



Hopfological Algebra for Infinite Dimensional Hopf Algebras

Marco A. Farinati¹

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Abstract

We consider “Hopfological” techniques as in Khovanov, M., J. Knot Theory Ramificat **25**(3), 26 (2016) but for infinite dimensional Hopf algebras, under the assumption of being co-Frobenius. In particular, $H = k[\mathbb{Z}]\#k[x]/x^2$ is the first example, whose corepresentations category is d.g. vector spaces. Motivated by this example we define the “Homology functor” (we prove it is homological) for any co-Frobenius algebra, with coefficients in H -comodules, that recover usual homology of a complex when $H = k[\mathbb{Z}]\#k[x]/x^2$. Another easy example of co-Frobenius Hopf algebra gives the category of “mixed complexes” and we see (by computing an example) that this homology theory differs from cyclic homology, although there exists a long exact sequence analogous to the SBI-sequence. Finally, because we have a tensor triangulated category, its K_0 is a ring, and we prove a “last part of a localization exact sequence” for K_0 that allows us to compute -or describe- K_0 of some family of examples, giving light of what kind of rings can be categorified using this techniques.

Keywords Co-Frobenius Hopf algebras · Tensor triangulated categories · Homology theories · K_0 · Categorification

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1 Introduction

This paper has mainly 3 contributions:

- (1) The “Hopfological algebra” can be developed not only for finite dimensional Hopf algebras but also for infinite dimensional ones, provided they are co-Frobenius. The language of comodules is better adapted than the language of modules.

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✉ Marco A. Farinati
mfarinat@dm.uba.ar

¹ Dpto de Matemática FCEyN UBA - IMAS (Conicet), Buenos Aires, Argentina

- (2) The formula “Ker d /Im d ” can be written in Hopf-co-Frobenius language.
- (3) Some K -theoretical results allow us to compute K_0 of the stable categories associated to co-Frobenius Hopf algebras of the form $H_0\#\mathfrak{B}$, with H_0 cosemisimple and \mathfrak{B} finite dimensional.

The paper is organized as follows: In Sections 2 and 3 we show points (1) and (2) respectively. In Section 4 we develop some tools to understand the triangulated structure. In Section 5 we exhibit the first examples. Section 6 deals with K_0 . Section 7 illustrate the first step on how to develop -in the setting of co-Frobenius Hopf algebras- the direction taken in [8] for finite dimensional Hopf algebras.

2 Integrals, Co-Frobenius and Triangulated Structure

k will be a field, H a Hopf algebra over k , all comodules will be right comodules. The category of H -comodules is denoted \mathcal{M}^H and the subcategory of finite dimensional comodules is denoted \mathfrak{m}^H .

2.1 Integrals

Definition 2.1 (Hochschild, 1965; G. I. Kac, 1961; Larson-Sweedler, 1969). A (left) integral is a linear map $\Lambda : H \rightarrow k$ such that

$$(\text{id} \otimes \Lambda)\Delta h = \Lambda(h)1 \quad \forall h \in H$$

that is, $h_1\Lambda(h_2) = \Lambda(h)1$.

It is well-known that the dimension of the space of (left) integral is ≤ 1 . In case H admits a non-zero (left) integral $\Lambda \in H^*$, H will be called co-Frobenius. The following is well-known, we refer to [2] and [3] and references therein for the proofs:

Theorem 2.2 *If H is co-Frobenius then, in the category of (say right) H -comodules*

1. *there exists enough projectives;*
2. *every finite dimensional comodule is a quotient of a finite dimensional projective, and embeds into a finite dimensional injective;*
3. *being projective is the same as being injective.*

We will use very often the fact that a finite collection of elements in a comodule is contained in a finite-dimensional comodule. This is sometimes referred as *the fundamental theorem for comodules* (see [4, Theorem 2.1.7]). For expository purpose we just remind that if $m \in M$ is an element in an H -comodule M , a way to produce a finite dimensional subcomodule $M' \subset M$ containing m is the following: If $\rho : M \rightarrow M \otimes H$ is the comodule structure map and $\{h_i\}_{i \in I}$ is a k -basis of H , then write

$$\rho(m) = \sum_{i \in I} m_i \otimes h_i$$

where almost all m_i are zero. This says that $M' :=$ the k -span of the m_i 's is finite dimensional. The counit axiom implies $m \in M'$, because

$$m = (\text{id} \otimes \epsilon)\rho(m) = \sum_{i \in I} \epsilon(h_i)m_i$$

A manipulation of the coassociativity axiom shows that in fact M' is a subcomodule.

Finally if M and N are two objects in a category \mathcal{B} , denote $\text{Hom}_{\mathcal{B}}(M, N)$ set of morphisms and $I_{\mathcal{B}}(M, N)$ the subset of $\text{Hom}_{\mathcal{B}}(M, N)$ consisting on morphisms that factors through an injective object of \mathcal{B} . Denote

$$\underline{\text{Hom}}_{\mathcal{B}}(M, N) := \frac{\text{Hom}_{\mathcal{B}}(M, N)}{I_{\mathcal{B}}(M, N)}$$

The category whose objects are H -comodules and morphism $\underline{\text{Hom}}^H$ is called the *stable* category and it is denoted $\underline{\mathcal{M}}^H$. Similarly $\underline{\mathfrak{m}}^H$ is the stable category associated to \mathfrak{m}^H . By the above theorem, $\underline{\mathfrak{m}}^H$ is embedded fully faithfully in $\underline{\mathcal{M}}^H$. With these preliminaries, one can prove the following main construction:

Theorem 2.3 *If H is a co-Frobenius Hopf algebra then $\underline{\mathcal{M}}^H$ has a natural structure of triangulated category, $\underline{\mathfrak{m}}^H$ is a triangulated subcategory.*

Proof We apply directly Happel’s Theorem 2.6 of [5]. The only thing to do is to notice that \mathcal{M}^H (and \mathfrak{m}^H) are Frobenius exact categories. Using Happel’s notation, let \mathcal{B} be an additive category embedded as a full and extension-closed subcategory in some abelian category \mathcal{A} , and \mathcal{S} the set of short exact sequences in \mathcal{A} with terms in \mathcal{B} . For as, since both \mathcal{M}^H and \mathfrak{m}^H are already abelian, we have $\mathcal{A} = \mathcal{B}$ and the notion of \mathcal{S} -projective and \mathcal{S} -injective is the same as usual projectives and injectives. Maybe we just remark that an object in \mathfrak{m}^H is injective in \mathfrak{m}^H if and only if it is injective in \mathcal{M}^H and similarly for projectives (see Lemma 2.4 as an illustration).

An exact category $(\mathcal{B}, \mathcal{S})$ is called a Frobenius category if $(\mathcal{B}, \mathcal{S})$ has enough \mathcal{S} -projectives and enough \mathcal{S} -injectives and if moreover the \mathcal{S} -projectives coincide with the \mathcal{S} -injectives. In our case, $\mathcal{B} = \mathcal{M}^H$ or $\mathcal{B} = \mathfrak{m}^H$ are clearly Frobenius categories if H is a co-Frobenius Hopf algebra. Theorem 2.6 in [5] just state that the stable category $\underline{\mathcal{B}}$ is triangulated. □

Lemma 2.4 *If $P \in \mathfrak{m}^H$ then P is projective in \mathfrak{m}^H if and only if it is projective in \mathcal{M}^H .*

Proof If P is projective in \mathcal{M}^H then then it has the lifting property for all comodules, in particular for the finite dimensional ones. Assume P is projective in \mathfrak{m}^H and consider a diagram of comodules

$$\begin{array}{ccc} Z & \xrightarrow{\pi} & Y \\ & & \uparrow f \\ & & P \end{array}$$

where Z and Y are not necessarily finite dimensional. Since P is finite dimensional, one can consider $Y' = f(P) \subseteq Y$, clearly Y' is a finite dimensional comodule, with generators say $\{y_1, \dots, y_n\}$. Since π is surjective, one may found z_i ($i = 1, \dots, n$) with $\pi(z_i) = y_i$ and *there exists a finite dimensional subcomodule $Z' \subseteq Z$ containing all z_i 's*, hence we have a diagram

$$\begin{array}{ccc} Z' & \xrightarrow{\pi|_{Z'}} & Y' \\ \swarrow \exists \bar{f} & & \uparrow f \\ & & P \end{array}$$

Now all comodules are finite dimensional, and because P is projective within finite dimensional comodules, there exists a lifting $\bar{f} : P \rightarrow Z' \subseteq Z$ of f , hence, a lifting of the original f . □

For clarity of the exposition we recall the definition of suspension, desuspension and triangles in $\underline{\mathcal{M}}^H$. For this particular case of comodules over a co-Frobenius Hopf algebra, the general definitions can be more explicitly realized. Moreover, for $H = H_0\#\mathfrak{B}$ as in Section 4, concrete and functorial constructions can be done in \mathfrak{m}^H . The reader familiar with Happel’s results may go directly to Section 3.

2.2 Suspension and Desuspension Functors

In [6], when H is finite dimensional and Λ is an integral in H (not in H^*), the author embeds an H -module X via $X \otimes \Lambda \subset X \otimes H$ and define $T(X)$ as $(X \otimes H)/(X \otimes \Lambda)$. For us, $\Lambda \in H^*$ and this definition makes no sense, but (even without using the integral) one can always embed an H -comodule M into $M \otimes H$ by means of its structural map. The structure map ρ is H -colinear provided we use the (co)free H -comodule structure on $M \otimes H$ (and not the diagonal one).

Definition 2.5 For a right H comodule M with structure $\rho : M \rightarrow M \otimes H$, define

$$T(M) := (M \otimes H)/\rho(M)$$

If H is finite dimensional this definition also makes sense in \mathfrak{m}^H . If H is co-Frobenius and $0 \neq M$ is finite dimensional, $(M \otimes H)/\rho(M)$ is not finite dimensional, however, there exists a finite dimensional injective $I(M)$ and a monomorphism $M \rightarrow I(M)$, so, one can define $I(M)/M$ in \mathfrak{m}^H and we know $T(M) \cong I(M)/M$ in $\underline{\mathcal{M}}^H$. Moreover, for co-Frobenius Hopf algebras, one can give functorial embeddings $M \rightarrow I(M)$ in \mathcal{M}^H that works in \mathfrak{m}^H (see Corollary 3.16).

Remark 2.6 If the notation $M \otimes H$ is confusing because H is Hopf and one also has the diagonal action, one may consider another injecting embedding:

$$\begin{aligned} i_M : M &\rightarrow M \otimes H \\ m &\mapsto m \otimes 1 \end{aligned}$$

This map is clearly an embedding, and it is H -colinear if one uses the diagonal action on $M \otimes H$. Both embeddings are ok because $M \otimes H$ with diagonal action and $M \otimes H$ with structure coming only from H are isomorphic (see Lemma 3.14).

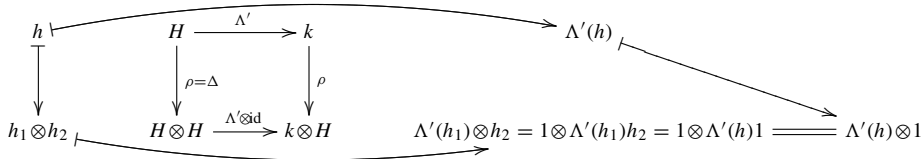
Similar (or dually) to [6] one can define desuspension. Consider the map $\Lambda' = \Lambda \circ S$

$$\Lambda' : H \rightarrow k$$

Recall that H co-Frobenius implies S is bijective (see for instance [4]) and it is easy to prove that $\Lambda' = \Lambda \circ S$ is a *right* integral:

$$\begin{aligned} \Lambda'(h_1)h_2 &= \Lambda(S(h_1))h_2 = \Lambda(S(h_1))S^{-1}S(h_2) \\ &= S^{-1}\left(\Lambda(S(h_1))S(h_2)\right) = S^{-1}\left(\Lambda(S(h_2))S(h_1)\right) = S^{-1}\left(\Lambda(S(h))1\right) = \Lambda(S(h))1 = \Lambda'(h)1 \end{aligned}$$

We use Λ' because Λ' is right colinear:



Hence, $\text{Ker}(\Lambda')$ is a right H -comodule and we have, for any M , a short exact sequence

$$0 \rightarrow M \otimes \text{Ker}(\Lambda') \rightarrow M \otimes H \rightarrow M \rightarrow 0$$

Definition 2.7 The desuspension functor is $T'(M) := M \otimes \text{Ker}(\Lambda') \in \underline{\mathcal{M}}^H$

Remark 2.8 When considering \mathfrak{m}^H , we know every finite dimensional comodule M has a finite dimensional projective cover $P(M) \rightarrow M$, so we can consider $T''(M) := \text{Ker}(P(M) \rightarrow M) \in \underline{\mathfrak{m}}^H$ and this is isomorphic to $T'(M)$ in the stable category. Also (see Corollary 3.16), one can define $P(M)$ in \mathcal{M}^H and in \mathfrak{m}^H in a functorial way.

As an illustration of the need of stabilization for having a triangulated category one see that, for any M a comodule, we have a short exact sequence in \mathcal{M}^H

$$0 \rightarrow M \rightarrow M \otimes H \rightarrow TM \rightarrow 0$$

In particular, considering $T'M$ instead of M , there is a short exact sequence

$$0 \rightarrow T'M \rightarrow T'M \otimes H \rightarrow TT'M \rightarrow 0$$

But there is also a short exact sequence

$$0 \rightarrow T'M \rightarrow M \otimes H \rightarrow M \rightarrow 0$$

So “ M computes $TT'M$ using another injective embedding”. Usually $TT'M \not\cong M$ in \mathcal{M}^H but $M \cong TT'M$ in $\underline{\mathcal{M}}^H$. Similar argument for $T'T$, hence these are mutually inverse functors in the *stable* category, but not in \mathcal{M}^H .

2.3 Triangles

One of the axioms of triangulated categories is that any map $f : X \rightarrow Y$ is a part of a triangle $X \xrightarrow{f} Y \rightarrow Z \rightarrow TX \rightarrow$. Triangles are defined via the mapping cone construction. For $f : X \rightarrow Y$, $Co(f)$ is defined in the following way:

Choose an injective embedding $i : X \rightarrow I(X)$ and define $Co(f)$ by the square

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \downarrow i & & \downarrow \\
 I(X) & \longrightarrow & Co(f) := I(X) \amalg_X Y
 \end{array}$$

One can see that this definition does not depend -in the stable category- on the choice of the injective embedding $X \rightarrow I(X)$. Notice also a well defined map $Co(f) \rightarrow T(X)$ given by the universal property of the push-out:

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \downarrow i & & \downarrow \\
 I(X) & \longrightarrow & I(X) \amalg_X Y \\
 & \searrow \pi & \nearrow 0 \\
 & & I(X)/X
 \end{array}$$

Triangles $X \rightarrow Y \rightarrow Z \rightarrow TX$ in $\underline{\mathcal{M}}^H$ are (by definition) all sequences isomorphic (in $\underline{\mathcal{M}}^H$) to some sequence of the form $X \xrightarrow{f} Y \rightarrow Co(f) \rightarrow T(X)$. Next two Lemmas emphasize the strong relation between the exact structure of \mathcal{M}^H (resp \mathfrak{m}^H) and the triangulated structure of $\underline{\mathcal{M}}^H$ (resp $\underline{\mathfrak{m}}^H$)

Lemma 2.9 *If $0 \rightarrow X \xrightarrow{u} Y \xrightarrow{\pi} Z \rightarrow 0$ is a short exact sequence in \mathcal{M}^H then the sequence $X \rightarrow Y \rightarrow Z$ is isomorphic to $X \rightarrow Y \rightarrow Co(u)$ in the stable category.*

Proof We assume $Z = Y/u(X)$. Consider the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & X & \xrightarrow{u} & Y & \xrightarrow{\pi} & Y/u(X) \longrightarrow 0 \\
 & & \parallel & & \parallel & & \\
 & & X & \xrightarrow{u} & Y & \xrightarrow{v} & Co(u)
 \end{array}$$

Let $X \rightarrow I(X)$ be an embedding into an injective object, for simplicity we assume $X \subseteq I(X)$. We define the map

$$\begin{aligned}
 I(X) \oplus Y &\longrightarrow Z \\
 (e, y) &\mapsto \pi(y)
 \end{aligned}$$

It has the property that, for any $x \in X$,

$$(-x, u(x)) \mapsto \pi(u(x)) = 0$$

So, it induces a well defined map

$$\begin{aligned}
 Co(u) &= \frac{I(X) \oplus Y}{(x, 0) \sim (0, u(x))} \longrightarrow Z \\
 \overline{(e, y)} &\mapsto \pi(y)
 \end{aligned}$$

Now from the injectivity of $I(X)$ we know there exists a map fitting into the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & X & \xrightarrow{u} & Y & \xrightarrow{\pi} & Y/u(X) \longrightarrow 0 \\
 & & \downarrow & \nearrow U & & & \\
 & & I(X) & & & &
 \end{array}$$

So, define the map

$$\begin{aligned}
 Y &\rightarrow Co(u) \\
 y &\mapsto \overline{(U(y), y)}
 \end{aligned}$$

It has the property

$$u(x) \mapsto \overline{(-U(u(x)), u(x))} = \overline{(-x, u(x))} = 0$$

so, it induces a well defined map

$$Z = Y/u(X) \rightarrow Co(u)$$

One composition is the identity:

$$\begin{aligned} Z &\rightarrow Co(u) \rightarrow Z \\ z = \pi(y) &\mapsto \overline{(-U(y), y)} \mapsto \pi(y) = z \end{aligned}$$

The other composition is

$$\begin{aligned} Co(u) &\rightarrow Z \rightarrow Co(u) \\ \overline{(e, y)} &\mapsto \pi(y) \mapsto \overline{(-U(y), y)} \end{aligned}$$

so, the Kernel is

$$\{\overline{(e, y)} : y \in u(X)\} \cong \frac{I(X) \oplus u(X)}{(x, 0) \sim (0, u(x))} \cong I(X)$$

that is an injective comodule, so, these morphisms are mutually inverses in $\underline{\mathcal{M}}^H$. □

The second Lemma is a useful one, maybe it is folklore but it is not usually written:

Lemma 2.10 *If $X \twoheadrightarrow Y \twoheadrightarrow Z \twoheadrightarrow TX$ is a triangle in the stable category then there exists a short exact sequence $0 \rightarrow X' \rightarrow Y' \rightarrow Z' \rightarrow 0$ in \mathcal{M}^H such that the sequence $X \twoheadrightarrow Y \twoheadrightarrow Z$ is isomorphic to $X' \twoheadrightarrow Y' \twoheadrightarrow Z'$ in the stable category.*

Proof One of the axioms of triangulated categories says that $X \twoheadrightarrow Y \twoheadrightarrow Z \twoheadrightarrow TX$ is a triangle if and only if $T^{-1}Z \twoheadrightarrow X \twoheadrightarrow Y \twoheadrightarrow Z$ is so. Hence, $T^{-1}Z \twoheadrightarrow X \twoheadrightarrow Y \twoheadrightarrow Z$ is isomorphic to a distinguished triangle, that is, there is an isomorphism (in the stable category) of t-uples

$$\begin{array}{ccccccc} T^{-1}Z & \twoheadrightarrow & X & \twoheadrightarrow & Y & \twoheadrightarrow & Z \\ \cong \downarrow & & \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ A & \xrightarrow{u} & B & \twoheadrightarrow & Co(u) & \twoheadrightarrow & T(A) \end{array}$$

In particular, there is a commutative diagram in the stable category

$$\begin{array}{ccccccc} X & \twoheadrightarrow & Y & \twoheadrightarrow & Z \\ \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ B & \twoheadrightarrow & Co(u) & \twoheadrightarrow & T(A) \end{array}$$

and clearly $0 \rightarrow B \rightarrow Co(u) \rightarrow T(A) \rightarrow 0$ -or equivalently

$$0 \rightarrow B \rightarrow I(A) \coprod_A B \rightarrow I(A)/A \rightarrow 0,$$

is a short exact sequence in \mathcal{M}^H . Notice that if A and B are finite dimensional, one can find a finite dimensional injective hull $I(A)$ and hence the short exact sequence also belongs to \mathfrak{m}^H . □

3 Integrals and Coinvariants

If C is a coalgebra and M a right C -comodule then M is a left C^* module via

$$\phi \cdot m := \phi(m_1)m_0 \quad (\phi \in C^*, m \in M)$$

where, as usual, if M is a right H -comodule, we denote $\rho : M \rightarrow M \otimes H$ its structural map and we use Sweedler-type notation $\rho(m) = m_0 \otimes m_1 \in M \otimes H$. In particular, for $C = H$ and $\phi = \Lambda \in H^*$, being left integral means $\Lambda \cdot h = \Lambda(h)1$. Moreover, multiplication by Λ in M has the following standard and main property:

$$\rho(\Lambda \cdot m) = \rho(\Lambda(m_1)m_0) = \Lambda(m_2)m_0 \otimes m_1 = m_0 \otimes \Lambda(m_2)m_1 = m_0 \otimes \Lambda(m_1)1 = (\Lambda \cdot m) \otimes 1$$

That is, $\Lambda \cdot M \subseteq M^{coH}$. We list some examples, keeping in mind the above formula.

3.1 Examples

1. If H is co-semisimple (e.g. $H = \mathcal{O}(G)$ with G an affine reductive group) then the inclusion $k \rightarrow H$ split as H -comodules. One can check that an H -colinear splitting is an integral. In the cosemisimple case, the inclusion $\Lambda \cdot M \subseteq M^{coH}$ is actually an equality (this will be clear in Section 3.2). Nevertheless, the integral may not be so explicitly described. An easy and explicit example is:
2. If G is a finite group and $H = k^G$, then $\Lambda = \sum_{g \in G} g \in k[G] \cong (k^G)^*$ is an integral. For any $f \in k^G$:

$$\Lambda(f) = \sum_{g \in G} f(g)$$

Actually, every finite dimensional Hopf algebra is (Frobenius and) co-Frobenius. Notice that k^G is co-semisimple if and only if $k[G]$ is semisimple, if and only if the characteristic of the ground field does not divide the order of G .

3. Let G be a group (possibly infinite, e.g. $G = \mathbb{Z}$) and $H = k[G]$, define

$$\Lambda\left(\sum_{g \in G} \lambda_g g\right) := \lambda_{1_G}$$

A right H -comodule M is the same as G -graded vector space $M = \bigoplus_{g \in G} M_g$. The action of Λ gives the projection into M_{1_G} .

4. Tensor product of co-Frobenius algebras is co-Frobenius, the integral can be computed using tensor products of integrals.
5. Let H be a Hopf algebra and H_0 its coradical. Notice that H_0 does not need to be a Hopf subalgebra in general. Nevertheless, one of the main results in [2] is that H is co-Frobenius if and only if the coradical filtration is finite. A particular case is illustrated in the following:
6. Let H_0 be a cosemisimple Hopf algebra and let $V \in {}_{H_0}\mathcal{YD}^{H_0}$ be a finite dimensional Yetter-Drinfel'd module such that its Nichols algebra $\mathfrak{B} = \mathfrak{B}(V)$ is finite dimensional. Then $H = H_0 \# \mathfrak{B}$ is co-Frobenius. The integral is essentially given by the ‘‘volume form’’, or ‘‘Fermionic integration’’ in \mathfrak{B} (see Remark 4.5).
 - (a) The simplest example is: H generated by x and $g^{\pm 1}$ with relations $x^2 = 0$ and $gx = -xg$. Comultiplication given by

$$\begin{aligned} \Delta g &= g \otimes g \\ \Delta x &= x \otimes g + 1 \otimes x \end{aligned}$$

The antipode is

$$S(g) = g^{-1}, S(x) = -xg^{-1} = g^{-1}x$$

We have $H \cong k[\mathbb{Z}] \# k[x]/x^2$. An element of h may be uniquely written as

$$h = \sum_{n \in \mathbb{Z}} a_n g^n + \sum_{n \in \mathbb{Z}} b_n g^n x \quad (a_n, b_n \in k)$$

A left integral is given by

$$\Lambda(h) := b_0$$

This particular example motivates all definitions of this paper. The second simplest example of this kind is the following:

- (b) H generated by x, y and $g^{\pm 1}$ with relations $x^2 = 0 = y^2, xy = -yx, gx = -xg, gy = -yg$ and comultiplication given by

$$\Delta g = g \otimes g$$

$$\Delta x = x \otimes g + 1 \otimes x$$

$$\Delta y = y \otimes g^{-1} + 1 \otimes y$$

If we write an element $h \in H$ as

$$h = \sum_{n \in \mathbb{Z}} a_n g^n + \sum_{n \in \mathbb{Z}} b_n g^n x + \sum_{n \in \mathbb{Z}} c_n g^n y + \sum_{n \in \mathbb{Z}} d_n g^n xy$$

then a left integral is given by $\Lambda(h) = d_0$. We will compute some invariants of the (stable) comodule category associated to this $H \cong k[\mathbb{Z}] \# \Lambda(x, y)$.

One of the main goals of this paper is to translate into Hopf-co-Frobenius language the notion of homology “ $\text{Ker } d / \text{Im } d$ ”. The definition is very natural:

3.2 Hopf Homology for Algebras with a Non-zero Integral

Definition 3.1 Given a co-Frobenius Hopf algebra H and $M \in \mathcal{M}^H$, denote

$$\mathcal{H}_0^H(M) := \frac{M^{coH}}{\Lambda \cdot M}$$

For $n \in \mathbb{N}$, we define

$$\mathcal{H}_{-n}^H(M) := \mathcal{H}_0^H(T^{-n}M)$$

and

$$\mathcal{H}_n^H(M) := \mathcal{H}_0^H(T^nM)$$

Example 3.2 If $M = k$ and H is co-Frobenius with $\Lambda(1) = 0$, then $\Lambda \cdot k = 0$, hence $\mathcal{H}_0^H(k) = k$ and the functor \mathcal{H}_0^H is non trivial.

Example 3.3 For $M = H, \Lambda \cdot H = k1_H = H^{coH} \Rightarrow \mathcal{H}_0^H(H) = 0$.

Example 3.4 The condition “ $M^{coH} / \Lambda \cdot M = 0$ ” is stable under arbitrary direct sums and direct summands, so $I^{coH} / \Lambda \cdot I = 0$ for any injective module I .

As a corollary, if $f : M \rightarrow N$ is an H -colinear map such that it factors through an injective:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ & \searrow & \nearrow \\ & I & \end{array}$$

then the induced map

$$\begin{array}{ccc} \mathcal{H}_0^H(M) & \xrightarrow{\mathcal{H}_0^H(f)} & \mathcal{H}_0^H(N) \\ & \searrow & \nearrow \\ & \mathcal{H}_0^H(I) = 0 & \end{array}$$

is necessarily zero. So, the functor $\mathcal{H}_0^H(-)$ is actually defined in the stable category

$$\mathcal{H}_0^H : \underline{\mathcal{M}}^H \rightarrow {}_k \text{Vect}$$

Remark 3.5 For all $n \in \mathbb{Z}$, the functors $\mathcal{H}_n^H(-)$ are defined in the stable category.

Corollary 3.6 If H is co-semisimple then every H -comodule is injective, hence $\mathcal{H}_0^H(M) = 0$ for all comodule M . In other words, $\Lambda \cdot M = M^{coH}$ for all comodule M .

Remark 3.7 From the point of view of invariant theory, $\Lambda \cdot M = M^{coH}$ is the more convenient situation, but from the point of view of homological algebra, $\mathcal{H}_0^H(M) \neq 0$ is most interesting.

Lemma 3.8 Let H be a Hopf algebra with nonzero integral Λ and denote $\underline{\text{Hom}}^H$ the Hom space in the stable category of H comodules, then there exists an **epimorphism**

$$\underline{\text{Hom}}^H(k, M) \rightarrow \mathcal{H}_0^H(M)$$

Proof Notice that

$$\begin{aligned} \text{Hom}^H(k, M) &\rightarrow M^{coH} \\ f &\mapsto f(1) \end{aligned}$$

is an isomorphism. We will show that this map fits into a commutative square

$$\begin{array}{ccc} \text{Hom}^H(k, M) & \xrightarrow[\substack{\cong \\ f \mapsto f(1)}} & M^{coH} \\ \downarrow & & \downarrow \\ \underline{\text{Hom}}^H(k, M) & \twoheadrightarrow & M^{coH} / \Lambda \cdot M \end{array}$$

Assume $f : k \rightarrow M$ factors through an injective object

$$\begin{array}{ccc} k & \xrightarrow{f} & M \\ & \searrow & \nearrow \\ & I & \end{array}$$

then one may consider the unit map $k \xrightarrow{\eta} H$ and the diagram

$$\begin{array}{ccc} k & \xrightarrow{f} & M \\ \eta \downarrow & \searrow & \nearrow \\ H & \dashrightarrow & I \end{array}$$

Since η is a monomorphism and I is injective, one may find a dashed morphism making a commutative diagram, so, it is enough to consider the case $I = H$.

$$\begin{array}{ccc} k & \xrightarrow{f} & M \\ & \searrow & \nearrow b \\ & & H \end{array}$$

Now if $x \in H$ is such that $\Lambda(x) = 1$, then

$$f(1) = b(1) = b(\Lambda(x)1) = b(\Lambda \cdot x) = \Lambda \cdot b(x) \in \Lambda \cdot M$$

so $f(1) \in \Lambda \cdot M$. This proves that the induced map

$$\underline{\text{Hom}}^H(k, M) \rightarrow M^{coH} / \Lambda \cdot M$$

is both well-defined, and clearly surjective. \square

Remark 3.9 One may wonder if the epimorphism of the above Lemma is in fact an isomorphism. This will be the case (see Theorem 3.12). For finite dimensional Hopf algebras it is due to You Qi [9], where he proves actually for finite dimensional Frobenius algebras, in particular for finite dimensional Hopf algebras. It is not clear for the author how to adapt Qi's arguments to our case, maybe one can find a simpler proof, but we provide a proof with some homological machinery first.

Remark 3.10 Lemmas 2.9 and 2.10 gives an alternative proof that the composition of two consecutive morphisms in a triangle is zero (in the stable category), and so, every functor defined in the stable category sends triangles to complexes. For the particular case of $\mathcal{H}_\bullet^H(-)$, without knowing that it is representable or not, we have the expected result:

Theorem 3.11 *If $X \rightarrow Y \rightarrow Z \rightarrow TX$ is a triangle in the stable category then there is a long exact sequence of vector spaces*

$$\cdots \rightarrow \mathcal{H}_{n+1}^H(Z) \rightarrow \mathcal{H}_n^H(X) \rightarrow \mathcal{H}_n^H(Y) \rightarrow \mathcal{H}_n^H(Z) \rightarrow \mathcal{H}_{n-1}^H(X) \rightarrow \cdots$$

Proof We will prove that

$$\mathcal{H}_0^H(X) \rightarrow \mathcal{H}_0^H(Y) \rightarrow \mathcal{H}_0^H(Z)$$

is exact in $\mathcal{H}_0^H(Y)$ when $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ is a short exact sequence. The general result follows from Lemma 2.10 and the shifting axiom of triangles. So assume $0 \rightarrow X \rightarrow$

$Y \rightarrow Z \rightarrow 0$ is a short exact sequence in \mathcal{M}^H , then multiplication by the integral gives as a commutative diagram (of vector spaces) with exact rows

$$\begin{array}{ccccccc}
 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z \longrightarrow 0 \\
 & & \downarrow \Lambda \cdot & & \downarrow \Lambda \cdot & & \downarrow \Lambda \cdot \\
 0 & \longrightarrow & X^{coH} & \longrightarrow & Y^{coH} & \longrightarrow & Z^{coH}
 \end{array}$$

So, even forgetting that $X \rightarrow Y$ is injective, the snake Lemma gives in particular that

$$X^{coH} / \Lambda \cdot X \longrightarrow Y^{coH} / \Lambda \cdot Y \longrightarrow Z^{coH} / \Lambda \cdot Z$$

is exact. □

Now, the above Theorem together with Lemma 3.8 gives the following:

Theorem 3.12 *Let H be a Hopf algebra with nonzero integral Λ and denote $\underline{\text{Hom}}^H$ the Hom space in the stable category of H comodules, then the natural map*

$$\underline{\text{Hom}}^H(k, M) \rightarrow \mathcal{H}_0^H(M)$$

is an isomorphism.

Proof First recall the following version of the “5”-lemma: given a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D \\
 \downarrow a & & \downarrow b & & \downarrow c & & \downarrow d \\
 A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D'
 \end{array}$$

if b and d are monomorphisms and a is an epimorphism, then c is a monomorphism.

Consider \mathcal{S} the class of H -comodules S such that the map $\underline{\text{Hom}}^H(k, S) \rightarrow \mathcal{H}_0^H(S)$ is an isomorphism. Because short exact sequences in \mathcal{M}^H gives both long exact sequences for $\mathcal{H}_n^H(-)$ and $\underline{\text{Hom}}^H(k, T^{-n}(-))$, given a short exact sequence of comodules

$$0 \rightarrow S_1 \rightarrow M \rightarrow S_2 \rightarrow 0$$

where S_1 and S_2 are in \mathcal{S} , then we have a diagram

$$\begin{array}{ccccccc}
 \underline{\text{Hom}}^H(k, T^{-1}S_2) & \longrightarrow & \underline{\text{Hom}}^H(k, S_1) & \longrightarrow & \underline{\text{Hom}}^H(k, M) & \longrightarrow & \underline{\text{Hom}}^H(k, S_2) \\
 \downarrow a & & \downarrow b & & \downarrow c & & \downarrow d \\
 \mathcal{H}_0(T^{-1}S_2) & \longrightarrow & \mathcal{H}_0(S_1) & \longrightarrow & \mathcal{H}_0(M) & \longrightarrow & \mathcal{H}_0(S_2)
 \end{array}$$

Every vertical map is an epimorphism (Lemma 3.8) and both b and d are monomorphism because they are isomorphisms ($S_i \in \mathcal{S}$), so c is monomorphism, hence, an isomorphism.

We conclude that the theorem is true for any finite dimensional comodule M , provided it is true on simple comodules.

If $S = k$ and k is not injective then $\Lambda \cdot k = 0$ and $\mathcal{H}_0^H(k) = k \cong \underline{\text{Hom}}^H(k, k)$. (If k is injective, the theorem is noninteresting, but still true).

If S is simple and $S \not\cong k$ then $S^{coH} = 0$, so trivially $\mathcal{H}_0^H(S) = 0 = \underline{\text{Hom}}(k, S)$.

Now let M be a possibly infinite dimensional comodule and $f : k \rightarrow M$ such that $f(1) = \Lambda \cdot m$ for some $m \in M$. Consider $M' \subset M$ a finite dimensional subcomodule

containing $f(1)$ and m . Then, the class of $f(1)$ in $\mathcal{H}_0^H(M')$ is zero. But because M' is finite dimensional we know $\mathcal{H}_0^H(M') = \underline{\text{Hom}}^H(k, M')$ and so there exists a factorization

$$\begin{array}{ccccc}
 k & \xrightarrow{f} & M' & \hookrightarrow & M \\
 \downarrow & & \nearrow & & \\
 I & & & &
 \end{array}$$

with I injective. So, f is zero in $\underline{\text{Hom}}^H(k, M)$. □

3.3 Multiplicative structure

Because H is Hopf, the categories \mathcal{M}^H and \mathfrak{m}^H are tensor categories, and the tensor structure descends to the stable category, as one can see after these standard facts:

- Lemma 3.13** 1. *If C is a coalgebra and V a vector space, the right C comodule $V \otimes C$ with structure map $\rho = \text{id}_V \otimes \Delta$ is an injective comodule.*
 2. *Every injective comodule is a direct summand of one as above. The category of comodules has enough injectives.*

Proof 1. It follows from the adjunction formula

$$\begin{aligned}
 \text{Hom}^C(M, V \otimes C) &\cong \text{Hom}_k(M, V) \\
 f &\mapsto (\text{id}_V \otimes \epsilon) \circ f
 \end{aligned}$$

and that every vector space is an injective object in $k\text{-Vect}$.

2. If M is a comodule, the structure morphism

$$\rho_M : M \rightarrow M \otimes C$$

gives an embedding into an injective object: C -colinearity is by coassociativity and injectivity is because of counitarity. If $M = I$ is injective, then the monomorphism $\rho_I : I \rightarrow I \otimes C$ splits, hence, I is a direct summand of $V \otimes C$ where V is the underlying vector space of I . □

Lemma 3.14 *Let H be a Hopf algebra, $M \in \mathcal{M}^H$. Denote by V_M the underlying vector space of M .*

1. $M \otimes H$ (with diagonal coaction) is isomorphic to $V_M \otimes H$ (with $\rho = \text{id}_{V_M} \otimes \Delta_H$).
2. Also $H \otimes M \cong V_M \otimes H$
3. If I is injective then $M \otimes I$ and $I \otimes M$ are both injectives.

Proof 1. We only exhibit the maps:

$$\begin{aligned}
 M \otimes H &\rightarrow V_M \otimes H \\
 m \otimes h &\mapsto m_0 \otimes m_1 h
 \end{aligned}$$

with inverse

$$m \otimes h \mapsto m_0 \otimes S(m_1)h$$

The composition is

$$m \otimes h \mapsto m_0 \otimes S(m_1)m_2h = m_0 \otimes \epsilon(m_1)h = m \otimes h$$

The other composition is similar. The surprising part is that these maps are H -colinear. For instance:

$$\begin{array}{ccc}
 m \otimes h & \xrightarrow{\quad} & m_0 \otimes m_1 h \\
 \downarrow & \begin{array}{c} H \otimes H \longrightarrow V_H \otimes H \\ \downarrow \rho_{diag} \quad \downarrow \text{id}_V \otimes \Delta \end{array} & \downarrow \\
 m_0 \otimes h_1 \otimes m_1 h_2 & \xrightarrow{\quad} & m_0 \otimes m_1 h_1 \otimes m_2 h_2 = m_0 \otimes (m_1 h)_1 \otimes (m_1 h)_2
 \end{array}$$

2. The maps are similar: consider

$$H \otimes M \rightarrow V_M \otimes H$$

$$h \otimes m \mapsto m_0 \otimes m_1 h$$

with inverse

$$m \otimes h \mapsto S(m_1)h \otimes m_0$$

The composition is

$$h \otimes m \mapsto m_0 \otimes m_1 h = S(m_1)m_2 h \otimes m_0 = \epsilon(m_1)h \otimes m_0 = h \otimes m$$

The other composition is similar. The colinearity follows the same lines.

3. If I is injective then it is isomorphic to a direct summand of $V \otimes H$ for some vector space V (e.g. $V = V_I$), and so $M \otimes I$ is isomorphic to a direct summand of

$$M \otimes (V \otimes H) \cong V \otimes (M \otimes H) \cong (V \otimes V_M) \otimes H$$

and $I \otimes M$ is a direct summand of

$$(V \otimes H) \otimes M \cong V \otimes (H \otimes M) \cong (V \otimes V_M) \otimes H$$

In any case, a direct summand of a comodule of the form $W \otimes H$ for some vector space W . □

There are several corollaries:

Corollary 3.15 *The tensor product is well defined in the stable category. In particular, $K_0(\underline{\mathfrak{m}}^H)$ is an associative ring.*

Let $E := E(k)$ be the injective hull of k in \mathcal{M}^H . It is well-known that H is co-Frobenius if and only if $E(k)$ is finite dimensional (see Theorem 2.1 in [3]). Also, for co-Frobenius Hopf algebras, there exists a finite dimensional projective comodule $P = P(k)$ with a surjective map $P \rightarrow k$.

Corollary 3.16 *Define $E(M) := M \otimes E$. The map $M \rightarrow E(M)$ ($m \mapsto m \otimes 1$) is a functorial injective embedding, if $M \in \mathfrak{m}^H$ then $E(M) \in \mathfrak{m}^H$ as well. Also, $P(M) := M \otimes P$ gives a functorial projective surjection $P(M) \rightarrow M$, if $P \in \mathfrak{m}^H$ then $P(M) \in \mathfrak{m}^H$ as well.*

Proof The injective part is clear. Let us prove the existence of a surjective map $P \rightarrow k$ with P finite dimensional:

Since $\Lambda' : H \rightarrow k$ is surjective, there exists $h_0 \in H$ such that $\Lambda'(h_0) = 1$, and there exists a finite dimensional subcomodule $M_0 \subset H$ containing h_0 . In particular, $\Lambda'(M_0) \neq 0$.

Because H is co-Frobenius, there exists a finite dimensional injective hull of M_0 , let's call it $I(M_0)$. Looking at the diagram

$$\begin{array}{ccccc}
 0 & \longrightarrow & M & \longrightarrow & I(M) \\
 & & \downarrow & \nearrow \exists & \\
 & & H & &
 \end{array}$$

Because H is injective there exist the dashed arrow. Because $I(M_0)$ is the injective hull and H is injective, the map $I(M) \rightarrow H$ is injective and $I(M_0)$ is a direct summand of H . Eventually changing M_0 by $I(M_0)$ we get a finite dimensional direct summand of H such that the restriction of Λ' is non-zero, hence, a surjection $P \rightarrow k$ with P projective and finite dimensional. □

Corollary 3.17 *For any M, N in \mathcal{M}^H , there are isomorphisms in the stable category*

$$T(M \otimes N) \cong TM \otimes N \cong M \otimes TN$$

and similarly for T' . Hence, $\underline{\mathcal{M}}^H$ and $\underline{\mathcal{m}}^H$ are tensor triangulated categories.

Proof Let $i : M \rightarrow I(M)$ and $j : N \rightarrow I(N)$ be embeddings into injective comodules, then $I(M) \otimes I(N)$ is injective and one can compute $T(M \otimes N)$ via

$$0 \rightarrow M \otimes N \xrightarrow{i \otimes j} I(M) \otimes I(N) \rightarrow T(M \otimes N) \rightarrow 0$$

But $I(M) \otimes N$ and $M \otimes I(M)$ are injectives too, and we have the following short exact sequences with injective objects in the middle:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & M \otimes N & \xrightarrow{i \otimes \text{id}} & I(M) \otimes N & \longrightarrow & TM \otimes N \longrightarrow 0 \\
 & & \parallel & & \downarrow \text{id} \otimes j & & \downarrow \\
 0 & \longrightarrow & M \otimes N & \xrightarrow{i \otimes j} & I(M) \otimes I(N) & \longrightarrow & T(M \otimes N) \longrightarrow 0 \\
 & & \parallel & & \uparrow i \otimes \text{id} & & \uparrow \\
 0 & \longrightarrow & M \otimes N & \xrightarrow{\text{id} \otimes j} & M \otimes I(N) & \longrightarrow & M \otimes TN \longrightarrow 0
 \end{array}$$

□

Notice that the morphisms are not canonical in the category of comodules, but they are canonically determined in the stable category

Corollary 3.18 *For any $m, n \in \mathbb{Z}$ there is an isomorphism in the stable category*

$$T^n M \otimes T^n N \cong T^{n+m}(M \otimes N)$$

3.4 Künneth Map

Let M and N be two comodules. It is clear that $M^H \otimes N^H \subseteq (M \otimes N)^H$ and also one can easily check that

$$\Lambda \cdot (M \otimes N^{coH}) = \Lambda \cdot M \otimes N^{coH}$$

and

$$\Lambda \cdot (M^{coH} \otimes N) = M^{coH} \otimes \Lambda \cdot N$$

So, there is a canonical map

$$\mathcal{H}_0^H M \otimes \mathcal{H}_0^H N = \frac{M^{coH}}{\Lambda \cdot M} \otimes \frac{N^{coH}}{\Lambda \cdot N} \cong \frac{M^{coH} \otimes N^{coH}}{\Lambda \cdot M \otimes N^{coH} + M^{coH} \otimes \Lambda \cdot N} \rightarrow \mathcal{H}_0^H (M \otimes N)$$

Moreover, using Corollary 3.18 on can define maps

$$\begin{array}{ccc} \mathcal{H}_p^H(M) \otimes \mathcal{H}_q^H(N) & & \\ \parallel & \searrow & \\ \mathcal{H}_H^0(T'^p M) \otimes \mathcal{H}_H^0(T'^q N) & & \\ \downarrow & & \\ \mathcal{H}_H^0(T'^p M \otimes T'^q N) \xrightarrow{\cong} \mathcal{H}_H^0(T'^{p+q}(M \otimes N)) = \mathcal{H}_{p+q}^H(M \otimes N) \end{array}$$

(If a number is negative, we use the convention $(T')^{-n} = T^n$.) In this way, one can assemble all those maps and get, for any fixed n , a map that we call "Künneth map"

$$\bigoplus_{p+q=n} \mathcal{H}_p^H(M) \otimes \mathcal{H}_q^H(N) \rightarrow \mathcal{H}_n^H(M \otimes N)$$

For $M = N = k$, from concrete computations (see Corollary 5.10) we know this map cannot be an isomorphism in general. It would be interesting to know their general properties. In any case, $\mathcal{H}_\bullet^H(k) = \bigoplus_{n \in \mathbb{Z}} \mathcal{H}_n^H(k)$ is a graded algebra.

4 Small injective embeddings for $m^{H_0 \# \mathfrak{B}}$

During this section we assume

1. H_0 is a co-semisimple Hopf algebra,
2. $V \in {}_{H_0} \mathcal{YD}^{H_0}$ is such that $\mathfrak{B}(V)$, the Nichols algebra associated to the braided vector space V , is finite dimensional.

Let us recall briefly the conditions above and set notations and conventions. First, ${}_{H_0} \mathcal{YD}^{H_0}$ is the category whose objects are left H_0 -modules and right H_0 -comodules with the compatibility

$$h_1 m_0 \otimes h_2 m_1 = (h_2 m)_0 \otimes (h_2 m)_1 h_1$$

where $h_1 \otimes h_2 = \Delta h$, $h \in H_0$ and $m \in M$, $\rho(m) = m_0 \otimes m_1 \in M \otimes H_0$. Morphisms are H_0 -linear and colinear maps. For any Hopf algebra A , the category ${}_A \mathcal{YD}^A$ is braided with

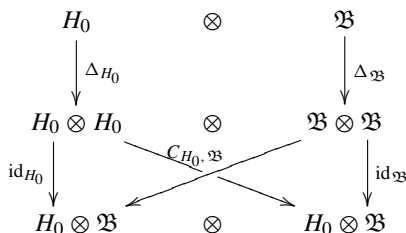
$$\begin{aligned} c_{V,W} : V \otimes W &\rightarrow W \otimes V \\ v \otimes w &\mapsto w_0 \otimes w_1 \cdot v \end{aligned}$$

Recall that if V is a braided vector space (e.g. $V \in \mathcal{YD}_{H_0}^{H_0}$) then both TV (the tensor algebra) and $T^c V$ (the tensor coalgebra) are *braided* Hopf algebras. TV has free product and braided-shuffle coproduct, while $T^c V$ has deconcatenation coproduct and braided-shuffle product. The Nichols algebra $\mathfrak{B}(V)$ is, by definition, the image of the unique (bi)algebra map $TV \rightarrow T^c V$ that is the identity on V :

$$\begin{array}{ccc} TV & \xrightarrow{\quad} & T^c V \\ & \searrow & \nearrow \\ & \mathfrak{B}(V) & \end{array}$$

It happens to be, degree by degree, the image of the quantum symmetrizer map associated to the braiding. We refer to Andruskievitch’s notes [1] for a gentle introduction and full discussion on Nichols algebras. The reader may keep in mind the easy example $\mathfrak{B}(V) = \Lambda V$ when the braiding is -flip. The braided bialgebra $\mathfrak{B}(V)$ is actually a braided Hopf algebra, and the bicross product $H_0\#\mathfrak{B}(V)$ is a usual Hopf algebra. Since there is a lot of structures around $\mathfrak{B}(V)$ we recall them:

- \mathfrak{B} is a coalgebra, we denote $\Delta(b) = b_1 \otimes b_2$,
- $\mathfrak{B} \in \mathcal{M}^{H_0}$, we denote the structure $\rho(b) = b_0 \otimes b_1$,
- $H_0\#\mathfrak{B}$ is a coalgebra, the comultiplication is given by the following diagram (recall the underlying vector space of $H_0\#\mathfrak{B}$ is $H_0 \otimes \mathfrak{B}$):



In Sweedler-type notation:

$$\Delta(h\#b) = h_1\#(b_1)_0 \otimes (b_1)_1h_2\#b_2$$

- In particular $\Delta(1\#b) = 1\#(b_1)_0 \otimes (b_1)_1\#b_2$. Denoting $H := H_0\#\mathfrak{B}$, we have that $\mathfrak{B} \cong 1\#\mathfrak{B}$ is a right H -subcomodule of H . With this structure we consider \mathfrak{B} as an object in \mathcal{M}^H . To emphasize the difference with $\rho : \mathfrak{B} \rightarrow \mathfrak{B} \otimes H_0$ we call it ρ_H .

Example 4.1 Let $x \in V \subset \mathfrak{B}$, $\Delta x = x \otimes 1 + 1 \otimes x$. Assume $\rho(x) = x \otimes g \in \mathfrak{B} \otimes H_0$. In order to compute $\rho_H(x)$ we proceed as follows:

$$\begin{aligned}
 \rho_H(x) &\leftrightarrow \Delta_H(1\#x) = 1\#(x_1)_0 \otimes (x_1)_1\#x_2 \\
 &= 1\#(x)_0 \otimes (x)_1\#1 + 1\#(1)_0 \otimes (1)_1\#x \\
 &= 1\#x \otimes g\#1 + 1\#1 \otimes 1\#x \leftrightarrow x \otimes g + 1 \otimes x
 \end{aligned}$$

The main fact of this section is the following:

Proposition 4.2 $\mathfrak{B} \in \mathcal{M}^H$ is an injective object.

Proof Since H_0 is cosemisimple, the inclusion $k \rightarrow H_0$ splits as H_0 -comodule. Choose a splitting $\Lambda'_0 : H_0 \rightarrow k$. This is actually right integral for H_0 , that is, it satisfies

$$\Lambda'_0(h_1)h_2 = \Lambda'_0(h)1$$

and additionally $\Lambda'_0(1) = 1$.

Now we define a splitting $H \rightarrow \mathfrak{B}$ of the inclusion $\mathfrak{B} \cong 1\#\mathfrak{B} \subset H_0\#\mathfrak{B} = H$ via

$$h\#b \mapsto \Lambda'_0(h)b$$

We need to see that it is H -colinear. Recall the H -structure in \mathfrak{B} is given by the identification $\mathfrak{B} \cong 1\#\mathfrak{B} \subset H$, so

$$\rho_H(b) = b_{1_0} \otimes b_{1_1}\#b_2$$

We check H -colinearity:

$$\begin{array}{ccccc}
 h\#b & \xrightarrow{\quad\quad\quad} & H_0\#\mathfrak{B} & \xrightarrow{\quad\pi\quad} & \mathfrak{B} & \xrightarrow{\quad\quad\quad} & \Lambda'_0(h)b \\
 \downarrow & & \downarrow \Delta_{H_0\#\mathfrak{B}} & & \downarrow \rho_H & & \downarrow \\
 h_1\#b_{1_0} \otimes b_{1_1} h_2\#b_2 & & H_0\#\mathfrak{B} \otimes H_0\#\mathfrak{B} & \longrightarrow & \mathfrak{B} \otimes (H_0\#\mathfrak{B}) & & \Lambda'_0(h)b_{1_0} \otimes b_{1_1} h_2\#b_2 \\
 & \searrow & & & & & \parallel \\
 & & & & & & \Lambda'_0(h_1)b_{1_0} \otimes b_{1_1} h_2\#b_2 = b_{1_0} \otimes b_{1_1} \Lambda'_0(h_1)h_2\#b_2
 \end{array}$$

□

Remark 4.3 The proof is independent of the fact of \mathfrak{B} being finite dimensional, but we are interested in the case $\dim \mathfrak{B} < \infty$ so that $H_0\#\mathfrak{B}$ is co-Frobenius.

As a corollary we have

Corollary 4.4 For any $M \in \mathcal{M}^H$, the map $i_M : M \rightarrow M \otimes \mathfrak{B}$ defined by

$$m \mapsto m \otimes 1$$

is a functorial injective embedding. In particular, if \mathfrak{B} is finite dimensional then $M \rightarrow M \otimes \mathfrak{B}$ ($m \mapsto m \otimes 1$) is a finite dimensional embedding working in \mathfrak{m}^H . From the short exact sequence

$$0 \rightarrow M \rightarrow M \otimes \mathfrak{B} \rightarrow (M \otimes \mathfrak{B})/M \rightarrow 0$$

we have $T M \cong (M \otimes \mathfrak{B})/M$. Recall \mathfrak{B} is graded (with the tensor grading) and $\mathfrak{B}_{<top}$ (its maximal degree) has dimension 1. The Kernel of $\Lambda'|_{\mathfrak{B}}$ is $\mathfrak{B}_{<top} = \bigoplus_{i=0}^{top-1} \mathfrak{B}_i$. From $\pi : \mathfrak{B} \rightarrow \mathfrak{B}/\mathfrak{B}_{<top}$ we have the short exact sequence

$$0 \rightarrow M \otimes \mathfrak{B}_{<top} \rightarrow M \otimes \mathfrak{B} \rightarrow M \otimes (\mathfrak{B}/\mathfrak{B}_{<top}) \rightarrow 0$$

hence $T'(M \otimes (\mathfrak{B}/\mathfrak{B}_{<top})) \cong M \otimes \mathfrak{B}_{<top}$.

Remark 4.5 $\mathfrak{B}/\mathfrak{B}_{<top}$ is not isomorphic to k in general, but it is 1-dimensional. So, in order to compute $T'M$ one should "twist $M \otimes \mathfrak{B}_{<top}$ by the inverse of the quantum determinant":

If \bar{b} is a generator of the 1-dimensional vector space \mathfrak{B}_{top} then $k\bar{b}$ is not in general an H -subcomodule of \mathfrak{B} , but $\mathfrak{B}/\mathfrak{B}_{<top} = k\bar{b}$ is so, hence

$$\rho_H(\bar{b}) = \bar{b} \otimes D$$

for a unique group-like element $D \in H$, that we call "quantum determinant". From the surjective map

$$\mathfrak{B} \xrightarrow{\pi} \mathfrak{B}/\mathfrak{B}_{<top} \cong kD$$

we get a surjective map into the trivial comodule k :

$$\mathfrak{B} \otimes kD^{-1} \rightarrow k$$

If we call $P := \mathfrak{B} \otimes kD^{-1}$, it is a projective H -comodule that surjects into k and from it one has functorial projective surjections for any comodule M :

$$P(M) := M \otimes P \rightarrow M$$

and functorial T' , since from:

$$0 \rightarrow M \otimes \mathfrak{B}_{<top} \otimes kD^{-1} \rightarrow M \otimes P \rightarrow M \rightarrow 0$$

we get $T'(M) := M \otimes B_{<top} \otimes kD^{-1}$ is a functor in \mathcal{M}^H (resp. in \mathfrak{m}^H if M is finite dimensional) that gives the desuspension functor in $\underline{\mathcal{M}}^H$ (resp. in $\underline{\mathfrak{m}}^H$).

Before going into K_0 rings, we look at some examples.

5 First Examples

5.1 The Example $k[\mathbb{Z}] \# k[x]/x^2$

Let H be the k -algebra generated by x and $g^{\pm 1}$ with relations

$$gx = -xg, x^2 = 0$$

It is a Hopf algebra if one defines the comultiplication by

$$\Delta g = g \otimes g, \Delta x = x \otimes g + 1 \otimes x$$

(to be compared with Example 4.1). That is, g is group-like and x is 1- g -primitive. Notice that $k[x]/x^2$ is not a Hopf algebra in the usual sense (unless characteristic=2), but it is a super Hopf algebra. Nevertheless, $H = k[\mathbb{Z}] \# (k[x]/x^2)$ is a Hopf algebra in the usual sense. Maybe all computations in this example are folklore, but for clearness we include them.

For an element

$$\omega = \sum_{n \in \mathbb{Z}} a_n g^n + \sum_{n \in \mathbb{Z}} b_n g^n x$$

define

$$\Lambda(\omega) := b_0$$

The main fact about the category \mathcal{M}^H , noticed by Bodo Pareigis [7], is

$$\boxed{\text{A right } H\text{-comodule } M \text{ is the same as a d.g. structure on } M}$$

Notice that evaluation at $x = 0$ gives a map $H \rightarrow k[\mathbb{Z}]$, so any H -comodule is a $k[\mathbb{Z}]$ -comodule (i.e. a \mathbb{Z} -graded object), but the presence of x keep track of a square-zero differential. We just write the correspondence: if $M = (\oplus_{n \in \mathbb{Z}} M_n, \partial)$ with $\partial(M_n) \subseteq M_{n-1}$ and $\partial^2 = 0$ then, for $m \in M_n$, the right comodule structure is

$$\boxed{\rho(m) = m \otimes g^n + \partial(m) \otimes x g^{n-1}}$$

and every right H -comodule is of this form.

It is a pleasant exercise to check that the standard differential on the tensor products of complexes with the usual Koszul $\partial \otimes 1 \pm 1 \otimes \partial$ agree with the standard H -comodule structure on the tensor product of H -comodules.

Notice that

$$\begin{aligned} \Delta 1 &= 1 \otimes 1 \\ \Delta x &= x \otimes g + 1 \otimes x \end{aligned}$$

means that $k[x]/x^2$ is a right H -subcomodule of H . As d.g. vector space is the complex

$$\dots \rightarrow 0 \rightarrow kx \xrightarrow{x \mapsto 1} k \rightarrow 0 \rightarrow \dots$$

where $|x| = 1, |1| = 0$.

5.1.1 Smaller injective embeddings for $H = k[\mathbb{Z}]\#k[x]/x^2$

In this case we have $\mathfrak{B} = k \oplus kx$, considered as H -comodule via

$$\rho(1) = 1 \otimes 1$$

$$\rho(x) = x \otimes g + 1 \otimes x$$

The general argument developed in the previous section gives us that, for any $M \in \mathcal{M}^H$, the map $M \rightarrow I(M) := M \otimes \mathfrak{B} = M \otimes k[x]/x^2$ given by $i(m) = m \otimes 1$ is an embedding of M into an injective object. In particular $I(M)/\rho(M) = M \otimes x$ and we have proven the following:

Corollary 5.1 *In the stable category of \mathcal{M}^H for $H = k[\mathbb{Z}]\#(k[x]/x^2)$,*

$$TM = I(M)/M \cong M \otimes kx \cong M[1]$$

We leave as an exercise the following:

Corollary 5.2 *Identifying $d.g.Vect_k$ and \mathcal{M}^H , the comodule $M \otimes k[x]/x^2$ identifies with the mapping cone of the identity of M . Moreover, the “stable H -comodule mapping cone” of a colinear map identifies with the classical mapping cone of a map between complexes.*

5.1.2 Homology

For a d.g. vector space $M = (\oplus_n M_n, \partial)$ viewed as $k[\mathbb{Z}]\#k[x]/x^2$ -comodule, the coinvariants are

$$M^{coH} = \{m : \rho(m) = m \otimes 1_H = m \otimes g^0\}$$

But “ $\rho(m) = m \otimes g^0$ ” means that $m \in M_0$ and $\partial m = 0$, so

$$M^{coH} = \text{Ker}(\partial : M_0 \rightarrow M_{-1})$$

On the other side, the action of the integral on an element m gives

$$\begin{aligned} \Lambda \cdot m &= (\text{id} \otimes \Lambda)\rho(m) = (\text{id} \otimes \Lambda)\rho\left(\sum_n m_n\right) \\ &= (\text{id} \otimes \Lambda)\left(\sum_n (m_n \otimes g^n + \partial(m_n) \otimes g^{n-1}x)\right) = \partial(m_1) \in M_0 \end{aligned}$$

That is

$$\Lambda \cdot (\oplus_n M_n, \partial) = \text{Im}(\partial : M_1 \rightarrow M_0) \subseteq \text{Ker}(\partial : M_0 \rightarrow M_{-1}) \subseteq M_0$$

Hence

$$\mathcal{H}_0^{k[\mathbb{Z}]\#k[x]/x^2}(M) = M^{coH} / \Lambda \cdot M = H_0(M, \partial)$$

5.2 The Example $k[\mathbb{Z}]\#k[x]/x^N$ and N -Complexes

Fix $N \in \mathbb{N}, N \geq 2$. Let H be the algebra generated by $g^{\pm 1}$ and x with the relations

$$x^N = 0, \quad gx = \xi_N xg$$

where ξ_N is an N -primitive root of unity. This algebra is Hopf with comultiplication

$$\Delta g = g \otimes g, \quad \Delta x = x \otimes g + 1 \otimes x$$

To have an H -comodule is the same as a \mathbb{Z} -graded vector space together with a degree -1 map ∂ satisfying $\partial^N = 0$. The tensor structure for $(M_\bullet, \partial_M) \otimes (M'_\bullet, \partial_{M'})$ is given by the usual total grading in $M \otimes M'$, and the differential on homogeneous elements is

$$\partial(m \otimes m') = \partial(m) \otimes m' + \xi_N^{|m|} m \otimes \partial(m')$$

For an homogeneous element $m \in M$ of degree n , the coaction is given by

$$\begin{aligned} \rho(m) &= m \otimes g^n + \partial(m) \otimes x g^{n-1} + \frac{1}{[2]_\xi} \partial^2(m) \otimes x^2 g^{n-2} + \dots + \frac{1}{[N-1]_\xi!} \partial^{N-1}(m) \otimes x^{N-1} g^{n-N+1} \\ &= \sum_{i=1}^{N-1} \frac{1}{[i]_\xi!} \partial^i(m) \otimes x^i g^{n-i} \end{aligned}$$

where as usual $[0]_\xi! = [1]_\xi! = 1$, $[n]_\xi = 1 + \xi + \dots + \xi^{n-1}$ and $[n+1]_\xi! = [n+1]_\xi \cdot [n]_\xi!$.

If (M_\bullet, ∂) is an N -complex, there are several ways to associate an ‘‘homology’’ in degree n . For each $0 < i < N$, since $0 = \partial^N = \partial^i \partial^{N-i}$, one may consider $\text{Ker}(\partial^i) / \text{Im}(\partial^{N-i})$. The general machinery of co-Frobenius algebras and stable categories, however, choose one particular i . Since $M^{coH} = \text{Ker}(\partial) \cap M_0$ and $\Lambda \cdot M = M_0 \cap \text{Im}(\partial^{N-1})$ we have

$$\mathcal{H}_0^H(M) = \frac{\{m \in M_0 : \partial(m) = 0\}}{\partial^{N-1}(M_{-N+1})}$$

The other (homological) degrees are not the \mathcal{H}_0 of the degree-shiftings of the N -complex. As an illustration we compute $\mathcal{H}_1(M)$ in terms of the N -complex data:

Proposition 5.3 $H_1(M) \cong \frac{\text{Ker}(\partial^{N-1}) \cap M_{N-1}}{\text{Im}(\partial : M_N \rightarrow M_{N-1})}$

Remark 5.4 From Corollary 4.4 and the isomorphism $\mathfrak{B} / \mathfrak{B}_{<top} \cong kg^{N-1}$ (notice $\mathfrak{B} / \mathfrak{B}_{<top}$ is generated by the class of x^{N-1} and $\rho_H(x^{N-1}) = x^{N-1} \otimes g^{N-1}$ + lower degree terms) it follows that

$$T'(M) \cong M \otimes kg^{-N+1} \otimes \mathfrak{B}_{<top}$$

Proof Recall $\mathfrak{B} = \bigoplus_{i=0}^{N-1} kx^i$, the structure is given by

$$|x^i| = i, \quad \partial(x^i) = [i]x^{i-1}$$

$$\mathcal{H}_1(M) = \mathcal{H}_0(T'M) = \mathcal{H}_0(M \otimes kg^{-N+1} \otimes (k \oplus kx \oplus kx^2 \oplus \dots \oplus kx^{N-2}))$$

The degree zero part of $T'M$, if $M = \bigoplus_n M_n$, is

$$T'(M)_0 = \bigoplus_{i=0}^{N-2} M_{N-i-1} \otimes g^{-N+1} \otimes x^i$$

A typical element is of the form

$$\sum_{i=0}^{N-2} m_i \otimes g^{N-1} \otimes x^i$$

where $|m_i| = -N - i + 1$. The differential has the form

$$\partial\left(\sum_{i=0}^{N-2} m_i \otimes g^{N-1} \otimes x^i\right) = \sum_{i=0}^{N-2} \partial(m_i) \otimes g^{N-1} \otimes x^i + \sum_{i=0}^{N-2} \xi_N^{-N-i+1+N-1} m_i \otimes g^{N-1} \otimes \partial(x^i)$$

$$\begin{aligned}
 &= \sum_{i=0}^{N-2} \partial(m_i) \otimes g^{N-1} \otimes x^i + \sum_{i=1}^{N-2} \xi_N^{-i} m_i \otimes g^{N-1} \otimes [i]x^{i-1} \\
 &= \partial(m_{N-2}) \otimes g^{N-1} \otimes x^{N-2} + \sum_{i=0}^{N-3} \left(\partial(m_i) + \xi_N^{-i-1} [i+1]m_{i+1} \right) \otimes g^{N-1} \otimes x^i
 \end{aligned}$$

This expression is equal to zero if and only if

$$\partial(m_{N-2}) = 0 \quad \text{and} \quad m_i = \frac{-\xi_N^i}{[i]} \partial(m_{i-1}) \quad (i = N - 3, N - 4, \dots, 1)$$

From the second set of equalities we see that the only parameter is m_0 , because m_i is, up to scalar, $\partial^i(m_0)$. The equation $\partial(m_{N-2}) = 0$ means $\partial^{N-1}(m_0) = 0$. We conclude

$$(T^*M)^{coH} \cong \text{Ker}(\partial^{N-1}) \cap M_{N-1}$$

We leave to the reader to check that, under this bijection, $\Lambda \cdot (M \otimes g^{N-1} \otimes \mathfrak{B}_{<top})$ corresponds to $\partial(M_N)$. □

5.3 The Example $k[\mathbb{Z}]\#\Lambda(x, y)$ and Mixed Complexes

Denote

$$\Lambda(x, y) := k\{x, y\}/(x^2, y^2, xy + yx)$$

It is not a Hopf algebra in the usual sense, but it is a Hopf algebra in the (signed) graded sense. The algebra

$$k[\mathbb{Z}]\#\Lambda(x, y) = k\{g^{\pm 1}, x, y\}/(gx = -xg, gy = -yg, 0 = x^2 = y^2 = xy + yx)$$

is a Hopf algebra with comultiplication

$$\Delta g = g \otimes g, \quad \Delta x = x \otimes g + 1 \otimes x, \quad \Delta y = y \otimes g^{-1} + 1 \otimes y$$

Notice that x produce a differential of degree -1 , while y produce a differential of degree $+1$. This Hopf algebra H is isomorphic to $H_0\#\mathfrak{B}(V)$ where $H_0 = k[\mathbb{Z}]$ and $V = kx \oplus ky \in H_0\mathcal{D}^{H_0}$. Writing \mathbb{Z} multiplicatively $\mathbb{Z} \cong \{g^n : n \in \mathbb{Z}\}$, the action is given by

$$gv = -v, \quad \forall v \in V$$

and the coaction is determined by

$$\rho x = x \otimes g, \quad \rho y = y \otimes g^{-1}$$

Lemma 5.5 \mathcal{M}^H identifies with objects (M, d, B) where M is a \mathbb{Z} -graded vector space, d and B are square zero differentials with $|d| = -1, |B| = 1$, and $dB + Bd = 0$. In other words, \mathcal{M}^H are **mixed complexes**.

The proof is straightforward, we only indicate the correspondence: for a mixed complex (M, d, B) , the corresponding right comodule structure

$$\rho : M \rightarrow M \otimes H$$

for an homogeneous m , is given by

$$\rho(m) = m \otimes g^{|m|} + d(m) \otimes xg^{|m|-1} + B(m) \otimes yg^{|m|+1} + dB(m) \otimes yxg^{|m|}$$

It is clear that $M^{coH} = M_0 \cap \text{Ker } d \cap \text{Ker } B$. Also, an easy computation shows (see Example 6(b) of Section 3 for the expression of the integral)

$$\Lambda \cdot M = d(B(M_0)) = B(d(M_0)) \subseteq M^{coH}$$

So,

$$\mathcal{H}^0(M) = \frac{M_0 \cap \text{Ker } d \cap \text{Ker } B}{d(B(M_0))}$$

Remark 5.6 In this (stable) category, the suspension functor is *not* the shifting degree in general. However, we have the following lemma:

Lemma 5.7 $k[x]/x^2$ and $k[y]/y^2$ are H -subcomodules of H . $\mathfrak{B} = \Lambda(x, y) \cong k[x]/x^2 \otimes k[y]/y^2$ as objects in \mathcal{M}^H . For $M \in \mathcal{M}^H$ denote $M(x) := M \otimes k[x]/x^2$ and $M(y) := M \otimes k[y]/y^2$. The following assertions follows:

- $M(x)(y) \cong M(y)(x) \cong M \otimes \Lambda(x, y)$ is an injective object in \mathcal{M}^H .
- $T(M(x)) \cong M(x)[-1]$,
- $T(M(y)) \cong M(y)[1]$
- $\mathcal{H}_\bullet(M(x)) = H_{-\bullet}(M, B)$
- $\mathcal{H}_\bullet(M(y)) = H_\bullet(M, d)$

Proof The first item follows from the obvious isomorphism

$$k[x]/x^2 \otimes k[y]/y^2 \cong \Lambda(x, y)$$

Observe that $kx = (k[x]/x^2)/k = kx \cong k[1]$ and $ky = (k[y]/y^2)/k = ky \cong k[-1]$. Now from the short exact sequence

$$0 \rightarrow M(x) \rightarrow M(x) \otimes k[y]/y^2 \rightarrow M(x) \otimes y \rightarrow 0$$

we get

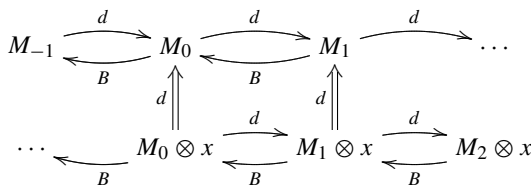
$$0 \rightarrow M(x) \rightarrow M \otimes \Lambda(x, y) \rightarrow M(x)[-1] \rightarrow 0$$

Since $M \otimes \Lambda(x, y)$ is injective, we conclude $T(M(x)) \cong M(x)[-1]$. Similarly for $M(y)$.

In order to compute cohomology we consider first

$$M(x)^{coH} = M(x)_0 \cap \text{Ker } d \cap \text{Ker } B = M(x)_0^{d,B}$$

We visualize it using the following diagram



So,

$$\begin{aligned}
 M(x)^d &= \{(m_0, m_1 \otimes x) : d(m_0) + m_1 = 0, d(m_1) = 0\} \\
 &= \{(m_0, -dm_0 \otimes x) : m_0 \in M_0\} \cong M_0
 \end{aligned}$$

$$M(x)^{d,B} = \{(m_0, -dm_0|x) : Bm_0 = 0, B(-dm_0) = 0\}$$

but $B(-dm_0) = dBm_0 = 0$, so $M(x)^{d,B} \cong M_0^B$.

We also must compute $B(d(M(x)_0))$:

$$\begin{aligned}
 Bd(m_0, m_1 \otimes x) &= B(dm_0 + m_1, dm_1 \otimes x) = (Bdm_0 + Bm_1, Bdm_1 \otimes x) \\
 &= (B(m_1 + dm_0), -d(B(m_1 + dm_0)) \otimes x) = (B\tilde{m}, -d(B\tilde{m}) \otimes x)
 \end{aligned}$$

So, under the isomorphism $M(x)^{d,B} \cong M_0^B$, the subspace $\Lambda \cdot M(x) = Bd(M(x)_0)$ corresponds to $B(M_1) \subset M_0$. We conclude $\mathcal{H}^0(M(x)) \cong H_0(M, B)$.

Now from the second item we get

$$\mathcal{H}_n(M(x)) = \mathcal{H}_0(T^{-n}(M(x))) = \mathcal{H}_0(M(x)[n]) = H_0(M(x)[n]) = H_{-n}(M, B)$$

The parts with $M(y)$ instead of $M(x)$ is completely analogous. □

Corollary 5.8 *For any mixed complex (M, d, B) there are long exact sequences*

$$\dots \rightarrow \mathcal{H}_\bullet(M) \rightarrow H_\bullet(M, d) \rightarrow \mathcal{H}_\bullet(M[-1]) \rightarrow \mathcal{H}_{\bullet-1}(M) \rightarrow \dots$$

and

$$\dots \rightarrow \mathcal{H}_\bullet(M) \rightarrow H_{-\bullet}(M, B) \rightarrow \mathcal{H}_\bullet(M[1]) \rightarrow \mathcal{H}_{\bullet-1}(M) \rightarrow \dots$$

Proof We consider the short exact sequences in \mathcal{M}^H :

$$0 \rightarrow M \rightarrow M \otimes k[y]/y^2 \rightarrow M \otimes y \rightarrow 0$$

and

$$0 \rightarrow M \rightarrow M \otimes k[x]/x^2 \rightarrow M \otimes x \rightarrow 0$$

Recall $M \otimes y \cong M[-1]$ and $M \otimes x \cong M[1]$. These short exact sequences in \mathcal{M}^H gives triangles in the stable category; their log exact sequences together with the previous Lemma gives the result. □

Corollary 5.9 $H_n(M, d) = 0 \forall n \Rightarrow \mathcal{H}_n(M) = \mathcal{H}_0(M[-n]);$
 $H_n(M, B) = 0 \forall n \Rightarrow \mathcal{H}_n(M) = \mathcal{H}_0(M[n]).$

Another corollary is the following computation:

Corollary 5.10 *Considering k as trivial mixed complex concentrated in degree zero,*

$$\mathcal{H}_\bullet(k) = \begin{cases} k & \bullet = 0 \\ k & \bullet = -1 \\ 0 & \text{otherwise} \end{cases}$$

Proof Specializing the long exact sequence

$$\dots \rightarrow \mathcal{H}_\bullet(M) \rightarrow H_\bullet(M, d) \rightarrow \mathcal{H}_\bullet(M[1]) \rightarrow \mathcal{H}_{\bullet-1}(M) \rightarrow H_{\bullet-1}(M, d) \rightarrow \dots$$

at $M = k[p]$ and $\bullet = q + 1$ gives

$$\dots \rightarrow H_{q+1}(k[p], d) \rightarrow \mathcal{H}_{q+1}(k[p + 1]) \rightarrow \mathcal{H}_q(k[p]) \rightarrow H_q(k[p], d) \rightarrow \dots$$

If $p \neq q, q + 1$ we have

$$\mathcal{H}_{q+1}(k[p + 1]) \cong \mathcal{H}_q(k[p])$$

Inductively, for $n \neq 0, 1$

$$\mathcal{H}_n(k) = \mathcal{H}_n(k[0]) \cong \mathcal{H}_{n-1}(k[-1]) \cong \dots \cong \mathcal{H}_0(k[-n]) = 0$$

because $k[n]$ do not have 0-degree component if $n \neq 0$. It remains to compute $\mathcal{H}_0(k)$. and $\mathcal{H}_1(k)$.

Clearly $\mathcal{H}_0(k) = k$. For \mathcal{H}_1 , since $\mathfrak{B}/\mathfrak{B}_{<top} \cong k$ (notice $xy \in \mathfrak{B}_{top}$ has degree zero), the formula for T' is

$$T'(k) = k \oplus kx \oplus ky$$

We have $(T'k)_0 = k = (T'k)^{coH}$ and $dB = Bd = 0$ in $T'k$, so $\mathcal{H}_1(k) = \mathcal{H}_0(T'k) = k$. □

Remark 5.11 Notice the asymmetry in the gradings, $\mathcal{H}_1(k) = k$ but $\mathcal{H}_{-1}(k) = 0$, as we can see from the general argument above, or compute directly:

$$T(k) = \mathfrak{B}/k = kx \oplus ky \oplus kxy$$

The degree zero component is kxy , but $d(xy) \neq 0$ (also $B(xy) \neq 0$), so $T(k)^{coH} = 0$ and

$$\mathcal{H}_{-1}(k) = \mathcal{H}_0(T(k)) = \frac{T(k)^{coH}}{\Lambda \cdot T(k)} = 0$$

6 (De)Categorification: Computation of K_0

6.1 K_0 of Exact and Triangulated Categories

The main result of this section is Theorem 6.1, that gives a general presentation of $K_0(\underline{\mathfrak{m}}^H)$. We recall the main constructions:

If \mathcal{A} is an *exact* category such that the isomorphism classes of objects is a set, then $K_0(\mathcal{A})$ is defined as the free abelian group on the set of isomorphism classes of objects module the relations

$$[X] + [Z] = [Y]$$

whenever there is a short exact sequence

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

We remark that $[X] + [Y] = [X \oplus Y]$ and, for $n \in \mathbb{N}$, $n[X] = [X^n]$, so, every element in $K_0(\mathcal{A})$ can be written in the form $[X] - [Y]$ for some objects X, Y in \mathcal{A} . For *triangulated* categories, K_0 is defined similarly, taking the free abelian group on isomorphism classes of objects modulo the relations

$$[X] + [Z] = [Y]$$

whenever there is a triangle

$$X \rightarrow Y \rightarrow Z \rightarrow TX$$

By $K_0(\mathfrak{m}^H)$ we understand the K-theory of the category of finite dimensional H -comodules, that is an exact category with usual short exact sequences. We denote $K_0(\underline{\mathfrak{m}}^H)$ the K-theory of the stable category $\underline{\mathfrak{m}}^H$ as triangulated category.

Almost by definition, if \mathcal{I} denotes the full subcategory of injective objects in \mathfrak{m}^H , there is a short exact exact sequence of categories

$$0 \rightarrow \mathcal{I} \rightarrow \mathfrak{m}^H \rightarrow \underline{\mathfrak{m}}^H \rightarrow 0$$

One could expect a general result in K-theory concluding a long exact sequence ending with

$$K_0(\mathcal{I}) \rightarrow K_0(\mathfrak{m}^H) \rightarrow K_0(\underline{\mathfrak{m}}^H) \rightarrow 0$$

This is actually the case for short exact sequences of *exact* categories where the left hand side is also a Serre subcategory. Recall a Serre (sub)category is closed under quotients, sub-objects and extensions. In our case, \mathfrak{m}^H is an exact category but \mathcal{I} is not a Serre subcategory in general. Also, $\underline{\mathfrak{m}}^H$ is not in general an exact category, it is triangulated, but the other two are not triangulated.

For exact sequences of Waldhausen categories there is also a long exact sequence in K-theory, both \mathcal{I} and \mathfrak{m}^H are Waldhausen categories, but it is not clear that $\underline{\mathfrak{m}}^H$ is so, in any case one should prove it. Instead, one can prove directly the following:

Theorem 6.1 *The natural functors $\mathcal{I} \rightarrow \mathfrak{m}^H$ and $\mathfrak{m}^H \rightarrow \underline{\mathfrak{m}}^H$ induce an exact sequence*

$$K_0(\mathcal{I}) \rightarrow K_0(\mathfrak{m}^H) \rightarrow K_0(\underline{\mathfrak{m}}^H) \rightarrow 0$$

In particular, the ring $K_0(\underline{\mathfrak{m}}^H)$ can be presented as the quotient of $K_0(\mathfrak{m}^H)$ by the ideal generated by injective objects.

Proof The functor $\mathfrak{m}^H \rightarrow \underline{\mathfrak{m}}^H$ is the identity on objects, so the induced map $K_0(\mathfrak{m}^H) \rightarrow K_0(\underline{\mathfrak{m}}^H)$ is surjective. By definition of $\underline{\mathfrak{m}}^H$, the composition $\mathcal{I} \rightarrow \mathfrak{m}^H \rightarrow \underline{\mathfrak{m}}^H$ is zero, so the composition $K_0(\mathcal{I}) \rightarrow K_0(\mathfrak{m}^H) \rightarrow K_0(\underline{\mathfrak{m}}^H)$ is zero as well. Let us denote

$$K_0(\mathfrak{m}^H)/K_0(\mathcal{I}) := \frac{K_0(\mathfrak{m}^H)}{\text{Im}(K_0(\mathcal{I}) \rightarrow K_0(\mathfrak{m}^H))}$$

We have a surjective ring homomorphism

$$K_0(\mathfrak{m}^H)/K_0(\mathcal{I}) \rightarrow K_0(\underline{\mathfrak{m}}^H)$$

To see injectivity of this map we argue as follows: Assume $\omega = [M] - [N]$ in $K_0(\mathfrak{m}^H)$ goes to zero in $K_0(\underline{\mathfrak{m}}^H)$. From the short exact sequence

$$0 \rightarrow N \rightarrow I(N) \rightarrow TN \rightarrow 0$$

in \mathfrak{m}^H with $I(N)$ injective, $[TN] = [I(N)] - [N]$ in $K_0(\mathfrak{m}^H)$, hence $[TN] = -[N]$ in $K_0(\mathfrak{m}^H)/K_0(\mathcal{I})$. So, we have

$$\omega = [M] - [N] = [M \oplus TN] \text{ Mod } \mathcal{I}$$

Eventually changing $\omega = [M] - [N]$ by $\omega' := [M \oplus TN]$, we can assume that, modulo \mathcal{I} , the element ω is equal to $[M]$ for some object M . Now if $[M]$ is zero in $K_0(\underline{\mathfrak{m}}^H)$, then there exists integers n_i and triangles in the stable category

$$X_i \rightarrow Y_i \rightarrow Z_i \rightarrow TX_i$$

such that

$$[M] = \sum_i m_i ([X_i] + [Z_i] - [Y_i])$$

But, using that direct sum of triangles is a triangle, for the positive m_i 's we get

$$\sum_i m_i ([X_i] + [Z_i] - [Y_i]) = [\oplus_{m_i > 0} X_i^{m_i}] + [\oplus_{m_i > 0} Z_i^{m_i}] - [\oplus_{m_i > 0} Y_i^{m_i}]$$

and similarly for the negative m_i 's. From this, we may assume that there are two triangles $X_i \rightarrow Y_i \rightarrow Z_i \rightarrow TX_i$, $i = 1, 2$ such that

$$[M] = ([X_1] + [Z_1] - [Y_1]) - ([X_2] + [Z_2] - [Y_2])$$

But because $X \xrightarrow{\text{id}} X \rightarrow 0 \rightarrow TX$ is a triangle, then so is $X \rightarrow 0 \rightarrow TX \rightarrow TX \rightarrow$, hence $[TX] = -[X]$ in $K_0(\underline{\mathfrak{m}}^H)$, and for $X \rightarrow Y \rightarrow Z \rightarrow TX$ a triangle, we have $TX \rightarrow TY \rightarrow TZ \rightarrow T^2X$ is also a triangle and

$$-([X] + [Z] - [Y]) = [TX] + [TZ] - [TY]$$

So, we can conclude that there exists a single triangle $X \rightarrow Y \rightarrow Z \rightarrow$ such that

$$[M] = [X] + [Z] - [Y]$$

But we know (Lemma 2.10) that any triangle in the stable category $X \rightarrow Y \rightarrow Z \rightarrow TX$ is isomorphic, in the stable category, to a short exact sequence

$$0 \rightarrow X' \rightarrow Y' \rightarrow Z' \rightarrow 0$$

Recall also that $X \cong X'$ in $\underline{\mathfrak{m}}^H$ if and only if there exist injectives I and J such that $X \oplus I \cong X' \oplus J$ in $\underline{\mathfrak{m}}^H$. But Modulo \mathcal{I} , clearly $[X] = [X] + [I] = [X \oplus I] = [X' \oplus J] = [X'] + [J] = [X']$. So, we finally get that

$$[M] = [X'] + [Z'] - [Y'] \text{ Mod } \mathcal{I}$$

Hence, $[M]$ in $K_0(\underline{\mathfrak{m}}^H)$ is zero Mod $K_0(\mathcal{I})$. □

Remark 6.2 It could be interesting to know if this is the last part of a long exact sequence for higher K -groups.

6.2 K_0 and the Coradical

Let H be a Hopf algebra and H_0 its coradical. Since H_0 is a subcoalgebra, every H_0 -comodule is an H -comodule. Consider the category $\mathcal{A} = \underline{\mathfrak{m}}^H$ and $\mathcal{B} = \underline{\mathfrak{m}}^{H_0}$; \mathcal{B} is a non-empty full subcategory closed under taking subobjects, quotient objects, and finite products in \mathcal{A} . Also \mathcal{B} is an abelian category and the inclusion functor $\mathcal{B} \rightarrow \mathcal{A}$ is exact, so Quillen’s theorem gives.

Theorem 6.3 ([10], Theorem 4. (Devissage)) *Let \mathcal{B} and \mathcal{A} be as above. Suppose that every object M of \mathcal{A} has a finite filtration $0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_n = M$ such that M_j/M_{j-1} is in \mathcal{B} for each j . Then the inclusion $\mathcal{B} \rightarrow \mathcal{A}$ induces an isomorphism $K_\bullet(\mathcal{B}) \cong K_\bullet(\mathcal{A})$*

If $M \in \mathcal{M}^H$ is a nonzero comodule, then its socle $\text{soc}(M)$ is a nonzero subcomodule that is actually an H_0 -comodule (see Exercise 3.1.2. of [4], page 117, its solution on page 140). If in addition M is finite dimensional, considering $M/\text{soc}(M)$ and induction in the dimension of M one can easily define a finite filtration

$$0 = \text{soc}(M) \subseteq M_2 \subseteq \dots \subseteq M_n = M$$

such that $M_j/M_{j-1} = \text{soc}(M_j/M_{j-1})$, hence $M_j/M_{j-1} \in \underline{\mathfrak{m}}^{H_0}$. Now Quillen’s theorem implies the following:

Corollary 6.4 *As abelian groups, $K_0(\underline{\mathfrak{m}}^H) \cong K_0(\underline{\mathfrak{m}}^{H_0})$. If H_0 is a Hopf subalgebra then this isomorphism is also a ring isomorphism.*

6.3 Smash Products

Let $H = H_0 \# \mathfrak{B}$ where H_0 is cosemisimple and \mathfrak{B} a finite dimensional braided Hopf algebra in ${}_{H_0}\mathcal{YD}^{H_0}$. For an element $M \in \mathcal{M}^H$, denote $\text{gr}M$ the associated graded with respect to the "socle filtration". Recall that the assignment $[M] \mapsto [\text{gr}M]$ implements the isomorphism $K_0(\underline{\mathfrak{m}}^H) \cong K_0(\underline{\mathfrak{m}}^{H_0})$. If $\{S_i : i \in I\}$ denote the set of (isomorphism classes of) simple objects in \mathcal{M}^{H_0} , then, for $M \in \underline{\mathfrak{m}}^H$,

$$\text{gr}M \cong \bigoplus_{i \in I} S_i^{m_i}$$

for unques (and finite non zero) multiplicity integers $m_i = m_i(M)$. We define

$$[M]_{H_0} := \sum_i m_i [S_i] = [\text{gr}M] \in K_0(\underline{\mathfrak{m}}^{H_0}) = \bigoplus_{i \in I} \mathbb{Z}[S_i]$$

In particular \mathfrak{B} is a finite dimensional H -comodule, so it makes sense

$$[\mathfrak{B}]_{H_0} \in K_0(\underline{\mathfrak{m}}^{H_0})$$

In the case $H_0 = k[G]$ with G a group, the isomorphism classes of simple comodules can be parametrized by $\{kg\}_{g \in G}$ and $kg \otimes kh \cong kgh$, so we identify $K_0(\underline{m}^{k[G]}) \cong \mathbb{Z}[G]$. The main result of this section is the following:

Theorem 6.5 *Let G be a group and $H_0 = k[G]$. Assume $H = H_0 \# \mathfrak{B}$, with finite dimensional \mathfrak{B} . The assignment $[M] \mapsto [M]_{H_0}$ induces an isomorphism of rings*

$$K_0(\underline{m}^H) \cong \mathbb{Z}[G]/([\mathfrak{B}]_{H_0})$$

Proof From Theorem 6.1 it follows that $K_0(\underline{m}^H) \cong K_0(\underline{m}^H)/K_0(\mathcal{I})$. But from Corollary 6.4 we know

$$\begin{aligned} K_0(\underline{m}^H) &\cong K_0(\underline{m}^{H_0}) \\ M &\mapsto [\text{gr}M] \end{aligned}$$

For $H_0 = k[G]$ we also know $K_0(\underline{m}^{k[G]}) \cong \mathbb{Z}[G]$. We need to identify $K_0(\mathcal{I})$ inside $K_0(\underline{m}^H) \cong \mathbb{Z}[G]$.

Recall that \mathfrak{B} is injective and $\mathfrak{B}^{coH} = k$. Now let I be a finite dimensional indecomposable injective H -comodule. Because I is indecomposable and injective, $\text{soc}(I)$ is an indecomposable H_0 -comodule (here, injectivity of I is essential), hence simple and

$$\text{soc}(I) \cong kg$$

for some $g \in G$. Clearly $\tilde{I} := I \otimes kg^{-1}$ is an injective indecomposable H -comodule and $(\tilde{I})^{coH} = k$. Since \tilde{I} is injective, there exist a dashed morphism in the diagram:

$$\begin{array}{ccc} k & \xlongequal{\quad} & \text{soc} \tilde{I} \hookrightarrow \tilde{I} \\ \downarrow & & \nearrow \text{---} \\ \mathfrak{B} & & \end{array}$$

This map restricted to the socle is injective, so the map is injective and we have $\dim \mathfrak{B} \leq \dim \tilde{I}$. But \mathfrak{B} is injective, so the same argument in the opposite direction gives $\dim \tilde{I} \leq \dim \mathfrak{B}$ and so $B \cong \tilde{I}$. In other words,

$$I \cong \mathfrak{B} \otimes kg$$

for some $g \in G$. We can conclude that if I is finite dimensional injective (non necessarily indecomposable) H -comodule, then there exists integers $\{m_g : g \in G\}$ with

$$I \cong \bigoplus_{g \in G} m_g \mathfrak{B} \otimes kg$$

That is, $\text{Im}(K_0(\mathcal{I}) \rightarrow K_0(\underline{m}^H))$ is the ideal generated by $[\mathfrak{B}]$. □

6.4 Examples

The first two examples are well-known:

1. $H = k[\mathbb{Z}] \# k[x]/x^2 :$

$$K_0(\underline{m}^H) \cong K_0(k[\mathbb{Z}]) / \langle k[x]/x^2 \rangle \cong \mathbb{Z}[z^{\pm 1}] / (1+z) \cong \mathbb{Z}$$

2. (Khovanov) $H = k[\mathbb{Z}] \# k[x]/x^N :$

$$K_0(\underline{m}^H) \cong K_0(k[\mathbb{Z}]) / \langle k[x]/x^N \rangle \cong \mathbb{Z}[z^{\pm 1}] / (1+z+\dots+z^{N-1})$$

If N is a prime p then $K_0(\underline{m}^H) \cong \mathbb{Z}[\xi_p]$.

3. $H = k[\mathbb{Z}]\#\Lambda(x, y)$ where $|x| = 1, |y| = -1$, then

$$\Lambda(x, y) = k \oplus kx \oplus ky \oplus kxy$$

hence $[\text{gr}\Lambda(x, y)] = 1 + z + z^{-1} + 1 = z^{-1} + 2 + z = z^{-1}(1 + z)^2$ and so

$$K_0(\underline{\mathfrak{m}}^H) \cong \mathbb{Z}[z^{\pm 1}]/(1 + z)^2 = \mathbb{Z}[z]/(1 + z)^2 \cong \mathbb{Z}[t]/t^2$$

4. If $N_1, \dots, N_k \in \mathbb{N}$, for $1 \leq i < j \leq k$, a list of nonzero scalars $q_{ij} \in k^\times$ is given, then define H as the algebra generated by $x_1, \dots, x_k, g_1^{\pm 1}, \dots, g_k^{\pm 1}$ with relations

$$\begin{aligned} g_i g_j &= g_j g_i & (\forall i, j) \\ x_i x_j &= q_{ij} x_j x_i & (i < j) \\ x_i g_j &= q_{ij} g_j x_i & (i < j) \\ g_i x_j &= q_{ij} x_j g_i & (i < j) \\ x_i^{N_i} &= 0 \end{aligned}$$

It is a Hopf algebra with comultiplication given by

$$\Delta g_i = g_i \otimes g_i$$

$$\Delta x_i = x_i \otimes g_i + 1 \otimes x_i$$

Then H is a Hopf algebra of the form $k[\mathbb{Z}^k]\#\mathfrak{B}$. The algebra \mathfrak{B} has monomial basis $\{x_1^{n_1} \cdots x_k^{n_k}, 0 \leq n_i < N_i\}$, so, writing $\mathbb{Z}[\mathbb{Z}^k] = \mathbb{Z}[z_1^{\pm 1}, \dots, z_k^{\pm 1}]$,

$$[\mathfrak{B}]_{H_0} = \prod_{i=1}^k (1 + z_i + z_i^2 + \cdots + z_i^{N_i-1})$$

Hence,

$$K_0(\underline{\mathfrak{m}}^H) \cong \mathbb{Z}[z_1^{\pm 1}, \dots, z_k^{\pm 1}]/\prod_{i=1}^k (1 + z_i + \cdots + z_i^{N_i-1})$$

Remark 6.6 It would be interesting to compute $K_0(H_0\#\mathfrak{B})$ for some non pointed cosemisimple H_0 , for instance, $H_0 = \mathcal{O}(G)$ with G non abelian reductive affine group.

7 H-comodule Algebras and the Category ${}_A\mathcal{M}^H$

An H -comodule algebra A is a k -algebra A together with an H -comodule structure such that the multiplication map

$$A \otimes A \rightarrow A$$

and the unit

$$k \rightarrow A$$

are H -colinear. Usual examples are:

- $H = k[G]$: comodule algebra = G -graded algebra.
- G finite, $H = k^G$: comodule algebra = algebra with a G -action by ring homomorphisms.
- G affine group, $H = \mathcal{O}(G)$: comodule algebra = algebra with a rational G -action.
- $H = U\mathfrak{g}$, comodule algebra = algebra with a \mathfrak{g} -action acting by derivations.

For our purpose, the motivating example is $H = k[\mathbb{Z}]\#k[x]/x^2$. In this case, an H -comodule algebra = d.g. algebra.

Also, if H is any Hopf algebra and A is any algebra, then A viewed as trivial H -comodule is an H -comodule algebra.

The main fact for our interest is the following:

$$M \in {}_A\mathcal{M}^H, V \in \mathcal{M}^H \Rightarrow M \otimes V \in {}_A\mathcal{M}^H$$

where A -module structure in $M \otimes V$ is the one coming from M and the H -comodule structure is the diagonal one. Moreover, if M is finitely generated as A -module and V is finite dimensional, then $A \otimes V$ is finitely generated as A -module. In this way, the subcategory of ${}_A\mathcal{M}^H$ consisting on A -finitely generated modules, denoted by ${}_A\mathfrak{m}^H$, is naturally a module over the category \mathfrak{m}^H . Following [6], we consider the restriction functor

$${}_A\mathcal{M}^H \rightarrow \mathcal{M}^H$$

and define $M \in {}_A\mathcal{M}^H \rightarrow \mathcal{M}^H$ to be *acyclic* (or H -acyclic to emphasize the role of H) if M is injective as H -comodule. In other words, if $M \cong 0$ in $\underline{\mathcal{M}}^H$. A map $f : M \rightarrow N$ is called *quasi-isomorphism* (qis) if f becomes an isomorphism in $\underline{\mathcal{M}}^H$. Denote \mathcal{I}_A the class of objects in ${}_A\mathcal{M}^H$ that are injective as H -comodules.

Example 7.1 Let $M \in {}_A\mathcal{M}^H$ be an arbitrary object and $I \in \mathcal{M}^H$ an injective H -comodule. In virtue of Lemma 3.14, $M \otimes I \in \mathcal{I}_A$.

If $M, N \in {}_A\mathcal{M}^H$, denote $\mathcal{I}_A(M, N)$ the set of maps that factors through an object in \mathcal{I}_A . The stable category - or the H -derived category-, denoted by ${}_A\underline{\mathcal{M}}^H$ and also by $\mathcal{D}_H(A)$, is defined as the category with same objects as ${}_A\mathcal{M}^H$ but morphism

$$\text{Hom}_{\mathcal{D}_H(A)}(M, N) := \frac{\text{Hom}_A^H(M, N)}{\mathcal{I}_A(M, N)}$$

The subcategory of $\mathcal{D}_H(A)$ whose objects are in ${}_A\mathfrak{m}^H$ (i.e. are finitely generated as A -modules) is denoted by $\mathcal{D}_H^c(A)$.

Recall that if $E = E(k)$ is the injective hull of k , E is a finite dimensional injective H -comodule (because H is co-Frobenius), and for any $M \in {}_A\mathcal{M}^H$, then

$$M \rightarrow M \otimes E$$

is an embedding of M into an acyclic object in ${}_A\mathcal{M}^H$. If $P := P(k)$ is a (finite dimensional) projective cover of k , then

$$M \otimes P \rightarrow M$$

is an epimorphism from an H -acyclic object in ${}_A\mathcal{M}^H$ into M . If M is finitely generated as A -module, then so is $M \otimes E$ and $M \otimes P$. The definition of TM , of $T'M$ and of the mapping cone of objects and maps in ${}_A\mathcal{M}^H$ (resp. in ${}_A\mathfrak{m}^H$) actually gives objects in ${}_A\mathcal{M}^H$ (resp. in ${}_A\mathfrak{m}^H$). One can easily see that all constructions and proofs of Happel's Theorem 2.6 in [5], when starting with objects in ${}_A\mathcal{M}^H$ (resp. in ${}_A\mathfrak{m}^H$) always stay in ${}_A\mathcal{M}^H$ (resp. in ${}_A\mathfrak{m}^H$). So $\mathcal{D}_H(A)$ and $\mathcal{D}_H^c(A)$ are triangulated categories, and by Example 7.1, they are modules over $\underline{\mathcal{M}}^H$ and $\underline{\mathfrak{m}}^H$ respectively.

Remark 7.2 $K_0(\mathcal{D}_H^c(A))$ is a module over the ring $K_0(\underline{\mathfrak{m}}^H)$.

Example 7.3 If $A = k$ then $\mathcal{D}_H(k) = \underline{\mathcal{M}}^H$ and $\mathcal{D}_H^c(k) = \underline{\mathfrak{m}}^H$.

Example 7.4 If $H = k[\mathbb{Z}]\#k[x]/x^2$ and A is an ordinary algebra viewed as trivial H -comodule algebra then $\mathcal{D}_H(A) = \mathcal{D}(A)$, the (unbounded) derived category of A .

Example 7.5 If A is a semisimple Hopf algebra and H is a co-Frobenius Hopf algebra, we view A as trivial H -comodule algebra, then

$${}_A\mathcal{M}^H \cong \mathcal{M}^{A^* \otimes H}$$

Since A is semisimple, A^* is co-semisimple and $A^* \otimes H$ is co-Frobenius. In this case we have $\mathcal{D}_H(A) = \underline{\mathbf{m}}^{(A^* \otimes H)}$. Also if $H = k[G]\#\mathfrak{B}$ as in Section 6.3 then $K_0(\mathcal{D}_H(A)) = K_0(\underline{\mathbf{m}}^{A^* \otimes k[G]\#\mathfrak{B}})$ is a quotient of $K_0(\underline{\mathbf{m}}^{A^* \otimes k[G]})$. Assuming k algebraically closed, every simple corepresentation of the tensor product $A^* \otimes k[G]$ is given by the tensor product of a simple A^* -comodule and a simple $k[G]$ -comodule, hence $K_0(\underline{\mathbf{m}}^{A^* \otimes k[G]}) = K_0(\underline{\mathbf{m}}^{A^*}) \otimes_{\mathbb{Z}} \mathbb{Z}[G] = K_0(A) \otimes_{\mathbb{Z}} \mathbb{Z}[G]$.

7.1 Enriched Hom

If $M, N \in {}_A\mathcal{M}^H$, there are several Hom spaces that one can consider. We begin with the discussion for d.g. A -modules:

If M and N are d.g. A -modules, then one may consider

- Chain maps: $\text{Hom}_A^H(M, N)$ = maps preserving degree and commuting with the differential.
- Chain maps up to homotopy: $\text{Hom}_A^H(M, N)/\sim$, where $f \sim g$ if $f - g = dh + hd$ for some degree $+1$ A -linear map h .
- The HOM complex: $\text{HOM}_A(M, N) = \bigoplus_{n \in \mathbb{Z}} \text{HOM}_A(M, N)_n$ where $\text{HOM}_A(M, N)_n = A$ -linear maps of degree n . If A is concentrated in degree zero (i.e. A is a trivial $k[\mathbb{Z}]\#k[x]/x^2$ -comodule) then $\text{HOM}_A(M, N)_n = \prod_{q \in \mathbb{Z}} \text{Hom}_A(M_q, N_{q+n})$
- Morphisms in the derived category: $\text{Hom}_{\mathcal{D}_H(A)}(M, N)$.

In general $\text{HOM}_A(M, N)$ is different from $\text{Hom}_A(M, N)$. Assume for simplicity A is an ordinary algebra (i.e. d.g. algebra concentrated in degree zero), if M and N have infinite nonzero degrees, then

$$\text{Hom}_A(M, N) = \text{Hom}_A(\bigoplus_p M_p, \bigoplus_q N_q) \not\cong \bigoplus_n \left(\prod_q \text{Hom}_A(M_q, N_{q+n}) \right)$$

For instance, if $M = \bigoplus_n A[n]$ and $N = A$, then

$$\text{Hom}_A(\bigoplus_n A[n], A) \not\cong \bigoplus_n \text{Hom}_A(A[n], A)$$

Nevertheless, the set of chain maps agree with $B^0(\text{HOM}_A(M, N))$ and the set of chain maps up to homotopy is the same as $H_0(\text{HOM}_A(M, N))$.

For general co-Frobenius Hopf algebras (i.e. not necessarily finite dimensional ones) one has the same "problems" but also analogous solutions. First of all, if H is a (not finite dimensional) Hopf algebra, A an H -comodule algebra and $M, N \in {}_A\mathcal{M}^H$, then $\text{Hom}_A(M, N)$ is not an H -comodule in general. For instance, if $A = k = N$ and $M = H$, then H^* is a not rational H^* -module, so it is not an H -comodule. In this way, if one consider

$$\text{Hom}_A(M, N)$$

it is not expectable to have an object in \mathcal{M}^H .

It is not clear to the author how to get an object in \mathcal{M}^H analogous to Hom_A (maybe $\text{Hom}_A(M, N) := \lim_{\rightarrow \mu} \text{Hom}_A(M_\mu, N)$, where M_μ runs over all A -finitely generated sub-objects?). To have an object $\text{Hom}_A(M, N) \in \mathcal{M}^H$ would provide the notion of map up to homotopy just by taking H_0 . Nevertheless, we have the following

Proposition 7.6 *$\text{Hom}_A(M, N)$ is a (left) H^* -module and the definition of \mathcal{H}_0^H can be naturally extended to H^* -modules.*

Proof The first statement is probably well-known, for completeness we exhibit the proof: First recall that if K is a finite dimensional Hopf algebra and M, N are K -modules (e.g. $K = H^*$ if H is finite dimensional and $M, N \in \mathcal{M}^H$) then the standard action of an element $x \in K$ in a map f , acting on an element $m \in M$ is given by

$$(x \cdot f)(m) := x_1 f(S(x_2) \cdot m)$$

If $K = H^*$ and $M, N \in \mathcal{M}^H$ then the above formula is

$$\begin{aligned} (x \cdot f)(m) &= x_1 \cdot f((S(x_2)(m_1))m_0) = x_2(S(m_1))x_1 \cdot f(m_0) = x_1(f(m_0)_1)x_2(S(m_1))f(m_0)_0 \\ &= x(f(m_0)_1 S(m_1))f(m_0)_0 \end{aligned}$$

and the last term in the equality make sense for $x \in H^*$, independently on the dimension of H , so one defines the H^* -action of $x \in H^*$ on $f : M \rightarrow N$ via

$$(x \cdot f)(m) := x(f(m_0)_1 S(m_1))f(m_0)_0$$

In other words,

$$x \cdot f = (1 \otimes m_H^*(x))(\rho_N \otimes 1)(f \otimes S)\rho_M$$

One can proof by standard diagrammatic methods that this is an action, and f is H -colinear (if and only if it is H^* -linear) if and only if

$$x \cdot f = \epsilon(x)f = x(1)f \quad \forall x \in H^*$$

Concerning the second statement, if W is an H^* -module, one may define

$$W^{H^*} = \{w \in W : x \cdot w = x(1)w\} \cong \text{Hom}_H(k, W)$$

If W is a right H -comodule then it is clear that $W^{coH} = \{w : \rho(w) = w \otimes 1\} = W^{H^*}$, so one can extend the definition of \mathcal{H}_0^H on $H^* \mathcal{M}$ simply by

$$\mathcal{H}_0^H(W) := \frac{W^{H^*}}{\Lambda \cdot W}$$

If $W = \text{Hom}_A(M, N)$ then $W^{H^*} = A$ linear and H^* -linear maps $= \text{Hom}_A^H(M, N)$, and a definition of "chain maps up to homotopy" is available defining

$$\mathcal{H}_0^H(\text{Hom}_A(M, N)) = \frac{\text{Hom}_A^H(M, N)}{\Lambda \cdot \text{Hom}_A(M, N)}$$

This recover the definition given in [8] for finite dimensional Hopf algebras and when M and N are \mathbb{Z} -graded vector spaces, but we emphasizes that this definition makes sense in full generality for H a co-Frobenius algebra (whose coradical is not necessarily finite over $k[\mathbb{Z}]$). □

A warning on the notation in [8], we call \mathcal{H}_0^H what he calls \mathcal{H} in the ungraded case. He defines \mathcal{H}_n only in the graded case but using the degree shifting, and not the triangulated structure, so \mathcal{H}_n in [8] is different from our \mathcal{H}_n^H .

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