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Weighted Leavitt Path Algebras that are Isomorphic to Unweighted Leavitt Path Algebras

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Abstract
Let K be a field. We characterise the row-finite weighted graphs (E, w) such that the weighted Leavitt path algebra $L_K(E, w)$ is isomorphic to an unweighted Leavitt path algebra. Moreover, we prove that if $L_K(E, w)$ is locally finite, or Noetherian, or Artinian, or von Neumann regular, or has finite Gelfand-Kirillov dimension, then $L_K(E, w)$ is isomorphic to an unweighted Leavitt path algebra.

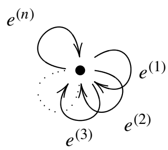
Keywords Leavitt path algebra · Weighted Leavitt path algebra

Mathematics Subject Classification (2000) 16S10 · 16W10 · 16W50 · 16D70

1 Introduction

Let m and n be positive integers such that $m \leq n$. In a series of papers [9–12] William Leavitt studied K-algebras that are now denoted by $L_K(m, n)$ and have been coined Leavitt algebras. Let $X = (x_{ij})$ and $Y = (y_{ji})$ be $n \times m$ and $m \times n$ matrices consisting of symbols x_{ij} and y_{ji}, respectively. Then $L_K(m, n)$ is the unital K-algebra generated by all x_{ij} and y_{ji} subject to the relations $XY = I_n$ and $YX = I_m$. The algebra $L_K(m, n)$ can be described as the K-algebra A with a universal left A-module isomorphism $A^m \to A^n$, cf. [4, second paragraph on p. 35].

(Unweighted) Leavitt path algebras are algebras associated to directed graphs. They were introduced by G. Abrams and G. Aranda Pino in 2005 [1] and independently by P. Ara, M. Moreno and E. Pardo in 2007 [3]. For the directed graph



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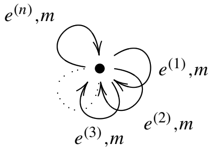
with one vertex and n loops one recovers the Leavitt algebra $L_K(1,n)$. The definition and the development of the theory were inspired on the one hand by Leavitt's construction of $L_K(1,n)$ and on the other hand by the Cuntz algebras $\mathcal{O}_n$ [6] and the Cuntz-Krieger algebras in C^*-algebra theory [17]. The Cuntz algebras and later Cuntz-Krieger type C^*-algebras revolutionised C^*-theory, leading ultimately to the astounding Kirchberg-Phillips classification theorem [13]. The Leavitt path algebras have created the same type of stir in the algebraic community.

As mentioned in the previous paragraph, the class of Leavitt path algebras contains the Leavitt algebras $L_K(1,n)$, $1 \leq n$. If $1 < m \leq n$, then the Leavitt algebra $L_K(m,n)$ is a noncommutative domain, see [5, §5] (the noncommutativity follows from the normal form for $L_K(m,n)$ obtained in [5, §5]). The only Leavitt path algebras over K which are domains are $L_K(\,\bullet\,) \cong K$ and $L_K(\,\bullet\circlearrowright\,) \cong K[x,x^{-1}]$ and they are both commutative. Thus, for $1 < m \leq n$, the algebra $L_K(m,n)$ is not K-algebra isomorphic to a Leavitt path algebra (in fact, in this case $L_K(m,n)$ is not even ring isomorphic to a Leavitt path algebra over a coefficient field).

In 2013, R. Hazrat [7] introduced weighted Leavitt path algebras. These are algebras associated to weighted graphs, i.e. directed graphs with a weight map that assigns to each edge a positive integer. If E is a directed graph and w is the map that assigns to each edge in E the weight 1, then the weighted Leavitt path algebra $L_K(E,w)$ is isomorphic to the unweighted Leavitt path algebra $L_K(E)$. Hence the weighted Leavitt path algebras generalise their unweighted cousins in a natural way. On the other hand, for the weighted graph



(1)

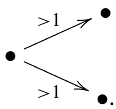
with one vertex and n loops of weight m one recovers the Leavitt algebra $L_K(m,n)$. Thus the class of weighted Leavitt path algebras contains *all* Leavitt algebras $L_K(m,n)$.

Let (E,w) be a row-finite weighted graph. We call an edge in (E,w) *unweighted* if its weight is 1 and *weighted* otherwise. We use E^1_w to denote the set of all weighted edges in (E,w). If X is a set of vertices in (E,w), we use $T(X)$ to denote the union of all the trees of the elements of X. Two edges e and f are called *in line* if they are equal or there is a path from $r(e)$ to $s(f)$ or there is a path from $r(f)$ to $s(e)$, cf. Section 2. Consider the following conditions:

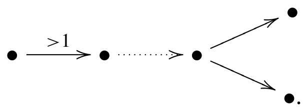
(LPA1) Any vertex $v \in E^0$ emits at most one weighted edge.
(LPA2) Any vertex $v \in T(r(E^1_w))$ emits at most one edge.
(LPA3) If two weighted edges $e, f \in E^1_w$ are not in line, then $T(r(e)) \cap T(r(f)) = \emptyset$.
(LPA4) If $e \in E^1_w$ and c is a cycle based at some vertex $v \in T(r(e))$, then e belongs to c.

Each of the conditions above "forbids" a certain constellation in the weighted graph (E,w). The pictures below illustrate these forbidden constellations. Symbols above or below edges indicate the weight. A dotted arrow stands for a path.

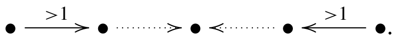
(LPA1)



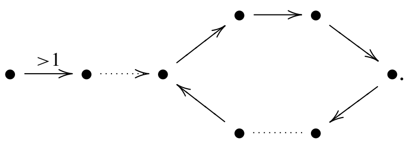
(LPA2)



(LPA3)



(LPA4)



Conditions (LPA1)-(LPA3) already appeared in [15, Section 6], where finite-dimensional weighted Leavitt path algebras were investigated. Conditions (LPA1)-(LPA3) were independently found by N. T. Phuc in an unpublished work on finite-dimensional weighted Leavitt path algebras. In [16], it was shown that $L_K(E, w)$ is finite-dimensional iff (E, w) is finite, acyclic and moreover Conditions (LPA1)-(LPA3) and Conditions (iv),(v) in [16, Definition 19] are satisfied. It was also shown that $L_K(E, w)$ is locally finite with respect to its standard grading iff (E, w) is finite, no cycle has an exit and moreover Conditions (LPA1)-(LPA3) and Conditions (iv),(v) in [16, Definition 19] are satisfied. Furthermore, it was shown that if $L_K(E, w)$ is locally finite, then it is K-algebra isomorphic to the unweighted Leavitt path algebra $L_K(F)$ of some finite graph F. On the other hand, it was shown in [14, Corollary 16], that the class of weighted Leavitt path algebras contains infinitely many domains which are neither isomorphic to an unweighted Leavitt path algebra nor to a Leavitt algebra $L_K(m, n)$.

In this paper we characterise the row-finite weighted graphs (E, w) such that the weighted Leavitt path algebra $L_K(E, w)$ is isomorphic to an unweighted Leavitt path algebra. We say that a weighted graph (E, w) *satisfies Condition (LPA)* if it satisfies Conditions (LPA1)-(LPA4). The main results of this paper are the following three theorems.

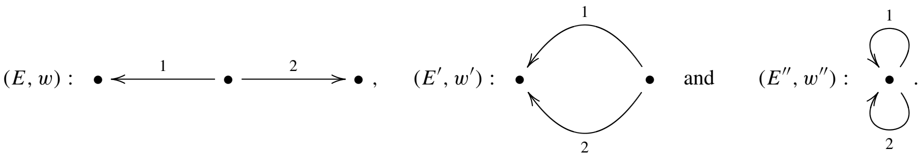
Theorem 1 *Let K be a field and (E, w) a row-finite weighted graph. If (E, w) satisfies Condition (LPA), then there is a row-finite graph F such that $L_K(E, w) \cong L_K(F)$ as K-algebras.*

Theorem 2 *Let K be a field and (E, w) a row-finite weighted graph. If (E, w) does not satisfy Condition (LPA), then there is no field K' and graph F (row-finite or not) such that $L_K(E, w) \cong L_{K'}(F)$ as rings.*

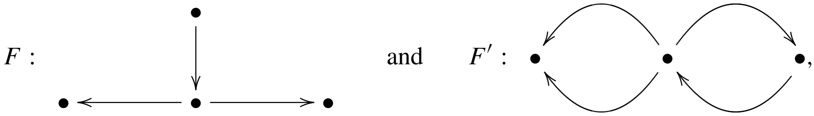
Theorem 3 *Let K be a field and (E, w) a row-finite weighted graph. If $L_K(E, w)$ is locally finite with respect to its standard grading, or Noetherian, or Artinian, or von Neumann regular, or has finite Gelfand-Kirillov dimension, then there is a row-finite graph F such that $L_K(E, w) \cong L_K(F)$ as K-algebras.*

A part of Theorem 3 had already been proved in [16], see the previous paragraph. Since the paper [16] was never published in a journal, we prove this part again.

For example, consider the weighted graphs



(E, w) and (E', w') satisfy Condition (LPA) but (E'', w'') does not. It follows from the proof of Theorem 1 that $L_K(E, w) \cong L_K(F)$ and $L_K(E', w') \cong L_K(F')$ as K-algebras where F and F' are the graphs



respectively (cf. [15, Example 40], [14, Example 21]). Theorem 2 implies that $L_K(E'', w'')$ cannot be ring isomorphic to an unweighted Leavitt path algebra over a coefficient field.

The rest of the paper is organised as follows. In Section 2, we recall the definitions of the unweighted and weighted Leavitt path algebras. In Section 3, we prove Theorem 1. In Section 4, we prove Theorems 2 and 3.

Throughout the paper K denotes a field. By a K-algebra we mean an associative (but not necessarily commutative or unital) K-algebra. By an ideal we mean a two-sided ideal. $\mathbb{N}$ denotes the set of positive integers, $\mathbb{N}_0$ the set of nonnegative integers, $\mathbb{Z}$ the set of integers and $\mathbb{R}_+$ the set of positive real numbers.

2 Unweighted and weighted Leavitt path algebras

A *(directed) graph* is a quadruple $E = (E^0, E^1, s, r)$ where E^0 and E^1 are sets and $s, r : E^1 \to E^0$ maps. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. If e is an edge, then $s(e)$ is called its *source* and $r(e)$ its *range*. If v is a vertex and e an edge, we say that v *emits* e if $s(e) = v$ and v *receives* e if $r(e) = v$. A vertex v is called a *sink* if it emits no edges and an *infinite emitter* if it emits infinitely many edges. A vertex which is neither a sink nor an infinite emitter is called *regular*. The subset of E^0 consisting of all regular vertices is denoted by E^0_{reg}. The graph E is called *finite* if E^0 and E^1 are finite sets, and *row-finite* if E^0 contains no infinite emitters.

Let E be a graph. A *path* is a nonempty word $p = x_1 \dots x_n$ over the alphabet $E^0 \cup E^1$ such that either $x_i \in E^1$ $(i = 1, \dots, n)$ and $r(x_i) = s(x_{i+1})$ $(i = 1, \dots, n-1)$ or $n = 1$ and $x_1 \in E^0$. By definition, the *length* $|p|$ of p is n in the first case and 0 in the latter case. We set $s(p) := s(x_1)$ and $r(p) := r(x_n)$ (here we use the convention $s(v) = v = r(v)$ for any $v \in E^0$). A *closed path (based at v)* is a path p such that $|p| > 0$ and $s(p) = r(p) = v$. A *cycle (based at v)* is a closed path $p = x_1 \dots x_n$ based at v such that $s(x_i) \neq s(x_j)$ for any $i \neq j$. If $u, v \in E^0$ and there is a path p such that $s(p) = u$ and $r(p) = v$, then we write $u \geq v$. If $u \in E^0$, then $T(u) := \{v \in E^0 \mid u \geq v\}$ is called the *tree* of u. If $X \subseteq E^0$, we define $T(X) := \bigcup_{v \in X} T(v)$. Two edges $e, f \in E^1$ are called *in line* if $e = f$ or $r(e) \geq s(f)$ or $r(f) \geq s(e)$.

Definition 4 Let E be a graph. The K-algebra $L_K(E)$ presented by the generating set $\{v, e, e^* \mid v \in E^0, e \in E^1\}$ and the relations

(i) $uv = \delta_{uv}u \quad (u, v \in E^0)$,
(ii) $s(e)e = e = er(e),\ r(e)e^* = e^* = e^*s(e) \quad (e \in E^1)$,
(iii) $e^*f = \delta_{ef}r(e) \quad (e, f \in E^1)$ and
(iv) $\sum_{e \in s^{-1}(v)} ee^* = v \quad (v \in E^0_{\text{reg}})$

is called the *(unweighted) Leavitt path algebra* of E.

Let E be a graph and A a K-algebra. An *E-family* in A is a subset $X = \{\alpha_v, \beta_e, \gamma_e \mid v \in E^0, e \in E^1\} \subseteq A$ such that

(i) the α_v's are pairwise orthogonal idempotents,
(ii) $\alpha_{s(e)}\beta_e = \beta_e = \beta_e\alpha_{r(e)},\ \alpha_{r(e)}\gamma_e = \gamma_e = \gamma_e\alpha_{s(e)} \quad (e \in E^1)$,
(iii) $\gamma_e\beta_f = \delta_{ef}\alpha_{r(e)} \quad (e, f \in E^1)$ and
(iv) $\sum_{e \in s^{-1}(v)} \beta_e\gamma_e = \alpha_v \quad (v \in E^0_{\text{reg}})$.

By the relations defining $L_K(E)$, there exists a unique K-algebra homomorphism $\phi : L_K(E) \to A$ such that $\phi(v) = \alpha_v$, $\phi(e) = \beta_e$ and $\phi(e^*) = \gamma_e$ for all $v \in E^0$ and $e \in E^1$. We will refer to this as the *Universal Property of* $L_K(E)$.

A *weighted graph* is a pair (E, w) where E is a graph and $w : E^1 \to \mathbb{N}$ is a map. If $e \in E^1$, then $w(e)$ is called the *weight* of e. An edge $e \in E^1$ is called *unweighted* if $w(e) = 1$ and *weighted* if $w(e) > 1$. The subset of E^1 consisting of all unweighted edges is denoted by E^1_{uw} and the subset consisting of all weighted edges by E^1_w. For a regular vertex $v \in E^0_{\text{reg}}$ we set $w(v) := \max\{w(e) \mid e \in s^{-1}(v)\}$. A weighted graph (E, w) is called *finite* (resp. *row-finite*) if the graph E is finite (resp. row-finite).

Definition 5 Let (E, w) be a weighted graph. The K-algebra $L_K(E, w)$ presented by the generating set $\{v, e_i, e_i^* \mid v \in E^0, e \in E^1, 1 \leq i \leq w(e)\}$ and the relations

(i) $uv = \delta_{uv}u \quad (u, v \in E^0)$,
(ii) $s(e)e_i = e_i = e_ir(e),\ r(e)e_i^* = e_i^* = e_i^*s(e) \quad (e \in E^1, 1 \leq i \leq w(e))$,
(iii) $\sum_{1 \leq i \leq \max\{w(e), w(f)\}} e_i^* f_i = \delta_{ef}r(e) \quad (e, f \in E^1)$ and
(iv) $\sum_{e \in s^{-1}(v)} e_ie_j^* = \delta_{ij}v \quad (v \in E^0_{\text{reg}}, 1 \leq i, j \leq w(v))$

is called the *weighted Leavitt path algebra* of (E, w). In relations (iii) and (iv) we set e_i and e_i^* zero whenever $i > w(e)$.

Example 6 If (E, w) is a weighted graph such that $w(e) = 1$ for all $e \in E^1$, then $L_K(E, w)$ is isomorphic to the unweighted Leavitt path algebra $L_K(E)$.

Example 7 Let $1 \leq m \leq n$. Let (E, w) be the weighted graph (1) with one vertex v and n edges $e^{(1)}, \ldots, e^{(n)}$ each of which has weight m. Then $L_K(E, w)$ is isomorphic to the Leavitt algebra $L_K(m, n)$, for details see [7, Example 5.5] or [8, Example 4].

Let (E, w) be a weighted graph and A a K-algebra. An (E, w)*-family* in A is a subset $X = \{\alpha_v, \beta_{e,i}, \gamma_{e,i} \mid v \in E, e \in E^1, 1 \leq i \leq w(e)\} \subseteq A$ such that

(i) the α_v's are pairwise orthogonal idempotents,
(ii) $\alpha_{s(e)}\beta_{e,i} = \beta_{e,i} = \beta_{e,i}\alpha_{r(e)},\ \alpha_{r(e)}\gamma_{e,i} = \gamma_{e,i} = \gamma_{e,i}\alpha_{s(e)} \quad (e \in E^1, 1 \leq i \leq w(e))$,
(iii) $\sum\limits_{1\leq i\leq \max\{w(e),w(f)\}} \gamma_{e,i}\beta_{f,i} = \delta_{ef}\alpha_{r(e)} \quad (e, f \in E^1)$ and
(iv) $\sum\limits_{e\in s^{-1}(v)} \beta_{e,i}\gamma_{e,j} = \delta_{ij}\alpha_v \quad (v \in E^0_{\text{reg}}, 1 \leq i, j \leq w(v))$.

In relations (iii) and (iv) we set $\beta_{e,i}$ and $\gamma_{e,i}$ zero whenever $i > w(e)$. By the relations defining $L_K(E, w)$, there exists a unique K-algebra homomorphism $\phi : L_K(E, w) \to A$ such that $\phi(v) = \alpha_v$, $\phi(e_i) = \beta_{e,i}$ and $\phi(e_i^*) = \gamma_{e,i}$ for all $v \in E^0$, $e \in E^1$ and $1 \leq i \leq w(e)$. We will refer to this as the *Universal Property of* $L_K(E, w)$. The proof of the Universal Property for a weighted Leavitt path algebra is analogous to the proof in the unweighted case.

Let (E, w) be a row-finite weighted graph. Then $L_K(E, w)$ has the properties (a)-(d) below (cf. [7, Proposition 5.7]).

(a) If E^0 is a finite set, then $L_K(E, w)$ is a unital ring (with $\sum\limits_{v\in E^0} v$ as multiplicative identity).
(b) $L_K(E, w)$ has a set of local units, namely the set of all finite sums of distinct elements of E^0. Recall that an associative ring R is said to have a *set of local units* X in case X is a set of idempotents in R having the property that for each finite subset $S \subseteq R$ there exists an $x \in X$ such that $xsx = s$ for any $s \in S$.
(c) There is an involution $*$ on $L_K(E, w)$ mapping $k \mapsto k$, $v \mapsto v$, $e_i \mapsto e_i^*$ and $e_i^* \mapsto e_i$ for any $k \in K$, $v \in E^0$, $e \in E^1$ and $1 \leq i \leq w(e)$.
(d) Set $\lambda := \sup\{w(e) \mid e \in E^1\}$ if this supremum is finite and otherwise $\lambda := \omega$ where ω is the smallest infinite ordinal. Let $\mathbb{Z}^\lambda$ denote the sum of λ-many copies of $\mathbb{Z}$. One can define a $\mathbb{Z}^\lambda$-grading on $L_K(E, w)$ by setting $\deg(v) := 0$, $\deg(e_i) := \epsilon_i$ and $\deg(e_i^*) := -\epsilon_i$ for any $v \in E^0$, $e \in E^1$ and $1 \leq i \leq w(e)$. Here ϵ_i denotes the element of $\mathbb{Z}^\lambda$ whose i-th component is 1 and whose other components are 0. We will refer to this grading as the *standard grading* of $L_K(E, w)$.

3 Presence of Condition (LPA)

Lemma 8 *Let (E, w) be a row-finite weighted graph that satisfies Condition (LPA). If e and f are distinct edges such that $s(e), s(f) \in T(r(E^1_w))$, then $r(e) \neq r(f)$.*

Proof Let $e, f \in E^1$ such that $s(e), s(f) \in T(r(E^1_w))$ and $r(e) = r(f)$. We will show that $e = f$. Since $s(e), s(f) \in T(r(E^1_w))$, there are $g, h \in E^1_w$ such that $s(e) \in T(r(g))$ and $s(f) \in T(r(h))$. It follows that $r(e) = r(f) \in T(r(g)) \cap T(r(h))$. Since (E, w) satisfies Condition (LPA3), g and h are in line. It follows that $s(e), s(f) \in T(r(g))$ or $s(e), s(f) \in T(r(h))$. Without loss of generality assume that $s(e), s(f) \in T(r(g))$.

Case 1. Assume that there is a cycle c based at some vertex $v \in T(r(g))$. Since (E, w) satisfies (LPA4), g belongs to c. Write $c = \alpha^{(1)} \dots \alpha^{(n)}$ where $\alpha^{(1)}, \dots, \alpha^{(n)} \in E^1$. Set $x_i := s(\alpha^{(i)})$ $(1 \leq i \leq n)$. Then, in view of (LPA2), we have $T(r(g)) = \{x_1, \dots, x_n\}$. Moreover, each x_i emits precisely one edge, namely $\alpha^{(i)}$. Since

$s(e), s(f) \in T(r(g))$, we get that $s(e) = x_i$ and $s(f) = x_j$ for some $1 \leq i, j \leq n$. Hence $e = \alpha^{(i)}$ and $f = \alpha^{(j)}$. Since $r(e) = r(f)$, it follows that $i = j$ and hence $e = f$.

Case 2. Assume that no cycle is based at a vertex in $T(r(g))$. Since $s(e), s(f) \in T(r(g))$, there are paths p and q such that $s(p) = r(g) = s(q)$, $r(p) = s(e)$ and $r(q) = s(f)$. Clearly pe and qf are paths starting at $r(g)$ and ending at $r(e) = r(f)$. It follows from (LPA2) and the assumption that no cycle is based at a vertex in $T(r(g))$, that $pe = qf$. Hence $e = f$. □

Lemma 9 *Let (E, w) be a row-finite weighted graph that satisfies Condition (LPA). Then there is a row-finite weighted graph $(\tilde{E}, \tilde{w})$ such that the ranges of the weighted edges in $(\tilde{E}, \tilde{w})$ are sinks, no vertex in $(\tilde{E}, \tilde{w})$ emits or receives two distinct weighted edges, and $L_K(\tilde{E}, \tilde{w}) \cong L_K(E, w)$ as K-algebras.*

Proof Set $Z := T(r(E^1_w))$. Define a weighted graph $(\tilde{E}, \tilde{w})$ by $\tilde{E}^0 = E^0$, $\tilde{E}^1 = \tilde{E}^1_Z \sqcup \tilde{E}^1_{Z^c}$ where

$$\tilde{E}^1_Z = \{e^{(1)}, \dots, e^{(w(e))} \mid e \in E^1, s(e) \in Z\} \text{ and } \tilde{E}^1_{Z^c} = \{e \mid e \in E^1, s(e) \notin Z\},$$

$\tilde{s}(e^{(i)}) = r(e)$, $\tilde{r}(e^{(i)}) = s(e)$ and $\tilde{w}(e^{(i)}) = 1$ for any $e^{(i)} \in \tilde{E}^1_Z$ and $\tilde{s}(e) = s(e)$, $\tilde{r}(e) = r(e)$ and $\tilde{w}(e) = w(e)$ for any $e \in \tilde{E}^1_{Z^c}$. Clearly $(\tilde{E}, \tilde{w})$ is row-finite. We have divided the rest of the proof into three parts. In Part I we show that the ranges of the weighted edges in $(\tilde{E}, \tilde{w})$ are sinks, in Part II we show that no vertex in $(\tilde{E}, \tilde{w})$ emits or receives two distinct weighted edges, and in Part III we show that $L_K(\tilde{E}, \tilde{w}) \cong L_K(E, w)$.

Part I. Let $\tilde{e} \in \tilde{E}^1_w$. We will show that $\tilde{r}(\tilde{e})$ is a sink in $(\tilde{E}, \tilde{w})$. Clearly $\tilde{e} \in \tilde{E}^1_{Z^c}$ since all the edges in $\tilde{E}^1_Z$ have weight one in $(\tilde{E}, \tilde{w})$. Hence there is an $e \in E^1, s(e) \notin Z$ such that $\tilde{e} = e$. Clearly $w(e) = \tilde{w}(e) = \tilde{w}(\tilde{e}) > 1$. Now suppose that there is an $\tilde{f} \in \tilde{E}^1$ such that $\tilde{s}(\tilde{f}) = \tilde{r}(\tilde{e})$.

Case 1. Assume that $\tilde{f} \in \tilde{E}^1_Z$. Then there is an $f \in E^1, s(f) \in Z$ and an $i \in \{1, \dots, w(f)\}$ such that $\tilde{f} = f^{(i)}$ (note that $e \neq f$, since $s(e) \notin Z$). It follows that $r(e) = \tilde{r}(e) = \tilde{r}(\tilde{e}) = \tilde{s}(\tilde{f}) = \tilde{s}(f^{(i)}) = r(f)$. Since $s(f) \in Z = T(r(E^1_w))$, there is a $g \in E^1_w$ such that $s(f) \in T(r(g))$. It follows that $r(f) \in T(r(e)) \cap T(r(g))$. Since (E, w) satisfies Condition (LPA3), we have that e and g are in line and hence $e = g$ or $r(e) \geq s(g)$ or $r(g) \geq s(e)$.

Case 1.1. Assume that $e = g$. Since $s(f) \in T(r(g)) = T(r(e))$, there is a path p such that $s(p) = r(e)$ and $r(p) = s(f)$. Since $r(f) = r(e)$, we have a closed path pf based at $r(e)$. That implies the existence of a cycle c based at $r(e)$. Since (E, w) satisfies (LPA4), e belongs to c and therefore $s(e) \in T(r(e))$. Now we obtain a contradiction $s(e) \in T(r(e)) \subseteq T(r(E^1_w)) = Z$.

Case 1.2. Assume that $r(e) \geq s(g)$. Then there is a path p such that $s(p) = r(e)$ and $r(p) = s(g)$. Since $s(f) \in T(r(g))$, there is a path q such that $s(q) = r(g)$ and $r(q) = s(f)$. Since $r(f) = r(e)$, we have a closed path $pgqf$ based at $r(e)$. Now we can proceed as in Case 1.1 to obtain a contradiction.

Case 1.3. Assume that $r(g) \geq s(e)$. Then, we obtain a contradiction $s(e) \in T(r(g)) \subseteq T(r(E^1_w)) = Z$.

Case 2. Assume that $\tilde{f} \in \tilde{E}^1_{Z^c}$. Then there is an $f \in E^1, s(f) \notin Z$ such that $\tilde{f} = f$. It follows that $r(e) = \tilde{r}(e) = \tilde{r}(\tilde{e}) = \tilde{s}(\tilde{f}) = \tilde{s}(f) = s(f)$. Hence, we obtain a contradiction $s(f) = r(e) \in T(r(E^1_w)) \subseteq Z$.

Thus the ranges of the weighted edges in $(\tilde{E}, \tilde{w})$ are sinks.

Part II. Assume that there are distinct $\tilde{e}, \tilde{f} \in \tilde{E}^1_w$ such that $\tilde{s}(\tilde{e}) = \tilde{s}(\tilde{f})$. Clearly $\tilde{e}, \tilde{f} \in \tilde{E}^1_{Z^c}$ since all the edges in $\tilde{E}^1_Z$ have weight one in $(\tilde{E}, \tilde{w})$. Hence there are distinct $e, f \in E^1, s(e), s(f) \notin Z$ such that $\tilde{e} = e$ and $\tilde{f} = f$. It follows that $s(e) = \tilde{s}(e) = \tilde{s}(\tilde{e}) = \tilde{s}(\tilde{f}) = \tilde{s}(f) = s(f)$ which contradicts the assumption that (E, w) satisfies Condition (LPA1) (note that $w(e) = \tilde{w}(\tilde{e}) > 1$ and $w(f) = \tilde{w}(\tilde{f}) > 1$). Thus no vertex emits two distinct weighted edges in $(\tilde{E}, \tilde{w})$.

Now assume that there are distinct $\tilde{e}, \tilde{f} \in \tilde{E}^1_w$ such that $\tilde{r}(\tilde{e}) = \tilde{r}(\tilde{f})$. Clearly $\tilde{e}, \tilde{f} \in \tilde{E}^1_{Z^c}$ since all the edges in $\tilde{E}^1_Z$ have weight one in $(\tilde{E}, \tilde{w})$. Hence there are distinct $e, f \in E^1, s(e), s(f) \notin Z$ such that $\tilde{e} = e$ and $\tilde{f} = f$. It follows that $r(e) = \tilde{r}(e) = \tilde{r}(\tilde{e}) = \tilde{r}(\tilde{f}) = \tilde{r}(f) = r(f)$. Since (E, w) satisfies Condition (LPA3), we have that e and f are in line. Since e and f are distinct, it follows that $r(e) \geq s(f)$ or $r(f) \geq s(e)$. In the first case we obtain a contradiction $s(f) \in Z$ and in the second case a contradiction $s(e) \in Z$. Thus no vertex receives two distinct weighted edges in $(\tilde{E}, \tilde{w})$.

Part III. It remains to show that $L_K(\tilde{E}, \tilde{w}) \cong L_K(E, w)$. Set $X := \{v, e_i, e_i^* \mid v \in E^0, e \in E^1, 1 \leq i \leq w(e)\}$ and $\tilde{X} := \{\tilde{v}, \tilde{e}_i, \tilde{e}_i^* \mid \tilde{v} \in \tilde{E}^0, \tilde{e} \in \tilde{E}^1, 1 \leq i \leq \tilde{w}(\tilde{e})\}$. Let $K\langle X\rangle$ and $K\langle\tilde{X}\rangle$ be the free K-algebras generated by X and $\tilde{X}$, respectively. Then the bijection $X \to \tilde{X}$ mapping

$$\begin{aligned}
v &\mapsto v && (v \in E^0),\\
e_i &\mapsto (e_1^{(i)})^* && (e \in E^1, s(e) \in Z, 1 \leq i \leq w(e)),\\
e_i^* &\mapsto e_1^{(i)} && (e \in E^1, s(e) \in Z, 1 \leq i \leq w(e)),\\
e_i &\mapsto e_i && (e \in E^1, s(e) \notin Z, 1 \leq i \leq w(e)),\\
e_i^* &\mapsto e_i^* && (e \in E^1, s(e) \notin Z, 1 \leq i \leq w(e))
\end{aligned}$$

induces an isomorphism $\phi : K\langle X\rangle \to K\langle\tilde{X}\rangle$. Let I and $\tilde{I}$ be the ideals of $K\langle X\rangle$ and $K\langle\tilde{X}\rangle$ generated by the relations (i)-(iv) in Definition 3, respectively (hence $L_K(E, w) \cong K\langle X\rangle/I$ and $L_K(\tilde{E}, \tilde{w}) \cong K\langle\tilde{X}\rangle/\tilde{I}$). In order to show that $L_K(E, w) \cong L_K(\tilde{E}, \tilde{w})$ it suffices to show that $\phi(I) = \tilde{I}$. Set

$$A^{(i)} := \{uv - \delta_{uv}u \mid u, v \in E^0\},$$

$$A^{(ii)} := \{s(e)e_i - e_i,\ e_i r(e) - e_i,\ r(e)e_i^* - e_i^*,\ e_i^* s(e) - e_i^* \mid e \in E^1, 1 \leq i \leq w(e)\},$$

and for any $v \in E^0_{\text{reg}}$

$$A_v^{(iii)} := \Big\{ \sum_{1 \leq i \leq w(v)} e_i^* f_i - \delta_{ef} r(e) \mid e, f \in s^{-1}(v)\Big\}$$

and

$$A_v^{(iv)} := \Big\{ \sum_{e \in s^{-1}(v)} e_i e_j^* - \delta_{ij} v \mid 1 \leq i, j \leq w(v)\Big\}.$$

Then I is generated by $A^{(i)}$, $A^{(ii)}$, the $A_v^{(iii)}$'s and the $A_v^{(iv)}$'s (note that relation (iii) in Definition 3 follows from relations (i) and (ii) if $s(e) \neq s(f)$). Analogously define subsets $B^{(i)}$, $B^{(ii)}$, $B_v^{(iii)}$ $(v \in \tilde{E}^0_{\mathrm{reg}})$, $B_v^{(iv)}$ $(v \in \tilde{E}^0_{\mathrm{reg}})$ of $K\langle \tilde{X} \rangle$. Then $\tilde{I}$ is generated by $B^{(i)}$, $B^{(ii)}$, the $B_v^{(iii)}$'s and the $B_v^{(iv)}$'s. Clearly $\phi(A^{(i)}) = B^{(i)}$ and $\phi(A^{(ii)}) = B^{(ii)}$. One directly checks that $\phi(A_v^{(iii)}) = B_v^{(iii)}$ and $\phi(A_v^{(iv)}) = B_v^{(iv)}$ if $v \in E^0_{\mathrm{reg}} \setminus Z = \tilde{E}^0_{\mathrm{reg}} \setminus Z$.

Let now $v \in E^0_{\mathrm{reg}} \cap Z$. Then, we have $s^{-1}(v) = \{e\}$ for some $e \in E^1$ since (E, w) satisfies Condition (LPA2). Set $\bar{v} := r(e)$. Clearly

$$A_v^{(iii)} = \Big\{ \sum_{1 \le i \le w(e)} e_i^* e_i - \bar{v} \Big\}$$

and

$$A_v^{(iv)} = \big\{ e_i e_j^* - \delta_{ij} v \mid 1 \le i, j \le w(e) \big\}.$$

It follows from Lemma 8 that $\tilde{s}^{-1}(\bar{v}) = \{e^{(1)}, \ldots, e^{(w(e))}\}$. Hence

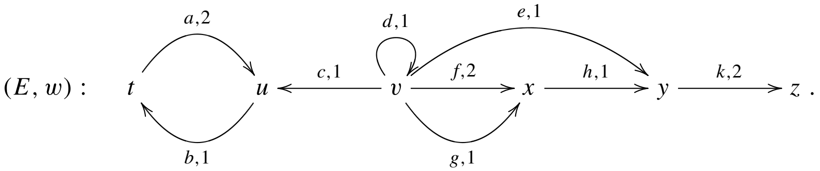
$$B_{\bar{v}}^{(iii)} = \big\{ (e_1^{(i)})^* e_1^{(j)} - \delta_{ij} v \mid 1 \le i, j \le w(e) \big\}$$

and

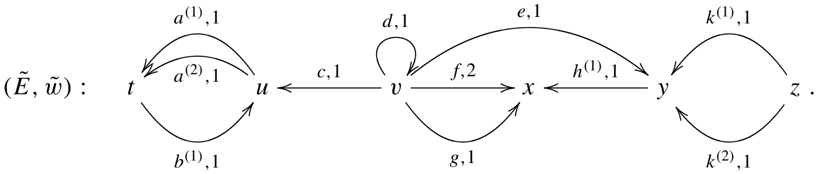
$$B_{\bar{v}}^{(iv)} = \Big\{ \sum_{1 \le i \le w(e)} e_1^{(i)} (e_1^{(i)})^* - \bar{v} \Big\}$$

Clearly $\phi(A_v^{(iii)}) = B_{\bar{v}}^{(iv)}$ and $\phi(A_v^{(iv)}) = B_{\bar{v}}^{(iii)}$. It follows from Lemma 8 that the map $\bar{\ } : v \mapsto \bar{v}$ defines a bijection between $E^0_{\mathrm{reg}} \cap Z$ and $\tilde{E}^0_{\mathrm{reg}} \cap Z$. Hence $\phi(I) = \tilde{I}$ and thus $L_K(\tilde{E}, \tilde{w}) \cong L_K(E, w)$. □

Example 10 Consider the weighted graph



One directly checks that (E, w) satisfies Condition (LPA) (note that $T(r(E^1_w)) = \{t, u, x, y, z\}$). Let $(\tilde{E}, \tilde{w})$ be defined as in the proof of Lemma 9. Then $(\tilde{E}, \tilde{w})$ is the weighted graph



There is only one weighted edge in $(\tilde{E}, \tilde{w})$, namely f, and its range is a sink. The proof of Lemma 9 shows that $L_K(E, w) \cong L_K(\tilde{E}, \tilde{w})$.

Lemma 11 *Let (E, w) be a row-finite weighted graph such that the ranges of the weighted edges are sinks and no vertex emits or receives two distinct weighted edges. Then there is a row-finite graph $\tilde{E}$ such that $L_K(E, w) \cong L_K(\tilde{E})$ as K-algebras.*

Proof If $v \in r(E^1_w)$, then there is a unique edge $g^v \in E^1_w$ such that $r(g^v) = v$ (since no vertex in (E, w) receives two distinct weighted edges). Define a graph $\tilde{E}$ by

$$\begin{aligned}
\tilde{E}^0 &= M \sqcup N \text{ where}\\
M &= E^0 \setminus r(E^1_w),\\
N &= \{v^{(1)}, \dots, v^{(w(g^v))} \mid v \in r(E^1_w)\},\\
\tilde{E}^1 &= A \sqcup B \sqcup C \sqcup D \text{ where}\\
A &= \{e \mid e \in E^1_{uw}, r(e) \notin r(E^1_w)\},\\
B &= \{e^{(1)}, \dots, e^{(w(g^{r(e)}))} \mid e \in E^1_{uw}, r(e) \in r(E^1_w)\},\\
C &= \{e^{(1)} \mid e \in E^1_w\},\\
D &= \{e^{(2)}, \dots, e^{(w(e))} \mid e \in E^1_w\},\\
\tilde{s}(e) &= s(e),\ \tilde{r}(e) = r(e) \quad (e \in A),\\
\tilde{s}(e^{(i)}) &= s(e),\ \tilde{r}(e^{(i)}) = r(e)^{(i)} \quad (e^{(i)} \in B),\\
\tilde{s}(e^{(1)}) &= s(e),\ \tilde{r}(e^{(1)}) = r(e)^{(1)} \quad (e^{(1)} \in C),\\
\tilde{s}(e^{(i)}) &= r(e)^{(i)},\ \tilde{r}(e^{(i)}) = s(e) \quad (e^{(i)} \in D),
\end{aligned}$$

(note that if $e \in E^1$, then $s(e) \in E^0 \setminus r(E^1_w)$ since the elements of $r(E^1_w)$ are sinks). Clearly $\tilde{E}$ is row-finite. We have divided the rest of the proof into three parts. In Part I we define a homomorphism $\phi : L_K(E, w) \to L_K(\tilde{E})$, in Part II we define a homomorphism $\tilde{\phi} : L_K(\tilde{E}) \to L_K(E, w)$, and in Part III we show that ϕ and $\tilde{\phi}$ are inverse to each other.

Part I. Set

$$\alpha_v := \begin{cases} v, & \text{if } v \notin r(E^1_w),\\ \sum\limits_{i=1}^{w(g^v)} v^{(i)}, & \text{if } v \in r(E^1_w), \end{cases}$$

$$\beta_{e,i} := \begin{cases} e, & \text{if } e \in E^1_{uw}, r(e) \notin r(E^1_w), i = 1,\\ \sum\limits_{j=1}^{w(g^{r(e)})} e^{(j)}, & \text{if } e \in E^1_{uw}, r(e) \in r(E^1_w), i = 1,\\ e^{(1)}, & \text{if } e \in E^1_w, i = 1,\\ (e^{(i)})^*, & \text{if } e \in E^1_w, i > 1, \end{cases}$$

$$\gamma_{e,i} := \begin{cases} e^*, & \text{if } e \in E^1_{uw}, r(e) \notin r(E^1_w), i = 1,\\ \sum\limits_{j=1}^{w(g^{r(e)})} (e^{(j)})^*, & \text{if } e \in E^1_{uw}, r(e) \in r(E^1_w), i = 1,\\ (e^{(1)})^*, & \text{if } e \in E^1_w, i = 1,\\ e^{(i)}, & \text{if } e \in E^1_w, i > 1. \end{cases}$$

In order to show that $X := \{\alpha_v, \beta_{e,i}, \gamma_{e_i} \mid v \in E^0, e \in E^1, 1 \le i \le w(e)\}$ is an (E, w)-family in $L_K(\tilde{E})$, one has to show that the relations (i)-(iv) below Example 7 are satisfied. We leave (i) and (ii) to the reader and show (iii) and (iv).

First we check (iii). Let $v \in E^0$ and $e, f \in s^{-1}(v)$ (for $e, f \in E^1$ such that $s(e) \neq s(f)$ relation (iii) follows from relations (i) and (ii)). We have to show that $\sum\limits_{1 \le i \le w(v)} \gamma_{e,i}\beta_{f,i} = \delta_{ef}\alpha_{r(e)}$.

Case 1. Assume that $e, f \in E^1_{uw}$.

Case 1.1. Assume that $r(e), r(f) \notin r(E^1_w)$. Then

$$\sum_{1\leq i\leq w(v)} \gamma_{e,i}\beta_{f,i} = e^* f = \delta_{ef}\tilde{r}(e) = \delta_{ef} r(e) = \delta_{ef}\alpha_{r(e)}.$$

Case 1.2. Assume that $r(e) \notin r(E^1_w)$ and $r(f) \in r(E^1_w)$. Then

$$\sum_{1\leq i\leq w(v)} \gamma_{e,i}\beta_{f,i} = e^* \sum_{j=1}^{w(g^{r(f)})} f^{(j)} = \sum_{j=1}^{w(g^{r(f)})} e^* f^{(j)} = 0 = \delta_{ef}\alpha_{r(e)}.$$

Case 1.3. Assume that $r(e) \in r(E^1_w)$ and $r(f) \notin r(E^1_w)$. Then

$$\sum_{1\leq i\leq w(v)} \gamma_{e,i}\beta_{f,i} = \sum_{j=1}^{w(g^{r(e)})} (e^{(j)})^* f = 0 = \delta_{ef}\alpha_{r(e)}.$$

Case 1.4. Assume that $r(e), r(f) \in r(E^1_w)$. Then

$$\begin{aligned}\sum_{1\leq i\leq w(v)} \gamma_{e,i}\beta_{f,i} &= \sum_{j=1}^{w(g^{r(e)})} (e^{(j)})^* \sum_{k=1}^{w(g^{r(f)})} f^{(k)} \\ &= \sum_{j=1}^{w(g^{r(e)})} \sum_{k=1}^{w(g^{r(f)})} (e^{(j)})^* f^{(k)} = \delta_{ef} \sum_{j=1}^{w(g^{r(e)})} r(e)^{(j)} \\ &= \delta_{ef}\alpha_{r(e)}.\end{aligned}$$

Case 2. Assume that $e \in E^1_{uw}$ and $f \in E^1_w$.

Case 2.1. Assume that $r(e) \notin r(E^1_w)$. Then

$$\sum_{1\leq i\leq w(v)} \gamma_{e,i}\beta_{f,i} = e^* f^{(1)} = 0 = \delta_{ef}\alpha_{r(e)}.$$

Case 2.2. Assume that $r(e) \in r(E^1_w)$. Then

$$\sum_{1\leq i\leq w(v)} \gamma_{e,i}\beta_{f,i} = \sum_{j=1}^{w(g^{r(e)})} (e^{(j)})^* f^{(1)} = 0 = \delta_{ef}\alpha_{r(e)}.$$

Case 3. Assume that $e \in E^1_w$ and $f \in E^1_{uw}$. This case is similar to Case 2 and therefore is ommitted.

Case 4. Assume that $e, f \in E^1_w$. Since no vertex emits two distinct weighted edges in (E, w), it follows that $e = f$ and $w(v) = w(e)$. Clearly

$$\sum_{1\leq i\leq w(v)} \gamma_{e,i}\beta_{f,i} = (e^{(1)})^* e^{(1)} + \sum_{j=2}^{w(e)} e^{(j)}(e^{(j)})^* = r(e)^{(1)} + \sum_{j=2}^{w(e)} r(e)^{(j)} = \delta_{ef}\alpha_{r(e)}$$

(note that $e^{(j)}$ is the only edge emitted by $r(e)^{(j)}$ in $\tilde{E}$).

Thus (iii) holds.

Next we check (iv). Let $v \in E^0_{\text{reg}}$ and $1 \le i, j \le w(v)$. It follows that $v \in E^0 \setminus r(E^1_w)$. We have to show that $\sum_{e \in s^{-1}(v)} \beta_{e,i}\gamma_{e,j} = \delta_{ij}\alpha_v$.

Case (a). Assume that $i = j = 1$. Clearly

$$\begin{aligned}
&\sum_{e \in s^{-1}(v)} \beta_{e,1}\gamma_{e,1}\\
&= \sum_{\substack{e \in s^{-1}(v)\cap E^1_{uw},\\ r(e)\notin r(E^1_w)}} \beta_{e,1}\gamma_{e,1} + \sum_{\substack{e \in s^{-1}(v)\cap E^1_{uw},\\ r(e)\in r(E^1_w)}} \beta_{e,1}\gamma_{e,1} + \sum_{e \in s^{-1}(v)\cap E^1_{w}} \beta_{e,1}\gamma_{e,1}\\
&= \sum_{\substack{e \in s^{-1}(v)\cap E^1_{uw},\\ r(e)\notin r(E^1_w)}} ee^* + \sum_{\substack{e \in s^{-1}(v)\cap E^1_{uw},\\ r(e)\in r(E^1_w)}} \sum_{j=1}^{w(g^{r(e)})} e^{(j)} \sum_{k=1}^{w(g^{r(e)})} (e^{(k)})^*\\
&\quad + \sum_{e \in s^{-1}(v)\cap E^1_{w}} e^{(1)}(e^{(1)})^*\\
&= \underbrace{\sum_{\substack{e \in s^{-1}(v)\cap E^1_{uw},\\ r(e)\notin r(E^1_w)}} ee^* + \sum_{\substack{e \in s^{-1}(v)\cap E^1_{uw},\\ r(e)\in r(E^1_w)}} \sum_{j,k=1}^{w(g^{r(e)})} e^{(j)}(e^{(k)})^* + \sum_{e \in s^{-1}(v)\cap E^1_{w}} e^{(1)}(e^{(1)})^*}_{T_1:=}.
\end{aligned}$$

Since $\tilde{r}(e^{(j)}) = r(e)^{(j)}$ for any $e \in E^1_{uw}, r(e) \in r(E^1_w)$, we have $e^{(j)}(e^{(k)})^* = 0$ in $L_K(\tilde{E})$ whenever $j \neq k$. Hence

$$T_1 = \underbrace{\sum_{\substack{e \in s^{-1}(v)\cap E^1_{uw},\\ r(e)\notin r(E^1_w)}} ee^* + \sum_{\substack{e \in s^{-1}(v)\cap E^1_{uw},\\ r(e)\in r(E^1_w)}} \sum_{j=1}^{w(g^{r(e)})} e^{(j)}(e^{(j)})^* + \sum_{e \in s^{-1}(v)\cap E^1_{w}} e^{(1)}(e^{(1)})^*}_{T_2:=}.$$

One directly checks that

$$\begin{aligned}
\tilde{s}^{-1}(v) = &\{e \mid e \in s^{-1}(v) \cap E^1_{uw}, r(e) \notin r(E^1_w)\}\\
&\sqcup \{e^{(j)} \mid e \in s^{-1}(v) \cap E^1_{uw}, r(e) \in r(E^1_w), 1 \le j \le w(g^{r(e)})\}\\
&\sqcup \{e^{(1)} \mid e \in s^{-1}(v) \cap E^1_{w}\}.
\end{aligned}$$

Hence $T_2 = v = \delta_{11}\alpha_v$.

Case (b). Assume that $i = 1$ and $j > 1$. Then $w(v) \ge j > 1$ and hence v emits precisely one weighted edge f. Since $\gamma_{e,j} = 0$ whenever $j \ge w(e)$, we have

$$\sum_{e \in s^{-1}(v)} \beta_{e,1}\gamma_{e,j} = \beta_{f,1}\gamma_{f,j} = f^{(1)}f^{(j)} = 0 = \delta_{1j}\alpha_v$$

(note that $\tilde{r}(f^{(1)}) = r(f)^{(1)} \neq r(f)^{(j)} = \tilde{s}(f^{(j)})$).

Case (c). Assume that $i > 1$ and $j = 1$. Then v emits precisely one weighted edge f. Clearly

$$\sum_{e \in s^{-1}(v)} \beta_{e,i}\gamma_{e,1} = \beta_{f,i}\gamma_{f,1} = (f^{(i)})^*(f^{(1)})^* = 0 = \delta_{i1}\alpha_v$$

(note that $\tilde{s}(f^{(i)}) = r(f)^{(i)} \neq r(f)^{(1)} = \tilde{r}(f^{(1)})$).

Case (d). Assume that $i, j > 1$. Then v emits precisely one weighted edge f. Clearly

$$\sum_{e \in s^{-1}(v)} \beta_{e,i}\gamma_{e,j} = \beta_{f,i}\gamma_{f,j} = (f^{(i)})^* f^{(j)} = \delta_{ij}\tilde{r}(f^{(i)}) = \delta_{ij}v = \delta_{ij}\alpha_v.$$

Thus (iv) holds too and hence X is an (E, w)-family in $L_K(\tilde{E})$. By the Universal Property of $L_K(E, w)$ there is a unique K-algebra homomorphism $\phi : L_K(E, w) \to L_K(\tilde{E})$ such that $\phi(v) = \alpha_v$, $\phi(e_i) = \beta_{e,i}$ and $\phi(e_i^*) = \gamma_{e,i}$ for all $v \in E^0$, $e \in E^1$ and $1 \leq i \leq w(e)$.

Part II. Set

$$\tilde{\alpha}_{\tilde{v}} := \begin{cases} v, & \text{if } \tilde{v} = v \in M, \\ (g_i^v)^* g_i^v, & \text{if } \tilde{v} = v^{(i)} \in N, \end{cases}$$

$$\tilde{\beta}_{\tilde{e}} := \begin{cases} e_1, & \text{if } \tilde{e} = e \in A, \\ e_1(g_i^{r(e)})^* g_i^{r(e)}, & \text{if } \tilde{e} = e^{(i)} \in B, \\ e_1, & \text{if } \tilde{e} = e^{(1)} \in C, \\ e_i^*, & \text{if } \tilde{e} = e^{(i)} \in D, \end{cases}$$

$$\tilde{\gamma}_{\tilde{e}} := \begin{cases} e_1^*, & \text{if } \tilde{e} = e \in A, \\ (g_i^{r(e)})^* g_i^{r(e)} e_1^*, & \text{if } \tilde{e} = e^{(i)} \in B, \\ e_1^*, & \text{if } \tilde{e} = e^{(1)} \in C, \\ e_i, & \text{if } \tilde{e} = e^{(i)} \in D. \end{cases}$$

In order to show that $\tilde{X} := \{\tilde{\alpha}_{\tilde{v}}, \tilde{\beta}_{\tilde{e}}, \tilde{\gamma}_{\tilde{e}} \mid \tilde{v} \in \tilde{E}^0, \tilde{e} \in \tilde{E}^1\}$ is an $\tilde{E}$-family in $L_K(E, w)$, one has to show that the relations (i)-(iv) below Definition 4 are satisfied. We leave (i) and (ii) to the reader and show (iii) and (iv).

First we check (iii). Let $\tilde{v} \in \tilde{E}^0$ and $\tilde{e}, \tilde{f} \in \tilde{s}^{-1}(\tilde{v})$ (for $\tilde{e}, \tilde{f} \in \tilde{E}^1$ such that $\tilde{s}(\tilde{e}) \neq \tilde{s}(\tilde{f})$ relation (iii) follows from relations (i) and (ii)). We have to show that $\tilde{\gamma}_{\tilde{e}}\tilde{\beta}_{\tilde{f}} = \delta_{\tilde{e}\tilde{f}}\tilde{\alpha}_{\tilde{r}(\tilde{e})}$.

Case 1. Assume that $\tilde{v} \in M$. Then $\tilde{e}, \tilde{f} \in A \cup B \cup C$ since $\tilde{s}^{-1}(D) \subseteq N$.

Case 1.1. Assume that $\tilde{e}, \tilde{f} \in A$. Then there are $e, f \in E^1_{uw}, r(e), r(f) \notin r(E^1_w)$ such that $\tilde{e} = e$ and $\tilde{f} = f$. Clearly $\tilde{\gamma}_{\tilde{e}}\tilde{\beta}_{\tilde{f}} = e_1^* f_1 = \delta_{ef} r(e) = \delta_{\tilde{e}\tilde{f}}\tilde{\alpha}_{\tilde{r}(\tilde{e})}$.

Case 1.2. Assume that $\tilde{e} \in A$ and $\tilde{f} \in B$. Then there is an $e \in E^1_{uw}, r(e) \notin r(E^1_w)$ such that $\tilde{e} = e$. Moreover, there is an $f \in E^1_{uw}, r(f) \in r(E^1_w)$ and an $1 \leq i \leq w(g^{r(f)})$ such that $\tilde{f} = f^{(i)}$. Clearly $e \neq f$ and $\tilde{e} \neq \tilde{f}$. Hence $\tilde{\gamma}_{\tilde{e}}\tilde{\beta}_{\tilde{f}} = e_1^* f_1 (g_i^{r(f)})^* g_i^{r(f)} = \delta_{ef}(g_i^{r(f)})^* g_i^{r(f)} = 0 = \delta_{\tilde{e}\tilde{f}}\tilde{\alpha}_{\tilde{r}(\tilde{e})}$.

Case 1.3. Assume that $\tilde{e} \in A$ and $\tilde{f} \in C$. Then there is an $e \in E^1_{uw}, r(e) \notin r(E^1_w)$ such that $\tilde{e} = e$. Moreover, there is an $f \in E^1_w$ such that $\tilde{f} = f^{(1)}$. Clearly $e \neq f$ and $\tilde{e} \neq \tilde{f}$. Hence $\tilde{\gamma}_{\tilde{e}}\tilde{\beta}_{\tilde{f}} = e_1^* f_1 = \delta_{ef} r(e) = 0 = \delta_{\tilde{e}\tilde{f}}\tilde{\alpha}_{\tilde{r}(\tilde{e})}$.

Case 1.4. Assume that $\tilde{e} \in B$ and $\tilde{f} \in A$. Then there is an $e \in E^1_{uw}, r(e) \in r(E^1_w)$ and an $1 \leq i \leq w(g^{r(e)})$ such that $\tilde{e} = e^{(i)}$. Moreover, there there is an $f \in E^1_{uw}, r(f) \notin r(E^1_w)$ such that $\tilde{f} = f$. Clearly $e \neq f$ and $\tilde{e} \neq \tilde{f}$. Hence $\tilde{\gamma}_{\tilde{e}}\tilde{\beta}_{\tilde{f}} = (g_i^{r(e)})^* g_i^{r(e)} e_1^* f_1 = \delta_{ef}(g_i^{r(e)})^* g_i^{r(e)} = 0 = \delta_{\tilde{e}\tilde{f}}\tilde{\alpha}_{\tilde{r}(\tilde{e})}$.

Case 1.5. Assume that $\tilde{e}, \tilde{f} \in B$. Then there are $e, f \in E^1_{uw}, r(e), r(f) \in r(E^1_w)$ and $1 \leq i \leq w(g^{r(e)}), 1 \leq j \leq w(g^{r(f)})$ such that $\tilde{e} = e^{(i)}$ and $\tilde{f} = f^{(j)}$. Clearly $\tilde{\gamma}_{\tilde{e}}\tilde{\beta}_{\tilde{f}} = (g_i^{r(e)})^* g_i^{r(e)} e_1^* f_1 (g_j^{r(f)})^* g_j^{r(f)} = \delta_{ef}(g_i^{r(e)})^* g_i^{r(e)} (g_j^{r(f)})^* g_j^{r(f)} = \delta_{ef}\delta_{ij}(g_i^{r(e)})^* g_i^{r(e)} = \delta_{\tilde{e}\tilde{f}}\tilde{\alpha}_{\tilde{r}(\tilde{e})}$.

Case 1.6. Assume that $\tilde{e} \in B$ and $\tilde{f} \in C$. Then there is an $e \in E^1_{uw}, r(e) \in r(E^1_w)$ and a $1 \leq i \leq w(g^{r(e)})$ such that $\tilde{e} = e^{(i)}$. Moreover, there is an $f \in E^1_w$ such that $\tilde{f} = f^{(1)}$. Clearly $e \neq f$ and $\tilde{e} \neq \tilde{f}$. Hence $\tilde{\gamma}_{\tilde{e}}\tilde{\beta}_{\tilde{f}} = (g_i^{r(e)})^* g_i^{r(e)} e_1^* f_1 = \delta_{ef}(g_i^{r(e)})^* g_i^{r(e)} = 0 = \delta_{\tilde{e}\tilde{f}}\tilde{\alpha}_{\tilde{r}(\tilde{e})}$.

Case 1.7. Assume that $\tilde{e} \in C$ and $\tilde{f} \in A$. Then there is an $e \in E^1_w$ such that $\tilde{e} = e^{(1)}$. Moreover, there is an $f \in E^1_{uw}, r(f) \notin r(E^1_w)$ such that $\tilde{f} = f$. Clearly $e \neq f$ and $\tilde{e} \neq \tilde{f}$. Hence $\tilde{\gamma}_{\tilde{e}}\tilde{\beta}_{\tilde{f}} = e_1^* f_1 = \delta_{ef} r(e) = 0 = \delta_{\tilde{e}\tilde{f}}\tilde{\alpha}_{\tilde{r}(\tilde{e})}$.

Case 1.8. Assume that $\tilde{e} \in C$ and $\tilde{f} \in B$. Then there is an $e \in E^1_w$ such that $\tilde{e} = e^{(1)}$. Moreover, there is an $f \in E^1_{uw}, r(f) \in r(E^1_w)$ and an $1 \leq i \leq w(g^{r(f)})$ such that $\tilde{f} = f^{(i)}$. Clearly $e \neq f$ and $\tilde{e} \neq \tilde{f}$. Hence $\tilde{\gamma}_{\tilde{e}}\tilde{\beta}_{\tilde{f}} = e_1^* f_1 (g_i^{r(f)})^* g_i^{r(f)} = \delta_{ef}(g_i^{r(f)})^* g_i^{r(f)} = 0 = \delta_{\tilde{e}\tilde{f}}\tilde{\alpha}_{\tilde{r}(\tilde{e})}$.

Case 1.9. Assume that $\tilde{e}, \tilde{f} \in C$. Then there are $e, f \in E^1_w$ such that $\tilde{e} = e^{(1)}$ and $\tilde{f} = f^{(1)}$. Since $s(e) = \tilde{s}(e^{(1)}) = \tilde{s}(\tilde{e}) = \tilde{s}(\tilde{f}) = \tilde{s}(f^{(1)}) = s(f)$, we have $e = f$ (because no vertex in (E, w) emits two distinct weighted edges). It follows that $\tilde{e} = \tilde{f}$. Clearly $\tilde{\gamma}_{\tilde{e}}\tilde{\beta}_{\tilde{f}} = e_1^* e_1 = \delta_{\tilde{e}\tilde{f}}\tilde{\alpha}_{\tilde{r}(\tilde{e})}$.

Case 2. Assume that $\tilde{v} \in N$. Then $\tilde{v} = v^{(i)}$ for some $v \in r(E^1_w)$ and $1 \leq i \leq w(g^v)$. One directly checks that $\tilde{s}^{-1}(\tilde{v}) = \emptyset$ if $i = 1$ and $\tilde{s}^{-1}(\tilde{v}) = \{(g^v)^{(i)}\}$ if $i > 1$. It follows that $i > 1$ and $\tilde{e} = \tilde{f} = (g^v)^{(i)}$. Hence $\tilde{\gamma}_{\tilde{e}}\tilde{\beta}_{\tilde{f}} = g_i^v (g_i^v)^* = s(g^v) = \delta_{\tilde{e}\tilde{f}}\tilde{\alpha}_{\tilde{r}(\tilde{e})}$.

Thus (iii) holds.

Next we check (iv). Let $\tilde{v} \in \tilde{E}^0_{\text{reg}}$. We have to show that $\sum\limits_{\tilde{e} \in \tilde{s}^{-1}(\tilde{v})} \tilde{\beta}_{\tilde{e}}\tilde{\gamma}_{\tilde{e}} = \tilde{\alpha}_{\tilde{v}}$.

Case (a). Assume that $\tilde{v} \in M$. Then $\tilde{v} = v$ for some $v \in E^0 \setminus r(E^1_w)$. One directly checks that

$$\begin{aligned}\tilde{s}^{-1}(\tilde{v}) = &\{e \mid e \in s^{-1}(v) \cap E^1_{uw}, r(e) \notin r(E^1_w)\}\\ &\sqcup\{e^{(i)} \mid e \in s^{-1}(v) \cap E^1_{uw}, r(e) \in r(E^1_w), 1 \leq i \leq w(g^{r(e)})\}\\ &\sqcup\{e^{(1)} \mid e \in s^{-1}(v) \cap E^1_w\}.\end{aligned}$$

Hence

$$\sum_{\tilde e\in\tilde s^{-1}(\tilde v)}\tilde\beta_{\tilde e}\tilde\gamma_{\tilde e}$$

$$= \sum_{\substack{e\in s^{-1}(v)\cap E^1_{uw},\\ r(e)\notin r(E^1_w)}}\tilde\beta_e\tilde\gamma_e + \sum_{\substack{e\in s^{-1}(v)\cap E^1_{uw},\\ r(e)\in r(E^1_w)}}\sum_{i=1}^{w(g^{r(e)})}\tilde\beta_{e^{(i)}}\tilde\gamma_{e^{(i)}} + \sum_{e\in s^{-1}(v)\cap E^1_w}\tilde\beta_{e^{(1)}}\tilde\gamma_{e^{(1)}}$$

$$= \sum_{\substack{e\in s^{-1}(v)\cap E^1_{uw},\\ r(e)\notin r(E^1_w)}} e_1e_1^* + \sum_{\substack{e\in s^{-1}(v)\cap E^1_{uw},\\ r(e)\in r(E^1_w)}}\sum_{i=1}^{w(g^{r(e)})} e_1(g_i^{r(e)})^*g_i^{r(e)}(g_i^{r(e)})^*g_i^{r(e)}e_1^*$$
$$+ \sum_{e\in s^{-1}(v)\cap E^1_w} e_1e_1^*$$

$$= \sum_{\substack{e\in s^{-1}(v)\cap E^1_{uw},\\ r(e)\notin r(E^1_w)}} e_1e_1^* + \sum_{\substack{e\in s^{-1}(v)\cap E^1_{uw},\\ r(e)\in r(E^1_w)}}\sum_{i=1}^{w(g^{r(e)})} e_1(g_i^{r(e)})^*g_i^{r(e)}e_1^* + \sum_{e\in s^{-1}(v)\cap E^1_w} e_1e_1^*$$

$$= \sum_{\substack{e\in s^{-1}(v)\cap E^1_{uw},\\ r(e)\notin r(E^1_w)}} e_1e_1^* + \sum_{\substack{e\in s^{-1}(v)\cap E^1_{uw},\\ r(e)\in r(E^1_w)}} e_1\Big(\sum_{i=1}^{w(g^{r(e)})}(g_i^{r(e)})^*g_i^{r(e)}\Big)e_1^*$$
$$+ \sum_{e\in s^{-1}(v)\cap E^1_w} e_1e_1^*$$

$$= \sum_{\substack{e\in s^{-1}(v)\cap E^1_{uw},\\ r(e)\notin r(E^1_w)}} e_1e_1^* + \sum_{\substack{e\in s^{-1}(v)\cap E^1_{uw},\\ r(e)\in r(E^1_w)}} e_1e_1^* + \sum_{e\in s^{-1}(v)\cap E^1_w} e_1e_1^*$$
$$= v = \tilde\alpha_{\tilde v}.$$

Case (b). Assume that $\tilde v\in N$. Then $\tilde v = v^{(i)}$ for some $v\in r(E^1_w)$ and $1\le i\le w(g^v)$. As mentioned above we have $\tilde s^{-1}(\tilde v)=\emptyset$ if $i=1$ and $\tilde s^{-1}(\tilde v)=\{(g^v)^{(i)}\}$ if $i>1$. Since by assumption $\tilde s^{-1}(\tilde v)\neq\emptyset$, it follows that $i>1$ and $\sum_{\tilde e\in\tilde s^{-1}(\tilde v)}\tilde\beta_{\tilde e}\tilde\gamma_{\tilde e} =$

$$\tilde\beta_{(g^v)^{(i)}}\tilde\gamma_{(g^v)^{(i)}} = (g_i^v)^*g_i^v = \tilde\alpha_{\tilde v}.$$

Thus (iv) holds too and hence $\tilde X$ is an $\tilde E$-family in $L_K(E,w)$. By the Universal Property of $L_K(\tilde E)$ there is a unique K-algebra homomorphism $\tilde\phi : L_K(\tilde E)\to L_K(E,w)$ such that $\tilde\phi(\tilde v)=\tilde\alpha_{\tilde v}$, $\tilde\phi(\tilde e)=\tilde\beta_{\tilde e}$ and $\tilde\phi(\tilde e^*)=\tilde\gamma_{\tilde e}$ for all $\tilde v\in\tilde E^0$ and $\tilde e\in\tilde E^1$.

Part III. First we show that $\tilde\phi\circ\phi=\mathrm{id}_{L_K(E,w)}$. Clearly it suffices to show that $\tilde\phi\circ\phi$ fixes all elements of $\{v, e_i, e_i^* \mid v\in E^0, e\in E^1, 1\le i\le w(e)\}$ since these elements generate $L_K(E,w)$ as a K-algebra. One directly checks that $\tilde\phi\circ\phi$ fixes all elements v, e_i, e_i^* where $v\in E^0$ and $e\in E^1_w$ or $e\in E^1_{uw}, r(e)\notin r(E^1_w)$. Let now $e\in E^1_{uw}, r(e)\in r(E^1_w)$. Then

$$\tilde\phi(\phi(e_1)) = \tilde\phi\Big(\sum_{j=1}^{w(g^{r(e)})} e^{(j)}\Big) = \sum_{j=1}^{w(g^{r(e)})} e_1(g_j^{r(e)})^*g_j^{r(e)} = e_1\sum_{j=1}^{w(g^{r(e)})}(g_j^{r(e)})^*g_j^{r(e)} = e_1r(e) = e_1.$$

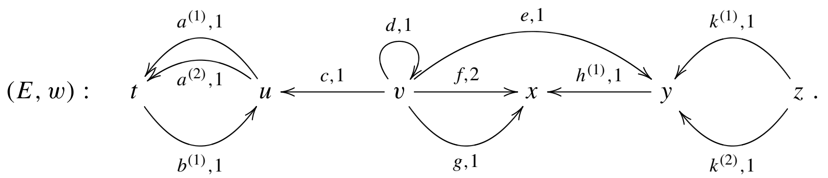
Similarly one can show that $\phi(\phi(e_1^*)) = e_1^*$ in this case. Hence $\tilde{\phi} \circ \phi = \mathrm{id}_{L_K(E,w)}$.

Now we show that $\phi\circ\tilde{\phi} = \mathrm{id}_{L_K(\tilde{E})}$. Clearly it suffices to show that $\phi\circ\tilde{\phi}$ fixes all elements of $\{\tilde{v}, \tilde{e}, \tilde{e}^* \mid \tilde{v} \in \tilde{E}^0, \tilde{e} \in \tilde{E}^1\}$ since these elements generate $L_K(\tilde{E})$ as a K-algebra. One directly checks that $\phi \circ \tilde{\phi}$ fixes all elements $\tilde{v}, \tilde{e}, \tilde{e}^*$ where $\tilde{v} \in \tilde{E}^0$ and $\tilde{e} \in \tilde{E}^1 \setminus B$. Let now $\tilde{e} \in B$. Then $\tilde{e} = e^{(i)}$ for some $e \in E^1_{uw}$, $r(e) \in r(E^1_w)$ and $1 \leq i \leq w(g^{r(e)})$. Clearly

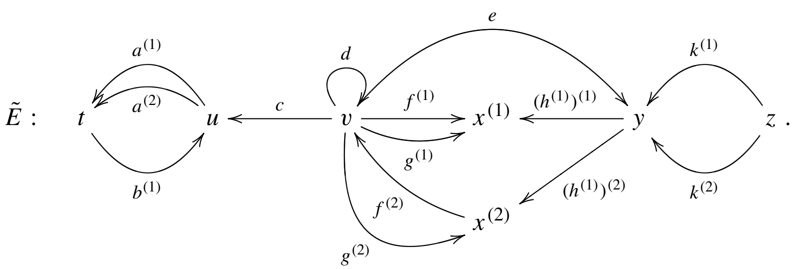
$$\phi(\tilde{\phi}(\tilde{e})) = \tilde{\phi}(e_1(g_i^{r(e)})^* g_i^{r(e)}) = \begin{cases} \sum\limits_{j=1}^{w(g^{r(e)})} e^{(j)}((g^{r(e)})^{(1)})^*(g^{r(e)})^{(1)}, & \text{if } i = 1, \\ \sum\limits_{j=1}^{w(g^{r(e)})} e^{(j)}(g^{r(e)})^{(i)}((g^{r(e)})^{(i)})^*, & \text{if } i > 1. \end{cases}$$

But $((g^{r(e)})^{(1)})^*(g^{r(e)})^{(1)} = \tilde{r}((g^{r(e)})^{(1)}) = r(g^{r(e)})^{(1)} = r(e)^{(1)}$ in $L_K(\tilde{E})$. Since $\tilde{r}(e^{(j)}) = r(e)^{(j)}$, it follows that $\sum\limits_{j=1}^{w(g^{r(e)})} e^{(j)}((g^{r(e)})^{(1)})^*(g^{r(e)})^{(1)} = e^{(1)} = \tilde{e}$ if $i = 1$. Now assume that $i > 1$. One directly checks that $\tilde{s}^{-1}(r(e)^{(i)}) = \{(g^{r(e)})^{(i)}\}$. Hence $(g^{r(e)})^{(i)}((g^{r(e)})^{(i)})^* = r(e)^{(i)}$ in $L_K(\tilde{E})$. Since $\tilde{r}(e^{(j)}) = r(e)^{(j)}$, it follows that $\sum\limits_{j=1}^{w(g^{r(e)})} e^{(j)}(g^{r(e)})^{(i)}((g^{r(e)})^{(i)})^* = e^{(i)} = \tilde{e}$. Hence, we have shown that $\phi(\tilde{\phi}(\tilde{e})) = \tilde{e}$ if $\tilde{e} \in B$. Similarly one can show that $\phi(\tilde{\phi}(\tilde{e}^*)) = \tilde{e}^*$ in this case. Hence $\phi \circ \tilde{\phi} = \mathrm{id}_{L_K(\tilde{E})}$ and thus $L_K(E, w) \cong L_K(\tilde{E})$. □

Example 12 Consider the weighted graph



Let $\tilde{E}$ be defined as in the proof of Lemma 11. Then $\tilde{E}$ is the graph



The proof of Lemma 11 shows that $L_K(E, w) \cong L_K(\tilde{E})$.

Lemmas 9 and 11 directly imply Theorem 1.

4 Abscence of Condition (LPA)

Throughout this subsection (E, w) denotes a row-finite weighted graph. We start by recalling the basis result of [8]. Set $X := \{v, e_i, e_i^* \mid v \in E^0, e \in E^1, 1 \le i \le w(e)\}$, let $\langle X \rangle$ the set of all nonempty words over X and set $\overline{\langle X \rangle} := \langle X \rangle \cup \{\text{empty word}\}$. Together with juxtaposition of words $\langle X \rangle$ becomes a semigroup and $\overline{\langle X \rangle}$ a monoid. If $A, B \in \overline{\langle X \rangle}$, then B is called a *subword of* A if there are $C, D \in \overline{\langle X \rangle}$ such that $A = CBD$ and a *suffix of* A if there is a $C \in \overline{\langle X \rangle}$ such that $A = CB$.

Definition 13 Let $p = x_1 \dots x_n \in \langle X \rangle$. Then p is called *a d-path* if either $x_1, \dots, x_n \in X \setminus E^0$ and $r(x_i) = s(x_{i+1})$ $(1 \le i \le n-1)$ or $x_1 \in E^0$ and $n = 1$. Here we use the convention $s(v) := v$, $r(v) := v$, $s(e_i) := s(e)$, $r(e_i) := r(e)$, $s(e_i^*) := r(e)$ and $r(e_i^*) := s(e)$ for any $v \in E^0$, $e \in E^1$ and $1 \le i \le w(e)$.

Remark 14 Let $\hat{E}$ be the directed graph associated to (E, w) and $\hat{E}_d$ the double graph of $\hat{E}$ (see [15, Definitions 2 and 8]). The d-paths are precisely the paths in the double graph $\hat{E}_d$.

For any $v \in E^0_{\text{reg}}$, fix an edge $e^v \in s^{-1}(v)$ such that $w(e^v) = w(v)$. The e^v's are called *special edges*.

Definition 15 The words $e_i^v (e_j^v)^*$ $(v \in E^0_{\text{reg}}, 1 \le i, j \le w(v))$ and $e_1^* f_1$ $(e, f \in E^1)$ in $\langle X \rangle$ are called *forbidden*. A *normal d-path* or *nod-path* is a d-path p such that none of its subwords is forbidden.

Let $K\langle X \rangle$ the free K-algebra generated by X (i.e. the K-vector space with basis $\langle X \rangle$ which becomes a K-algebra by linearly extending the juxtaposition of words). Then $L_K(E, w)$ is the quotient of $K\langle X \rangle$ by the ideal generated by the relations (i)-(iv) in Definition 3. Let $K\langle X \rangle_{\text{nod}}$ be the linear subspace of $K\langle X \rangle$ spanned by the nod-paths.

Theorem 16 ([8, Theorem 16]) *The canonical map $K\langle X \rangle_{\text{nod}} \to L_K(E, w)$ is an isomorphism of K-vector spaces. In particular the images of the nod-paths under this map form a linear basis for $L_K(E, w)$.*

The following lemma will be used in the proofs of Theorems 2 and 3.

Lemma 17 *Suppose that (E, w) does not satisfy Condition (LPA). Then there is a nod-path whose first letter is e_2 and whose last letter is e_2^* for some $e \in E^1_w$.*

Proof [15, Proof of Lemma 35] shows that if one of the Conditions (LPA1), (LPA2) and (LPA3) is not satisfied, then then there is a nod-path whose first letter is e_2 and whose last letter is e_2^* for some $e \in E^1_w$. Assume now that (E, w) does not satisfy Condition (LPA4). Then there is an $e \in E^1_w$, a path p and a cycle c such that $s(p) = r(e)$, $r(p) = s(c)$ and e does not belong to c. Write $c = f^{(1)} \dots f^{(m)}$ where $f^{(1)}, \dots, f^{(m)} \in E^1$. If $p = r(e)$, then $e_2 f_1^{(1)} \dots f_1^{(m)} e_2^*$ is a nod-path (since $f^{(m)} \ne e$). Now assume that $p = g^{(1)} \dots g^{(n)}$ where $g^{(1)}, \dots, g^{(n)} \in E^1$. Clearly we assume that no letter of p is a letter of c. One directly checks that $e_2 g_1^{(1)} \dots g_1^{(n)} f_1^{(1)} \dots f_1^{(m)} (g_1^{(n)})^* \dots (g_1^{(1)})^* e_2^*$ is a nod-path (note that $f^{(m)} \ne g^{(n)}$). □

Corollary 18 (cf. [15, Section 6],[8, Section 3]) *If $L_K(E, w)$ is finite-dimensional as a K-vector space, or simple, or graded simple with respect to its standard grading, then there is a row-finite graph F such that $L_K(E, w) \cong L_K(F)$ as K-algebras.*

Proof If (E, w) satisfies Condition (LPA), then there is a row-finite graph F such that $L_K(E, w) \cong L_K(F)$ as K-algebras by Theorem 1. Suppose now that (E, w) does not satisfy Condition (LPA). By Lemma 17, we can choose a nod-path p whose first letter is e_2 and whose last letter is e_2^* for some $e \in E_w^1$. Clearly p^n $(n \in \mathbb{N})$ are pairwise distinct nod-paths. It follows from Theorem 16 that $\dim_K(L_K(E, w)) = \infty$. Next we show $L_K(E, w)$ is neither simple nor graded simple. One directly checks that the ideal I generated by p equals the linear span of all nod-paths that contain p as a subword (note that e_2 is not the second letter of a forbidden word and e_2^* not the first letter of a forbidden word). It follows that I is a proper ideal of $L_K(E, w)$ (it is not the zero ideal since it contains the basis element p and it is not equal to $L_K(E, w)$ since it does not contain any vertex). Since I is generated by a homogeneous element, it is a graded ideal. □

Recall that a group graded K-algebra $A = \bigoplus_{g \in G} A_g$ is called *locally finite* if $\dim_K A_g < \infty$ for every $g \in G$.

We recall some general facts on the growth of algebras. Let $A \neq \{0\}$ be a finitely generated K-algebra. Let V be a *finite-dimensional generating subspace* of A, i.e. a finite-dimensional subspace of A that generates A as a K-algebra. For $n \geq 1$ let V^n denote the linear span of the set $\{v_1 \dots v_k \mid k \leq n, v_1, \dots, v_k \in V\}$. Then

$$V = V^1 \subseteq V^2 \subseteq V^3 \subseteq \dots, \quad A = \bigcup_{n \in \mathbb{N}} V^n \text{ and } d_V(n) := \dim V^n < \infty.$$

Given functions $f, g : \mathbb{N} \to \mathbb{R}^+$, we write $f \preccurlyeq g$ if there is a $c \in \mathbb{N}$ such that $f(n) \leq cg(cn)$ for all n. If $f \preccurlyeq g$ and $g \preccurlyeq f$, then the functions f, g are called *asymptotically equivalent* and we write $f \sim g$. If W is another finite-dimensional generating subspace of A, then $d_V \sim d_W$. The *Gelfand-Kirillov dimension* or *GK dimension* of A is defined as

$$\text{GKdim } A := \limsup_{n \to \infty} \log_n d_V(n).$$

The definition of the GK dimension does not depend on the choice of the finite-dimensional generating subspace V. If $d_V \preccurlyeq n^m$ for some $m \in \mathbb{N}$, then A is said to have *polynomial growth* and we have GKdim $A \leq m$. If $d_V \sim a^n$ for some real number $a > 1$, then A is said to have *exponential growth* and we have GKdim $A = \infty$. If A does not happen to be finitely generated over K, then the GK dimension of A is defined as

$$\text{GKdim}(A) := \sup\{\text{GKdim}(B) \mid B \text{ is a finitely generated subalgebra of } A\}.$$

For the algebra $A = \{0\}$ we set GKdim $A := 0$.

Proof of Theorem 3 If (E, w) satisfies Condition (LPA), then there is a row-finite graph F such that $L_K(E, w) \cong L_K(F)$ as K-algebras by Theorem 1. Suppose now that (E, w) does not satisfy Condition (LPA). In Part I we prove that $L_K(E, w)$ is not locally finite, in Part II that $L_K(E, w)$ is not Noetherian, in Part III that $L_K(E, w)$ is not Artinian, in Part IV that $L_K(E, w)$ is not von Neumann regular and in Part V that GKdim$(L_K(E, w)) = \infty$. By Lemma 17, we can choose a nod-path $p = x_1 \dots x_n$ whose first letter is e_2 and whose last letter is e_2^* for some $e \in E_w^1$.

Part I. Set $p^* := x_n^* \dots x_1^*$ (where $(f_i^*)^* = f_i$ for any $f \in E^1$ and $1 \le i \le w(f)$). One directly checks that for any $n \in \mathbb{N}$, $(pp^*)^n$ is a nod-path that lies in the homogeneous 0-component $L_K(E, w)_0$. It follows from Theorem 16 that $\dim_K(L_K(E, w)_0) = \infty$.

Part II. Let q be the nod-path one obtains by replacing the first letter of p by e_1. For any $n \in \mathbb{N}$ let I_n be the left ideal generated by the nod-paths $p, pq, \dots, pq^n$. One directly checks that I_n equals the linear span of all nod-paths o such that one of the words $p, pq, \dots, pq^n$ is a suffix of o. It follows that $I_n \subsetneq I_{n+1}$ (clearly none of the words $p, pq, \dots, pq^n$ is a suffix of pq^{n+1} since p and q have the same length but are distinct; hence $pq^{n+1} \notin I_n$).

Part III. For any $n \in \mathbb{N}$ let I_n be the left ideal generated by p^n. One directly checks that I_n equals the linear span of all nod-paths o such that p^n is a suffix of o. Hence $I_n \supsetneq I_{n+1}$ (clearly p^{n+1} is not a suffix of p^n and hence $p^n \notin I_{n+1}$).

Part IV. One directly checks that for any $x \in L_K(E, w)$, pxp is a linear combination of nod-paths of length $\ge 2|p|$. Hence the equation $pxp = p$ has no solution $x \in L_K(E, w)$.

Part V. Suppose first that (E, w) is finite. Let q be the nod-path one obtains by replacing the first letter of p by e_1. Let $n \in \mathbb{N}$. Consider the nod-paths

$$p^{i_1}q^{i_2} \dots p^{i_{k-1}}q^{i_k} \ (k \text{ even}), \text{ and } p^{i_1}q^{i_2} \dots p^{i_{k-2}}q^{i_{k-1}}p^{i_k} \ (k \text{ odd}) \tag{2}$$

where $k, i_1, \dots, i_k \in \mathbb{N}$ satisfy

$$(i_1 + \dots + i_k)|p| \le n. \tag{3}$$

Clearly different solutions $(k, i_1, \dots, i_k)$ and $(k', i'_1, \dots, i'_{k'})$ of inequality (3) correspond to different nod-paths in (2) since $|p| = |q|$. Let V denote the finite-dimensional subspace of $L_K(E, w)$ spanned by $\{v, f_i, f_i^* \mid v \in E^0, f \in E_1, 1 \le i \le w(f)\}$. By Theorem 16 the nod-paths in (2) are linearly independent in V^n. The number of solutions of (3) is $\sim 2^n$ and hence $L_K(E, w)$ has exponential growth.

Now suppose that (E, w) is not finite. One directly checks that there is a finite complete weighted subgraph $(\tilde{E}, \tilde{w})$ of (E, w) that does not satisfy Condition (LPA) (see [7, p. 884 and Proof of Lemma 5.19]). By the previous paragraph $L_K(\tilde{E}, \tilde{w})$ has exponential growth. Clearly the inclusion $(\tilde{E}, \tilde{w}) \hookrightarrow (E, w)$ induces an algebra monomorphism $L_K(\tilde{E}, \tilde{w}) \to L_K(E, w)$ since one can choose the special edges such that distinct nod-paths are mapped to distinct nod-paths. Hence $L_K(E, w)$ has a finitely generated subalgebra with exponential growth. It follows from the definition of the GK dimension that $\operatorname{GKdim} L_K(E, w) = \infty$.

We need two more lemmas in order to prove Theorem 2.

Lemma 19 *Let p be a nod-path starting with e_2 and ending with e_2^* for some $e \in E_w^1$. Then the ideal I of $L_K(E, w)$ generated by p contains no nonzero idempotent.*

Proof For a nod-path $q = x_1 \dots x_n$ define $m(q)$ as the largest nonnegative integer m such that there are indices $i_1, \dots, i_m \in \{1, \dots, n\}$ such that $i_j + |p| - 1 < i_{j+1}$ $(1 \le j \le m-1)$, $i_m + |p| - 1 \le n$ and $x_{i_j} \dots x_{i_j+|p|-1} = p$ $(1 \le j \le m)$. Hence $m(q)$ is maximal with the property that q contains $m(q)$ not overlapping copies of p.

Now let $a \in I\setminus\{0\}$. By Theorem 16 we can write $a = \sum_{r=1}^{t} k_r q_r$ where $k_1, \dots, k_t \in K\setminus\{0\}$ and $q_1, \dots, q_t$ are pairwise distinct nod-paths. Clearly $m(q_r) \geq 1$ for any $1 \leq r \leq t$, since I consists of all linear combinations of nod-paths containing p as a subword. Using the fact that e_2 is not the second letter of a forbidden word and e_2^* not the first letter of a forbidden word, it easy to show that for any $1 \leq r, s \leq t$ the product $q_r q_s$ is a linear combination of nod-paths o such that $m(o) \geq m(q_r) + m(q_s)$ (cf. [8, Proof of Proposition 40]). It follows that $a^2 = \sum_{r,s=1}^{t} k_r k_s q_r q_s$ is a linear combination of nod-paths o such that $m(o) \geq 2m(q_{r_{\min}}) > m(q_{r_{\min}})$ where $1 \leq r_{\min} \leq t$ is chosen such that $m(q_{r_{\min}})$ is minimal. Hence a^2 is a linear combination of nod-paths none of which equals $q_{r_{\min}}$. Thus a^2 cannot be equal to a. □

If Λ is an infinite set and S is a unital ring, then we denote by $M_\Lambda(S)$ the K-algebra consisting of all square matrices M, with rows and columns indexed by Λ, with entries from S, for which there are at most finitely many nonzero entries in M (cf. [2, Notation 2.6.3]).

Lemma 20 *Let Λ be an infinite set and S a left Noetherian, unital ring. Let $I_1 \subseteq I_2 \subseteq \dots$ be an ascending chain of left ideals of $M_\Lambda(S)$. Suppose there is a finite subset Λ^{fin} of Λ such that $\sigma_{\lambda\mu} = 0$ for any $n \in \mathbb{N}$, $\sigma \in I_n$, $\lambda \in \Lambda$ and $\mu \in \Lambda \setminus \Lambda^{\text{fin}}$. Then the chain $I_1 \subseteq I_2 \subseteq \dots$ eventually stabilises.*

Proof Write $\Lambda^{\text{fin}} = \{\lambda_1, \dots, \lambda_m\}$. Fix a $\tau \in \Lambda$. For any $n \in \mathbb{N}$, let N_n be the left S-submodule of S^m consisting of all row vectors $(\sigma_{\tau\lambda_1}, \dots, \sigma_{\tau\lambda_m})$ where σ varies over all matrices in I_n. Then I_n equals the set of all matrices $\sigma \in M_\Lambda(S)$ such that $\sigma_{\lambda\mu} = 0$ for any $\lambda \in \Lambda$, $\mu \in \Lambda \setminus \Lambda^{\text{fin}}$ and $(\sigma_{\lambda\lambda_1}, \dots, \sigma_{\lambda\lambda_m}) \in N_n$ for any $\lambda \in \Lambda$. Since S is a left Noetherian ring, S^m is a Noetherian module. It follows that the chain $N_1 \subseteq N_2 \subseteq \dots$ eventually stabilises and thus the chain $I_1 \subseteq I_2 \subseteq \dots$ eventually stabilises. □

Proof of Theorem 2 Assume there is a field K', a graph F and a ring isomorphism $\phi : L_K(E, w) \to L_{K'}(F)$. By Lemma 17, there is a nod-path p whose first letter is e_2 and whose last letter is e_2^* for some $e \in E_w^1$. Let q be the nod-path one obtains by replacing the last letter of p by e_1^*. By Lemma 19, the ideal I of $L_K(E, w)$ generated by p contains no nonzero idempotent. Similarly, for any $n \in \mathbb{N}$, the ideal I_n of $L_K(E, w)$ generated by qp^n contains no nonzero idempotent. It follows from [2, Proposition 2.7.9], that $\phi(I), \phi(I_n) \subseteq I(P_c(F))$ $(n \in \mathbb{N})$ where $I(P_c(F))$ is the ideal of $L_{K'}(F)$ generated by all vertices in F^0 which belong to a cycle without an exit. It follows that $\phi(p), \phi(qp^n) \in I(P_c(F))$ $(n \in \mathbb{N})$. By [2, Theorem 2.7.3] we have

$$I(P_c(F)) \cong \bigoplus_{i\in\Gamma} M_{\Lambda_i}(K'[x, x^{-1}]) \tag{4}$$

as a K'-algebra and hence also as a ring. The sets Γ and Λ_i $(i \in \Gamma)$ in (4) might be infinite if F is not finite.

It follows from the previous paragraph that there is a subring A of $L_K(E, w)$ such that $p, qp^n \in A$ $(n \in \mathbb{N})$ and $A \cong \bigoplus_{i\in\Gamma} M_{\Lambda_i}(K'[x, x^{-1}])$. For any $n \in \mathbb{N}$ let J_n be the left ideal of A generated by $qp^2, \dots, qp^{n+1}$. Then J_n is contained in the linear span of all nod-paths o such that one of the words $qp^2, \dots, qp^{n+1}$ is a suffix of o. It follows that $J_n \subsetneq J_{n+1}$ (clearly none of the words $qp^2, \dots, qp^{n+1}$ is a suffix of qp^{n+2} since p and q have the same length but are distinct). If the sets Γ and Λ_i $(i \in \Gamma)$ are finite, then we already have a contradiction since it is well-known that $\bigoplus_{i\in\Gamma} M_{\Lambda_i}(K'[x, x^{-1}])$ is Noetherian in this case. Hence the next two paragraphs are only needed if one of the sets Γ and Λ_i $(i \in \Gamma)$ is infinite.

If $a \in A$, then we identify a with its image in $\bigoplus_{i\in\Gamma} M_{\Lambda_i}(K'[x, x^{-1}])$ and write a_i for the i-th component of a. Set $\Gamma^{\text{fin}} := \{i \in \Gamma \mid p_i \neq 0\}$. Then Γ^{fin} is a finite subset of Γ. Clearly $(qp^n)_i = 0$ for any $i \in \Gamma \setminus \Gamma^{\text{fin}}$ and $n \geq 2$ (since $(qp^n)_i = (qp^{n-1}p)_i = (qp^{n-1})_i p_i$ for any $n \geq 2$). Hence, we can reduce our consideration to the case that Γ is finite.

For any $n \in \mathbb{N}$ and $i \in \Gamma$, let $J_{n,i}$ be the left ideal of $M_{\Lambda_i}(K'[x, x^{-1}])$ generated by $(qp^2)_i, \dots, (qp^{n+1})_i$. Then $J_n = \bigoplus_{i\in\Gamma} J_{n,i}$ since each $M_{\Lambda_i}(K'[x, x^{-1}])$ has local units. Now fix an $i \in \Gamma$. Let Λ_i^{fin} be the finite subset of Λ_i consisting of all $\lambda \in \Lambda_i$ such that the λ-th column of p_i has a nonzero entry. Then clearly $\sigma_{\lambda\mu} = 0$ for any $n \in \mathbb{N}$, $\sigma \in J_{n,i}$, $\lambda \in \Lambda_i$ and $\mu \in \Lambda_i \setminus \Lambda_i^{\text{fin}}$ (since any element of $J_{n,i}$ is a left multiple of p_i). Hence, by Lemma 20, the chain $J_{1,i} \subseteq J_{2,i} \subseteq \dots$ eventually stabilises. Since this holds for any $i \in \Gamma$, the chain $J_1 \subseteq J_2 \subseteq \dots$ eventually stabilises so we obtain a contradiction.

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