–7 (2008)
18. Kaplansky, I.: Rings with a polynomial identity. Bull. Amer. Math. Soc. **54**, 575–580 (1948)
19. Knus, M.-A., Merkurjev, A., Rost, M., Tignol, J.-P.: The book of involutions, Colloquium Publications, vol. 44 American Mathematical Society (1998)
20. Köthe, G.: Schiefkörper unendlichen Range über dem Zentrum. Math. Ann. **105**, 15–39 (1931)
21. Lam, T.Y.: Lectures on Modules and Rings, LNM 189, Springer-Verlag (1999)
22. Lam, T.Y.: Multiples, modules with isomorphic matrix rings - a survey, monographie 35 de L'Enseignement Mathématiqueu, Genève (1999)
23. Maltcev, V., Mazorchuk, V.: Presentation of the singular part of the Brauer monoid. Math. Bohem. **132**(3), 297–323 (2007)
24. Matzri, E.: Symbol length in the Brauer group of a field. Trans. Amer. Math. Soc. **368**, 413–427 (2016)
25. McCrimmon, K.: Deep matrices and their Frankenstein actions, Non-associative algebra and its applications. Lect. Notes Pure Appl. Math. **246**, 261–274 (2006). Chapman & Hall/CRC, Boca Raton FL
26. Novikov, B.V.: On the Brauer monoid (Russian). Mat. Zametki **57**(4), 633–636 (1995). translation in Math. Notes 57(3–4), (1995), 440–442
27. Ribes, L., Zalesskii, P.: Profinite Groups. Springer, Berlin (2010)
28. Rowen, L.H.: Ring Theory, vol. I. Academic Press, Cambridge (1988)
29. Saltman, D.: Lectures on Division Algebras, CBMS Regional Conference Series in Mathematics 94, AMS (1998)
30. Solian, A.: Theorey of Modules. Wiley, New York (1977)
31. Tignol, J.P., Wadsworth, A.: Value functions on simple algebras, and associated graded rings. Springer Monographs in Mathematics (2015)
32. Zelmanov, E.I.: Lie algebras and torsion groups with identity. J. Combin. Alg. **1**(3), 289–340 (2017)