



# Bounded t-Structures on the Bounded Derived Category of Coherent Sheaves over a Weighted Projective Line

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Received: 12 March 2019 / Accepted: 9 October 2019 / Published online: 9 December 2019  
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## Abstract

We use recollement and HRS-tilt to describe bounded t-structures on the bounded derived category  $\mathcal{D}^b(\mathbb{X})$  of coherent sheaves over a weighted projective line  $\mathbb{X}$  of domestic or tubular type. We will see from our description that the combinatorics in the classification of bounded t-structures on  $\mathcal{D}^b(\mathbb{X})$  can be reduced to that in the classification of bounded t-structures on the bounded derived categories of finite dimensional right modules over representation-finite finite dimensional hereditary algebras.

**Keywords** Weighted projective line · Derived category · T-structure · Derived equivalence

**Mathematics Subject Classification (2010)** Primary 14F05; Secondary 18E30

## 1 Introduction

### 1.1 Background and Aim

In an attempt to give a geometric treatment of Ringel's canonical algebras [43], Geigle and Lenzing introduced in [17] a class of noncommutative curves, called weighted projective lines, and each canonical algebra is realized as the endomorphism algebra of a tilting bundle in the category of coherent sheaves over some weighted projective line. A stacky point of view to weighted projective lines is that for a weighted projective line  $\mathbb{X}$  defined over a field  $k$ , there is a smooth algebraic  $k$ -stack  $\mathcal{X}$  with the projective line over  $k$  as its coarse moduli space such that  $\text{coh}\mathcal{X} \simeq \text{coh}\mathbb{X}$  and  $\text{Qcoh}\mathcal{X} \simeq \text{Qcoh}\mathbb{X}$ , where  $\text{coh}$  (resp.  $\text{Qcoh}$ ) denotes the category of coherent (resp. quasi-coherent) sheaves. As an indication of the importance of the notion of weighted projective lines, a famous theorem of Happel [20] states that if  $\mathcal{A}$  is a connected hereditary category linear over an algebraically closed field  $k$  with finite

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dimensional morphism and extension spaces such that its bounded derived category  $\mathcal{D}^b(\mathcal{A})$  admits a tilting object then  $\mathcal{D}^b(\mathcal{A})$  is triangle equivalent to the bounded derived category of finite dimensional modules over a finite dimensional hereditary algebra over  $k$  or to the bounded derived category of coherent sheaves on a weighted projective line defined over  $k$ .

The notion of t-structures is introduced by Beilinson, Bernstein and Deligne in [7] to serve as a categorical framework for defining perverse sheaves in the derived category of constructible sheaves over a stratified space. Recently, there has been a growing interest in t-structures ever since Bridgeland [12] introduced the notion of stability conditions. To give a stability condition on a triangulated category requires specifying a bounded t-structure. On the other hand, there are many works on bounded t-structures on the bounded derived category  $\mathcal{D}^b(\Lambda)$  of finite dimensional modules over a finite dimensional algebra  $\Lambda$  in recent years. Remarkably, König and Yang proved the existence of bijective correspondences, which we call König-Yang correspondences, between several concepts among which are bounded t-structures with length heart on  $\mathcal{D}^b(\Lambda)$ , equivalence classes of simple-minded collections in  $\mathcal{D}^b(\Lambda)$ , equivalence classes of silting objects in  $\mathcal{K}^b(\text{proj}\Lambda)$ , and bounded co-t-structures on  $\mathcal{K}^b(\text{proj}\Lambda)$ , where  $\mathcal{K}^b(\text{proj}\Lambda)$  denotes the bounded homotopy category of finite dimensional projective modules over  $\Lambda$ .

This article is devoted to describing bounded t-structures on the bounded derived category of coherent sheaves over a weighted projective line. We mainly combine two classical tools to describe t-structures: recollement and HRS-tilt. Recollement is introduced at the same time with t-structures in [7]. A recollement stratifies a triangulated category into smaller ones and allows us to glue t-structures. HRS-tilt, introduced by Happel, Reiten and Smalø in [22], constructs a new t-structure from an old one via a torsion pair in the heart of the old t-structure. We will see that a large class of t-structures are glued from recollements. Given a t-structure, to build a recollement from which the t-structure can be glued, we rely on Ext-projectives. This concept was introduced by Auslander and Smalø to investigate almost split sequences in subcategories [5]. Assem, Salario and Trepode introduced a triangulated version in [2] to study t-structures. Our small observation is that an exceptional Ext-projective object helps us to build a desired recollement under some condition (see Lemma 2.15). Almost all recollements in this article are built in this way (plus induction). There do exist bounded t-structures without any available Ext-projective. Fortunately, in our situation, these are up to shift HRS-tilts with respect to some torsion pair in the standard heart and they can be described explicitly.

### 1.2 Main Results

Let  $\mathbb{X}$  be a weighted projective line defined over an algebraically closed field  $k$ , and  $\mathcal{O}$  its structure sheaf (see Section 3.1). Depending on its weight function  $\mathbf{w} : \mathbb{P}^1 \rightarrow \mathbb{Z}_{\geq 1}$ , where  $\mathbb{P}^1$  is (the set of closed points of) the projective line over  $k$  and  $\mathbb{Z}_{\geq 1}$  is the set of positive integers,  $\mathbb{X}$  is of domestic type, of tubular type, or of wild type. Denote by  $\text{vect}\mathbb{X}$  resp.  $\text{coh}_0\mathbb{X}$  the category of vector bundles resp. torsion sheaves over  $\mathbb{X}$ , by  $\mathcal{A} = \text{coh}\mathbb{X}$  the category of coherent sheaves and by  $\mathcal{D} = \mathcal{D}^b(\mathbb{X})$  the bounded derived category of  $\text{coh}\mathbb{X}$ .  $\text{coh}_0\mathbb{X}$  consists exactly of finite length objects in  $\text{coh}\mathbb{X}$  and  $\text{coh}_0\mathbb{X}$  decomposes as a coproduct  $\text{coh}_0\mathbb{X} = \coprod_{\lambda \in \mathbb{P}^1} \text{coh}_\lambda\mathbb{X}$ , where  $\text{coh}_\lambda\mathbb{X}$  consists of those coherent sheaves supported at  $\lambda$ . For  $P \subset \mathbb{P}^1$ , denote by  $(\mathcal{T}_P, \mathcal{F}_P)$  the torsion pair in  $\text{coh}\mathbb{X}$

$$(\text{add}\{\text{coh}_\lambda\mathbb{X} \mid \lambda \in P\}, \text{add}\{\text{vect}\mathbb{X}, \text{coh}_\lambda\mathbb{X} \mid \lambda \in \mathbb{P}^1 \setminus P\}).$$

The number of isoclasses of simple sheaves in  $\text{coh}_\lambda\mathbb{X}$  is  $\mathbf{w}(\lambda)$ . A (possibly empty) collection  $\mathcal{S}$  of simple sheaves over  $\mathbb{X}$  is called proper if for each  $\lambda \in \mathbb{P}^1$ ,  $\mathcal{S}$  does not contain a complete

set of simple sheaves in  $\text{coh}_\lambda \mathbb{X}$  and if simple sheaves in  $\mathcal{S}$  are pairwise non-isomorphic. Two such collections are equivalent if they yield the same isoclasses of simple sheaves. A t-structure on  $\mathcal{D}^b(\mathbb{X})$  is said to be compatible with a given a recollement if it is glued from the recollement (see Section 2.4). See Section 1.4 for the notation  $\langle - \rangle_{\mathcal{D}}$ ,  $(-)^{\perp_{\mathcal{A}}}$ ,  $(-)^{\perp_{\mathcal{D}}}$  and  $\mathcal{D}^b(-)$ .

We are ready to state our theorem for a weighted projective line of domestic type.

**Theorem 1.1** (Theorem 4.18) *Suppose  $\mathbb{X}$  is of domestic type and let  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  be a bounded t-structure on  $\mathcal{D}^b(\mathbb{X})$  with heart  $\mathcal{B}$ . Then exactly one of the following holds:*

- (1) *up to the action of the Picard group  $\text{Pic}\mathbb{X}$  of  $\mathbb{X}$ ,  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the recollement*

$$\mathcal{O}^{\perp_{\mathcal{D}}} \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{i_*} \\ \xleftarrow{\quad} \end{array} \mathcal{D} = \mathcal{D}^b(\mathbb{X}) \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \langle \mathcal{O} \rangle_{\mathcal{D}},$$

where  $i_*$ ,  $j_!$  are the inclusion functors, in which case  $\mathcal{B}$  is of finite length;

- (2) *for a unique (up to equivalence) proper collection  $\mathcal{S}$  of simple sheaves and a unique  $P \subset \mathbb{P}^1$ ,  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the recollement*

$$\mathcal{D}^b(\mathcal{S}^{\perp_{\mathcal{A}}}) = \mathcal{S}^{\perp_{\mathcal{D}}} \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{i_*} \\ \xleftarrow{\quad} \end{array} \mathcal{D} = \mathcal{D}^b(\mathbb{X}) \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \langle \mathcal{S} \rangle_{\mathcal{D}},$$

where  $i_*$ ,  $j_!$  are the inclusion functors, such that the corresponding t-structure on  $\mathcal{D}^b(\mathcal{S}^{\perp_{\mathcal{A}}})$  is a shift of the HRS-tilt with respect to the torsion pair  $(\mathcal{S}^{\perp_{\mathcal{A}}} \cap \mathcal{T}_P, \mathcal{S}^{\perp_{\mathcal{A}}} \cap \mathcal{F}_P)$  in  $\mathcal{S}^{\perp_{\mathcal{A}}}$ , in which case  $\mathcal{B}$  is not of finite length and  $\mathcal{B}$  is noetherian resp. artinian iff  $P = \emptyset$  resp.  $P = \mathbb{P}^1$ .

To state our theorem for a weighted projective line of tubular type, we need to introduce more notation (see Section 3.3). Let  $\mathbb{R}$  (resp.  $\mathbb{Q}$ ) be the set of real (resp. rational) numbers and let  $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ ,  $\bar{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ . Let  $\mathbb{X}$  be of tubular type. Denote by  $\text{coh}^\mu \mathbb{X}$  the category of semistable coherent sheaves over  $\mathbb{X}$  with slope  $\mu \in \bar{\mathbb{Q}}$  (we deem torsion sheaves to be semistable and thus  $\text{coh}^\infty \mathbb{X} = \text{coh}_0 \mathbb{X}$ ).  $\mathcal{D}^b(\mathbb{X})$  admits an exact autoequivalence  $\Phi_{q',q}$  for each  $q', q \in \bar{\mathbb{Q}} \cup \{\infty\}$ , which is called a telescopic functor, such that  $\Phi_{q',q}(\text{coh}^q \mathbb{X}) = \text{coh}^{q'} \mathbb{X}$ . For  $\mu \in \bar{\mathbb{Q}}$ , denote  $\text{coh}_\lambda^\mu \mathbb{X} = \Phi_{\mu,\infty}(\text{coh}_\lambda \mathbb{X})$ . The category  $\text{coh}^\mu \mathbb{X}$  decomposes as  $\text{coh}^\mu \mathbb{X} = \coprod_{\lambda \in \mathbb{P}^1} \text{coh}_\lambda^\mu \mathbb{X}$ . For  $\mu \in \bar{\mathbb{R}}$ ,  $\text{coh}^{>\mu} \mathbb{X}$  (resp.  $\text{coh}^{<\mu} \mathbb{X}$ ) denotes the subcategory of  $\text{coh} \mathbb{X}$  consisting of those sheaves whose semistable factors have slope  $> \mu$  (resp.  $< \mu$ ).

**Theorem 1.2** (Theorem 4.20) *Suppose  $\mathbb{X}$  is of tubular type and let  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  be a bounded t-structure on  $\mathcal{D}^b(\mathbb{X})$  with heart  $\mathcal{B}$ . Then exactly one of the following holds:*

- (1) *for a unique  $\mu \in \mathbb{R} \setminus \mathbb{Q}$ ,  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is a shift of the HRS-tilt with respect to the torsion pair  $(\text{coh}^{>\mu} \mathbb{X}, \text{coh}^{<\mu} \mathbb{X})$  in  $\text{coh} \mathbb{X}$ , in which case  $\mathcal{B}$  is neither noetherian nor artinian;*
- (2) *for a unique  $\mu \in \bar{\mathbb{Q}}$  and a unique  $P \subset \mathbb{P}^1$ ,  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is a shift of the HRS-tilt with respect to the torsion pair*

$$(\text{add}\{\text{coh}^{>\mu} \mathbb{X}, \text{coh}_\lambda^\mu \mathbb{X} \mid \lambda \in P\}, \text{add}\{\text{coh}_\lambda^\mu \mathbb{X}, \text{coh}^{<\mu} \mathbb{X} \mid \lambda \in \mathbb{P}^1 \setminus P\})$$

in  $\text{coh} \mathbb{X}$ , in which case  $\mathcal{B}$  is not of finite length and  $\mathcal{B}$  is noetherian resp. artinian iff  $P = \emptyset$  resp.  $P = \mathbb{P}^1$ ;

- (3) for a unique  $q \in \bar{\mathbb{Q}}$ , a unique (up to equivalence) nonempty proper collection  $\mathcal{S}$  of simple sheaves and a unique  $P \subset \mathbb{P}^1$ ,  $\Phi_{\infty,q}((\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}))$  is compatible with the recollement

$$\mathcal{D}^b(\mathcal{S}^{\perp_{\mathcal{A}}}) = \mathcal{S}^{\perp_{\mathcal{D}}} \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{i_*} \\ \xleftarrow{\quad} \end{array} \mathcal{D} = \mathcal{D}^b(\mathbb{X}) \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \langle \mathcal{S} \rangle_{\mathcal{D}},$$

where  $i_*$ ,  $j_!$  are the inclusion functors, such that the corresponding t-structure on  $\mathcal{D}^b(\mathcal{S}^{\perp_{\mathcal{A}}})$  is a shift of the HRS-tilt with respect to the torsion pair  $(\mathcal{S}^{\perp_{\mathcal{A}}} \cap \mathcal{T}_P, \mathcal{S}^{\perp_{\mathcal{A}}} \cap \mathcal{F}_P)$  in  $\mathcal{S}^{\perp_{\mathcal{A}}}$ , in which case  $\mathcal{B}$  is not of finite length and  $\mathcal{B}$  is noetherian resp. artinian iff  $P = \emptyset$  resp.  $P = \mathbb{P}^1$ ;

- (4) for some  $q \in \bar{\mathbb{Q}}$  and some exceptional simple sheaf  $S$ ,  $\Phi_{\infty,q}((\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}))$  is compatible with the recollement

$$\mathcal{D}^b(\mathcal{S}^{\perp_{\mathcal{A}}}) = \mathcal{S}^{\perp_{\mathcal{D}}} \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{i_*} \\ \xleftarrow{\quad} \end{array} \mathcal{D} = \mathcal{D}^b(\mathbb{X}) \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \langle S \rangle_{\mathcal{D}},$$

where  $i_*$ ,  $j_!$  are the inclusion functors, such that the corresponding t-structure on  $\mathcal{D}^b(\mathcal{S}^{\perp_{\mathcal{A}}})$  has a length heart, in which case  $\mathcal{B}$  is of finite length.

We obtain from the two theorems above certain bijective correspondence for those bounded t-structures whose heart is not of finite length. Note that any group  $G$  of exact autoequivalences of  $\mathcal{D}^b(\mathbb{X})$  acts on the set of bounded t-structures on  $\mathcal{D}^b(\mathbb{X})$  by  $\Phi((\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})) := (\Phi(\mathcal{D}^{\leq 0}), \Phi(\mathcal{D}^{\geq 0}))$  for  $\Phi \in G$  and a bounded t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  on  $\mathcal{D}^b(\mathbb{X})$ . In the following corollary, we deem  $\mathbb{Z}$  as the group of exact autoequivalences generated by the translation functor of  $\mathcal{D}^b(\mathbb{X})$ , which acts freely on the set of bounded t-structures on  $\mathcal{D}^b(\mathbb{X})$ .

**Corollary 1.3** (Corollary 4.21) (1) If  $\mathbb{X}$  is of domestic type then there is a bijection

$$\{\text{bounded t-structures on } \mathcal{D}^b(\mathbb{X}) \text{ whose heart is not of finite length}\} / \mathbb{Z} \longleftrightarrow \bigsqcup_{\mathcal{S}} \left( \{P \mid P \subset \mathbb{P}^1\} \times \{\text{bounded t-structures on } \langle \mathcal{S} \rangle_{\mathcal{D}}\} \right), \quad (1.2.1)$$

where  $\mathcal{S}$  runs through all equivalence classes of proper collections of simple sheaves.

- (2) If  $\mathbb{X}$  is of tubular type then there is a bijection

$$\{\text{bounded t-structures on } \mathcal{D}^b(\mathbb{X}) \text{ whose heart is not of finite length}\} / \mathbb{Z} \longleftrightarrow \mathbb{R} \setminus \mathbb{Q} \bigsqcup \left( \bar{\mathbb{Q}} \times \bigsqcup_{\mathcal{S}} \left( \{P \mid P \subset \mathbb{P}^1\} \times \{\text{bounded t-structures on } \langle \mathcal{S} \rangle_{\mathcal{D}}\} \right) \right), \quad (1.2.2)$$

where  $\mathcal{S}$  runs through all equivalence classes of proper collections of simple sheaves.

Recall that an equioriented  $\mathbb{A}_s$ -quiver refers to the quiver

$$\bullet_1 \longrightarrow \bullet_2 \longrightarrow \dots \longrightarrow \bullet_{s-1} \longrightarrow \bullet_s$$

(Since only such an orientation is involved in this article,  $\vec{\mathbb{A}}_s$  will always denote an equioriented  $\mathbb{A}_s$ -quiver.) For convenience, we also define  $\vec{\mathbb{A}}_0$  to be the empty quiver and define  $\text{mod } k\vec{\mathbb{A}}_0$  to be the zero category. Given a nonempty proper collection  $\mathcal{S}$  of simple sheaves on  $\mathbb{X}$ , there are positive integers  $m, k_1, \dots, k_m$  such that  $\langle \mathcal{S} \rangle_{\mathcal{A}} \simeq \bigsqcup_{i=1}^m \text{mod } k\vec{\mathbb{A}}_{k_i}$ , where  $\text{mod } k\vec{\mathbb{A}}_l$  is the category of finite dimensional right modules over the path algebra of the

equioriented  $\mathbb{A}_1$ -quiver, and we have an exact equivalence  $\langle \mathcal{S} \rangle_{\mathcal{D}} \simeq \coprod_{i=1}^m \mathcal{D}^b(\text{mod} k\bar{\mathbb{A}}_{k_i})$ . By Corollary 1.3, if  $\mathbb{X}$  is a weighted projective line of domestic or tubular type then to classify bounded t-structures on  $\mathcal{D}^b(\mathbb{X})$  whose heart is not of finite length, it suffices to classify bounded t-structures on each  $\mathcal{D}^b(\text{mod} k\bar{\mathbb{A}}_{k_i})$ . Since bounded t-structures on  $\mathcal{D}^b(\text{mod} k\bar{\mathbb{A}}_1)$  have length hearts, one can achieve this by calculating silting objects or simple-minded collections in  $\mathcal{D}^b(\text{mod} k\bar{\mathbb{A}}_{k_i})$  by virtue of König-Yang correspondences. We know that  $\mathcal{D}^b(\mathbb{X})$  is triangle equivalent to the bounded derived category of finite dimensional right modules over a canonical algebra whose global dimension is at most 2. So to obtain a bijective correspondence for bounded t-structures on  $\mathcal{D}^b(\mathbb{X})$  with length heart, we can again utilize König-Yang correspondences and try to compute collections of simple objects in the heart (using Proposition 2.11) or silting objects in  $\mathcal{D}^b(\mathbb{X})$  (using [36, Corollary 3.4]) from the recollements in Theorem 1.1(1) and Theorem 1.2(4). As illustrated after Corollary 4.21 in Section 4.4, the two theorems reduce the combinatorics in the classification of bounded t-structures on  $\mathcal{D}^b(\mathbb{X})$  to the combinatorics in the classification of bounded t-structures on bounded derived categories of finite dimensional modules over representation-finite finite dimensional hereditary algebras.

To give an application of our description of bounded t-structures, we prove in Section 5 a characterization of when the heart of a bounded t-structure on  $\mathcal{D}^b(\mathbb{X})$  is derived equivalent to the standard heart  $\text{coh}\mathbb{X}$ , which is inspired by the work [44] of Stanley and van Roosmalen.

**Theorem 1.4** (Theorem 5.2) *Let  $\mathbb{X}$  be a weighted projective line of domestic or tubular type and  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  a bounded t-structure on  $\mathcal{D}^b(\mathbb{X})$  with heart  $\mathcal{B}$ . Then the inclusion  $\mathcal{B} \rightarrow \mathcal{D}^b(\mathbb{X})$  extends to a derived equivalence  $\mathcal{D}^b(\mathcal{B}) \xrightarrow{\sim} \mathcal{D}^b(\mathbb{X})$  iff the Serre functor of  $\mathcal{D}^b(\mathbb{X})$  is right t-exact with respect to  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ .*

Here we say that the inclusion  $\mathcal{B} \rightarrow \mathcal{D}^b(\mathbb{X})$  extends to a derived equivalence  $\mathcal{D}^b(\mathcal{B}) \xrightarrow{\sim} \mathcal{D}^b(\mathbb{X})$  if some realization functor  $\mathcal{D}^b(\mathcal{B}) \rightarrow \mathcal{D}^b(\mathbb{X})$  is an equivalence (see Section 5). As a corollary (see Corollary 5.4), a similar assertion holds for the bounded derived category of finite dimensional right modules over a tubular algebra in the sense of Ringel [43].

### 1.3 Sketch of this Article

This article is organized as follows.

In Section 2, we collect preliminaries on t-structures and some facts on hereditary categories. In Section 2.1–2.2, we recall basic definitions and properties of t-structures and introduce width-bounded t-structures and HRS-tilt. In Section 2.3–2.5, we recall recollements of triangulated categories, admissible subcategories, gluing t-structures and properties of glued t-structures. In Section 2.6 we recall Ext-projective objects, and use an exceptional Ext-projective object to establish a recollement with which the given t-structure is compatible. In Section 2.7, we recall some facts on hereditary categories, including Happel-Ringel Lemma. In Section 2.8, we recall and prove some facts on t-structures on the bounded derived category of finitely generated modules over a finite dimensional algebra, including a part of König-Yang correspondences. In Section 2.9, we describe bounded t-structures on the bounded derived category of finite dimensional nilpotent representations of a cyclic quiver.

In Section 3, we collect preparatory materials and results on weighted projective lines. In Section 3.1, we recall basic definitions and facts on weighted projective lines. In Section 3.2, we recap Auslander-Reiten theory. In Section 3.3, we recall the classification and important properties of vector bundles over a weighted projective line of domestic or tubular

type. In Section 3.4, we recall descriptions of perpendicular categories of some exceptional sequences. In Section 3.5, we recall and prove the non-vanishing of some morphism spaces in the category  $\text{coh}\mathbb{X}$  of coherent sheaves over a weighted projective line  $\mathbb{X}$ . In Section 3.6, we investigate full exceptional sequences in  $\text{coh}\mathbb{X}$ , and prove the existence of certain nice terms in some cases. In Section 3.7, we give some preliminary descriptions of some torsion pairs in  $\text{coh}\mathbb{X}$ , and establish bijections between isoclasses of basic tilting sheaves, certain torsion pairs in  $\text{coh}\mathbb{X}$  and certain bounded t-structures on the bounded derived category  $\mathcal{D}^b(\mathbb{X})$  of  $\text{coh}\mathbb{X}$ , and finally we investigate the noetherianness and the artinianness of tilted hearts given by certain torsion pairs in  $\text{coh}\mathbb{X}$ .

In Section 4, we describe bounded t-structures on the bounded derived category  $\mathcal{D}^b(\mathbb{X})$  of coherent sheaves over a weighted projective line  $\mathbb{X}$  of domestic or tubular type. In Section 4.1, we investigate and describe bounded t-structures that restrict to bounded t-structures on the bounded derived category  $\mathcal{D}^b(\text{coh}_0\mathbb{X})$  of the category  $\text{coh}_0\mathbb{X}$  of torsion sheaves. In Section 4.2, we investigate those bounded t-structures on  $\mathcal{D}^b(\mathbb{X})$  that cannot restrict to t-structures on  $\mathcal{D}^b(\text{coh}_0\mathbb{X})$  even up to the action of the group of exact autoequivalences of  $\mathcal{D}^b(\mathbb{X})$ . In particular, we prove that the heart of such a bounded t-structure is necessarily of finite length and possesses only finitely many indecomposable objects, all of which are exceptional. In Section 4.3, we prove some properties possessed by silting objects in  $\mathcal{D}^b(\mathbb{X})$ . This is mainly acquired via properties of full exceptional sequences obtained earlier and will yield information on bounded t-structures by virtue of König-Yang correspondences. In Section 4.4, we complete our description of bounded t-structures on  $\mathcal{D}^b(\mathbb{X})$ , in which we mainly use HRS-tilt and recollement. In Section 4.5, we use our description of bounded t-structures to give a description of torsion pairs in  $\text{coh}\mathbb{X}$ .

In Section 5, we prove a characterization of when the heart of a bounded t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  on  $\mathcal{D}^b(\mathbb{X})$  is derived equivalent to  $\text{coh}\mathbb{X}$  for a domestic or tubular  $\mathbb{X}$ , which is pertinent to the right t-exactness of the Serre functor of  $\mathcal{D}^b(\mathbb{X})$  and gives an application of our main result (i.e., description of bounded t-structures). We conjecture that this result holds for arbitrary weighted projective line and propose a potential approach at the end of Section 5.

### 1.4 Notation and Conventions

We denote by  $\mathbb{R}$  (resp.  $\mathbb{Q}, \mathbb{Z}, \mathbb{Z}_{\geq 1}$ ) the set of real numbers (resp. rational numbers, integers, positive integers). Pose  $\tilde{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$  and  $\tilde{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ .

For a finite dimensional algebra  $\Lambda$  over a field  $k$ ,  $\text{mod}\Lambda$  denotes the category of finite dimensional right modules over  $\Lambda$  and  $\mathcal{D}^b(\Lambda)$  the bounded derived category of  $\text{mod}\Lambda$ .

A subcategory of a category is tacitly a full subcategory. If  $\mathcal{B}$  is a subcategory of a category  $\mathcal{A}$  (typically abelian or triangulated in our setup), denote

$$\mathcal{B}^{\perp_{0,\mathcal{A}}} = \{X \in \mathcal{A} \mid \text{Hom}_{\mathcal{A}}(\mathcal{B}, X) = 0\},$$

which we will simply write as  $\mathcal{B}^{\perp_0}$  if there is no confusion. Dually we have  ${}^{\perp_{0,\mathcal{A}}}\mathcal{B}$  or  ${}^{\perp_0}\mathcal{B}$ .

For an abelian category  $\mathcal{A}$ , its bounded derived category is denoted by  $\mathcal{D}^b(\mathcal{A})$ . Let  $\mathcal{B}$  be an additive subcategory of  $\mathcal{A}$ . Following [18], we call  $\mathcal{B}$  an *exact subcategory*<sup>1</sup> of  $\mathcal{A}$  if  $\mathcal{B}$  is an abelian category and the inclusion functor  $\iota : \mathcal{B} \rightarrow \mathcal{A}$  is exact.  $\mathcal{B}$  is called a *thick subcategory* of  $\mathcal{A}$  if  $\mathcal{B}$  is closed under kernels, cokernels and extensions. A thick subcategory of  $\mathcal{A}$  is an exact subcategory of  $\mathcal{A}$ . Given a collection  $\mathcal{C}$  of objects in  $\mathcal{A}$ , we denote by  $\langle \mathcal{C} \rangle_{\mathcal{A}}$

<sup>1</sup>Note the difference with a subcategory that is an exact category.

the smallest thick subcategory of  $\mathcal{A}$  containing  $\mathcal{C}$ . The *right perpendicular category*  $\mathcal{C}^{\perp\mathcal{A}}$  and the *left perpendicular category*  ${}^{\perp\mathcal{A}}\mathcal{C}$  of  $\mathcal{C}$  in the sense of [18] are

$$\begin{aligned} \mathcal{C}^{\perp\mathcal{A}} &= \{X \in \mathcal{A} \mid \text{Hom}_{\mathcal{A}}(C, X) = 0 = \text{Ext}_{\mathcal{A}}^1(C, X) \text{ for all } C \in \mathcal{C}\}, \\ {}^{\perp\mathcal{A}}\mathcal{C} &= \{X \in \mathcal{A} \mid \text{Hom}_{\mathcal{A}}(X, C) = 0 = \text{Ext}_{\mathcal{A}}^1(X, C) \text{ for all } C \in \mathcal{C}\}. \end{aligned}$$

It's shown in [18, Proposition 1.1] that if objects in  $\mathcal{C}$  have projective dimension at most 1, that is,  $\text{Ext}_{\mathcal{A}}^2(X, -) = 0$  for all  $X \in \mathcal{C}$ , then  $\mathcal{C}^{\perp\mathcal{A}}$  and  ${}^{\perp\mathcal{A}}\mathcal{C}$  are exact subcategories of  $\mathcal{A}$  closed under extensions.

Let  $\mathcal{D}$  be a triangulated category. We denote by  $\text{Aut}\mathcal{D}$  the group of exact autoequivalences of  $\mathcal{D}$ . A triangle in  $\mathcal{D}$  refers always to a distinguished triangle. For two subcategories  $\mathcal{D}_1, \mathcal{D}_2$  of  $\mathcal{D}$ , define a subcategory  $\mathcal{D}_1 * \mathcal{D}_2$  of  $\mathcal{D}$  by

$$\mathcal{D}_1 * \mathcal{D}_2 = \{X \in \mathcal{D} \mid \exists \text{ a triangle } Y \rightarrow X \rightarrow Z \rightsquigarrow, Y \in \mathcal{D}_1, Z \in \mathcal{D}_2\}.$$

By the octahedral axiom,  $*$  is associative. Given a triangulated category  $\mathcal{D}$  and a collection  $\mathcal{C}$  of objects in  $\mathcal{D}$ , we denote by  $\langle \mathcal{C} \rangle_{\mathcal{D}}$  the thick closure of  $\mathcal{C}$  in  $\mathcal{D}$ , that is, the smallest triangulated subcategory of  $\mathcal{D}$  containing  $\mathcal{C}$  and closed under direct summands. We say that  $\mathcal{C}$  classically generates  $\mathcal{D}$  if  $\langle \mathcal{C} \rangle_{\mathcal{D}}$  coincides with  $\mathcal{D}$ . Moreover, we denote

$$\mathcal{C}^{\perp} = \mathcal{C}^{\perp\mathcal{D}} := \{X \in \mathcal{D} \mid \text{Hom}_{\mathcal{D}}^n(\mathcal{C}, X) = 0, \forall n \in \mathbb{Z}\} = \langle \mathcal{C} \rangle_{\mathcal{D}}^{\perp 0}.$$

Dually one defines  ${}^{\perp}\mathcal{C} = {}^{\perp\mathcal{D}}\mathcal{C}$ .  $\mathcal{C}^{\perp\mathcal{D}}$  and  ${}^{\perp\mathcal{D}}\mathcal{C}$  are thick subcategories of  $\mathcal{D}$ . If  $\mathcal{D}$  is a triangulated category linear over a field  $k$ , we denote

$$\text{Hom}^{\bullet}(X, Y) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}^n(X, Y)[-n],$$

where the latter is deemed as a complex of  $k$ -spaces with zero differential.  $\mathcal{D}$  is said to be of *finite type* if  $\bigoplus_{n \in \mathbb{Z}} \text{Hom}^n(X, Y)$  is a finite dimensional  $k$ -space for each  $X, Y$  in  $\mathcal{D}$ .

If  $\mathcal{A}$  is a hereditary abelian category and  $\mathcal{B}$  is an exact subcategory of  $\mathcal{A}$  closed under extensions then  $\mathcal{B}$  is a hereditary abelian category and the inclusion functor  $\iota : \mathcal{B} \rightarrow \mathcal{A}$  induces a fully faithful exact functor  $\mathcal{D}^b(\iota) : \mathcal{D}^b(\mathcal{B}) \rightarrow \mathcal{D}^b(\mathcal{A})$  whose essential image consists of those objects in  $\mathcal{D}^b(\mathcal{A})$  with cohomologies in  $\mathcal{B}$ .<sup>2</sup> Denote  $\mathcal{D} = \mathcal{D}^b(\mathcal{A})$ . If  $\mathcal{C}$  is a collection of objects in  $\mathcal{A}$  then  $\mathcal{B} := \langle \mathcal{C} \rangle_{\mathcal{A}}$  (resp.  $\mathcal{B} := \mathcal{C}^{\perp\mathcal{A}}$ , resp.  $\mathcal{B} := {}^{\perp\mathcal{A}}\mathcal{C}$ ) is an exact subcategory of  $\mathcal{A}$  closed under extensions and the functor  $\mathcal{D}^b(\iota) : \mathcal{D}^b(\mathcal{B}) \rightarrow \mathcal{D}^b(\mathcal{A})$  identifies canonically

$$\mathcal{D}^b(\langle \mathcal{C} \rangle_{\mathcal{A}}) \text{ resp. } \mathcal{D}^b(\mathcal{C}^{\perp\mathcal{A}}) \text{ resp. } \mathcal{D}^b({}^{\perp\mathcal{A}}\mathcal{C})$$

with the subcategory

$$\langle \mathcal{C} \rangle_{\mathcal{D}} \text{ resp. } \mathcal{C}^{\perp\mathcal{D}} \text{ resp. } {}^{\perp\mathcal{D}}\mathcal{C}$$

of  $\mathcal{D}$ . We will often make this identification in this article.

<sup>2</sup>One can argue as follows for this simple fact. By [8, Lemma 3.2.3], we have an injection  $\text{Ext}_{\mathcal{B}}^2(X, Y) \hookrightarrow \text{Ext}_{\mathcal{A}}^2(X, Y)$  for  $X, Y \in \mathcal{B}$ . Since  $\mathcal{A}$  is hereditary,  $\text{Ext}_{\mathcal{B}}^2(X, Y) = 0$ . So  $\mathcal{B}$  is hereditary. Since the exact subcategory  $\mathcal{B}$  is closed under extensions, the inclusion  $\iota : \mathcal{B} \rightarrow \mathcal{A}$  induces an isomorphism  $\text{Ext}_{\mathcal{B}}^1(X, Y) \cong \text{Ext}_{\mathcal{A}}^1(X, Y)$  for any  $X, Y \in \mathcal{B}$ . Since  $\mathcal{B}$  classically generates  $\mathcal{D}^b(\mathcal{B})$ , the derived functor  $\mathcal{D}^b(\iota) : \mathcal{D}^b(\mathcal{B}) \rightarrow \mathcal{D}^b(\mathcal{A})$  is fully faithful. The essential image of  $\mathcal{D}^b(\iota)$  is clear.

## 2 Preliminaries

### 2.1 Basics on t-Structures

We recall basic definitions concerning t-structures in this subsection. The standard reference is [7].

Let  $\mathcal{D}$  be a triangulated category. A *t-structure* on  $\mathcal{D}$  is a pair  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  of strictly (=closed under isomorphisms) full subcategories  $(\mathcal{D}^{\leq n} := \mathcal{D}^{\leq 0}[-n], \mathcal{D}^{\geq n} := \mathcal{D}^{\geq 0}[-n])$

- $\text{Hom}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = 0$ ;
- $\mathcal{D}^{\leq -1} \subset \mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0}$ ;
- $\mathcal{D} = \mathcal{D}^{\leq 0} * \mathcal{D}^{\geq 1}$ , i.e., for any object  $X$  in  $\mathcal{D}$ , there exists a triangle  $A \rightarrow X \rightarrow B \rightsquigarrow$  with  $A \in \mathcal{D}^{\leq 0}$  and  $B \in \mathcal{D}^{\geq 1}$ .

For example, there is a *standard t-structure*  $(\mathcal{D}^b(\mathcal{A})^{\leq 0}, \mathcal{D}^b(\mathcal{A})^{\geq 0})$  on the bounded derived category  $\mathcal{D}^b(\mathcal{A})$  of an abelian category  $\mathcal{A}$  defined by

$$\mathcal{D}^b(\mathcal{A})^{\leq n} = \{K \in \mathcal{D}^b(\mathcal{A}) \mid H^i(K) = 0, \forall i > n\},$$

$$\mathcal{D}^b(\mathcal{A})^{\geq n} = \{K \in \mathcal{D}^b(\mathcal{A}) \mid H^i(K) = 0, \forall i < n\}.$$

Given a t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  on  $\mathcal{D}$ , the inclusion of  $\mathcal{D}^{\leq n}$  (resp.  $\mathcal{D}^{\geq n}$ ) into  $\mathcal{D}$  admits a right (resp. left) adjoint  $\tau_{\leq n}$  (resp.  $\tau_{\geq n}$ ), which are called *truncation functors*. Moreover,  $\mathcal{D}^{\leq n} = {}^{\perp_0}(\mathcal{D}^{\geq n+1}), \mathcal{D}^{\geq n} = (\mathcal{D}^{\leq n-1})^{\perp_0}$ .  $\mathcal{D}^{\leq n}$  is actually characterized by the property that it is a subcategory closed under suspensions and extensions for which the inclusion functor admits a right adjoint. A subcategory of  $\mathcal{D}$  with such a property is called an *aisle* [27]. A dual property characterizes  $\mathcal{D}^{\geq n}$  and a subcategory of  $\mathcal{D}$  with the dual property is called a *co-aisle*. There are bijections between t-structures, aisles and co-aisles, whence these notions are often used interchangeably.

The *heart*  $\mathcal{A}$  of  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is defined as the subcategory  $\mathcal{A} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ .  $\mathcal{A}$  is an abelian subcategory of  $\mathcal{D}$  and we have a system  $\{H^i\}$  of cohomological functors defined by

$$H^i = \tau_{\geq 0}\tau_{\leq 0}(-[i]) : \mathcal{D} \longrightarrow \mathcal{A}.$$

$\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}$  and  $\mathcal{A}$  are closed under extensions and direct summands. Given a sequence  $A \xrightarrow{f} B \xrightarrow{g} C$  of morphisms in  $\mathcal{A}$ ,  $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$  is a short exact sequence in  $\mathcal{A}$  iff  $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$  is a triangle in  $\mathcal{D}$  for some morphism  $h : C \rightarrow A[1]$  in  $\mathcal{D}$ .

Denote  $\mathcal{D}^{[m,n]} = \mathcal{D}^{\geq m} \cap \mathcal{D}^{\leq n}$ . A t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  on  $\mathcal{D}$  is called *bounded* if  $\mathcal{D} = \bigcup_{m,n \in \mathbb{Z}} \mathcal{D}^{[m,n]}$ . If t-structure *tstr* is bounded then an object  $X$  in  $\mathcal{D}$  lies in  $\mathcal{D}^{[m,n]}$  iff  $H^l(X) = 0$  for  $l < m$  and  $l > n$ . A bounded t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is determined by its heart  $\mathcal{A}$ . In fact,

$$\mathcal{D}^{\leq 0} = \cup_{n \geq 0} \mathcal{A}[n] * \mathcal{A}[n-1] * \dots * \mathcal{A},$$

$$\mathcal{D}^{\geq 0} = \cup_{n \leq 0} \mathcal{A} * \dots * \mathcal{A}[n+1] * \mathcal{A}[n].$$

We will also denote by  $(\mathcal{D}_{\mathcal{A}}^{\leq 0}, \mathcal{D}_{\mathcal{A}}^{\geq 0})$  the bounded t-structure with heart  $\mathcal{A}$ .

Any group of exact autoequivalences of  $\mathcal{D}$  acts on the set of t-structures. Given a t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  on  $\mathcal{D}$  and an exact autoequivalence  $\Phi$  of  $\mathcal{D}$ ,

$$\Phi((\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})) := (\Phi(\mathcal{D}^{\leq 0}), \Phi(\mathcal{D}^{\geq 0}))$$

is a t-structure on  $\mathcal{D}$ .  $\Phi((\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}))$  is bounded iff so is  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ .



Suppose  $F : \mathcal{D}_1 \rightarrow \mathcal{D}_2$  is an exact functor between two triangulated categories  $\mathcal{D}_i$  ( $i = 1, 2$ ) equipped with t-structures  $(\mathcal{D}_i^{\leq 0}, \mathcal{D}_i^{\geq 0})$ . We say that  $F$  is *right t-exact* if  $F(\mathcal{D}_1^{\leq 0}) \subset \mathcal{D}_2^{\leq 0}$ , *left t-exact* if  $F(\mathcal{D}_1^{\geq 0}) \subset \mathcal{D}_2^{\geq 0}$ , and *t-exact* if it is both right and left t-exact.

If  $\mathcal{C}$  is a triangulated subcategory of  $\mathcal{D}$  and  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is a t-structure on  $\mathcal{D}$ , the pair

$$(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}) := (\mathcal{C} \cap \mathcal{D}^{\leq 0}, \mathcal{C} \cap \mathcal{D}^{\geq 0})$$

gives a t-structure on  $\mathcal{C}$  iff  $\mathcal{C}$  is stable under some (equivalently, any)  $\tau_{\leq l}$ , i.e.,  $\tau_{\leq l}\mathcal{C} \subset \mathcal{C}$ . Such a t-structure on  $\mathcal{C}$  is called an *induced t-structure by restriction*.

### 2.2 Width-Bounded t-Structures, HRS-Tilt

Let  $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0}), (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  be two t-structures on a triangulated category  $\mathcal{D}$ . We say that  $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0})$  is *width bounded*<sup>3</sup> with respect to  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  if  $\mathcal{D}^{\leq m} \subset \mathcal{D}'^{\leq 0} \subset \mathcal{D}^{\leq n}$  for some  $m, n$ . Define a relation  $\sim$  on the set of t-structures:  $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0}) \sim (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  if  $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0})$  is width bounded with respect to  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ .

**Lemma 2.1**  $\sim$  is an equivalence relation.

*Proof* Reflexivity of  $\sim$  is clear. One sees the symmetry of  $\sim$  by noting that  $\mathcal{D}^{\leq m} \subset \mathcal{D}'^{\leq 0} \subset \mathcal{D}^{\leq n}$  iff  $\mathcal{D}'^{\leq -n} \subset \mathcal{D}^{\leq 0} \subset \mathcal{D}'^{\leq -m}$  and sees the transitivity of  $\sim$  by noting that  $\mathcal{D}^{\leq m} \subset \mathcal{D}'^{\leq 0} \subset \mathcal{D}^{\leq n}$  iff  $\mathcal{D}^{\leq m} \subset \mathcal{D}'^{\leq 0}$  and  $\mathcal{D}'^{\geq 0} \supset \mathcal{D}^{\geq n}$ . □

Obviously, if  $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0})$  is width bounded with respect to  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  then  $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0})$  is a bounded t-structure iff  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is. Hence  $\sim$  restricts to an equivalence relation on the set of bounded t-structures.

Observe that if  $\mathcal{A}$  and  $\mathcal{B}$  are the respective hearts of two bounded t-structures on  $\mathcal{D}$ , the t-structure  $(\mathcal{D}_{\mathcal{B}}^{\leq 0}, \mathcal{D}_{\mathcal{B}}^{\geq 0})$  is width bounded with respect to the t-structure  $(\mathcal{D}_{\mathcal{A}}^{\leq 0}, \mathcal{D}_{\mathcal{A}}^{\geq 0})$  iff  $\mathcal{B} \subset \mathcal{D}_{\mathcal{A}}^{[m,n]}$  for some  $m \leq n$ . Indeed, if  $\mathcal{D}_{\mathcal{A}}^{\leq m} \subset \mathcal{D}_{\mathcal{B}}^{\leq 0} \subset \mathcal{D}_{\mathcal{A}}^{\leq n}$  then  $\mathcal{B} \subset \mathcal{D}_{\mathcal{B}}^{\leq 0} \subset \mathcal{D}_{\mathcal{A}}^{\leq n}, \mathcal{B} \subset \mathcal{D}_{\mathcal{B}}^{\geq 0} \subset \mathcal{D}_{\mathcal{A}}^{\geq m}$  and so  $\mathcal{B} \subset \mathcal{D}_{\mathcal{A}}^{[m,n]}$ ; conversely, if  $\mathcal{B} \subset \mathcal{D}_{\mathcal{A}}^{[m,n]}$  then  $\mathcal{D}_{\mathcal{B}}^{\leq 0} \subset \mathcal{D}_{\mathcal{A}}^{\leq n}, \mathcal{D}_{\mathcal{B}}^{\geq 0} \subset \mathcal{D}_{\mathcal{A}}^{\geq m}$  since  $\mathcal{D}_{\mathcal{B}}^{\leq 0}$  (resp.  $\mathcal{D}_{\mathcal{B}}^{\geq 0}$ ) is the smallest subcategory of  $\mathcal{D}$  containing  $\mathcal{B}$  and closed under extensions and suspensions (resp. desuspensions).

*Example 2.2* (1) If  $\mathcal{D}$  admits a bounded t-structure with length heart containing finitely many (isoclasses of) simple objects, for example,  $\mathcal{D} = \mathcal{D}^b(\Lambda)$  for a finite dimensional algebra  $\Lambda$  over a field  $k$ , then bounded t-structures on  $\mathcal{D}$  are width bounded with respect to each other. By Lemma 2.1, it suffices to show that a bounded t-structure with length heart  $\mathcal{C}$  containing finitely many simple objects is width-bounded with respect to any given bounded t-structure  $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0})$  on  $\mathcal{D}$ . Let  $\{S_i \mid 1 \leq i \leq t\}$  be a complete set of simple objects in  $\mathcal{C}$ . Then  $S_i \in \mathcal{D}^{[k_i, l_i]}$  for each  $i$  and some  $k_i, l_i \in \mathbb{Z}$ . Take  $k = \min\{k_i, l_i \mid 1 \leq i \leq t\}, l = \max\{k_i, l_i \mid 1 \leq i \leq t\}$ .  $\mathcal{C} \subset \mathcal{D}^{[k,l]}$  shows our assertion.

(2) Let  $X$  be a smooth projective variety over a field  $k$  and  $\mathcal{D}^b(X)$  the bounded derived category of coherent sheaves over  $X$ . Then bounded t-structures on  $\mathcal{D}^b(X)$  are width bounded with respect to each other. It suffices to show that the standard t-structure  $(\mathcal{D}_{\text{std}}^{\leq 0}, \mathcal{D}_{\text{std}}^{\geq 0})$  is width bounded with respect to any given bounded t-structure

<sup>3</sup>I learnt this notion from Zeng-Qiang Lin's lectures on the paper [26] of Keller. Moreover, Example 2.2(1) strengthens slightly an example presented by him.

$(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  on  $\mathcal{D}^b(X)$ . Let  $\iota : X \rightarrow \mathbb{P}_k^n$  be a closed immersion, where  $\mathbb{P}_k^n$  is the  $n$ -dimensional projective space over  $k$ , and let  $\mathcal{O}_X(i) = \iota^* \mathcal{O}(i)$ . It follows from Beilinson’s theorem (see e.g. [39, Theorem 3.1.4]) that for each  $j < -n$ , we have an exact sequence

$$0 \rightarrow \mathcal{O}_X(j) \rightarrow V_n \otimes \mathcal{O}_X(-n) \rightarrow \dots \rightarrow V_0 \otimes \mathcal{O}_X \rightarrow 0,$$

where  $V_i = H^n(\mathbb{P}_k^n, \Omega_{\mathbb{P}_k^n}^i(i + j))$  ( $\Omega_{\mathbb{P}_k^n}^i$  is the  $i$ -th wedge product of the cotangent bundle  $\Omega_{\mathbb{P}_k^n}$ ). Since  $\bigoplus_{i=0}^n \mathcal{O}_X(-i)$  lies in some  $\mathcal{D}^{\leq l}$ ,  $\mathcal{O}_X(j)$  lies in  $\mathcal{D}^{\leq l+n}$  for any  $j \leq 0$ . Now that  $\mathcal{D}_{\text{std}}^{\leq 0}$  is the smallest aisle containing  $\{\mathcal{O}_X(j) \mid j \leq 0\}$ , we have  $\mathcal{D}_{\text{std}}^{\leq 0} \subset \mathcal{D}^{\leq l+n}$ . On the other hand, applying the duality functor  $\mathbb{D} = R\mathcal{H}om(-, \mathcal{O}_X)$ , we obtain a bounded t-structure  $(\mathbb{D}(\mathcal{D}^{\geq 0 \text{op}}), \mathbb{D}(\mathcal{D}^{\leq 0 \text{op}}))$  on  $\mathcal{D}^b(X)$ . By the discussion above,  $\mathbb{D}(\mathcal{D}^{\leq 0 \text{op}}) \subset \mathcal{D}_{\text{std}}^{\geq m}$  for some  $m$ . Since  $\mathcal{O}_X$  admits a finite injective resolution of quasi-coherent sheaves, we have  $(\mathbb{D}\mathcal{D}_{\text{std}}^{\geq m})^{\text{op}} \subset \mathcal{D}_{\text{std}}^{\leq r}$  for some  $r$ . So  $\mathcal{D}^{\leq 0} \subset \mathcal{D}_{\text{std}}^{\leq r}$ .  $\mathcal{D}^{\leq -r} \subset \mathcal{D}_{\text{std}}^{\leq 0} \subset \mathcal{D}^{\leq l+n}$  shows our assertion.

Given a bounded t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  on  $\mathcal{D}$  with heart  $\mathcal{A}$ , [22] gives a useful and important construction of a class of width-bounded t-structures with respect to  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  from torsion pairs in  $\mathcal{A}$ , which is called *HRS-tilt*. Now it is well-known (see e.g. [40, §1.1]) that

**Proposition 2.3** *Torsion pairs in the heart of a t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  are in bijective correspondence with t-structures  $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0})$  on  $\mathcal{D}$  satisfying  $\mathcal{D}^{\leq -1} \subset \mathcal{D}'^{\leq 0} \subset \mathcal{D}^{\leq 0}$ .*

Let us explain the correspondence. Assume that  $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0})$  is a t-structure with heart  $\mathcal{B}$  such that  $\mathcal{D}^{\leq -1} \subset \mathcal{D}'^{\leq 0} \subset \mathcal{D}^{\leq 0}$ . Then  $(\mathcal{A} \cap \mathcal{B}, \mathcal{A} \cap \mathcal{B}[-1])$  and  $(\mathcal{A}[1] \cap \mathcal{B}, \mathcal{A} \cap \mathcal{B})$  are torsion pairs in  $\mathcal{A}$  and  $\mathcal{B}$ , respectively. Conversely, let  $(\mathcal{T}, \mathcal{F})$  be a torsion pair in the abelian category  $\mathcal{A}$ . Denote

$$\mathcal{D}'^{\leq 0} = \mathcal{D}^{\leq -1} * \mathcal{T}, \quad \mathcal{D}'^{\geq 0} = \mathcal{F}[1] * \mathcal{D}^{\geq 0}.$$

Then  $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0})$  is a t-structure on  $\mathcal{D}$  with  $\mathcal{D}^{\leq -1} \subset \mathcal{D}'^{\leq 0} \subset \mathcal{D}^{\leq 0}$  and  $(\mathcal{F}[1], \mathcal{T})$  is a torsion pair in its heart  $\mathcal{B}$ . In particular,  $\mathcal{B} = \mathcal{F}[1] * \mathcal{T}$ . The t-structure  $(\mathcal{D}^{\leq -1} * \mathcal{T}, \mathcal{F}[1] * \mathcal{D}^{\geq 0})$  is so-called *HRS-tilt* with respect to the torsion pair  $(\mathcal{T}, \mathcal{F})$  in  $\mathcal{A}$  and  $\mathcal{B} = \mathcal{F}[1] * \mathcal{T}$  is called the *tilted heart*.

As noted before, such a t-structure  $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0})$  is bounded iff  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is. Moreover, if  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is bounded then  $\mathcal{D}^{\leq -1} \subset \mathcal{D}'^{\leq 0} \subset \mathcal{D}^{\leq 0}$  iff  $\mathcal{B} \subset \mathcal{A}[1] * \mathcal{A}$ .

### 2.3 Recollement, Admissible Subcategory, Exceptional Sequence

A *recollement* of triangulated categories [7, §1.4] is a diagram

$$\begin{array}{ccccc} & & & & \\ & & i^* & \xrightarrow{\quad} & \\ & & \xleftarrow{\quad} & & \\ & & i_* & \xrightarrow{\quad} & \\ & & \xleftarrow{\quad} & & \\ & & i^! & \xrightarrow{\quad} & \\ & & \xleftarrow{\quad} & & \\ \mathcal{X} & \xleftrightarrow{\quad} & \mathcal{D} & \xleftrightarrow{\quad} & \mathcal{Y} \\ & & j_! & \xrightarrow{\quad} & \\ & & \xleftarrow{\quad} & & \\ & & j_* & \xrightarrow{\quad} & \\ & & \xleftarrow{\quad} & & \end{array} \tag{2.3.1}$$

of three triangulated categories  $\mathcal{D}, \mathcal{X}, \mathcal{Y}$  and six exact functors  $i^*, i_*, i^!, j_!, j^*, j_*$  between them such that

- $(i^*, i_*, i^!), (j_!, j^*, j_*)$  are adjoint triples;
- $i_*, j_!, j_*$  are fully faithful;
- $\ker j^* = \text{im } i_*$ .

Given such a recollement, there are two functorial triangles in  $\mathcal{D}$ :

$$j_!j^* \rightarrow \text{id} \rightarrow i_*i^* \rightsquigarrow, \quad i_*i^! \rightarrow \text{id} \rightarrow j_*j^* \rightsquigarrow, \tag{2.3.2}$$

where the natural transformations between these functors are given by the respective unit or counit of the relevant adjoint pair.

A well-known equivalent notion is so-called admissible subcategories, due to [9]. Let us recall some classical results from [9]. For a triangulated category  $\mathcal{D}$ , a strictly full triangulated subcategory  $\mathcal{C}$  is called *right* (resp. *left*) *admissible* if the inclusion functor  $\mathcal{C} \hookrightarrow \mathcal{D}$  admits a right (resp. left) adjoint;  $\mathcal{C}$  is called *admissible* if it is both left and right admissible. If  $\mathcal{C}$  is right admissible then  ${}^\perp(\mathcal{C}^\perp) = \mathcal{C}$  and the inclusion functor  $\mathcal{C}^\perp \hookrightarrow \mathcal{D}$  admits a left adjoint. In particular,  $\mathcal{C}$  is closed under direct summands and thus is a thick subcategory of  $\mathcal{D}$ . Moreover, the projection  $\mathcal{C}^\perp \rightarrow \mathcal{D}/\mathcal{C}$  is an exact equivalence. One has dual results for left admissible subcategories. Hence if  $\mathcal{C}$  is admissible then we have

$${}^\perp\mathcal{C} \xrightarrow{\simeq} \mathcal{D}/\mathcal{C} \xleftarrow{\simeq} \mathcal{C}^\perp$$

and we can form (equivalent) recollements

$$\begin{array}{ccc} \mathcal{C} & \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{i_*} \\ \xleftarrow{j_!} \end{array} & \mathcal{D} & \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{j_!} \\ \xleftarrow{j_!} \end{array} & \perp\mathcal{C}, \\ \mathcal{C} & \begin{array}{c} \xleftarrow{i_*} \\ \xrightarrow{i_*} \\ \xleftarrow{i_*} \end{array} & \mathcal{D} & \begin{array}{c} \xleftarrow{\hat{j}^*} \\ \xrightarrow{\hat{j}^*} \\ \xleftarrow{\hat{j}^*} \end{array} & \mathcal{D}/\mathcal{C}, \\ \mathcal{C} & \begin{array}{c} \xleftarrow{i_*} \\ \xrightarrow{i_*} \\ \xleftarrow{i_*} \end{array} & \mathcal{D} & \begin{array}{c} \xleftarrow{\check{j}_*} \\ \xrightarrow{\check{j}_*} \\ \xleftarrow{\check{j}_*} \end{array} & \mathcal{C}^\perp, \end{array} \tag{2.3.3}$$

where  $i_*$ ,  $j_!$ ,  $\check{j}_*$  are the inclusion functors and  $\hat{j}^*$  is the Verdier quotient functor.

We will need the following well-known fact. Recall that a Serre functor of a triangulated category is always exact ([10, Proposition 3.3]; see also [41, Proposition I.1.8]).

**Proposition 2.4** *Let  $\mathcal{D}$  be a Hom-finite  $k$ -linear triangulated category with a Serre functor  $\mathbb{S}$ , where  $k$  is a field, and  $\mathcal{C}$  an admissible subcategory of  $\mathcal{D}$ . Denote by  $i_* : \mathcal{C} \rightarrow \mathcal{D}$  the inclusion functor and by  $i^! : \mathcal{D} \rightarrow \mathcal{C}$  (resp.  $i^* : \mathcal{D} \rightarrow \mathcal{C}$ ) the right (resp. left) adjoint of  $i_*$ . Then*

- (1)  $i^!\mathbb{S}i_*$  is a Serre functor of  $\mathcal{C}$  with a quasi-inverse  $i^*\mathbb{S}^{-1}i_*$ ;
- (2)  ${}^\perp\mathcal{C}$  and  $\mathcal{C}^\perp$  admit Serre functors;
- (3)  $\mathcal{C}^\perp$  and  ${}^\perp\mathcal{C}$  are admissible subcategories of  $\mathcal{D}$ .

*Proof* (1) One easily sees that  $i^!\mathbb{S}i_*$  (resp.  $i^*\mathbb{S}^{-1}i_*$ ) is a right (resp. left) Serre functor of  $\mathcal{C}$ . Thus  $i^!\mathbb{S}i_*$  is a Serre functor of  $\mathcal{C}$  with a quasi-inverse  $i^*\mathbb{S}^{-1}i_*$ .

(2) This is [10, Proposition 3.7].

(3) Recall the well-known fact that if  $\mathcal{D}_1, \mathcal{D}_2$  are two Hom-finite  $k$ -linear triangulated categories with Serre functors  $\mathbb{S}_1, \mathbb{S}_2$  respectively and  $F : \mathcal{D}_1 \rightarrow \mathcal{D}_2$  is an exact functor with a left (resp. right) adjoint  $G$  then  $F$  admits a right (resp. left) adjoint  $\mathbb{S}_1 \circ G \circ \mathbb{S}_2^{-1}$  (resp.  $\mathbb{S}_1^{-1} \circ G \circ \mathbb{S}_2$ ). Thus (3) follows from (2).  $\square$

Important examples of admissible subcategories are those generated by an exceptional sequence [9]. Recall that a sequence  $(E_1, \dots, E_n)$  of objects in a  $k$ -linear triangulated category  $\mathcal{D}$  of finite type, where  $k$  is a field, is called an *exceptional sequence* if

- each  $E_i$  is an exceptional object, i.e.,  $\text{Hom}^{\neq 0}(E_i, E_i) = 0$  and  $\text{End}(E_i) = k$ ;
- $\text{Hom}^\bullet(E_j, E_i) = 0$  if  $j > i$ .

An exceptional sequence  $(E_1, \dots, E_n)$  is said to be *full* if  $E_1, \dots, E_n$  classically generate  $\mathcal{D}$ .

Let  $\mathcal{C} = \langle E_1, \dots, E_n \rangle_{\mathcal{D}}$  be the thick closure of  $\{E_i \mid 1 \leq i \leq n\}$  and  $i_* : \mathcal{C} \rightarrow \mathcal{D}$  be the inclusion functor. The left and right adjoint functors of  $i_*$  exist, which we denote by  $i^*, i^!$  respectively. Let us recall from [9] how  $i^*$  maps an object. Suppose  $X \in \mathcal{D}$ . Denote  $X_0 = X$ . If  $X_i$  is defined for  $0 \leq i < n$ , let

$$X_{i+1} = \text{co-cone}(X_i \xrightarrow{\text{co-ev}} D\text{Hom}^\bullet(X_i, E_{i+1}) \otimes E_{i+1}).$$

Then  $X_{i+1} \in {}^\perp\{E_1, \dots, E_{i+1}\}$ . Define  $i^*X = X_n$ . We have  $i^*X \in {}^\perp\mathcal{C}$  and  $i^*X$  fits into a triangle  $i^*X \rightarrow X \rightarrow Y \rightsquigarrow$  where  $Y \in \mathcal{C}$ . This choice of  $i^*$  on objects actually defines a unique functor up to a unique isomorphism, which is left adjoint to  $i_*$ . Dually one defines  $i^!$ .

### 2.4 Gluing t-Structures

Now fix a recollement of triangulated categories of the form (2.3.1). As the following theorem shows, one can obtain a t-structure on  $\mathcal{D}$  from t-structures on  $\mathcal{X}$  and  $\mathcal{Y}$ , which is called a *glued t-structure*. Such a glued t-structure on  $\mathcal{D}$  from the recollement is also said to be *compatible with the recollement*.

**Theorem 2.5** [7, Théorème 1.4.10] *Given t-structures  $(\mathcal{X}^{\leq 0}, \mathcal{X}^{\geq 0})$  and  $(\mathcal{Y}^{\leq 0}, \mathcal{Y}^{\geq 0})$  on  $\mathcal{X}$  and  $\mathcal{Y}$  respectively, denote*

$$\begin{aligned} \mathcal{D}^{\leq 0} &= \{X \in \mathcal{D} \mid i^*X \in \mathcal{X}^{\leq 0}, j^*X \in \mathcal{Y}^{\leq 0}\}, \\ \mathcal{D}^{\geq 0} &= \{X \in \mathcal{D} \mid i^!X \in \mathcal{X}^{\geq 0}, j_*X \in \mathcal{Y}^{\geq 0}\}. \end{aligned} \tag{2.4.1}$$

*Then  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is a t-structure on  $\mathcal{D}$ .*

With the given t-structures on  $\mathcal{X}, \mathcal{Y}$  and the glued t-structure on  $\mathcal{D}$ ,  $i^*, j_!$  becomes right t-exact,  $i_*, j^*$  t-exact and  $i^!, j_*$  left t-exact.

The following proposition answers the natural question when a t-structure on  $\mathcal{D}$  is compatible with a given recollement.

**Proposition 2.6** [7, Proposition 1.4.12] *Given a t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  on  $\mathcal{D}$ , the following conditions are equivalent:*

- (1)  $j_!j^*$  is right t-exact;
- (2)  $j_*j^!$  is left t-exact;
- (3) the t-structure is compatible with the recollement (2.3.1).

Moreover, we have

**Lemma 2.7** [35, Corollary 3.4, Lemma 3.5] *There is a bijection*

$$\{t\text{-structures on } \mathcal{X}\} \times \{t\text{-structures on } \mathcal{Y}\} \longleftrightarrow \{t\text{-structures on } \mathcal{D} \text{ compatible with the recollement (2.3.1)}\}, \quad (2.4.2)$$

which restricts to a bijection between bounded t-structures.

Indeed, once the equivalent conditions in Proposition 2.6 are satisfied, to obtain  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  using formula (2.4.1), the unique choice of the t-structure on  $\mathcal{X}$  resp.  $\mathcal{Y}$  is

$$(i^* \mathcal{D}^{\leq 0}, i^! \mathcal{D}^{\geq 0}) \text{ resp. } (j^* \mathcal{D}^{\leq 0}, j^* \mathcal{D}^{\geq 0}). \quad (2.4.3)$$

This t-structure on  $\mathcal{X}$  resp.  $\mathcal{Y}$  will be called *the corresponding t-structure* on  $\mathcal{X}$  resp.  $\mathcal{Y}$  to the t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  on  $\mathcal{D}$ . Moreover we have

$$(i_* i^* \mathcal{D}^{\leq 0}, i_* i^! \mathcal{D}^{\geq 0}) = (\text{im } i_* \cap \mathcal{D}^{\leq 0}, \text{im } i_* \cap \mathcal{D}^{\geq 0}). \quad (2.4.4)$$

Since we can identify  $\mathcal{X}$  with  $\text{im } i_*$  via  $i_*$ , we know that the t-structure on  $\mathcal{X}$  is essentially induced by restriction.

Suppose  $\mathcal{C}$  is an admissible subcategory of  $\mathcal{D}$  and  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is a t-structure on  $\mathcal{D}$ . Let

$$\begin{array}{ccc} \mathcal{C} & \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!} \end{array} & \mathcal{D} & \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*} \end{array} & \mathcal{C}' \end{array} \quad (2.4.5)$$

be a recollement, where  $i_*$  is the inclusion functor. Since  $j_! j^* X = \text{co-cone}(X \rightarrow i_* i^* X)$  for each  $X \in \mathcal{D}$  by Eq. 2.3.2,  $j_! j^*$  is right t-exact iff  $\text{co-cone}(X \rightarrow i_* i^* X)$  lies in  $\mathcal{D}^{\leq 0}$  for each  $X \in \mathcal{D}^{\leq 0}$ . So given another recollement

$$\begin{array}{ccc} \mathcal{C} & \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!} \end{array} & \mathcal{D} & \begin{array}{c} \xleftarrow{k_!} \\ \xrightarrow{k^*} \\ \xleftarrow{k_*} \end{array} & \mathcal{C}'' \end{array}, \quad (2.4.6)$$

$(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the recollement (2.4.5) iff it is compatible with the (equivalent) recollement (2.4.6). Thus it makes sense to say that  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with  $\mathcal{C}$  if  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with any recollement of the form Eq. 2.4.5, for example, any one of the recollements (2.3.3). This is convenient for use. If  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the admissible subcategory  $\mathcal{C}$  then  $(\mathcal{D}^{\leq 0} \cap \mathcal{C}, \mathcal{D}^{\geq 0} \cap \mathcal{C})$  is a t-structure on  $\mathcal{C}$ . In general, consider a finite admissible filtration [10, Definition 4.1]

$$\mathcal{D}_n \subset \mathcal{D}_{n-1} \subset \dots \subset \mathcal{D}_0 = \mathcal{D}$$

of a triangulated category  $\mathcal{D}$ . That is, each  $\mathcal{D}_i$  ( $1 \leq i \leq n$ ) is an admissible subcategory of  $\mathcal{D}_{i-1}$ , equivalently, each  $\mathcal{D}_i$  is an admissible subcategory of  $\mathcal{D}$ . We say the t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the admissible filtration if it is compatible with each  $\mathcal{D}_i$ .

Clearly we have the following two facts.

**Lemma 2.8**  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the admissible filtration

$$\mathcal{D}_n \subset \dots \subset \mathcal{D}_1 \subset \mathcal{D}_0 = \mathcal{D}$$

of  $\mathcal{D}$  iff the t-structure  $(\mathcal{D}^{\leq 0} \cap \mathcal{D}_i, \mathcal{D}^{\geq 0} \cap \mathcal{D}_i)$  on  $\mathcal{D}_i$  is compatible with  $\mathcal{D}_{i+1}$  for each  $1 \leq i \leq n - 1$ .

Here by the statement that the t-structure  $(\mathcal{D}^{\leq 0} \cap \mathcal{D}_i, \mathcal{D}^{\geq 0} \cap \mathcal{D}_i)$  on  $\mathcal{D}_i$  is compatible with  $\mathcal{D}_{i+1}$  for each  $1 \leq i \leq n - 1$ , we actually mean that:  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with  $\mathcal{D}_1$  (hence  $(\mathcal{D}^{\leq 0} \cap \mathcal{D}_1, \mathcal{D}^{\geq 0} \cap \mathcal{D}_1)$  is a t-structure on  $\mathcal{D}_1$ );  $(\mathcal{D}^{\leq 0} \cap \mathcal{D}_1, \mathcal{D}^{\geq 0} \cap \mathcal{D}_1)$  is compatible

with  $\mathcal{D}_2$  (hence  $(\mathcal{D}^{\leq 0} \cap \mathcal{D}_2, \mathcal{D}^{\geq 0} \cap \mathcal{D}_2)$  is a t-structure on  $\mathcal{D}_2$ ); and so on. This situation arises naturally from reduction/induction argument.

**Lemma 2.9** *Suppose that the t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the admissible filtration*

$$\mathcal{D}_n \subset \mathcal{D}_{n-1} \subset \dots \subset \mathcal{D}_0 = \mathcal{D}$$

and let  $\Phi$  be an exact autoequivalence of  $\mathcal{D}$ . Then the t-structure  $(\Phi(\mathcal{D}^{\leq 0}), \Phi(\mathcal{D}^{\geq 0}))$  is compatible with the admissible filtration

$$\Phi(\mathcal{D}_n) \subset \Phi(\mathcal{D}_{n-1}) \subset \dots \subset \Phi(\mathcal{D}_0) = \mathcal{D}.$$

**2.5 On the Hearts of the t-Structures in a Recollement Context**

Fix a recollement of the form (2.3.1). Each t-structure  $(\mathcal{X}^{\leq 0}, \mathcal{X}^{\geq 0})$  on  $\mathcal{X}$  induces (up to shift) two t-structures on  $\mathcal{D}$  in the following fashion. For each  $p \in \mathbb{Z}$ , since the inclusion  $i_*\mathcal{X}^{\leq p} \hookrightarrow \mathcal{D}$  admits a right adjoint  $i_*\tau_{\leq p}i^!$ ,  $i_*\mathcal{X}^{\leq p}$  is an aisle in  $\mathcal{D}$  and

$$(i_*\mathcal{X}^{\leq p}, (i_*\mathcal{X}^{\leq p})^{\perp_{0,\mathcal{D}}}[1])$$

is a t-structure on  $\mathcal{D}$ . Denote by  $\check{\tau}_{\geq p+1}$  the left adjoint of the inclusion  $(i_*\mathcal{X}^{\leq p})^{\perp_{0,\mathcal{D}}} \hookrightarrow \mathcal{D}$ . Then we have a functorial triangle

$$i_*\tau_{\leq p}i^! \rightarrow \text{id} \rightarrow \check{\tau}_{\geq p+1} \rightsquigarrow$$

for each  $p \in \mathbb{Z}$ . Dually, the inclusion  $i_*\mathcal{X}^{\geq p} \hookrightarrow \mathcal{D}$  admits a left adjoint  $i_*\tau_{\geq p}i^*$ , and we have a t-structure

$$(({}^{\perp_{0,\mathcal{D}}}i_*\mathcal{X}^{\geq p})[-1], i_*\mathcal{X}^{\geq p})$$

and a functorial triangle

$$\hat{\tau}_{\leq p-1} \rightarrow \text{id} \rightarrow i_*\tau_{\geq p}i^* \rightsquigarrow$$

for each  $p \in \mathbb{Z}$ , where  $\hat{\tau}_{\leq p-1}$  is the right adjoint of the inclusion  $({}^{\perp_{0,\mathcal{D}}}i_*\mathcal{X}^{\geq p}) \hookrightarrow \mathcal{D}$ . A similar argument shows that a t-structure on  $\mathcal{Y}$  also induces two t-structures on  $\mathcal{D}$ .

*Remark 2.10* In [7, §1.4.13], these induced t-structures are described via gluing.

Suppose  $(\mathcal{X}^{\leq 0}, \mathcal{X}^{\geq 0}), (\mathcal{Y}^{\leq 0}, \mathcal{Y}^{\geq 0})$  are t-structures on  $\mathcal{X}, \mathcal{Y}$  respectively and let  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  be the glued t-structure. Denote the respective heart by  $\mathcal{B}_1, \mathcal{B}_2$  and  $\mathcal{B}$ . Let  $\epsilon$  be the inclusion functor from  $\mathcal{B}_1, \mathcal{B}_2$  resp.  $\mathcal{B}$  to  $\mathcal{X}, \mathcal{Y}$  resp.  $\mathcal{D}$ . For  $T \in \{i^*, i_*, i^!, j!, j^*, j_*\}$ , denote  ${}^pT = H^0 \circ T \circ \epsilon$ . Then  $({}^pi^*, {}^pi_*, {}^pi^!)$  and  $({}^pj!, {}^pj^*, {}^pj_*)$  are adjoint triples, the compositions  ${}^pj^* \circ {}^pi_*, {}^pi^* \circ {}^pj!, {}^pi^! \circ {}^pj_*$  vanish, and  ${}^pi_*, {}^pj!, {}^pj_*$  are fully faithful.  $\text{im } {}^pi_* = \ker {}^pj^*$  is a Serre subcategory of  $\mathcal{B}$ , the functor  ${}^pi_*$  identifies  $\mathcal{B}_1$  with  $\text{im } {}^pi_*$  and the functor  ${}^pj^*$  identifies the quotient category  $\mathcal{B}/\text{im } {}^pi_*$  with  $\mathcal{B}_2$ . The composition  ${}^pj! \circ {}^pj^* \rightarrow \text{id} \rightarrow {}^pj_* \circ {}^pj^*$  provides a unique morphism of functors  ${}^pj! \rightarrow {}^pj_*$ . Define

$$j_{!*} = \text{im} ({}^pj!(-) \rightarrow {}^pj_*(-)) : \mathcal{B}_2 \longrightarrow \mathcal{B}. \tag{2.5.1}$$

The following proposition describes simple objects in  $\mathcal{B}$ .

**Proposition 2.11** [7, Proposition 1.4.23, 1.4.26]

(1) For  $X \in \mathcal{B}_2$ , we have

$$j_{!*}X = \check{\tau}_{\geq 1}j!X = \hat{\tau}_{\leq -1}j_*X.$$

(2) Simple objects in  $\mathcal{B}$  are those  ${}^pi_*S$ , for  $S$  simple in  $\mathcal{B}_1$ , and those  $j_{!*}S$ , for  $S$  simple in  $\mathcal{B}_2$ .

For more details, see [7, §1.4], from which the above are taken.

Recall that an abelian category  $\mathcal{A}$  is said to be *noetherian* (resp. *artinian*) if for any object  $A$  in  $\mathcal{A}$ , any infinite ascending (resp. descending) chain of subobjects of  $A$  stabilizes;  $\mathcal{A}$  is said to be *of finite length* or simply *length* if it is both noetherian and artinian. The following lemma strengthens [35, Proposition 3.9].

**Lemma 2.12**  $\mathcal{B}$  is noetherian (or artinian, or of finite length) iff so are  $\mathcal{B}_1, \mathcal{B}_2$ .

*Proof* [14, Lemma 1.3.3] states that if  $\mathcal{A}_1$  is a Serre subcategory of an abelian category  $\mathcal{A}$  then  $\mathcal{A}$  is noetherian iff  $\mathcal{A}_1$  and  $\mathcal{A}/\mathcal{A}_1$  are noetherian and if each object in  $\mathcal{A}$  has a largest subobject that belongs to  $\mathcal{A}_1$ . We claim that in our setting, each  $B \in \mathcal{B}$  admits a largest subobject  ${}^p i_* {}^p i^! B$  in  ${}^p i_* \mathcal{B}_1$ . By [7, Lemme 1.4.19], we have an exact sequence

$$0 \rightarrow {}^p i_* {}^p i^! B \xrightarrow{\eta} B \rightarrow {}^p j_* {}^p j^* B \rightarrow {}^p i_* H^1 i^! B \rightarrow 0.$$

Suppose  $\mu : {}^p i_* Z \rightarrow B$  is a monomorphism in  $\mathcal{B}$ , where  $Z \in \mathcal{B}_1$ . Note that

$$\text{Hom}({}^p i_* Z, {}^p j_* {}^p j^* B) = \text{Hom}(Z, {}^p i^! {}^p j_* {}^p j^* B) = 0.$$

So there exists  $\nu : {}^p i_* Z \rightarrow {}^p i_* {}^p i^! B$  such that  $\mu = \eta\nu$ . Since  $\mu$  is a monomorphism,  $\nu$  is a monomorphism. So  ${}^p i_* Z$  is a subobject of  ${}^p i_* {}^p i^! B$ . This shows our claim that  ${}^p i_* {}^p i^! B$  is the largest subobject of  $B$  in  ${}^p i_* \mathcal{B}_1$ . Hence the assertion on noetherianness follows. By duality, we conclude the assertion on artinianness. Combining these two assertions, we know that  $\mathcal{B}$  is of finite length iff  $\mathcal{B}_1, \mathcal{B}_2$  are of finite length. □

An easy induction argument yields

**Corollary 2.13** Suppose a t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  on  $\mathcal{D}$  is compatible with the admissible filtration

$$0 = \mathcal{D}_{n+1} \subset \mathcal{D}_n \subset \dots \subset \mathcal{D}_1 \subset \mathcal{D}_0 = \mathcal{D}.$$

Then  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  has a noetherian resp. artinian resp. length heart iff the corresponding t-structure on each  $\mathcal{D}_{i+1}^{\perp \mathcal{D}_i}$  (or  ${}^{\perp \mathcal{D}_i} \mathcal{D}_{i+1}$ , or  $\mathcal{D}_i/\mathcal{D}_{i+1}$ )  $(0 \leq i \leq n)$  has a noetherian resp. artinian resp. length heart.

### 2.6 Recollement and Ext-Projectives

Let  $\mathcal{D}$  be a  $k$ -linear triangulated category of finite type, where  $k$  is a field, and  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  a t-structure on  $\mathcal{D}$ . Recall from [2, §1] that  $X \in \mathcal{D}$  is Ext-projective in  $\mathcal{D}^{\leq l}$ , or  $\mathcal{D}^{\leq l}$ -projective for short, if  $X \in \mathcal{D}^{\leq l}$  and  $\text{Hom}^1(X, \mathcal{D}^{\leq l}) = 0$ ; dually,  $X \in \mathcal{D}$  is Ext-injective in  $\mathcal{D}^{\geq l}$ , or  $\mathcal{D}^{\geq l}$ -injective, if  $X \in \mathcal{D}^{\geq l}$  and  $\text{Hom}^1(\mathcal{D}^{\geq l}, X) = 0$ .

We use the following criterion to identify Ext-projectives (and Ext-injectives) when  $\mathcal{D}$  admits a Serre functor.

**Lemma 2.14** [2, Lemma 1.5] Suppose  $\mathcal{D}$  admits a Serre functor  $\mathbb{S}$  and  $X$  is an object in  $\mathcal{D}$ . Then  $X$  is  $\mathcal{D}^{\leq 0}$ -projective iff  $X \in \mathcal{D}^{\leq 0}$  with  $\mathbb{S}X \in \mathcal{D}^{\geq 0}$  iff  $\mathbb{S}X$  is  $\mathcal{D}^{\geq 0}$ -injective.

The following easy observation is essential for us.

**Lemma 2.15** *Suppose  $E \in \mathcal{D}$  is an exceptional object. If  $E$  is Ext-projective in some  $\mathcal{D}^{\leq l}$  and  $E^{\perp \mathcal{D}}$  is right admissible then  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the recollement*

$$\begin{array}{ccc}
 E^{\perp \mathcal{D}} & \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{i_*} \\ \xleftarrow{j_!} \end{array} & \mathcal{D} & \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_!} \end{array} & \langle E \rangle_{\mathcal{D}}
 \end{array}$$

where  $i_*, j_!$  are the inclusion functors.

*Proof* Since  $E$  is an exceptional object,  $\langle E \rangle_{\mathcal{D}}$  is admissible and thus  $E^{\perp \mathcal{D}}$  is left admissible with  ${}^{\perp \mathcal{D}}(E^{\perp \mathcal{D}}) = \langle E \rangle_{\mathcal{D}}$ . If  $E^{\perp \mathcal{D}}$  is right admissible then  $E^{\perp \mathcal{D}}$  is admissible and the given diagram is indeed a diagram of recollement. To show that the t-structure is compatible, it suffices to show that  $j_!j^*$  is right t-exact, i.e., for each  $X \in \mathcal{D}^{\leq 0}$ ,  $j_!j^*(X) \in \mathcal{D}^{\leq 0}$ . Note that for  $m > -l$ ,  $\text{Hom}(E, \mathcal{D}^{\leq 0}[m]) = 0$  since  $E$  is  $\mathcal{D}^{\leq l}$ -projective. Therefore

$$\begin{aligned}
 j_!j^*(X) &= \text{Hom}^\bullet(E, X) \otimes E \\
 &= \oplus \text{Hom}(E, X[m]) \otimes E[-m] \\
 &= \oplus_{m \leq -l} \text{Hom}(E, X[m]) \otimes E[-m] \\
 &\in \mathcal{D}^{\leq 0}.
 \end{aligned}$$

□

*Remark 2.16* (1) There is a dual version for Ext-injectives.

(2) In our application,  $\mathcal{D}$  has a Serre functor and thus  $E^{\perp \mathcal{D}}$  and  ${}^{\perp \mathcal{D}}E$  are indeed admissible by Proposition 2.4.

Assume that  $\mathcal{D}$  has a Serre functor and  $(E_n, \dots, E_1)$  is an exceptional sequence such that each  $E_i$  is  $\mathcal{D}^{\leq 0}$ -projective. Let  $\mathcal{D}_0 = \mathcal{D}$ ; for  $1 \leq i \leq n$ , let  $\mathcal{D}_i = \{E_i, E_{i-1}, \dots, E_1\}^{\perp \mathcal{D}}$ . Note that  $\mathcal{D}_i = E_i^{\perp \mathcal{D}_{i-1}}$  for  $1 \leq i \leq n$ . We already know that  $\langle E_i, E_{i-1}, \dots, E_1 \rangle_{\mathcal{D}}$  is admissible in  $\mathcal{D}$  and thus  $\mathcal{D}_i$  is admissible in  $\mathcal{D}$  by Proposition 2.4. The following fact is immediate from Lemma 2.15 and Lemma 2.8. (We also have a similar result when each  $E_i$  is  $\mathcal{D}^{\geq 0}$ -injective.)

**Corollary 2.17** *With the above hypotheses and notation,  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the admissible filtration*

$$\mathcal{D}_n \subset \dots \subset \mathcal{D}_i (= \{E_i, \dots, E_1\}^{\perp \mathcal{D}} = E_i^{\perp \mathcal{D}_{i-1}}) \subset \dots \subset \mathcal{D}_1 \subset \mathcal{D}.$$

Now let us be given a recollement of the form (2.3.1). Suppose that  $\mathcal{X}$  resp.  $\mathcal{Y}$  is equipped with a t-structure  $(\mathcal{X}^{\leq 0}, \mathcal{X}^{\geq 0})$  resp.  $(\mathcal{Y}^{\leq 0}, \mathcal{Y}^{\geq 0})$ , and  $\mathcal{D}$  with the glued t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ . One easily verifies the following fact.

- Lemma 2.18** (1) *If  $X$  is  $\mathcal{D}^{\leq 0}$ -projective which does not lie in  $\ker i^* = \text{im } j_!$ , then  $i^*X$  is nonzero  $\mathcal{X}^{\leq 0}$ -projective.*  
 (2) *If  $Y$  is nonzero  $\mathcal{Y}^{\leq 0}$ -projective then  $j_!Y$  is nonzero  $\mathcal{D}^{\leq 0}$ -projective. Moreover,  $j_!$  induces a bijection between isoclasses of indecomposable Ext-projectives in  $\mathcal{Y}^{\leq 0}$  and isoclasses of indecomposable Ext-projectives in  $\mathcal{D}^{\leq 0}$  which lie in  $\ker i^* = \text{im } j_!$ .*



### 2.7 Some Facts on Hereditary Categories

Let  $\mathcal{A}$  be a hereditary category linear over an algebraically closed field  $k$  with finite-dimensional morphism and extension spaces. It's well-known that each object  $X \in \mathcal{D}^b(\mathcal{A})$  decomposes as  $X \cong \bigoplus_i H^i(X)[-i]$ . In particular, each indecomposable object in  $\mathcal{D}^b(\mathcal{A})$  is a shift of an indecomposable object in  $\mathcal{A}$ .

The following Happel-Ringel Lemma (see e.g. [31, Proposition 5.1]) is fundamental for hereditary categories.

**Proposition 2.19** (Happel-Ringel Lemma) *Let  $E$  and  $F$  be indecomposable objects of  $\mathcal{A}$  such that  $\text{Ext}^1(F, E) = 0$ . Then each nonzero morphism  $f : E \rightarrow F$  is a monomorphism or an epimorphism. In particular, each indecomposable object in  $\mathcal{A}$  without self-extension is exceptional.*

Recall that an object  $T$  in a triangulated category is a *partial silting object* if  $\text{Hom}^{>0}(T, T) = 0$  and  $T$  is *basic* if its indecomposable direct summands are pairwise non-isomorphic. The following fact shows that a basic partial silting object in  $\mathcal{D}^b(\mathcal{A})$  can yield an exceptional sequence. Note that  $\mathcal{D}^b(\mathcal{A})$  is a Krull-Schmidt category since  $\mathcal{A}$  is Hom-finite.

**Proposition 2.20** [1, Proposition 3.11] *Let  $X$  be a basic partial silting object in  $\mathcal{D}^b(\mathcal{A})$ . Then pairwise non-isomorphic indecomposable direct summands of  $X$  can be ordered to form an exceptional sequence.*

Although it is stated for specific hereditary categories in [1, Proposition 3.11], the above fact follows from Happel-Ringel Lemma.

We will need to relate Ext-projectives to an exceptional sequence.

**Proposition 2.21** [2, Theorem (A)] *Let  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  be a t-structure in  $\mathcal{D}^b(\mathcal{A})$ . Then finitely many pairwise non-isomorphic indecomposable  $\mathcal{D}^{\leq 0}$ -projectives can be ordered to form an exceptional sequence in  $\mathcal{D}^b(\mathcal{A})$ .*

Proposition 2.21 follows from Proposition 2.20 since the direct sum of finitely many pairwise non-isomorphic indecomposable  $\mathcal{D}^{\leq 0}$ -projectives is a basic partial silting object.

### 2.8 Bounded t-Structures on $\mathcal{D}^b(\Lambda)$ for a Finite Dimensional Algebra $\Lambda$

Recall from [1, 27, 46] that an object  $X$  in a triangulated category  $\mathcal{D}$  is called *silting* if it is partial silting, i.e.,  $\text{Hom}^{>0}(X, X) = 0$ , and if  $\langle X \rangle_{\mathcal{D}} = \mathcal{D}$ . It is *tilting* if additionally  $\text{Hom}^{<0}(X, X) = 0$ . Two silting objects  $X$  and  $Y$  are said to be equivalent if  $\text{add } X = \text{add } Y$ .

Let  $\Lambda$  be a finite dimensional algebra over a field  $k$ . Denote by  $\mathcal{K}^b(\text{proj } \Lambda)$  the bounded homotopy category of finite dimensional projective right modules over  $\Lambda$ . The following part of König-Yang correspondences will be used repeatedly in the sequel. See [28] for bijective correspondences between more concepts.

**Theorem 2.22** [28, Theorem 6.1] *Equivalence classes of silting objects in  $\mathcal{K}^b(\text{proj } \Lambda)$  are in bijective correspondence with bounded t-structures on  $\mathcal{D}^b(\Lambda)$  with length heart.*

Let us recall this correspondence from [28]. For a silting object  $M$  in  $\mathcal{K}^b(\text{proj}\Lambda)$ , the associated t-structure on  $\mathcal{D}^b(\Lambda)$  is given by the pair

$$\mathcal{D}^{\leq 0} = \{N \in \mathcal{D}^b(\Lambda) \mid \text{Hom}^{>0}(M, N) = 0\},$$

$$\mathcal{D}^{\geq 0} = \{N \in \mathcal{D}^b(\Lambda) \mid \text{Hom}^{<0}(M, N) = 0\}.$$

Moreover, the heart of  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is equivalent to  $\text{mod End}(M)$  [28, Lemma 5.3]. We refer the reader to [28, §5.6] for the general construction (essentially due to Rickard [42]) of a silting object associated to a given bounded t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  on  $\mathcal{D}^b(\Lambda)$  with length heart. When  $\Lambda$  has finite global dimension, in which case the natural inclusion  $\mathcal{K}^b(\text{proj}\Lambda) \rightarrow \mathcal{D}^b(\Lambda)$  is an exact equivalence, the associated basic silting object in  $\mathcal{K}^b(\text{proj}\Lambda) = \mathcal{D}^b(\Lambda)$  is just the direct sum of a complete set of indecomposable Ext-projectives in the aisle  $\mathcal{D}^{\leq 0}$ .

**Lemma 2.23** [35, Lemma 6.7] *If  $\Lambda$  is a representation-finite hereditary algebra then each bounded t-structure on  $\mathcal{D}^b(\Lambda)$  has a length heart.*

Hence by Theorem 2.22, to classify bounded t-structures on  $\mathcal{D}^b(\Lambda)$ , where  $\Lambda$  is a representation-finite hereditary algebra, it suffices, say, to classify silting objects in  $\mathcal{D}^b(\Lambda)$ , which is indeed computable.

The following fact characterizes when a silting object is a tilting object in the presence of a Serre functor.

**Lemma 2.24** [36, Lemma 4.6] *Assume that  $\Lambda$  has finite global dimension and  $\mathbb{S}$  is a Serre functor of  $\mathcal{D}^b(\Lambda)$ . Let  $T$  be a silting object in  $\mathcal{D}^b(\Lambda)$  and  $\mathcal{B}$  the heart of the corresponding t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ . Then  $T$  is tilting iff  $\mathbb{S}$  is right t-exact with respect to  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  iff  $\mathbb{S}T$  lies in  $\mathcal{B}$ .*

We will also need the next two facts.

**Lemma 2.25** *Let  $k\vec{\mathbb{A}}_s$  be the path algebra of the equioriented  $\mathbb{A}_s$ -quiver. Suppose  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is a bounded t-structure on  $\mathcal{D}^b(k\vec{\mathbb{A}}_s)$ . Then some simple  $k\vec{\mathbb{A}}_s$ -module is Ext-projective in some  $\mathcal{D}^{\leq l}$ .*

*Proof* Denote  $\mathcal{A} = \text{mod } k\vec{\mathbb{A}}_s$ ,  $\mathcal{D} = \mathcal{D}^b(k\vec{\mathbb{A}}_s)$  for short. It is well-known that  $\mathcal{A}$  is a uniserial hereditary abelian category, each indecomposable object in  $\mathcal{A}$  is exceptional, and  $\mathcal{D}$  has a Serre functor (isomorphic to the Nakayama functor). We use induction on  $s$  to show our assertion. If  $s = 1$ , we have  $\text{mod } k\vec{\mathbb{A}}_1 = \text{mod } k$  and the assertion obviously holds. Suppose  $s > 1$ . By Lemma 2.23, the heart  $\mathcal{B}$  of  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is of finite length. Take an indecomposable direct summand  $N[p]$  ( $N \in \mathcal{A}$ ) of the corresponding silting object. Then  $N$  is  $\mathcal{D}^{\leq p}$ -projective. If  $N$  is a simple module then  $N$  is the desired. Otherwise, let

$$\bar{\mathcal{A}}_1 = \langle \tau^m(\text{top}(N)) \mid 1 \leq m < l(N) \rangle_{\mathcal{A}}, \quad \bar{\mathcal{A}}_1 = \langle \tau^m(\text{top}(N)) \mid 0 \leq m < l(N) \rangle_{\mathcal{A}},$$

where  $\tau = D\text{Tr}$  represents the Auslander-Reiten translation and  $l(N)$  is the length of  $N$ . For a simple module  $S$ , denote by  ${}^{[l]}S$  the unique indecomposable module with top  $S$  and of length  $l$ . Since  $\bigoplus_{0 \leq i < l(N)} {}^{[l(N)-i]} \tau^i \text{top}(N)$  is a projective generator for  $\bar{\mathcal{A}}_1$  with endomorphism algebra isomorphic to  $k\vec{\mathbb{A}}_{l(N)}$ , we have  $\bar{\mathcal{A}}_1 \simeq \text{mod } k\vec{\mathbb{A}}_{l(N)}$ .

We know that  $N^{\perp \mathcal{A}}$  is an exact subcategory of  $\mathcal{A}$  closed under extensions. Take

$$\mathcal{A}_2 = \text{add}\{M \in N^{\perp \mathcal{A}} \mid M \text{ is indecomposable and } M \notin \mathcal{A}_1\}.$$

We claim  $N^{\perp \mathcal{A}} = \mathcal{A}_1 \amalg \mathcal{A}_2$ , which implies that  $\mathcal{A}_2$  is an exact subcategory of  $\mathcal{A}$  closed under extensions. Since  $N^{\perp \mathcal{A}} = \text{add}\mathcal{A}_1 \cup \mathcal{A}_2$ , it suffices to show that  $\text{Hom}(\mathcal{A}_1, \mathcal{A}_2) = 0 = \text{Hom}(\mathcal{A}_2, \mathcal{A}_1)$ . Note that

$$\begin{aligned} \mathcal{A}_1 &= \text{add}\{{}^{[l]}\tau^i \text{top}(N) \mid 1 \leq i < l(N), 1 \leq l \leq l(N) - i\}, \\ N^{\perp \mathcal{A}} &= \{M \in N^{\perp \mathcal{A}} \mid \text{Hom}(N, M) = 0 = \text{Ext}^1(N, M)\} \\ &= \{M \in N^{\perp \mathcal{A}} \mid \text{Hom}(N, M) = 0 = \text{Hom}(M, \tau N)\}. \end{aligned}$$

Let  $M$  be an indecomposable  $k\bar{\mathbb{A}}_s$ -module. Suppose  $\text{Hom}({}^{[l]}\tau^i \text{top}(N), M) \neq 0$  for some  $1 \leq i < l(N), 1 \leq l \leq l(N) - i$ . Then for some  $1 \leq k \leq l, {}^{[k]}\tau^i \text{top}(N)$  is a subobject of  $M$ . If  $M \notin \mathcal{A}_1$  then  ${}^{[k+i]}\text{top}(N)$  is a subobject of  $M$ . Meanwhile,  ${}^{[k+i]}\text{top}(N)$  is a quotient object of  $N$  and thus  $\text{Hom}(N, M) \neq 0$ . This shows that if  $\text{Hom}(N, M) = 0$  then  $\text{Hom}(\mathcal{A}_1, M) = 0$ . Similarly, if  $\text{Hom}(M, {}^{[l]}\tau^i \text{top}(N)) \neq 0$  for some  $1 \leq i < l(N), 1 \leq l \leq l(N) - i$ , then  $M$  has a nonzero quotient object which is moreover a subobject of  $\tau N$ ; so  $\text{Hom}(M, \mathcal{A}_1) = 0$  if  $\text{Hom}(M, \tau N) = 0$ . It follows that  $\text{Hom}(\mathcal{A}_1, M) = 0 = \text{Hom}(M, \mathcal{A}_1)$  for an indecomposable module  $M \in \mathcal{A}_2$ . This shows our claim.

By Proposition 2.4,  $N^{\perp \mathcal{D}}$  is admissible in  $\mathcal{D}$ . Since  $N$  is an exceptional Ext-projective object in  $\mathcal{D}^{\leq p}$ , by Lemma 2.15,  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the admissible subcategory  $N^{\perp \mathcal{D}}$  and  $(\mathcal{D}^{\leq 0} \cap N^{\perp \mathcal{D}}, \mathcal{D}^{\geq 0} \cap N^{\perp \mathcal{D}})$  is a bounded t-structure on  $N^{\perp \mathcal{D}}$ . Obviously, this t-structure is compatible with the admissible subcategory  $\mathcal{D}^b(\mathcal{A}_2)$  of  $N^{\perp \mathcal{D}} = \mathcal{D}^b(N^{\perp \mathcal{A}})$ . Hence by Lemma 2.8,  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the recollement

$$\begin{array}{ccc} & \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!} \end{array} & \mathcal{D} & \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*} \end{array} & {}^{\perp \mathcal{D}} \mathcal{D}^b(\mathcal{A}_2), \end{array}$$

where  $i_*, j_!$  are the inclusion functors. Note that

$$\begin{aligned} {}^{\perp \mathcal{D}} \mathcal{D}^b(\mathcal{A}_2) &= \langle N, \tau^m(\text{top}(N)) \mid 1 \leq m < l(N) \rangle_{\mathcal{D}} \\ &= \langle \tau^m(\text{top}(N)) \mid 0 \leq m < l(N) \rangle_{\mathcal{D}} \\ &= \mathcal{D}^b(\bar{\mathcal{A}}_1) \\ &\simeq \mathcal{D}^b(k\bar{\mathbb{A}}_{l(N)}). \end{aligned}$$

Consider the bounded t-structure  $(j^* \mathcal{D}^{\leq 0}, j^* \mathcal{D}^{\geq 0})$  on  $\mathcal{D}^b(\bar{\mathcal{A}}_1) \simeq \mathcal{D}^b(k\bar{\mathbb{A}}_{l(N)})$ . By the induction hypothesis, some  $\tau^m(\text{top}(N))$  ( $0 \leq m < l(N)$ ) is Ext-projective in some  $j^* \mathcal{D}^{\leq l}$ . Hence the simple module  $\tau^m(\text{top}(N)) = j_! \tau^m(\text{top}(N))$  is  $\mathcal{D}^{\leq l}$ -projective by Lemma 2.18, as desired.  $\square$

**Corollary 2.26** *Let  $k\bar{\mathbb{A}}_s$  be the path algebra of the equioriented  $\mathbb{A}_s$ -quiver. Each silting object in  $\mathcal{D}^b(k\bar{\mathbb{A}}_s)$  contains a shift of some simple module as its direct summand. Each full exceptional sequence in  $\text{mod}k\bar{\mathbb{A}}_s$  contains a simple module.*

*Proof* The first assertion follows from Lemma 2.25. For a full exceptional sequence  $(E_1, \dots, E_n)$  in  $\text{mod}k\bar{\mathbb{A}}_s$ , it is observed in [1, Proposition 3.5] that we can take suitable  $l_i$  ( $1 \leq i \leq n$ ), say  $l_i = i$  here, such that  $\bigoplus_{i=1}^n E_i[l_i]$  is a silting object in  $\mathcal{D}^b(k\bar{\mathbb{A}}_s)$ . So the second assertion follows.  $\square$

### 2.9 Bounded t-Structures on $\mathcal{D}^b(\text{nilpk}\tilde{\mathbb{A}}_{t-1})$

Let  $k$  be a field. Denote by  $\tilde{\mathbb{A}}_{t-1}$  the quiver which is an oriented cycle with  $t$  vertices and by  $\mathcal{A}_t = \text{nilpk}\tilde{\mathbb{A}}_{t-1}$  the category of finite dimensional nilpotent  $k$ -representations of  $\tilde{\mathbb{A}}_{t-1}$ . Let us recall some standard facts on  $\mathcal{A}_t$ .  $\mathcal{A}_t$  is a connected hereditary uniserial length abelian category and admits an autoequivalence  $\tau$  of period  $t$  such that  $\tau(-)[1]$  is the Serre functor of  $\mathcal{D}^b(\mathcal{A}_t)$ . Moreover,  $\mathcal{A}_t$  has almost split sequences with Auslander-Reiten translation given by  $[M] \dashrightarrow [\tau M]$ , and its Auslander-Reiten quiver is a tube of rank  $t$  (see Section 3.2 if one is unfamiliar with Auslander-Reiten theory). If  $S$  is a simple object in  $\mathcal{A}_t$  then each simple object is of the form  $\tau^i S$  for some  $i \in \mathbb{Z}/t\mathbb{Z}$ . Denote by  $S^{[n]}$  (resp.  ${}^{[n]}S$ ) the unique (up to isomorphism) indecomposable object in  $\mathcal{A}_t$  of length  $n$  and with socle (resp. top)  $S$ . For an indecomposable object  $X$  in  $\mathcal{A}_t$ , its length is denoted by  $l(X)$ , and its simple socle resp. top by  $\text{soc}(X)$  resp.  $\text{top}(X)$ . Then  $X = (\text{soc}(X))^{[l(X)]} = {}^{[l(X)]}(\text{top}(X))$ .  $X$  is exceptional iff  $l(X) < t$ .

Recall from [22] that for a torsion pair  $(\mathcal{T}, \mathcal{F})$  in an abelian category  $\mathcal{A}$ ,  $\mathcal{T}$  is called a *tilting torsion class* if  $\mathcal{T}$  is a cogenerator for  $\mathcal{A}$ , i.e, for each  $A \in \mathcal{A}$ , there is a monomorphism  $A \hookrightarrow T$  with  $T \in \mathcal{T}$ ; dually,  $\mathcal{F}$  is called a *cotilting torsion-free class* if  $\mathcal{F}$  is a generator for  $\mathcal{A}$ .

**Lemma 2.27** *For a torsion pair  $(\mathcal{T}, \mathcal{F})$  in  $\mathcal{A}_t$ , exactly one of the following holds*

- (1)  $\mathcal{T}$  is a tilting torsion class, equivalently,  $\mathcal{T}$  contains a non-exceptional indecomposable object;
- (2)  $\mathcal{F}$  is a cotilting torsion-free class, equivalently,  $\mathcal{F}$  contains a non-exceptional indecomposable object.

*Proof* Since there exists a nonzero morphism between two non-exceptional indecomposable objects in  $\mathcal{A}_t$ ,  $\mathcal{T}$  and  $\mathcal{F}$  cannot contain non-exceptional indecomposable objects in the meantime. If  $\mathcal{T}$  is a tilting torsion class then it's easy to see that  $\mathcal{T}$  contains a non-exceptional indecomposable object. Conversely, if  $\mathcal{T}$  contains a non-exceptional indecomposable object  $T$  then  ${}^{[l]} \text{top}(T) \in \mathcal{T}$  for all  $l \in \mathbb{Z}_{\geq 1}$  since  $\mathcal{T}$  is closed under quotients and extensions. Since any indecomposable object in  $\mathcal{A}_t$  is a subobject of  ${}^{[l]} \text{top}(T)$  for some  $l$ ,  $\mathcal{T}$  is a tilting torsion class. Dual argument applies to conclude the asserted equivalence for  $\mathcal{F}$ . □

We will need the following criterion to make sure that certain subcategory of  $\mathcal{D}^b(\mathcal{A}_t)$  contains a non-exceptional indecomposable object.

**Lemma 2.28** *Let  $\mathcal{C}$  be a subcategory of  $\mathcal{A}_t$  closed under extensions and direct summands. If each simple object in  $\mathcal{A}_t$  occurs as a composition factor of some indecomposable object in  $\mathcal{C}$ , equivalently, there is a sequence*

$$(X_0, X_1, \dots, X_{n-1}, X_n = X_0)$$

*of indecomposable objects in  $\mathcal{A}_t$  with  $\text{Ext}^1(X_i, X_{i-1}) \neq 0$  ( $1 \leq i \leq n$ ), then  $\mathcal{C}$  contains a non-exceptional indecomposable object.*

*Proof* We claim that if  $Y, Z$  are two exceptional objects in  $\mathcal{A}_t$  with  $\text{Ext}^1(Z, Y) \neq 0$ , then  $\mathcal{C}$  contains an indecomposable object  $C$  such that  $Y$  is a subobject of  $C$  in  $\mathcal{A}_t$  and  $Z$  a

quotient object of  $C$  in  $\mathcal{A}_t$ . Indeed, if  $\text{Ext}^1(Z, Y) \neq 0$  then there are two objects  $A, B$  in  $\mathcal{A}_t$  such that  $B$  is indecomposable,  $A$  is a quotient object of  $Y$  and  $A, B$  fit into the exact sequence  $0 \rightarrow A \rightarrow Z \rightarrow B \rightarrow 0$ . Let  $C$  be the unique (up to isomorphism) indecomposable object which fits into the exact sequence  $0 \rightarrow Y \rightarrow C \rightarrow B \rightarrow 0$ . Then  $Y$  (resp.  $Z$ ) is a subobject (resp. quotient object) of  $C$ . Since  $Y$  is an exceptional object in  $\mathcal{A}_t$ ,  $\text{soc}(\tau C) = \text{soc}(\tau Y)$  is not a composition factor of  $Y$ ; on the other hand, if  $A \neq 0$  then  $A$  is exceptional and each composition factor of  $A$  is a composition factor of  $Y$ . Thus  $\text{Ext}^1(C, A) \cong D\text{Hom}(A, \tau C) = 0$ , where  $D = \text{Hom}_k(-, k)$ . Then we have a pullback diagram

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & & & A & \xlongequal{\quad} & A & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & Y & \longrightarrow & A \oplus C & \longrightarrow & Z \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & Y & \longrightarrow & C & \longrightarrow & B \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow & \\
 & & & & 0 & & 0 & 
 \end{array}$$

Hence  $C \in \mathcal{C}$ . This shows our claim.

Now suppose that  $\mathcal{C}$  contains a sequence  $(X_0, X_1, \dots, X_{n-1}, X_n = X_0)$  with the given property. Assume for a contradiction that  $\mathcal{C}$  contains no non-exceptional indecomposable object. In particular, each  $X_i$  is exceptional. Applying our claim to  $Y = X_1, Z = X_2$ , we obtain an indecomposable object  $C_1 \in \mathcal{C}$  such that  $X_1$  (resp.  $X_2$ ) is a subobject (resp. quotient object) of  $C_1$ . Then  $\text{Ext}^1(X_1, X_0) \neq 0$  implies  $\text{Ext}^1(C_1, X_0) \neq 0$ ;  $\text{Ext}^1(X_3, X_2) \neq 0$  implies  $\text{Ext}^1(X_3, C_1) \neq 0$ . Hence we have a sequence  $(X_0, C_1, X_3, \dots, X_n)$  of length  $(n - 1)$  in  $\mathcal{C}$  which also satisfies the given property. By assumption,  $C_1$  is exceptional. Then repeating the above argument for  $n$  times will eventually give us a sequence  $(C)$  of length 1 with  $C$  indecomposable and  $\text{Ext}^1(C, C) \neq 0$ , whence  $C$  is a non-exceptional indecomposable object in  $\mathcal{C}$ , a contradiction. Hence  $\mathcal{C}$  must contain a non-exceptional object.  $\square$

We show an analogue of Lemma 2.25 to perform induction.

**Lemma 2.29** *For a bounded t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  on  $\mathcal{D}^b(\mathcal{A}_t)$ , which is not a shift of the standard t-structure, there is some simple object in  $\mathcal{A}_t$  that is Ext-projective in some  $\mathcal{D}^{\leq l}$ .*

*Proof* Let  $\mathcal{B}$  be the heart of  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ . Each bounded t-structure on  $\mathcal{D}^b(\mathcal{A}_t)$  is width-bounded with respect to the standard t-structure (see Example 2.2). Hence,  $\mathcal{B} \subset \mathcal{D}_{\mathcal{A}_t}^{[m,n]}$  for some  $m, n$ . We take  $m$  to be maximal and  $n$  minimal. Since there exists a nonzero morphism between two non-exceptional indecomposable objects in  $\mathcal{A}_t$  and since  $\text{Hom}(\mathcal{B}[-m], \mathcal{B}[-n]) = 0$ , either i)  $\mathcal{B}[-m] \cap \mathcal{A}_t$  or ii)  $\mathcal{B}[-n] \cap \mathcal{A}_t$  contains no non-exceptional indecomposable object. Suppose case i) occurs. Then  $\mathcal{B}[-m] \cap \mathcal{A}_t$  contains only finitely many indecomposables. Moreover, Lemma 2.28 implies that there is some indecomposable object  $X$  such that  $\text{Ext}^1(X, Y) = 0$  for indecomposable object  $Y \in \mathcal{B}[-m] \cap \mathcal{A}_t$

non-isomorphic to  $X$ . Then we have  $\text{Hom}^{>0}(X[m], \mathcal{B}) = 0$ , whence  $X$  is  $\mathcal{D}^{\leq m}$ -projective. If case ii) happens then similarly we find an indecomposable object  $Y \in \mathcal{A}_t$  which is  $\mathcal{D}^{\geq n}$ -injective. This gives us a  $\mathcal{D}^{\leq n}$ -projective  $\tau^{-1}Y[-1]$ . Anyway we have an exceptional object  $B \in \mathcal{A}_t$  that is Ext-projective in some  $\mathcal{D}^{\leq l}$ .

Similarly as in the proof of Lemma 2.25, one can show that  $B^{\perp_{\mathcal{A}_t}}$  decomposes as  $B^{\perp_{\mathcal{A}_t}} = \mathcal{B}_1 \coprod \mathcal{B}_2$ , where

$$\mathcal{B}_1 = \langle \tau^m(\text{top}(B)) \mid 1 \leq m < l(B) \rangle_{\mathcal{A}_t}$$

and  $\mathcal{B}_2$  is an exact subcategory of  $\mathcal{A}_t$  closed under extensions, that

$$\bar{\mathcal{B}}_1 := \langle \tau^m(\text{top}(B)) \mid 0 \leq m < l(B) \rangle_{\mathcal{A}_t} \simeq \text{mod}k\bar{\mathbb{A}}_{l(B)},$$

and that  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the recollement

$$\mathcal{D}^b(\mathcal{B}_2) \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!} \end{array} \mathcal{D} \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*} \end{array} \langle \bar{\mathcal{B}}_1 \rangle_{\mathcal{D}} = \mathcal{D}^b(\bar{\mathcal{B}}_1),$$

where  $i_*, j_!$  are inclusion functors. Moreover, we have a bounded t-structure  $(j^*\mathcal{D}^{\leq 0}, j^*\mathcal{D}^{\geq 0})$  on  $\mathcal{D}^b(\bar{\mathcal{B}}_1) \simeq \mathcal{D}^b(k\bar{\mathbb{A}}_{l(B)})$ . We know from Lemma 2.25 that some  $\tau^m(\text{top}(B))$  is Ext-projective in some  $j^*\mathcal{D}^{\leq l}$ , which gives us the desired Ext-projective object  $\tau^m(\text{top}(B))$  in  $\mathcal{D}^{\leq l}$  by Lemma 2.18. □

Let  $\mathcal{S}$  be a (possibly empty) proper collection of simple objects in  $\mathcal{A}_t$ , where properness means that  $\mathcal{S}$  does not contain a complete set of simple objects in  $\mathcal{A}_t$  and simple objects in  $\mathcal{S}$  are pairwise non-isomorphic. Two such collections are said to be equivalent if they yield the same isoclasses of simple objects. If  $\mathcal{S}$  is nonempty then there exist uniquely determined  $\{S_1, \dots, S_n\} \subset \mathcal{S}$  and positive integers  $l_1, \dots, l_n$  such that

$$\mathcal{S} = \bigsqcup_{i=1}^n \{ \tau^j S_i \mid 0 \leq j < l_i \}. \tag{2.9.1}$$

Since  $\bigoplus_{1 \leq i \leq n} \bigoplus_{0 \leq j < l_i} {}^{[l_i-j]}\tau^j S_i$  is a projective generator for  $\langle \mathcal{S} \rangle_{\mathcal{A}_t}$  whose endomorphism algebra is isomorphic to  $k\bar{\mathbb{A}}_{l_1} \times \dots \times k\bar{\mathbb{A}}_{l_n}$ , we have an equivalence

$$\langle \mathcal{S} \rangle_{\mathcal{A}_t} \simeq \prod_{i=1}^n \text{mod}k\bar{\mathbb{A}}_{l_i}, \tag{2.9.2}$$

where  $k\bar{\mathbb{A}}_l$  is the path algebra of the equioriented  $\mathbb{A}_l$ -quiver. In the sequel, we will also write in the form (2.9.2) when  $\mathcal{S}$  is empty by defining the right hand side of Eq. 2.9.2 to be the zero category. Since  $\mathcal{S}^{\perp_{\mathcal{A}_t}}$  is a uniserial length abelian  $k$ -category whose Ext-quiver is an oriented cycle with  $t - \#\mathcal{S}$  vertices and trivial valuation and since each simple object in  $\mathcal{S}^{\perp_{\mathcal{A}_t}}$  has endomorphism algebra  $k$ , we have an equivalence

$$\mathcal{S}^{\perp_{\mathcal{A}_t}} \simeq \mathcal{A}_{t-\#\mathcal{S}}. \tag{2.9.3}$$

Bounded t-structures on  $\mathcal{D}^b(\mathcal{A}_t)$  can be described as follows.

**Proposition 2.30** *Given a bounded t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  on  $\mathcal{D}^b(\mathcal{A}_t)$ , there is a unique (up to equivalence) proper collection  $\mathcal{S}$  of simple objects in  $\mathcal{A}_t$  such that*

- $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the recollement

$$\mathcal{D}^b(\mathcal{S}^{\perp_{\mathcal{A}_t}}) = \mathcal{S}^{\perp_{\mathcal{D}}} \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{i_*} \\ \xleftarrow{\quad} \end{array} \mathcal{D} = \mathcal{D}^b(\mathcal{A}_t) \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \langle \mathcal{S} \rangle_{\mathcal{D}},$$

where  $i_*, j_!$  are the inclusion functors;

- the corresponding t-structure on  $\mathcal{S}^{\perp_{\mathcal{D}}}$  has heart  $\mathcal{S}^{\perp_{\mathcal{A}_t}}[m]$  for some  $m$ .

In particular, each bounded t-structure on  $\mathcal{D}^b(\mathcal{A}_t)$  has a length heart.

*Proof* Since each bounded t-structure on  $\langle \mathcal{S} \rangle_{\mathcal{D}} = \mathcal{D}^b(\langle \mathcal{S} \rangle_{\mathcal{A}_t}) \simeq \mathcal{D}^b(\coprod_{i=1}^n \text{mod} k\bar{\mathbb{A}}_i)$  has a length heart (by Lemma 2.23) and  $\mathcal{S}^{\perp_{\mathcal{A}_t}}[m]$  is of finite length, by Lemma 2.12, the second assertion follows from the first. We use induction on  $t$  to prove the first assertion.

Suppose  $t = 1$ . We have a unique (up to isomorphism) simple object  $S$  in  $\mathcal{A}_1$ . So the asserted  $\mathcal{S}$  is the empty set. We need show that any bounded t-structure on  $\mathcal{D}^b(\mathcal{A}_1)$ , whose heart is denoted by  $\mathcal{B}$ , is a shift of the standard one. Note that each indecomposable object in  $\mathcal{D}^b(\mathcal{A}_1)$  is of the form  $S^{[r][l]}$  for some  $r \in \mathbb{Z}_{\geq 1}, l \in \mathbb{Z}$ . Since  $\text{Hom}(S^{[r][l]}, S^{[r'][l']}) \neq 0$  for  $l \leq l'$ , we have  $\mathcal{B} \subset \mathcal{A}_1[l]$  for some  $l$ . Then  $\mathcal{B} = \mathcal{A}_1[l]$ , as desired.

Now consider  $t > 1$ . If  $\mathcal{B}$  is a shift of  $\mathcal{A}_t$ , just take  $\mathcal{S} = \emptyset$ . Suppose that  $\mathcal{B}$  is not a shift of  $\mathcal{A}_t$ . By Lemma 2.29 and Lemma 2.15, for some simple  $S$  in  $\mathcal{A}_t$ ,  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the admissible subcategory  $\mathcal{D}_1 := \mathcal{S}^{\perp_{\mathcal{D}}} = \mathcal{D}^b(\mathcal{S}^{\perp_{\mathcal{A}_t}})$ .  $\mathcal{A} := \mathcal{S}^{\perp_{\mathcal{A}_t}}$  is equivalent to  $\mathcal{A}_{t-1}$ , and simple objects in  $\mathcal{A}$  are  $\tau S^{[2]}$  and those  $S'$ , which are simple in  $\mathcal{A}_t$  and non-isomorphic to  $\tau S$  and  $S$ . By the induction hypothesis, for a proper collection  $\mathcal{S}_1$  of simple objects in  $\mathcal{S}^{\perp_{\mathcal{A}_t}}$ , the corresponding t-structure on  $\mathcal{D}_1 = \mathcal{D}^b(\mathcal{S}^{\perp_{\mathcal{A}_t}})$  is compatible with the admissible subcategory  $\mathcal{S}_1^{\perp_{\mathcal{D}_1}}$  and the corresponding t-structure on  $\mathcal{S}_1^{\perp_{\mathcal{D}_1}}$  has heart  $\mathcal{S}_1^{\perp_{\mathcal{A}}}[m]$  for some  $m$ . If  $\tau S^{[2]} \in \mathcal{S}_1$ , take  $\mathcal{S} = \{\tau S, S\} \cup (\mathcal{S}_1 \setminus \tau S^{[2]})$ ; if  $\tau S^{[2]} \notin \mathcal{S}_1$ , take  $\mathcal{S} = \mathcal{S}_1 \cup \{S\}$ . Then  $\mathcal{S}_1^{\perp_{\mathcal{D}_1}} = \mathcal{S}^{\perp_{\mathcal{D}}}$  and  $\mathcal{S}_1^{\perp_{\mathcal{A}}} = \mathcal{S}^{\perp_{\mathcal{A}_t}}$ . By Lemma 2.8,  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the admissible subcategory  $\mathcal{S}^{\perp_{\mathcal{D}}}$  and the corresponding t-structure on  $\mathcal{S}^{\perp_{\mathcal{D}}}$  has heart  $\mathcal{S}^{\perp_{\mathcal{A}_t}}[m]$ .

Let  $(\mathcal{D}_1^{\leq 0}, \mathcal{D}_1^{\geq 0})$  and  $(\mathcal{D}_2^{\leq 0}, \mathcal{D}_2^{\geq 0})$  be the corresponding t-structures on  $\mathcal{S}^{\perp_{\mathcal{D}}}$  and  $\langle \mathcal{S} \rangle_{\mathcal{D}}$ , respectively. Note that  $\mathcal{D}_1^{\leq 0}$  contains no nonzero Ext-projective object. Let  $T$  be the direct sum of a complete set of indecomposable  $\mathcal{D}^{\leq 0}$ -projectives. Then by Lemma 2.18,  $T \in \langle \mathcal{S} \rangle_{\mathcal{D}}$  and  $T$  is the direct sum of a complete set of indecomposable  $\mathcal{D}_2^{\leq 0}$ -projectives. Thus  $T$  is a silting object in  $\langle \mathcal{S} \rangle_{\mathcal{D}} = \mathcal{D}^b(\langle \mathcal{S} \rangle_{\mathcal{A}_t})$ . In particular,  $\langle T \rangle_{\mathcal{D}} = \mathcal{D}^b(\langle \mathcal{S} \rangle_{\mathcal{A}_t})$ . As a complete set of simple objects in  $\langle \mathcal{S} \rangle_{\mathcal{A}_t}$ , the collection  $\mathcal{S}$  is uniquely determined. This finishes the proof.  $\square$

### 3 Weighted Projective Lines

For self-containedness, we review the basic theory of weighted projective lines in details in Sections 3.1–3.4. The materials in Section 3.1 are taken from the original article [17], which introduced the notion of weighted projective lines. For a recent survey of the theory, see [30]. We fix an algebraically closed field  $k$  in this section.

### 3.1 Basic Definitions and Properties

Given a sequence  $\underline{p} = (p_1, \dots, p_t) (t > 2)$  of positive integers, define an abelian group  $L(\underline{p})$  of rank one by

$$L(\underline{p}) = \langle \vec{x}_1, \dots, \vec{x}_t, \vec{c} \mid p_1 \vec{x}_1 = \dots = p_t \vec{x}_t = \vec{c} \rangle.$$

Denote  $\vec{\omega} = (t - 2)\vec{c} - \sum_{i=1}^t \vec{x}_i$ , which is called *the dualizing element*. Each  $\vec{x} \in L(\underline{p})$  can be written uniquely in the form

$$\vec{x} = \sum_{i=1}^t l_i \vec{x}_i + l \vec{c}, \quad 0 \leq l_i < p_i, l_i, l \in \mathbb{Z}.$$

$L(\underline{p})$  is an ordered group if we define  $\vec{x} \geq 0$  iff  $\vec{x} \in \sum_{i=1}^t \mathbb{Z}_{\geq 0} \vec{x}_i$ . Let  $p = \text{lcm}(p_1, \dots, p_t)$ . We have a group homomorphism, called a degree map,

$$\delta : L(\underline{p}) \rightarrow \mathbb{Z}, \quad \vec{x}_i \mapsto \frac{p}{p_i}.$$

Let  $\mathbb{P}^1 = \mathbb{P}^1(k)$  be (the set of closed points of) the projective line over  $k$ . Given a sequence  $\underline{p} = (p_1, \dots, p_t)$  of positive integers and a sequence  $\underline{\lambda} = (\lambda_1, \dots, \lambda_t)$  of distinct points in  $\mathbb{P}^1$  (normalized such that  $\lambda_1 = \infty, \lambda_2 = 0, \lambda_3 = 1$ ), we define an algebra

$$S = S(\underline{p}, \underline{\lambda}) = k[X_1, \dots, X_t] / (X_i^{p_i} - X_2^{p_2} + \lambda_i X_1^{p_i}, 3 \leq i \leq t).$$

Write  $x_i = \vec{X}_i \in S$ .  $S$  becomes  $L(\underline{p})$ -graded with the assignment  $\text{deg}(x_i) = \vec{x}_i$  and thus  $S = \bigoplus_{\vec{x} \in L(\underline{p})} S_{\vec{x}}$ , where  $S_{\vec{x}}$  consists of those homogeneous elements of degree  $\vec{x}$ . Using  $S$  as the homogeneous coordinate algebra, [17] introduced a weighted projective line  $\mathbb{X} = \mathbb{X}(\underline{p}, \underline{\lambda})$ .  $\mathbb{X}$  is defined to be the  $L(\underline{p})$ -graded projective spectrum of  $S$ , which is the set

$$\text{Proj}^{L(\underline{p})} S := \{L(\underline{p})\text{-graded prime ideal } \mathfrak{p} \text{ of } S \mid \mathfrak{p} \not\supseteq S_+ := \bigoplus_{\vec{x} > 0} S_{\vec{x}}\}$$

equipped with the Zariski topology and a  $L(\underline{p})$ -graded structure sheaf  $\mathcal{O} = \mathcal{O}_{\mathbb{X}}$ . There is a bijection

$$\mathbb{X}(k) \longrightarrow \mathbb{P}^1, \quad [x_1, \dots, x_t] \mapsto [x_1^{p_1}, x_2^{p_2}] \tag{3.1.1}$$

between the set of closed points of  $\mathbb{X}$  and  $\mathbb{P}^1$ . By virtue of this bijection, the weighted projective line  $\mathbb{X}$  is understood to be the usual projective line  $\mathbb{P}^1$ , where weights  $p_1, \dots, p_t$  are attached respectively to the  $t$  points  $\lambda_1, \dots, \lambda_t$ . We can define  $L(\underline{p})$ -graded  $\mathcal{O}_{\mathbb{X}}$ -modules and coherent  $L(\underline{p})$ -graded  $\mathcal{O}_{\mathbb{X}}$ -modules. The category  $\text{coh}\mathbb{X}$  of  $L(\underline{p})$ -graded coherent  $\mathcal{O}_{\mathbb{X}}$ -modules over  $\mathbb{X} = \mathbb{X}(\underline{p}, \underline{\lambda})$  is a noetherian hereditary abelian category with finite dimensional morphism and extension spaces. In particular,  $\text{coh}\mathbb{X}$  is a Krull-Schmidt category. We have an analogue of Serre’s theorem, that is, we have an equivalence

$$\text{coh}\mathbb{X} \simeq \frac{\text{mod}^{L(\underline{p})} S}{\text{mod}_0^{L(\underline{p})} S},$$

where  $\text{mod}^{L(\underline{p})} S$  is the abelian category of  $L(\underline{p})$ -graded finite generated modules over  $S$  and  $\text{mod}_0^{L(\underline{p})} S$  is the Serre subcategory of  $\text{mod}^{L(\underline{p})} S$  consisting of modules of finite length. One may as well take the latter quotient category as the definition of  $\text{coh}\mathbb{X}$ .

For  $\vec{x} \in L(\underline{p})$ , we have a natural  $k$ -linear autoequivalence of  $\text{mod}^{L(\underline{p})} S$  given by degree shifting by  $\vec{x} \in L(\underline{p})$  on  $L(\underline{p})$ -graded  $S$ -modules  $M: M(\vec{x})_{\vec{y}} = M_{\vec{x}+\vec{y}}$ . And this induces a  $k$ -linear autoequivalence  $-(\vec{x})$  of  $\text{coh}\mathbb{X}: F \mapsto F(\vec{x}), F \in \text{coh}\mathbb{X}$ . We denote by  $\tau$  the  $k$ -linear autoequivalence  $-(\vec{\omega})$  of  $\text{coh}\mathbb{X}$ , where  $\vec{\omega}$  is the dualizing element.



**Theorem 3.1** (Serre duality) *For  $X, Y \in \text{coh}\mathbb{X}$ , we have an isomorphism*

$$D\text{Ext}^1(X, Y) \cong \text{Hom}(Y, \tau X)$$

*functorial in  $X, Y$ , where  $D = \text{Hom}_k(-, k)$ .*

Consequently, the bounded derived category  $\mathcal{D}^b(\mathbb{X}) = \mathcal{D}^b(\text{coh}\mathbb{X})$  of  $\text{coh}\mathbb{X}$  has a Serre functor  $\tau(-)[1]$ .

There is a linear form  $\text{rk} : K_0(\mathbb{X}) \rightarrow \mathbb{Z}$  on the Grothendieck group  $K_0(\mathbb{X})$  of  $\text{coh}\mathbb{X}$ , called *rank*, which is preserved under the action of  $L(\underline{p})$ . As usual, we have the notion of a *locally free sheaf*, or a *vector bundle*. A *line bundle* is a vector bundle of rank 1. A coherent sheaf  $F$  over  $\mathbb{X}$  is called *torsion* if it is of finite length in  $\text{coh}\mathbb{X}$ , equivalently, if  $\text{rk}(F) = 0$ . Each coherent sheaf over  $\mathbb{X}$  decomposes as the direct sum of a torsion sheaf and a vector bundle. The subcategory of vector bundles resp. torsion sheaves over  $\mathbb{X}$  is denoted by  $\text{vect}\mathbb{X}$  resp.  $\text{coh}_0\mathbb{X}$ . We have  $\text{Hom}(\text{coh}_0\mathbb{X}, \text{vect}\mathbb{X}) = 0$ .

The function  $\mathbf{w} : \mathbb{P}^1 \rightarrow \mathbb{Z}_{\geq 1}, \lambda \mapsto \begin{cases} 1 & \text{if } \lambda \neq \lambda_i, \forall i \\ p_i & \text{if } \lambda = \lambda_i \end{cases}$  is called the *weight function* of  $\mathbb{X}$ . A weight function of  $\mathbb{X}$  obviously shares the same data as that given by the pair  $(\underline{p}, \underline{\lambda})$ .  $(p_1, \dots, p_t)$  is called the *weight sequence* of  $\mathbb{X}$ . For  $\lambda \in \mathbb{P}^1$ , by virtue of the bijection (3.1.1), we denote by  $\text{coh}_\lambda\mathbb{X}$  the category of those torsion sheaves supported at  $\lambda$ .

**Proposition 3.2** *The category  $\text{coh}_0\mathbb{X}$  of torsion sheaves decomposes into a coproduct  $\coprod_{\lambda \in \mathbb{P}^1} \text{coh}_\lambda\mathbb{X}$  of uniserial categories. The number of simple objects in  $\text{coh}_\lambda\mathbb{X}$  is  $\mathbf{w}(\lambda)$ .*

$\lambda_i$ 's are called *exceptional points* and the remaining points of  $\mathbb{P}^1$  *ordinary points*. For an ordinary point  $\lambda$ , the unique simple sheaf  $S$  supported at  $\lambda$  fits into the exact sequence

$$0 \longrightarrow \mathcal{O} \xrightarrow{X_2^{p_2} - \lambda X_1^{p_1}} \mathcal{O}(\bar{c}) \longrightarrow S \longrightarrow 0.$$

For an exceptional point  $\lambda_i$ , the exact sequences

$$0 \longrightarrow \mathcal{O}(j\vec{x}_i) \xrightarrow{X_i} \mathcal{O}((j+1)\vec{x}_i) \longrightarrow S_{i,j} \longrightarrow 0, \quad j \in \mathbb{Z}/p_i\mathbb{Z}$$

characterize the  $p_i$  pairwise non-isomorphic simple sheaves  $S_{i,j}$  supported at  $\lambda_i$ . The simple sheaf  $S$  supported at an ordinary point satisfies  $S(\vec{x}) \cong S$  for any  $\vec{x} \in L(\underline{p})$ ; the simple sheaves  $S_{i,j}$  supported at  $\lambda_i$  satisfies  $S_{i,j}(\vec{x}) \cong S_{i,j+l_i}$  if  $\vec{x} = \sum_{i=1}^n l_i \vec{x}_i$ . In particular,  $\tau S_{i,j} \cong S_{i,j-1}$ .  $S_{i,j}$  is an exceptional object iff  $p_i > 1$ .

*Remark 3.3* As a uniserial length abelian  $k$ -category whose Ext-quiver is an oriented cycle with  $\mathbf{w}(\lambda)$  vertices and trivial valuation,  $\text{coh}_\lambda\mathbb{X}$  is equivalent to the category  $\text{nilpk}\tilde{\mathbb{A}}_{\mathbf{w}(\lambda)-1}$  of nilpotent finite dimensional  $k$ -representations of the cyclic quiver  $\tilde{\mathbb{A}}_{\mathbf{w}(\lambda)-1}$  with  $\mathbf{w}(\lambda)$  vertices. So the algebra  $k\tilde{\mathbb{A}}_{\mathbf{w}(\lambda)-1}$  provides a local study of a weighted projective line. This accounts for the presence of Section 2.9.

Denote by  $\text{Pic}\mathbb{X}$  the Picard group of  $\mathbb{X}$ , i.e., the group of isoclasses of line bundles under tensor product.

**Proposition 3.4** (1) *The mapping*

$$L(\underline{p}) \longrightarrow \text{Pic}\mathbb{X}, \quad \vec{x} \mapsto \mathcal{O}(\vec{x})$$

is a group isomorphism. In particular, each line bundle over  $\mathbb{X}$  is isomorphic to  $\mathcal{O}(\vec{x})$  for some  $\vec{x} \in L(\underline{p})$ .

- (2) Each nonzero bundle over  $\mathbb{X}$  admits a line bundle filtration. That is, for a nonzero bundle  $E$ , there is a filtration

$$0 = E_0 \subset E_1 \subset \dots \subset E_n = E$$

with line bundle factors  $L_i = E_i/E_{i-1}$  ( $0 < i \leq n$ ).

The Grothendieck group  $K_0(\mathbb{X})$  of  $\text{coh}\mathbb{X}$  (and thus the Grothendieck group  $K_0(\mathcal{D}^b(\mathbb{X}))$  of  $\mathcal{D}^b(\mathbb{X})$ ) is a finitely generated free abelian group of rank  $\sum_{i=1}^t (p_i - 1) + 2$  with a basis  $\{[\mathcal{O}(\vec{x})] \mid 0 \leq \vec{x} \leq \vec{c}\}$ . We have a linear form  $\text{deg} : K_0(\mathbb{X}) \rightarrow \mathbb{Z}$ , called *degree*, such that  $\text{deg}\mathcal{O}(\vec{x}) = \delta(\vec{x})$  for  $\vec{x} \in L(\underline{p})$ . The Euler form on  $K_0(\mathbb{X})$  is given by

$$\chi(E, F) = \dim_k \text{Hom}(E, F) - \dim_k \text{Ext}^1(E, F)$$

and the averaged Euler form is defined by  $\bar{\chi}(E, F) = \sum_{j=0}^{p-1} \chi(\tau^j E, F)$ .

**Theorem 3.5** (Riemann-Roch Theorem) *For  $E, F \in \mathcal{D}^b(\mathbb{X})$ , we have*

$$\bar{\chi}(E, F) = p(1 - g_{\mathbb{X}}) \text{rk}(E) \text{rk}(F) + \text{deg}(F) \text{rk}(E) - \text{deg}(E) \text{rk}(F).$$

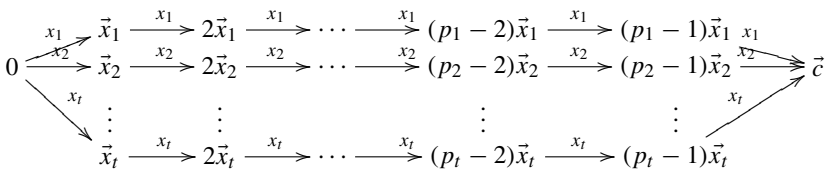
Here  $g_{\mathbb{X}} = 1 + \frac{1}{2} \delta(\vec{\omega})$  is the *virtual genus* of  $\mathbb{X}$ .  $\mathbb{X}$  is said to be of *domestic* (resp. *tubular*, resp. *wild*) type if  $g_{\mathbb{X}} < 1$  (resp.  $g_{\mathbb{X}} = 1$ , resp.  $g_{\mathbb{X}} > 1$ ), equivalently,  $\delta(\vec{\omega}) < 0$  (resp.  $\delta(\vec{\omega}) = 0$ , resp.  $\delta(\vec{\omega}) > 0$ ).  $\mathbb{X}$  is of domestic type iff the weight sequence is  $(1, p_1, p_2)$ ,  $(2, 2, n)$  ( $n \geq 2$ ),  $(2, 3, 3)$ ,  $(2, 3, 4)$ ,  $(2, 3, 5)$ , up to permutation;  $\mathbb{X}$  is of tubular type iff the weight sequence is  $(2, 2, 2, 2)$ ,  $(3, 3, 3)$ ,  $(2, 3, 6)$ ,  $(2, 4, 4)$ , up to permutation; weighted projective lines of wild type correspond to the remaining weight sequences.

A coherent sheaf  $T$  over  $\mathbb{X}$  is called a *tilting sheaf* if it is a tilting object as an object in  $\mathcal{D}^b(\mathbb{X})$ . A tilting sheaf  $T$  yields a derived equivalence  $\mathcal{D}^b(\mathbb{X}) \simeq \mathcal{D}^b(\text{End}T)$  and induces a torsion pair  $(\mathcal{T}, \mathcal{F})$  in  $\text{coh}\mathbb{X}$ , where

$$\mathcal{T} = \{E \in \text{coh}\mathbb{X} \mid \text{Ext}^1(T, E) = 0\}, \quad \mathcal{F} = \{E \in \text{coh}\mathbb{X} \mid \text{Hom}(T, E) = 0\}.$$

**Theorem 3.6** *There is a canonical tilting bundle  $T = \bigoplus_{0 \leq \vec{x} \leq \vec{c}} \mathcal{O}(\vec{x})$  over  $\mathbb{X}$ , whose endomorphism algebra is isomorphic to a canonical algebra  $\Lambda$  with the same parameter  $(\underline{p}, \underline{\lambda})$  in the sense of Ringel [43]. In particular, we have a derived equivalence  $\mathcal{D}^b(\Lambda) \simeq \mathcal{D}^b(\mathbb{X})$ .*

Recall from [43] that a canonical algebra  $\Lambda$  with parameter  $(\underline{p}, \underline{\lambda})$  is the path algebra of the quiver



with relations  $x_i^{p_i} = x_2^{p_2} - \lambda_i x_1^{p_1}$  ( $i = 3, \dots, t$ ).

### 3.2 A Glimpse of Auslander-Reiten Theory

Auslander-Reiten (=AR) theory is introduced by Auslander and Reiten to study representations of artin algebras. The standard reference is [4] (see also [3]). The central concept (i.e. an almost split sequence, or an Auslander-Reiten sequence) makes sense in any Krull-Schmidt category with short exact sequences (in the sense of [43, §2.3]) but there is a problem of existence. Later Happel introduced in [21] the notion of an Auslander-Reiten triangle, a triangulated version of Auslander-Reiten sequence. Reiten and Van den Bergh [41] investigated the close relationship between Serre duality (in the sense of [41]) and Auslander-Reiten sequences (as well as Auslander-Reiten triangles).

Here we recall some basic definitions and we follow [43]. Let  $\mathcal{A}$  be an essentially small Hom-finite abelian  $k$ -category. If  $X$  and  $Y$  are indecomposable,  $\text{rad}(X, Y)$  denotes the  $k$ -subspace of  $\text{Hom}(X, Y)$  consisting of non-invertible morphisms. If  $X = \bigoplus_{j=1}^m X_j, Y = \bigoplus_{i=1}^n Y_i$ , where  $X_j, Y_i$ 's are indecomposable, then  $\text{rad}(X, Y)$  denotes the  $k$ -subspace of  $\text{Hom}(X, Y)$  consisting of those  $f = (f_{ij})$  with  $f_{ij} \in \text{rad}(X_j, Y_i)$ .  $\text{rad}^2(X, Y)$  denotes the  $k$ -subspace of  $\text{Hom}(X, Y)$  consisting of morphisms of the form  $gf$  with  $f \in \text{rad}(X, M), g \in \text{rad}(M, Y)$  for some  $M$ . Let

$$\text{Irr}(X, Y) = \text{rad}(X, Y) / \text{rad}^2(X, Y).$$

A morphism  $h : X \rightarrow Y$  is called *irreducible* if  $h$  is neither a split monomorphism nor a split epimorphism and if  $h = ts$  for some  $s : X \rightarrow Z$  and  $t : Z \rightarrow Y$ , then  $s$  is a split monomorphism or  $t$  is a split epimorphism.  $h : X \rightarrow Y$  is irreducible iff  $h \in \text{rad}(X, Y) \setminus \text{rad}^2(X, Y)$ .

A morphism  $f : B \rightarrow C$  in  $\mathcal{A}$  is called a *sink map* (or a *minimal right almost split morphism*) if

- (1)  $f$  is right almost split, that is,  $f$  is not a split epimorphism and any morphism  $X \rightarrow C$  which is not a split epimorphism factors through  $f$ , and
- (2)  $f$  is right minimal, that is,  $\gamma \in \text{End}(B)$  satisfying  $f\gamma = f$  is an automorphism.

Dually, one defines a *source map* (or a *minimal left almost split morphism*). Sink (resp. source) maps with a fixed target (resp. source), if they exist, are obviously unique up to isomorphism. If  $f : B \rightarrow C$  is a sink (resp. source) map then  $C$  (resp.  $B$ ) is indecomposable. An exact sequence  $0 \rightarrow A \xrightarrow{g} B \xrightarrow{f} C \rightarrow 0$  in  $\mathcal{A}$  is called an *AR sequence* (or an *almost split sequence*) if  $g$  is a source map, equivalently, if  $f$  is a sink map (see [43, §2.2, Lemma 2] for the equivalence). If such an AR sequence exists, then each irreducible map  $f_1 : A \rightarrow B_1$  (or  $g_1 : B_1 \rightarrow C$ ) fits into an AR sequence

$$0 \longrightarrow A \xrightarrow{(f_1, f_2)^t} B_1 \oplus B_2 \xrightarrow{(g_1, g_2)} C \longrightarrow 0.$$

We say that  $\mathcal{A}$  has sink (resp. source) maps if for each indecomposable object  $A \in \mathcal{A}$ , there exists a sink map  $B \rightarrow A$  (resp. a source map  $A \rightarrow C$ ). We say that  $\mathcal{A}$  has AR sequences (or almost split sequences) if  $\mathcal{A}$  has both sink and source maps.

If  $\mathcal{A}$  has AR sequences then the *AR quiver*  $(\Gamma_{\mathcal{A}}, \sigma)$  of  $\mathcal{A}$ , which turns out to be a translation quiver, is defined as follows. The vertex set of  $\Gamma_{\mathcal{A}}$  is in bijection with a complete set of representatives of isoclasses of indecomposable objects in  $\mathcal{A}$ . Denote the vertex corresponding to an indecomposable object  $M$  by  $[M]$ . The number of arrows from a vertex  $[M]$  to another vertex  $[N]$  is  $\dim_k \text{Irr}(M, N)$ . By [43, §2.2, Lemma 3], if  $A \rightarrow B$  is a source map then there are  $d$  arrows from  $[A]$  to  $[D]$  iff the multiplicity of  $D$  as a direct summand of  $B$  is  $d$ . There is a dual fact for a sink map. So if  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is an AR sequence

then there are  $d$  arrows from  $[A]$  to  $[D]$  iff there are  $d$  arrows from  $[D]$  to  $[C]$ . The translation  $\sigma$ , called the *AR translation* of  $\mathcal{A}$ , is such that  $\sigma[C] = [A]$  if  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is an AR sequence.

The existence of AR sequences as well as the existence of AR triangles is closely related to the existence of a Serre functor. We refer the reader to [41] and here we only record the following fact (see [41, Theorem I.3.3]): if  $\mathcal{A}$  is a hereditary abelian  $k$ -category with finite dimensional morphism and extension spaces, then the existence of a Serre functor of  $\mathcal{D}^b(\mathcal{A})$  implies the existence of AR sequences in  $\mathcal{A}$ . Consequently, if  $\mathbb{X}$  is a weighted projective line then  $\text{coh}\mathbb{X}$  admits AR sequences.

**Proposition 3.7** [17, Corollary 2.3] *Let  $\mathbb{X}$  be a weighted projective line.  $\text{coh}\mathbb{X}$  has AR sequences with AR translation given by  $[M] \dashrightarrow [\tau M]$ .*

AR sequences are obtained in the following way. For each indecomposable sheaf  $E$  over  $\mathbb{X}$ , we have a distinguished exact sequence  $\eta_E : 0 \rightarrow \tau E \rightarrow F \rightarrow E \rightarrow 0$  whose class in  $\text{Ext}^1(E, \tau E)$  corresponds to  $\text{id}_{\tau E}$  under Serre duality  $D\text{Ext}^1(E, \tau E) \cong \text{Hom}(\tau E, \tau E)$ . The exact sequence  $\eta_E$  is an AR sequence. Since  $\tau$  is an autoequivalence of  $\text{coh}\mathbb{X}$ ,  $0 \rightarrow E \rightarrow \tau^{-1}F \rightarrow \tau^{-1}E \rightarrow 0$  is also an AR sequence.

An additive subcategory  $\mathcal{C}$  of  $\text{coh}\mathbb{X}$  closed under direct summands is said to be closed under the formation of AR sequences if for any AR sequence  $0 \rightarrow \tau E \rightarrow F \rightarrow E \rightarrow 0$ ,  $E \in \mathcal{C}$  implies  $F \in \mathcal{C}$  and  $\tau^i E \in \mathcal{C}$  for all  $i \in \mathbb{Z}$ . In this case, we can talk about the AR quiver of  $\mathcal{C}$  and the AR quiver of  $\mathcal{C}$  is a union of certain components of the AR quiver of  $\text{coh}\mathbb{X}$ . For each  $\lambda \in \mathbb{P}^1$ ,  $\text{coh}_\lambda\mathbb{X}$  is closed under the formation of AR sequences and the AR quiver of  $\text{coh}_\lambda\mathbb{X}$  is a tube of rank  $\mathbf{w}(\lambda)$ , where  $\mathbf{w}$  is the weight function of  $\mathbb{X}$ , and thus the AR quiver of  $\text{coh}_0\mathbb{X}$  is a family of tubes parametrized by  $\mathbb{P}^1$ .  $\text{vect}\mathbb{X}$  is also closed under the formation of AR sequences. We will see in the next subsection the shape of the AR quiver of  $\text{vect}\mathbb{X}$  for a domestic or tubular weighted projective line  $\mathbb{X}$ . We mention that for a wild weighted projective line  $\mathbb{X}$ , each AR component of  $\text{vect}\mathbb{X}$  has the shape  $\mathbb{Z}\mathbb{A}_\infty$  [34].

We introduce more definitions for the sake of the next subsection. Let  $E$  be an indecomposable object in  $\text{coh}\mathbb{X}$  lying in a component which is a tube of finite rank. The *quasi-length* of  $E$  is the largest integer  $l$  such that there exists a sequence

$$E = A_l \rightarrow A_{l-1} \rightarrow \dots \rightarrow A_2 \rightarrow A_1 = A$$

of irreducible epimorphisms, equivalently, there exists a sequence

$$B = B_1 \hookrightarrow B_2 \hookrightarrow \dots \hookrightarrow B_{l-1} \hookrightarrow B_l = E$$

of irreducible monomorphisms. In this case, we say  $A$  (resp.  $B$ ) is the *quasi-top* (resp. *quasi-socle*) of  $E$ .  $E$  is called *quasi-simple* if  $E$  is of quasi-length one, i.e.,  $E$  lies at the bottom of the tube. Note that the quasi-length of an indecomposable finite length sheaf coincides with its length and a quasi-simple torsion sheaf is just a simple sheaf. The  $\tau$ -*period* of  $E$  is the minimal positive integer  $n$  such that  $\tau^n E \cong E$ , which equals the rank of the tube.

### 3.3 Vector Bundles over a Domestic or Tubular Weighted Projective Line

We first recall the notion of stability of a vector bundle. For a nonzero bundle  $F$  over a weighted projective line  $\mathbb{X}$ , its slope  $\mu(F)$  is defined as  $\mu(F) = \text{deg}(F)/\text{rk}(F)$ .

**Lemma 3.8** [30, Lemma 2.5] *We have  $\mu(F(\vec{x})) = \mu(F) + \delta(\vec{x})$ . In particular,  $\mu(\tau F) = \mu(F) + \delta(\vec{\omega})$ .*

$F$  is called *semistable* (resp. *stable*) if  $\mu(E) \leq$  (resp.  $<$ )  $\mu(F)$  for any subbundle  $E$  of  $F$  with  $\text{rk}(E) < \text{rk}(F)$ . For  $\mu \in \mathbb{Q}$ , denote by  $\text{coh}^\mu \mathbb{X}$  the subcategory of  $\text{coh} \mathbb{X}$  consisting of semistable bundles of slope  $\mu$ .  $\text{coh}^\mu \mathbb{X}$  is a length abelian category whose simple objects are precisely stable bundles of slope  $\mu$ . For a torsion sheaf  $T$ , we define  $\mu(T) = \infty$  and denote  $\text{coh}^\infty \mathbb{X} = \text{coh}_0 \mathbb{X}$ . We have  $\text{Hom}(\text{coh}^\mu \mathbb{X}, \text{coh}^{\mu'} \mathbb{X}) = 0$  for  $\mu > \mu'$ .

Recall that a subsheaf  $E$  of a nonzero bundle  $F$  on  $\mathbb{X}$  is called a *maximal destabilizing subsheaf* if for any subsheaf  $G$  of  $F$ , we have  $\mu(E) \geq \mu(G)$  and  $\mu(E) = \mu(G)$  implies  $G$  is a subsheaf of  $E$ . As in the case of smooth projective curves, a maximal destabilizing subsheaf exists in our case, and the existence is unique up to isomorphism. It follows that each nonzero bundle admits a Harder-Narasimhan filtration, that is, a sequence

$$0 = F_0 \subset F_1 \subset \dots \subset F_m = F$$

such that all the factors  $A_i = F_i / F_{i-1}$  ( $0 < i \leq m$ ) are semistable bundles and

$$\mu(A_1) > \mu(A_2) > \dots > \mu(A_m).$$

Such a filtration is unique up to isomorphism.  $A_i$  are called the *semistable factors* of  $F$ . We will denote

$$\mu^+(F) = \mu(A_1), \quad \mu^-(F) = \mu(A_m).$$

Let  $\mu \in \bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ . Denote

$$\text{coh}^{\geq \mu} \mathbb{X} = \{E \in \text{coh} \mathbb{X} \mid \mu^-(E) \geq \mu\}, \quad \text{coh}^{< \mu} \mathbb{X} = \{E \in \text{coh} \mathbb{X} \mid \mu^+(E) < \mu\}.$$

Similarly one defines  $\text{coh}^{> \mu} \mathbb{X}, \text{coh}^{\leq \mu} \mathbb{X}$ . Then we have torsion pairs

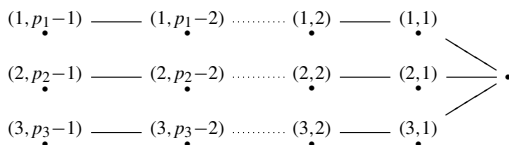
$$(\text{coh}^{\geq \mu} \mathbb{X}, \text{coh}^{< \mu} \mathbb{X}), \quad (\text{coh}^{> \mu} \mathbb{X}, \text{coh}^{\leq \mu} \mathbb{X})$$

for each  $\mu \in \bar{\mathbb{R}}$ .

Suppose  $\mathbb{X}$  is a weighted projective line of domestic type with weight sequence  $(p_1, p_2, p_3)$ . Then up to permutation,

$$(p_1, p_2, p_3) = (1, p_2, p_3), (2, 2, n) (n \geq 2), (2, 3, 3), (2, 3, 4), \text{ or } (2, 3, 5).$$

Let  $\Delta = \Delta(p_1, p_2, p_3)$  be the Dynkin diagram



Let  $\tilde{\Delta}$  be the extended Dynkin diagram attached to  $\Delta$ . We collect well-known and basic properties of vector bundles over a domestic weighted projective line in the following theorem.

**Theorem 3.9** *Let  $\mathbb{X}$  be a weighted projective line of domestic type with weight sequence  $(p_1, p_2, p_3)$ .*

- (1) *Each indecomposable bundle over  $\mathbb{X}$  is stable and exceptional. The rank function  $\text{rk}$  is bounded on indecomposable bundles over  $\mathbb{X}$ . If some  $p_i$  equals 1 then each indecomposable bundle is a line bundle.*
- (2) *The direct sum of a complete set of indecomposable bundles with slope in the interval  $(\delta(\tilde{\omega}), 0]$  is a tilting bundle and its endomorphism algebra is the path algebra  $k\tilde{\Delta}$  of an*



**Theorem 3.11** [37, Theorem 5.2.6] *Let  $\mathbb{X}$  be a weighted projective line of tubular type. For each  $q, q' \in \mathbb{Q}$ , there is an exact autoequivalence  $\Phi_{q,q'}$  of  $\mathcal{D}^b(\mathbb{X})$ , called a telescopic functor, such that  $\Phi_{q,q'}(\text{coh}^{q'}\mathbb{X}) = \text{coh}^q\mathbb{X}$ . Moreover, these functors satisfy the conditions  $\Phi_{q'',q} = \Phi_{q'',q'} \circ \Phi_{q',q}$  and  $\Phi_{q,q} = \text{id}$ .*

Denote  $\text{coh}_\lambda^\mu\mathbb{X} = \Phi_{\mu,\infty}(\text{coh}_\lambda\mathbb{X})$ . The next theorem summarizes well-known and basic properties of vector bundles over a tubular weighted projective line.

**Theorem 3.12** *Let  $\mathbb{X}$  be a weighted projective line of tubular type.*

- (1) *We have  $\text{coh}_\lambda^\mu\mathbb{X} \simeq \text{coh}_\lambda\mathbb{X}$  and  $\text{coh}^\mu\mathbb{X}$  decomposes as  $\text{coh}^\mu\mathbb{X} = \coprod_{\lambda \in \mathbb{P}^1} \text{coh}_\lambda^\mu\mathbb{X}$ . In particular, each  $\text{coh}_\lambda^\mu\mathbb{X}$  as well as  $\text{coh}^\mu\mathbb{X}$  is a uniserial abelian category.*
- (2) *Each indecomposable bundle over  $\mathbb{X}$  is semistable.  $\text{coh}_\lambda^\mu\mathbb{X}$  is closed under the formation of Auslander-Reiten sequences and the Auslander-Reiten quiver of  $\text{coh}_\lambda^\mu\mathbb{X}$  is a tube of rank  $\mathbf{w}(\lambda)$ , where  $\mathbf{w}$  is the weight function of  $\mathbb{X}$ . In particular, the Auslander-Reiten quiver of  $\text{vect}\mathbb{X}$  is a family of tubes parametrized by  $\mathbb{Q} \times \mathbb{P}^1$ .*
- (3) *An indecomposable bundle in  $\text{coh}_\lambda^\mu\mathbb{X}$  is exceptional iff its quasi-length is less than  $\mathbf{w}(\lambda)$ . An indecomposable bundle over  $\mathbb{X}$  is stable iff it is quasi-simple. A stable bundle in  $\text{coh}_\lambda^\mu\mathbb{X}$  has  $\tau$ -period  $\mathbf{w}(\lambda)$ .*

*Proof* The assertion that each indecomposable bundle is semistable is [17, Proposition 5.5(ii)]. The remaining assertions follow from facts on  $\text{coh}_0\mathbb{X}$  by applying a suitable telescopic functor. We remark that a telescopic functor commutes with  $\tau$  since any exact autoequivalence commutes with a Serre functor. □

Here we make an observation needed in the following two lemmas. Let  $(p_1, \dots, p_t)$  be the weight sequence of  $\mathbb{X}$ . Recall that we denote by  $p = \text{lcm}(p_1, \dots, p_t)$ . Since  $\mathbb{X}$  is of tubular type, there is some  $p_i$  equal to  $p$ . So there exists a simple sheaf  $S$  with  $\tau$ -period  $p$ .

For  $F \in \text{coh}(\mathbb{X})$  and  $n \in \mathbb{Z}$ , we define the slope  $\mu(F[n])$  of the object  $F[n] \in \mathcal{D}^b(\mathbb{X})$  to be  $\mu(F[n]) = \mu(F)$ . We will need to know the effect of the telescopic functor  $\Phi_{\infty,q}$  on slope and the essential image of  $\text{coh}^\mu\mathbb{X}$  under  $\Phi_{\infty,q}$ .

**Lemma 3.13** (1) *There is a fractional linear map*

$$\phi_q : \bar{\mathbb{R}} \rightarrow \bar{\mathbb{R}}, \mu \mapsto \frac{a\mu + b}{c\mu + d}, \tag{3.3.1}$$

where  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ , such that

$$\mu(\Phi_{\infty,q}(E)) = \phi_q(\mu(E))$$

for an indecomposable sheaf  $E$ .

(2) *For  $\mu \in \mathbb{Q}$ , we have*

$$\Phi_{\infty,q}(\text{coh}^\mu\mathbb{X}) = \begin{cases} \text{coh}^{\phi_q(\mu)}\mathbb{X} & \text{if } \mu \leq q, \\ \text{coh}^{\phi_q(\mu)}\mathbb{X}[1] & \text{if } \mu > q. \end{cases} \tag{3.3.2}$$

*Proof* Recall from [37, Chapter 5] that for a quasi-simple sheaf  $E$  over  $\mathbb{X}$  with  $\tau$ -period  $p_E$ , the tubular mutation functor  $T_{\tau \bullet E}$  with respect to the  $\tau$ -orbit of  $E$ , which is an exact autoequivalence of  $\mathcal{D}^b(\mathbb{X})$ , fits into a triangle

$$\bigoplus_{j=0}^{p_E-1} \text{Hom}^\bullet(\tau^j E, -) \otimes \tau^j E \rightarrow \text{id} \rightarrow T_{\tau \bullet E} \rightsquigarrow .$$

Define an action of  $SL(2, \mathbb{Z})$  on  $\bar{\mathbb{Q}}$  by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot q = \frac{aq + b}{cq + d}.$$

By [37, Corollary 5.2.3],  $T_{\tau \bullet \mathcal{O}}(\text{coh}^q \mathbb{X})$  is a shift of  $\text{coh}^{1-q} \mathbb{X}$  for each  $q \in \bar{\mathbb{Q}}$ . Let  $S$  be a simple sheaf with  $\tau$ -period  $p$ . From the triangle

$$\bigoplus_{j=0}^{p-1} \text{Hom}^\bullet(\tau^j S, -) \otimes \tau^j S \rightarrow \text{id} \rightarrow T_{\tau \bullet S} \rightsquigarrow,$$

we see that  $T_{\tau \bullet S}(\text{coh}^q \mathbb{X}) = \text{coh}^{1+q} \mathbb{X}$  for  $q \in \bar{\mathbb{Q}}$ . So  $T_{\tau \bullet S}$  ( $T_{\tau \bullet S}^{-1}$ ,  $T_{\tau \bullet \mathcal{O}}$ , respectively) acts on slopes by  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  ( $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ ,  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ , respectively). By definition,  $\Phi_{q, \infty} = \Phi_{\infty, q}^{-1}$  is a composition of a sequence of the functors  $T_{\tau \bullet S}$ ,  $T_{\tau \bullet S}^{-1}$ ,  $T_{\tau \bullet \mathcal{O}}$  (see [37, Theorem 5.2.6]). So we have a unique function  $\phi_q : \bar{\mathbb{Q}} \rightarrow \bar{\mathbb{Q}}$  such that  $\phi_q(\mu) = \frac{aq+b}{cq+d}$  for some  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$  and such that  $\Phi_{\infty, q}(\text{coh}^\mu \mathbb{X})$  is a shift of  $\text{coh}^{\phi_q(\mu)} \mathbb{X}$  for each  $\mu \in \bar{\mathbb{Q}}$ . We extend  $\phi_q$  to be the function

$$\phi_q : \bar{\mathbb{R}} \rightarrow \bar{\mathbb{R}}, \quad r \mapsto \frac{ar + b}{cr + d}.$$

By Riemann-Roch Theorem, we have  $\text{Hom}(\text{coh}^\mu \mathbb{X}, \text{coh}^{\mu'} \mathbb{X}) \neq 0$  for  $\mu < \mu'$ . Now that  $\Phi_{\infty, q}(\text{coh}^q \mathbb{X}) = \text{coh}^\infty \mathbb{X}$ , (2) follows immediately.  $\square$

It's well-known that a stable bundle over an elliptic curve defined over an algebraically closed field has coprime rank and degree. We have the following analogue<sup>4</sup> for a stable bundle over a tubular weighted projective line, which is implicit in [33]. Actually, there is a parallel proof for an elliptic curve.

**Lemma 3.14** *Let  $\mathbb{X}$  be a weighted projective line of tubular type and  $E$  a stable vector bundle over  $\mathbb{X}$  with  $\tau$ -period  $p_E$ . Then*

$$\text{gcd}(\text{rk}(E), \text{deg}(E)) = \frac{p}{p_E}.$$

*Proof* Let  $S$  be a simple sheaf with  $\tau$ -period  $p$ . By Riemann-Roch Theorem, the linear form  $\text{deg} : K_0(\mathbb{X}) \rightarrow \mathbb{Z}$  coincides with  $\bar{\chi}(\mathcal{O}, -)$  and the linear form  $\text{rk} : K_0(\mathbb{X}) \rightarrow \mathbb{Z}$  with  $\bar{\chi}(-, S)$ . So we have

$$\begin{aligned} \text{deg}(E) &= \bar{\chi}(\mathcal{O}, E) = \frac{p}{p_E} \sum_{i=0}^{p_E-1} \chi(\tau^i \mathcal{O}, E), \\ \text{rk}(E) &= \bar{\chi}(E, S) = \frac{p}{p_E} \sum_{i=0}^{p_E-1} \chi(\tau^i E, S), \end{aligned}$$

whence

$$\frac{p}{p_E} \mid \text{gcd}(\text{deg}(E), \text{rk}(E)).$$

<sup>4</sup>Prof. Lenzing informed me of this fact as an answer to my question.



Let  $S' = \Phi_{\infty, \mu(E)}(E)$ .  $S'$  is a simple sheaf with  $\tau$ -period  $p_{S'} = p_E$ . Observe that there exists  $\bar{x} \in L(\underline{p})$  such that  $\bar{\chi}(\mathcal{O}(\bar{x}), S') = \frac{p}{p_E}$ . Take  $F = \Phi_{\mu(E), \infty}(\mathcal{O}(\bar{x}))$ . Then we have

$$\deg(F)\text{rk}(E) - \deg(E)\text{rk}(F) = \bar{\chi}(F, E) = \bar{\chi}(\mathcal{O}(\bar{x}), S') = \frac{p}{p_E}.$$

Hence  $\text{gcd}(\text{rk}(E), \deg(E)) = \frac{p}{p_E}$ . □

### 3.4 Perpendicular Categories

Let  $\mathbb{X} = \mathbb{X}(\underline{p}, \lambda)$  be a weighted projective line with weight sequence  $\underline{p} = (p_1, \dots, p_t)$ . For convenience, we will denote  $\mathcal{A} = \text{coh}\mathbb{X}$ ,  $\mathcal{D} = \mathcal{D}^b(\mathbb{X})$ . For a collection  $\mathcal{S}$  of objects in  $\text{coh}\mathbb{X}$ , we have  $\mathcal{S}^{\perp \mathcal{A}} = {}^{\perp \mathcal{A}}\tau\mathcal{S}$  by Serre duality. So it suffices to describe right perpendicular categories. We are concerned about perpendicular categories of an exceptional sequence.

A (possibly empty) collection of simple sheaves over  $\mathbb{X}$  is called *proper* if it does not contain a complete set of simple sheaves supported at  $\lambda$  for each  $\lambda \in \mathbb{P}^1$  and simple sheaves in the collection are pairwise non-isomorphic. In particular, it contains only exceptional simple sheaves.

**Theorem 3.15** [18] *Let  $\mathcal{S} = \bigcup_{i=1}^t \mathcal{S}_i$  be a collection of simple sheaves, where  $\mathcal{S}_i$  is a proper collection of simple sheaves supported at  $\lambda_i$ .*

- (1) *We have an equivalence  $\mathcal{S}^{\perp \mathcal{A}} \simeq \text{coh}\mathbb{X}'$  preserving rank, where  $\mathbb{X}' = \mathbb{X}(\underline{p}', \lambda)$  is a weighted projective line with weight sequence*

$$\underline{p}' = (p_1 - \#\mathcal{S}_1, \dots, p_i - \#\mathcal{S}_i, \dots, p_t - \#\mathcal{S}_t).$$

- (2) *The inclusion of the exact subcategory  $\mathcal{S}^{\perp \mathcal{A}}$  into  $\mathcal{A} = \text{coh}\mathbb{X}$  admits an exact left adjoint and an exact right adjoint, both of which preserve rank.*

**Lemma 3.16** *Let  $E$  be an exceptional torsion sheaf. Denote*

$$\mathcal{S}_E = \{\tau^i \text{top}(E) \mid 0 \leq i < l(E)\}, \quad \mathcal{S}'_E = \mathcal{S}_E \setminus \{\text{top}(E)\}. \tag{3.4.1}$$

*Then  $E^{\perp \mathcal{A}}$  decomposes as*

$$E^{\perp \mathcal{A}} = \mathcal{S}_E^{\perp \mathcal{A}} \coprod \langle \mathcal{S}'_E \rangle_{\mathcal{A}},$$

*and we have an equivalence  $\mathcal{S}_E^{\perp \mathcal{A}} \simeq \text{coh}\mathbb{X}'$  preserving rank, where  $\mathbb{X}' = \mathbb{X}(\underline{p}', \lambda)$  is a weighted projective line with weight sequence*

$$\underline{p}' = (p_1, \dots, p_i - l(E), \dots, p_t),$$

*and an equivalence  $\langle \mathcal{S}'_E \rangle_{\mathcal{A}} \simeq \text{mod}k\vec{\mathbb{A}}_{l(E)-1}$ , where  $k\vec{\mathbb{A}}_l$  is the path algebra of the equi-oriented  $\mathbb{A}_l$ -quiver.*

Note that if  $\mathbb{X}$  is of tubular type then  $\mathbb{X}'$  is of domestic type.

*Proof* Suppose  $E$  is supported at  $\lambda$ . We have a decomposition

$$E^{\perp \mathcal{A}} \cap \text{coh}_{\lambda}\mathbb{X} = E^{\perp \text{coh}_{\lambda}\mathbb{X}} = (\mathcal{S}_E^{\perp \mathcal{A}} \cap \text{coh}_{\lambda}\mathbb{X}) \coprod \langle \mathcal{S}'_E \rangle_{\mathcal{A}}.$$

The argument for showing this is similar to that in showing  $N^\perp = \mathcal{A}_1 \coprod \mathcal{A}_2$  in the proof of Lemma 2.25. For  $\lambda \neq \lambda' \in \mathbb{P}^1$ , since  $\text{Hom}(\text{coh}_\lambda \mathbb{X}, \text{coh}_{\lambda'} \mathbb{X}) = 0$ , we have

$$E^{\perp \mathcal{A}} \cap \text{coh}_{\lambda'} \mathbb{X} = \text{coh}_{\lambda'} \mathbb{X} = \mathcal{S}_E^{\perp \mathcal{A}} \cap \text{coh}_{\lambda'} \mathbb{X}.$$

We continue to show

$$E^{\perp \mathcal{A}} \cap \text{vect} \mathbb{X} = \mathcal{S}_E^{\perp \mathcal{A}} \cap \text{vect} \mathbb{X}.$$

It suffices to show that each nonzero bundle  $F$  lying in  $E^{\perp \mathcal{A}}$  lies in  $\mathcal{S}_E^{\perp \mathcal{A}}$ . Assume for a contradiction that  $F \notin \mathcal{S}_E^{\perp \mathcal{A}}$ . Then for some  $S \in \mathcal{S}_E$ ,  $\text{Ext}^1(S, F) \neq 0$ , whence  $\text{Hom}(F, \tau S) \neq 0$  by Serre duality. Since  $\tau S$  is a composition factor of  $\tau E$  and since  $\text{Hom}(F, -) : \text{coh}_\lambda \mathbb{X} \rightarrow \text{mod} k$  is an exact functor,  $\text{Hom}(F, \tau S) \neq 0$  implies  $\text{Hom}(F, \tau E) \neq 0$ . Hence  $\text{Ext}^1(E, F) \neq 0$ , a contradiction to  $F \in E^{\perp \mathcal{A}}$ . So indeed we have

$$E^{\perp \mathcal{A}} \cap \text{vect} \mathbb{X} = \mathcal{S}_E^{\perp \mathcal{A}} \cap \text{vect} \mathbb{X}.$$

By Serre duality, this implies  $\text{Hom}(E^{\perp \mathcal{A}} \cap \text{vect} \mathbb{X}, \langle \mathcal{S}'_E \rangle_{\mathcal{A}}) = 0$ . Now that each coherent sheaf over  $\mathbb{X}$  is a direct sum of a bundle and a torsion sheaf and that  $\text{coh}_0 \mathbb{X} = \coprod_{\lambda \in \mathbb{P}^1} \text{coh}_\lambda \mathbb{X}$ , we can conclude

$$E^{\perp \mathcal{A}} = \mathcal{S}_E^{\perp \mathcal{A}} \coprod \langle \mathcal{S}'_E \rangle_{\mathcal{A}}.$$

One easily sees  $\langle \mathcal{S}'_E \rangle_{\mathcal{A}} \simeq \text{mod} k \bar{\mathbb{A}}_{\mathbb{A}(E)-1}$ . By Theorem 3.15, we have an equivalence  $\mathcal{S}_E^{\perp \mathcal{A}} \simeq \text{coh} \mathbb{X}'$  preserving rank, where  $\mathbb{X}'$  has a weight sequence as asserted.  $\square$

**Theorem 3.17** (1) ([25]; see also [24, Kapitel 5]) *Let  $E$  be an exceptional bundle over  $\mathbb{X}$ .*

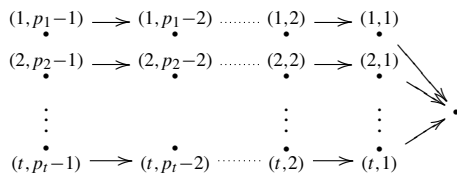
*Then  $E^{\perp \mathcal{A}} \simeq \text{mod} \Lambda$  for some finite dimensional hereditary algebra  $\Lambda$ .*

(2) ([25]; see also [30, Proposition 2.14]) *Let  $L$  be a line bundle in  $\text{coh} \mathbb{X}$ . Then*

$$L^{\perp \mathcal{A}} \simeq \text{mod} k[p_1, \dots, p_t],$$

*where  $k[p_1, \dots, p_t]$  is the path algebra of the equioriented star quiver  $[p_1, \dots, p_t]$ .*

Here, an equioriented star quiver  $[p_1, \dots, p_t]$  refers to the quiver



In certain cases, forming a perpendicular category can yield the module category of a representation-finite finite dimensional hereditary algebra.

**Lemma 3.18** (1) *If  $\mathbb{X}$  is of domestic type and  $E$  is an indecomposable bundle then  $E^{\perp \mathcal{A}}$  is equivalent to  $\text{mod} \Lambda$  for a representation-finite finite dimensional hereditary algebra  $\Lambda$ .*

(2) *If  $\mathbb{X}$  is of tubular type and  $(E, F)$  is an exceptional pair in  $\text{coh} \mathbb{X}$  with  $\mu(E) \neq \mu(F)$  then  $\{E, F\}^{\perp \mathcal{A}}$  is equivalent to  $\text{mod} \Lambda$  for a representation-finite finite dimensional hereditary algebra  $\Lambda$ .*

*Proof* (1) Let  $(p_1, p_2, p_3)$  be the weight sequence of  $\mathbb{X}$ . If some  $p_i = 1$ , say  $i = 1$ , then  $E$  is a line bundle and by Theorem 3.17(2) we have  $E^{\perp \mathcal{A}} \simeq \text{mod} k[p_2, p_3]$ . Otherwise

$p_i \geq 2$  for all  $i$ . Up to the action of some power of  $\tau$ , we can suppose  $\delta(\bar{\omega}) < \mu(E) \leq 0$ . Let  $T$  be the direct sum of a complete set of indecomposable bundles with slope in the interval  $(\delta(\bar{\omega}), 0]$  and suppose  $T = T_1 \oplus E$ . Recall that  $T$  is a tilting bundle and its endomorphism algebra  $\Gamma = \text{End}(T)$  is a tame hereditary algebra whose quiver has a bipartite orientation. Hence  $\Gamma_1 = \text{End}(T_1)$  is a representation-finite hereditary algebra. We already know  $E^{\perp_{\mathcal{A}}} \simeq \text{mod } \Lambda$  for a finite dimensional hereditary algebra  $\Lambda$ . Now that  $T_1$  is a tilting object in  $E^{\perp_{\mathcal{D}}}$ , we have exact equivalences  $\mathcal{D}^b(\Lambda) \simeq \mathcal{D}^b(E^{\perp_{\mathcal{A}}}) = E^{\perp_{\mathcal{D}}} \simeq \mathcal{D}^b(\Gamma_1)$ . Hence  $\Lambda$  is a representation-finite hereditary algebra, the underlying graph of whose quiver is the same as that of the quiver of  $\Gamma_1$ .

(2) By applying Lemma 3.16, we have an equivalence

$$F^{\perp_{\mathcal{D}}} \simeq \Phi_{\infty, \mu(F)}(F)^{\perp_{\mathcal{D}}} \simeq \mathcal{D}^b(\mathbb{X}') \coprod \mathcal{D}^b(k\bar{\mathbb{A}}_{l(F)-1}),$$

under which  $E \in F^{\perp_{\mathcal{A}}}$  corresponds to  $E'[m]$  for some exceptional bundle  $E'$  over  $\mathbb{X}'$  and some  $m \in \mathbb{Z}$ . Thus there are exact equivalences

$$\mathcal{D}^b(\{E, F\}^{\perp_{\mathcal{A}}}) = \{E, F\}^{\perp_{\mathcal{D}}} \simeq E'^{\perp_{\mathcal{D}^b(\text{coh } \mathbb{X}')}} \coprod \mathcal{D}^b(k\bar{\mathbb{A}}_{l(F)-1}) \simeq \mathcal{D}^b(\Gamma)$$

for a representation-finite finite dimensional hereditary algebra  $\Gamma$ . It follows that  $\{E, F\}^{\perp_{\mathcal{A}}}$  is equivalent to  $\text{mod } \Lambda$  for a representation-finite finite dimensional hereditary algebra  $\Lambda$ . □

*Remark 3.19* (1) There is a more direct proof of (1) using Theorem 3.22. The current proof has the advantage that it gives us additional information on the quiver of  $\Lambda$ .

(2) It can be shown that if  $\mathbb{X}$  is of tubular type and  $E$  is an exceptional bundle with quasi-length  $l$  then  $E^{\perp_{\mathcal{A}}} \simeq \text{mod } \Lambda \coprod \text{mod } k\bar{\mathbb{A}}_{l-1}$  for a tame hereditary algebra  $\Lambda$  and an equioriented  $\bar{\mathbb{A}}_{l-1}$ -quiver.

### 3.5 Some Nonvanishing Hom Spaces

The following two lemmas are well-known.

**Lemma 3.20** *Let  $E$  be a nonzero bundle over  $\mathbb{X}$  and  $F$  an non-exceptional indecomposable torsion sheaf. Then  $\text{Hom}(E, F) \neq 0, \text{Ext}^1(F, E) \neq 0$ .*

*Proof* Suppose  $F$  is supported at  $\lambda \in \mathbb{P}^1$ . Take a line bundle  $L$  such that there is an epimorphism  $E \twoheadrightarrow L$  and also a simple sheaf  $S$  supported at  $\lambda$  such that  $\text{Hom}(L, S) \neq 0$ . Then  $\text{Hom}(E, S) \neq 0$ . Since  $F$  is a non-exceptional indecomposable sheaf supported at  $\lambda$ ,  $S$  is a composition factor of  $F$ . Then there exist two exact sequences

$$0 \rightarrow F_1 \rightarrow F \rightarrow F_2 \rightarrow 0, \quad 0 \rightarrow S \rightarrow F_2 \rightarrow F_3 \rightarrow 0,$$

where  $F_i \in \text{coh}_{\lambda} \mathbb{X}$  ( $i = 1, 2, 3$ ). Applying  $\text{Hom}(E, -)$ , one has  $\text{Hom}(E, S) \hookrightarrow \text{Hom}(E, F_2)$  and  $\text{Hom}(E, F) \twoheadrightarrow \text{Hom}(E, F_2)$  therefore  $\text{Hom}(E, F) \neq 0$ . Note that  $\tau F$  is also a non-exceptional indecomposable sheaf and thus  $\text{Hom}(E, \tau F) \neq 0$ . This gives  $\text{Ext}^1(F, E) \neq 0$  by Serre duality. □

**Lemma 3.21** *Let  $\mathbb{X}$  be of tubular type. Suppose  $E, F$  are two nonzero bundles with  $\mu(E) < \mu(F)$ . Then  $\text{Hom}(E, \tau^i F) \neq 0$  for some  $i$ . If  $E$  or  $F$  is a non-exceptional indecomposable bundle,  $\text{Hom}(E, F) \neq 0$  always holds.*

*Proof* By Riemann-Roch Theorem, we have

$$\sum_{j=0}^{p-1} (\dim_k \text{Hom}(\tau^j E, F) - \dim_k \text{Ext}^1(\tau^j E, F)) = \bar{\chi}(E, F) = \text{rk}(E)\text{rk}(F)(\mu(F) - \mu(E)) > 0.$$

Since  $\text{Ext}^1(\tau^j E, F) = 0$  for each  $j$ ,  $\text{Hom}(\tau^m E, F) \neq 0$  for some  $0 \leq m < p$ , whereby  $\text{Hom}(E, \tau^i F) \neq 0$  for some  $i$ . If  $E$  is non-exceptional indecomposable bundle then  $E$  has a filtration with factors  $\tau^i G$  ( $0 \leq i < p_E$ ), where  $G$  is the quasi-top of  $E$  and  $p_E$  is the  $\tau$ -period of  $E$ . Now that  $\text{Hom}(\tau^i G, F) \neq 0$  for some  $i$ ,  $\text{Hom}(E, F) \neq 0$ . Similar argument applies to the case when  $F$  is a non-exceptional indecomposable bundle. □

Using stability argument, [34] showed the following fact.

**Theorem 3.22** [34, Theorem 2.7] *Let  $F, G$  be nonzero bundles on  $\mathbb{X}$  with  $\mu(G) - \mu(F) > \delta(\bar{c} + \bar{\omega}) = p + \delta(\bar{\omega})$  then  $\text{Hom}(F, G) \neq 0$ .*

For  $E[n] \in \mathcal{D}^b(\mathbb{X})$  ( $E \in \text{coh}\mathbb{X}$ ), we defined the slope of  $E[n]$  by  $\mu(E[n]) = \mu(E)$ . For a nonzero subcategory  $\mathcal{C}$  of  $\mathcal{D}$  closed under nonzero direct summands, define

$$\mu(\mathcal{C}) = \{\mu(E) \mid E \text{ an indecomposable object in } \mathcal{C}\}. \tag{3.5.1}$$

We emphasize that we only count in indecomposables. In the sequel, we will need to consider limit points<sup>5</sup> of  $\mu(\mathcal{C})$  in  $\bar{\mathbb{R}}$ , where  $\bar{\mathbb{R}}$  is equipped with the topology obtained via one point compactification of  $\mathbb{R}$ .

If  $\mathbb{X}$  is of tubular type, by Lemma 3.13, for each  $q \in \bar{\mathbb{Q}}$ , there is a fractional linear function  $\phi_q$  on  $\bar{\mathbb{R}}$  with integer coefficients such that  $\mu(\Phi_{\infty,q}(E)) = \phi_q(\mu(E))$ , where  $\Phi_{\infty,q}$  is a telescopic functor. Evidently,  $\phi_q$  is a homeomorphism of  $\bar{\mathbb{R}}$  and restricts to a homeomorphism of the subspace  $\bar{\mathbb{Q}}$ .

**Lemma 3.23** *Suppose  $\mathbb{X}$  is of tubular type and let  $E$  be an exceptional sheaf over  $\mathbb{X}$ . Then  $\mu(E)$  is the unique limit point of  $\mu(E^{\perp A})$  (and  $\mu({}^{\perp A}E)$ ).*

*Proof* First suppose that  $E$  is an exceptional torsion sheaf. By Lemma 3.16 (and with the notation there), we have

$$E^{\perp A} = \mathcal{S}_E^{\perp A} \coprod (S'_E)_{\mathcal{A}} \simeq \text{coh}\mathbb{X}' \coprod \text{mod}k\bar{\mathbb{A}}_{I(E)-1},$$

where  $\mathbb{X}'$  is a weighted projective line of domestic type, and the equivalence  $\mathcal{S}_E^{\perp A} \simeq \text{coh}\mathbb{X}'$  preserves rank. By Theorem 3.9, the rank function  $\text{rk}$  is bounded on indecomposable sheaves in  $E^{\perp A}$ . Moreover,  $L(n\bar{c}) \in E^{\perp A}$  for a line bundle  $L \in E^{\perp A}$  and  $n \in \mathbb{Z}$ . Thus  $\infty$  is the unique limit point of  $\mu(E^{\perp A})$ .

<sup>5</sup>Recall that if  $A$  is a subset of a topological space  $X$  and if  $x$  is a point of  $X$ ,  $x$  is called a *limit point* of  $A$  in  $X$  if every neighborhood of  $x$  intersects  $A$  in some point other than  $x$  itself.

Now consider an exceptional bundle  $E$  with slope  $q$ . Since  $\Phi_{\infty,q}(E)$  is an exceptional torsion sheaf,  $\infty$  is the unique limit point of  $\mu(\Phi_{\infty,q}(E)^{\perp\mathcal{A}})$ . Now that

$$\mu(E^{\perp\mathcal{A}}) = \mu(E^{\perp\mathcal{D}}) = \phi_q^{-1}(\mu(\Phi_{\infty,q}(E)^{\perp\mathcal{D}})) = \phi_q^{-1}(\mu(\Phi_{\infty,q}(E)^{\perp\mathcal{A}})),$$

$q = \phi_q^{-1}(\infty)$  is the unique limit point of  $\mu(E^{\perp\mathcal{A}})$ .

Recall that  ${}^{\perp\mathcal{A}}E = (\tau^{-1}E)^{\perp\mathcal{A}}$ . Hence  $\mu(E) = \mu(\tau^{-1}E)$  is the unique limit point of  $\mu({}^{\perp\mathcal{A}}E) = \mu((\tau^{-1}E)^{\perp\mathcal{A}})$ . □

**Corollary 3.24** *Suppose  $\mathbb{X}$  is of tubular type. Let  $E$  be an indecomposable sheaf and  $\mathcal{E} = \{E_i \mid i \in I\}$  a collection of indecomposable sheaves with  $\mu(\mathcal{E})$  a bounded subset of  $\mathbb{R}$ . Suppose  $\mu$  is a limit point of  $\mu(\mathcal{E})$ . If  $\mu < \mu(E)$  then there is some  $E_i$  with  $\text{Hom}(E_i, E) \neq 0$ ; if  $\mu > \mu(E)$  then there is some  $E_i$  with  $\text{Hom}(E, E_i) \neq 0$ .*

*Proof* We will consider the case  $\mu < \mu(E)$  and the other case is similar. If  $E$  is non-exceptional then our assertion follows from Lemma 3.20 and Lemma 3.21. So we consider exceptional  $E$ . We can assume that  $\mu(E_i) < \mu(E)$  for all  $i$  by dropping the other  $E_i$ 's. Then  $\text{Ext}^1(E_i, E) = 0$  for all  $i$ . If  $\text{Hom}(E_i, E) = 0$  for all  $i$  then  $E_i \in {}^{\perp\mathcal{A}}E$  for all  $i$  and thus  $\mu$  is a limit point of  $\mu({}^{\perp\mathcal{A}}E)$ . This is a contradiction to Lemma 3.23. Thus we have  $\text{Hom}(E_i, E) \neq 0$  for some  $i$ . □

### 3.6 Full Exceptional Sequences in $\text{coh}\mathbb{X}$

It's well-known that if a  $k$ -linear essentially small triangulated category  $\mathcal{D}$  of finite type contains an exceptional sequence of length  $n$  then the rank  $\text{rk}K_0(\mathcal{D})$  of the Grothendieck group  $K_0(\mathcal{D})$  of  $\mathcal{D}$  satisfies  $\text{rk}K_0(\mathcal{D}) \geq n$ . In general, the exceptional sequence is not full even if  $n = \text{rk}K_0(\mathcal{D})$ . But this is the case in our setup.

**Lemma 3.25** *An exceptional sequence  $(E_1, \dots, E_n)$  in  $\mathcal{D}^b(\mathbb{X})$  is full iff  $n = \text{rk}K_0(\mathbb{X})$ .*

*Proof* We always have  $n \leq \text{rk}K_0(\mathcal{D}^b(\mathbb{X})) = \text{rk}K_0(\mathbb{X})$ . Meltzer [37, Lemma 4.1.2] showed that an exceptional sequence in  $\mathcal{D}^b(\mathbb{X})$  of length  $\text{rk}K_0(\mathbb{X})$  generates  $\mathcal{D}^b(\mathbb{X})$ . So an exceptional sequence  $(E_1, \dots, E_n)$  in  $\mathcal{D}^b(\mathbb{X})$  is full iff  $n = \text{rk}K_0(\mathbb{X})$ . □

Observe that by Serre duality, if  $(E_1, \dots, E_n)$  is a full exceptional sequence in  $\text{coh}\mathbb{X}$  then

$$(\tau E_{i+1}, \dots, \tau E_n, E_1, \dots, E_i)$$

is also a full exceptional sequence. We show that a full exceptional sequence in  $\text{coh}\mathbb{X}$  can possess certain nice term.

**Lemma 3.26** *If a full exceptional sequence in  $\text{coh}\mathbb{X}$  contains a torsion sheaf then it contains a simple sheaf.*

*Proof* Let  $(E_1, \dots, E_n)$  be a full exceptional sequence with  $E_i$  a torsion sheaf. We can suppose  $i = n$ . Note that  $(E_1, \dots, E_{n-1})$  is a full exceptional sequence in  $E_n^{\perp\mathcal{A}}$ . If  $E_n$  is already simple then there is nothing to prove. Suppose  $l(E_n) > 1$ . Then by Lemma 3.16, we have an equivalence

$$E_n^{\perp\mathcal{A}} \simeq \text{coh}\mathbb{X}' \coprod \coprod \text{mod}k\bar{\mathbb{A}}_{l(E_n)-1} \tag{3.6.1}$$

for some weighted projective line  $\mathbb{X}'$  and an equioriented  $\mathbb{A}_{l(E_n)-1}$ -quiver. Via this equivalence, a subsequence of  $(E_1, \dots, E_{n-1})$  yields a full exceptional sequence in  $\text{mod}k\bar{\mathbb{A}}_{l(E_n)-1}$ , which contains a simple module by Corollary 2.26. Note that a simple  $k\bar{\mathbb{A}}_{l(E_n)-1}$ -module maps to a simple sheaf under the equivalence (3.6.1), which is clear from Lemma 3.16. So some  $E_i$  is a simple sheaf.  $\square$

**Proposition 3.27** *For  $\mathbb{X}$  of domestic type, each full exceptional sequence in  $\text{coh}\mathbb{X}$  contains a line bundle.*

*Proof* Let  $(E_1, \dots, E_n)$  be a full exceptional sequence in  $\text{coh}\mathbb{X}$ . We use induction to show our assertion. Consider the weight type  $(1, p_1, p_2)$ , in which case each indecomposable bundle over  $\mathbb{X}$  is a line bundle. Since  $(E_1, \dots, E_n)$  classically generates  $\mathcal{D}^b(\mathbb{X})$ , some  $E_i$  is an indecomposable bundle and thus a line bundle. We continue to consider a domestic weight type different than  $(1, p_1, p_2)$  even up to permutation. We claim that if each  $E_i$  is a bundle then the assertion holds, which is proved later. So consider the case that some  $E_i$  is a torsion sheaf. We can assume that  $i = n$ . Moreover,  $(E_1, \dots, E_{n-1})$  is a full exceptional sequence in  $E_n^{\perp A}$ . By Lemma 3.16 (and with the notation there), we have

$$E_n^{\perp A} = S_{E_n}^{\perp A} \coprod \langle S'_{E_n} \rangle \simeq \text{coh}\mathbb{X}' \coprod \text{mod}k\bar{\mathbb{A}}_{l(E_n)-1},$$

where  $\mathbb{X}'$  is a weighted projective line with a weight function dominated by the weight function of  $\mathbb{X}$  (in the sense of [18]), and the equivalence  $S_{E_n}^{\perp A} \simeq \text{coh}\mathbb{X}'$  preserves rank. By induction, we know that some  $E_i$  ( $i \in \{1, \dots, n - 1\}$ ) is a line bundle.

It remains to prove our claim that if each  $E_i$  is a bundle then some  $E_i$  is a line bundle. The proof is inspired by the proof of [37, Proposition 4.3.6]. As in [37, §4.3.6], for an exceptional sequence  $\underline{E} = (E_1, \dots, E_n)$ , define

$$\|\underline{E}\| = (\text{rk}(E_{\pi(1)}), \dots, \text{rk}(E_{\pi(n)})),$$

where  $\pi$  is a permutation on  $\{1, \dots, n\}$  such that  $\text{rk}(E_{\pi(1)}) \geq \dots \geq \text{rk}(E_{\pi(n)})$ .

Suppose for a contradiction that  $\text{rk}(E_i) \geq 2$  for each  $i$ . In particular,  $\bigoplus E_i$  is not a tilting bundle since each tilting bundle contains a line bundle summand for  $\mathbb{X}$  of domestic type by [30, Corollary 3.7] (reproved with Corollary 3.36(1)). Hence for some  $i < j$ ,  $\text{Ext}^1(E_i, E_j) \neq 0$ . We can assume that  $\text{Ext}^1(E_k, E_l) = 0$  for  $i \leq k < l \leq j$ . By [37, Lemma 3.2.4],  $\text{Hom}(E_i, E_j) = 0$ .

Consider  $i < k < j$  such that  $\text{Hom}(E_i, E_k) \neq 0$ . Let  $f : E_i \rightarrow E_k$  be a nonzero morphism, which is either a monomorphism or an epimorphism by Happel-Ringel Lemma (see Proposition 2.19).  $f$  being a monomorphism implies

$$0 = \text{Ext}^1(E_k, E_j) \twoheadrightarrow \text{Ext}^1(E_i, E_j) \neq 0,$$

a contradiction. Hence  $f$  is an epimorphism. Thus  $\text{Hom}(E_i, E_j) = 0$  implies  $\text{Hom}(E_k, E_j) = 0$ .

Let  $P$  be the subsequence of  $(E_{i+1}, \dots, E_{j-1})$  consisting of those  $E_k$  satisfying  $\text{Hom}(E_i, E_k) \neq 0$ . Then for each term  $E_k$  in  $P$ , we have an epimorphism in  $\text{Hom}(E_i, E_k)$  and  $\text{Hom}(E_k, E_j) = 0$ . Let  $Q$  be the subsequence of  $(E_{i+1}, \dots, E_{j-1})$  consisting of the remaining terms, i.e., those  $E_l$  satisfying  $\text{Hom}(E_i, E_l) = 0$ . We want to show that  $\text{Hom}(E_k, E_l) = 0$  for  $E_k \in P, E_l \in Q$ . Each nonzero morphism  $g : E_k \rightarrow E_l$  is either a monomorphism or an epimorphism by Happel-Ringel Lemma. If  $g$  is a monomorphism then  $\text{Hom}(E_i, E_k) \neq 0$  implies  $\text{Hom}(E_i, E_l) \neq 0$ , a contradiction to  $\text{Hom}(E_i, E_l) = 0$ ; if

$g$  is an epimorphism then composing with an epimorphism in  $\text{Hom}(E_i, E_k)$  yields an epimorphism in  $\text{Hom}(E_i, E_l)$ , again a contradiction to  $\text{Hom}(E_i, E_l) = 0$ . These show that  $\text{Hom}(E_k, E_l) = 0$  for  $E_k \in P, E_l \in Q$ . Moreover,  $\text{Hom}(E_k, E_j) = 0$  for  $E_k \in P$ . Therefore the sequence

$$(E_1, \dots, E_{i-1}, Q, E_i, E_j, P, E_{j+1}, \dots, E_n)$$

is a full exceptional sequence. This gives us a full exceptional sequence  $(F_1, F_2, \dots, F_n)$  such that  $\text{rk}(F_i) \geq 2$  for all  $1 \leq i \leq n$ ,  $\text{Ext}^1(F_j, F_{j+1}) \neq 0$  and  $\text{Hom}(F_j, F_{j+1}) = 0$  for some  $1 \leq j \leq n - 1$ . Without loss of generality, we can assume  $j = 1$ .

Now we use mutation of an exceptional sequence. Let  $L_{F_1} F_2$  be the universal extension:

$$0 \rightarrow F_2 \rightarrow L_{F_1} F_2 \rightarrow \text{Ext}^1(F_1, F_2) \otimes F_1 \rightarrow 0.$$

Then

$$\underline{F}' = (L_{F_1} F_2, F_1, F_3, \dots, F_n)$$

is a full exceptional sequence with  $\|\underline{F}'\| > \|\underline{F}\|$ . As before, since each bundle in the sequence has rank  $\geq 2$ , the direct sum of bundles in  $\underline{F}'$  is not a tilting bundle. This allows us to repeat the argument above. Successive repeating will give us indecomposable bundles with arbitrary large rank. This is a contradiction to the fact that the rank function is bounded on indecomposable bundles over a weighted projective line of domestic type. We have thus shown our claim that each full exceptional sequence  $(E_1, \dots, E_n)$  with each  $E_i$  a bundle indeed contains a line bundle. □

**Corollary 3.28** *Suppose  $\mathbb{X}$  is of tubular type. If a full exceptional sequence in  $\text{coh}\mathbb{X}$  contains a torsion sheaf then it contains a line bundle and a simple sheaf.*

*Proof* Let  $(E_1, \dots, E_n)$  be a full exceptional sequence in  $\text{coh}\mathbb{X}$ . By Lemma 3.26, if some  $E_i$  is torsion then some  $E_j$  is simple. Suppose  $j = n$ . Since  $(E_1, \dots, E_{n-1})$  is a full exceptional sequence in  $E_n^{\perp A} \simeq \text{coh}\mathbb{X}'$ , where  $\mathbb{X}'$  is a weighted projective line of domestic type and the equivalence preserves rank, it follows from Proposition 3.27 that some  $E_k$  is a line bundle. □

### 3.7 Torsion Pairs in $\text{coh}\mathbb{X}$

In this subsection, we discuss some properties of torsion pairs in  $\text{coh}\mathbb{X}$  and also give some preparatory descriptions of torsion pairs (see Section 4.5 for the final description). We first describe two simple classes of torsion pairs in  $\text{coh}\mathbb{X}$ . Obviously, any torsion pair in  $\text{coh}\mathbb{X}$  restricts to a torsion pair in  $\text{coh}_\lambda\mathbb{X}$  for each  $\lambda \in \mathbb{P}^1$ .

**Lemma 3.29** *Let  $(\mathcal{T}, \mathcal{F})$  be a pair of subcategories of  $\text{coh}\mathbb{X}$ .*

1.  *$(\mathcal{T}, \mathcal{F})$  is a torsion pair in  $\text{coh}\mathbb{X}$  with  $\mathcal{T} \subset \text{coh}_0\mathbb{X}$  iff for each  $\lambda \in \mathbb{P}^1$ , there is a torsion pair  $(\mathcal{T}_\lambda, \mathcal{F}_\lambda)$  in  $\text{coh}_\lambda\mathbb{X}$  such that*

$$\mathcal{T} = \text{add}\{\mathcal{T}_\lambda \mid \lambda \in \mathbb{P}^1\}, \quad \mathcal{F} = \text{add}\{\text{vect}\mathbb{X}, \mathcal{F}_\lambda \mid \lambda \in \mathbb{P}^1\}.$$

2.  *$(\mathcal{T}, \mathcal{F})$  is a torsion pair in  $\text{coh}\mathbb{X}$  with  $\mathcal{F} \subset \text{coh}_0\mathbb{X}$  iff*

$$\mathcal{F} = \text{add}\{\mathcal{F}_\lambda \mid \lambda \in \mathbb{P}^1\}, \quad \mathcal{T} = \{E \in \text{coh}\mathbb{X} \mid \text{Hom}(E, \mathcal{F}) = 0\},$$

where each  $\mathcal{F}_\lambda$  is a torsion-free class in  $\text{coh}_\lambda\mathbb{X}$  without non-exceptional indecomposable object.

*Proof* We prove (2) as (1) is clear.

( $\Rightarrow$ ) Suppose  $\mathcal{F} \subset \text{coh}_0 \mathbb{X}$ .  $\mathcal{F}$  restricts to a torsion-free class  $\mathcal{F}_\lambda$  in  $\text{coh}_\lambda \mathbb{X}$  for each  $\lambda \in \mathbb{P}^1$ . If  $\mathcal{F}_\lambda$  contains a non-exceptional indecomposable sheaf then by Lemma 3.20,  $\mathcal{T}$  contains no nonzero bundle and thus  $\text{vect} \mathbb{X} \subset \mathcal{F}$ , a contradiction. Hence each  $\mathcal{F}_\lambda$  contains no non-exceptional indecomposable sheaf.

( $\Leftarrow$ ) By the definition of  $\mathcal{T}$ ,  $\mathcal{T}$  is closed under quotients and extensions. Therefore  $\mathcal{T}$  is a torsion class in  $\text{coh} \mathbb{X}$  since  $\text{coh} \mathbb{X}$  is noetherian. Then  $(\mathcal{T}, \mathcal{T}^{\perp 0})$  is a torsion pair in  $\text{coh} \mathbb{X}$  and thus we need to show  $\mathcal{F} = \mathcal{T}^{\perp 0}$ .  $\text{Hom}(\mathcal{T}, \mathcal{F}) = 0$  implies  $\mathcal{F} \subset \mathcal{T}^{\perp 0}$  and it remains to show  $\mathcal{T}^{\perp 0} \subset \mathcal{F}$ . For each  $\lambda \in \mathbb{P}^1$ ,  $\mathcal{T} \cap \text{coh}_\lambda \mathbb{X} = {}^{\perp 0, \text{coh}_\lambda \mathbb{X}} \mathcal{F}_\lambda$  is the torsion class in  $\text{coh}_\lambda \mathbb{X}$  corresponding to the torsion-free class  $\mathcal{F}_\lambda$ , which implies  $\mathcal{T}^{\perp 0} \cap \text{coh}_\lambda \mathbb{X} \subset \mathcal{F}_\lambda$ . Hence  $\mathcal{T}^{\perp 0} \cap \text{coh}_0 \mathbb{X} \subset \mathcal{F}$ . We claim that  $\mathcal{T}^{\perp 0}$  contains no nonzero bundle, which implies  $\mathcal{T}^{\perp 0} \subset \mathcal{F}$ . Suppose for a contradiction that  $\mathcal{T}^{\perp 0}$  contains a nonzero bundle  $E$ . For each  $\lambda \in \mathbb{P}^1$ , by Lemma 2.28, it is impossible that each simple sheaf in  $\text{coh}_\lambda \mathbb{X}$  occurs as a composition factor of some indecomposable sheaf in  $\mathcal{F}_\lambda$ . Hence we have a line bundle  $L$  such that  $L(n\bar{c}) \in \mathcal{T}$  for all  $n \in \mathbb{Z}$ . But  $\text{Hom}(L(n\bar{c}), E) \neq 0$  for  $n \ll 0$ , a contradiction. This shows our claim.  $\square$

*Remark 3.30* For an ordinary point  $\lambda$ , either  $\mathcal{T}_\lambda = 0$  or  $\mathcal{F}_\lambda = 0$ .

Recall that for each  $\mu \in \bar{\mathbb{R}}$ , we have torsion pairs

$$(\text{coh}^{\geq \mu} \mathbb{X}, \text{coh}^{< \mu} \mathbb{X}), \quad (\text{coh}^{> \mu} \mathbb{X}, \text{coh}^{\leq \mu} \mathbb{X}).$$

These are very useful for our analysis.

A torsion pair in  $\text{coh} \mathbb{X}$  is either tilting or cotilting.

**Lemma 3.31** *Let  $(\mathcal{T}, \mathcal{F})$  be a torsion pair in  $\text{coh} \mathbb{X}$ .*

- (1) *If  $\mathcal{F}$  contains a nonzero bundle then  $\mathcal{F}$  is a cotilting torsion-free class and  $\text{coh}^{\leq \mu} \mathbb{X} \subset \mathcal{F}$  for some  $\mu \in \mathbb{R}$ .*
- (2) *If  $\mathcal{T}$  contains a nonzero bundle then  $\mathcal{T}$  is a tilting torsion class. If  $\text{coh}_0 \mathbb{X} \subsetneq \mathcal{T}$  then  $\text{coh}^{\geq \nu} \mathbb{X} \subset \mathcal{T}$  for some  $\nu \in \mathbb{R}$ .*

*Proof* Suppose that  $\mathcal{F}$  contains a nonzero bundle  $A$ . If  $\mathcal{T}$  contains no nonzero bundle, then  $\text{vect} \mathbb{X} \subset \mathcal{F}$ . Now suppose that  $\mathcal{T}$  contains a nonzero bundle  $T$ . Let  $\mu = \mu(A) - \delta(\bar{c} + \bar{w})$ . Then for each bundle  $B \in \mathcal{T}$ , we have  $\mu(B) > \mu$ . Indeed, if  $\mu(B) \leq \mu$  then  $\mu(A) - \mu(B) \geq \delta(\bar{c} + \bar{w})$  and  $\text{Hom}(B, A) \neq 0$  by Theorem 3.22, a contradiction to  $\text{Hom}(\mathcal{T}, \mathcal{F}) = 0$ . Since  $\mathcal{T}$  is closed under quotients, for each nonzero bundle  $E$  in  $\mathcal{T}$ , the last semistable factor of  $E$  lies in  $\mathcal{T}$  and hence  $\mu^-(E) > \mu$ . This shows  $\text{vect} \mathbb{X} \cap \mathcal{T} \subset \text{coh}^{> \mu} \mathbb{X}$ . Recall that a coherent sheaf over  $\mathbb{X}$  decomposes as a direct sum of a torsion sheaf and a vector bundle. So we have  $\mathcal{T} \subset \text{coh}^{> \mu} \mathbb{X}$  and thus  $\text{coh}^{\leq \mu} \mathbb{X} \subset \mathcal{F}$ . Similarly one shows that if  $\mathcal{T}$  contains a nonzero bundle then  $\text{vect} \mathbb{X} \cap \mathcal{F} \subset \text{coh}^{< \nu} \mathbb{X}$  for some  $\nu \in \mathbb{R}$ , which implies  $\text{coh}^{\geq \nu} \mathbb{X} \subset \mathcal{T}$  provided  $\text{coh}_0 \mathbb{X} \subsetneq \mathcal{T}$ .

Now we show that  $\mathcal{F}$  is a cotilting torsion-free class if  $\mathcal{F}$  contains a nonzero bundle. That is, we need to show that for each sheaf  $E$ , there is some sheaf  $F \in \mathcal{F}$  and an epimorphism  $F \twoheadrightarrow E$ . We do induction on  $\text{rk}(E)$ . We already have  $\text{coh}^{\leq \mu} \mathbb{X} \subset \mathcal{F}$  for some  $\mu \in \mathbb{R}$ . If  $E$  is an indecomposable torsion sheaf then we can take a line bundle  $L \in \mathcal{F}$  such that  $L \twoheadrightarrow E$ . If  $\text{rk}(E) > 0$ , take a line bundle  $L_1 \in \mathcal{F}$  with  $\mu(L) \ll \mu(E)$ . Then we have an exact sequence



$0 \rightarrow L_1 \rightarrow E \rightarrow E_1 \rightarrow 0$  with  $\text{rk}(E_1) < \text{rk}(E)$ . By the induction hypothesis, there is some  $F_1 \in \mathcal{F}$  and an epimorphism  $F_1 \twoheadrightarrow E_1$ . The pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1 & \longrightarrow & F & \longrightarrow & F_1 & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L_1 & \longrightarrow & E & \longrightarrow & E_1 & \longrightarrow & 0 \end{array}$$

gives us an object  $F \in \mathcal{F}$  and an epimorphism  $F \twoheadrightarrow E$ , as desired.

If  $\mathcal{T}$  contains a nonzero bundle, we show that  $\mathcal{T}$  is a tilting torsion class. For each  $\lambda \in \mathbb{P}^1$ , consider the torsion pair  $(\mathcal{T}_\lambda, \mathcal{F}_\lambda) = (\mathcal{T} \cap \text{coh}_\lambda \mathbb{X}, \mathcal{F} \cap \text{coh}_\lambda \mathbb{X})$  in  $\text{coh}_\lambda \mathbb{X}$ . By Lemma 3.20,  $\mathcal{F}_\lambda$  contains no non-exceptional object and thus  $\mathcal{T}_\lambda$  contains a non-exceptional object. Then  $S \in \mathcal{T}$  for a simple sheaf  $S$  supported at an ordinary point. Moreover,  $\mathcal{T}_\lambda$  is a tilting torsion class in  $\text{coh}_\lambda \mathbb{X}$  by Lemma 2.27. Hence each indecomposable torsion sheaf in  $\text{coh}_\lambda \mathbb{X}$  is a subobject of some object in  $\mathcal{T}_\lambda$ . Since  $\mathcal{T}$  is closed under quotients,  $\mathcal{T}$  contains a line bundle  $L$  by Proposition 3.4(2).  $L, S \in \mathcal{T}$  implies  $L(n\vec{c}) \in \mathcal{T}$  for  $n \geq 0$ . By [17, Corollary 2.7], for each  $E \in \text{vect} \mathbb{X}$ ,  $E$  is a subbundle of  $\bigoplus_{i=1}^m L_i$  for some line bundles  $L_1, \dots, L_m$ . Now that  $L_i$  is a subbundle of  $L(n\vec{c})$  for  $n \gg 0$ ,  $E$  is a subbundle of  $\bigoplus_{i=1}^m L(n\vec{c}) \in \mathcal{T}$ . This shows that  $\mathcal{T}$  is a tilting torsion class if  $\mathcal{T}$  contains a nonzero bundle.  $\square$

**Lemma 3.32** *Let  $(\mathcal{T}, \mathcal{F})$  be a torsion pair in  $\text{coh} \mathbb{X}$  with  $\text{coh}_0 \mathbb{X} \subsetneq \mathcal{T} \subsetneq \text{coh} \mathbb{X}$ .*

- (1) *If  $\mathbb{X}$  is of domestic type then the  $\tau$ -orbit of each line bundle contains some line bundle  $L$  such that  $L \in \mathcal{T}$  and  $\tau L \in \mathcal{F}$ .*
- (2) *If  $\mathbb{X}$  is of tubular type then exactly one of the following holds:*
  - (a) *there exists some quasi-simple bundle  $E$  in  $\mathcal{T}$  with  $\tau E \in \mathcal{F}$ ;*
  - (b) *for some  $\mu \in \mathbb{R} \setminus \mathbb{Q}$ ,  $(\mathcal{T}, \mathcal{F}) = (\text{coh}^{>\mu} \mathbb{X}, \text{coh}^{<\mu} \mathbb{X})$ ;*
  - (c) *for some  $\mu \in \mathbb{Q}$  and some  $P \subset \mathbb{P}^1$ ,*

$$(\mathcal{T}, \mathcal{F}) = (\text{add}\{\text{coh}^{>\mu} \mathbb{X}, \text{coh}_\lambda^\mu \mathbb{X} \mid \lambda \in P\}, \text{add}\{\text{coh}_\lambda^\mu \mathbb{X}, \text{coh}^{<\mu} \mathbb{X} \mid \lambda \notin P\}).$$

*Proof* Note that  $\text{coh}_0 \mathbb{X} \subsetneq \mathcal{T} \subsetneq \text{coh} \mathbb{X}$  implies  $\{0\} \subsetneq \mathcal{F} \subsetneq \text{vect} \mathbb{X}$ . By Lemma 3.31,  $\text{coh}^{\leq \mu_0} \mathbb{X} \subset \mathcal{F}$  for some  $\mu_0 \in \mathbb{R}$  and  $\text{coh}^{\geq \nu_0} \mathbb{X} \subset \mathcal{T}$  for some  $\nu_0 \in \mathbb{R}$ .

- (1) By Lemma 3.38,  $\mu(\tau^n L) = \mu(L) + n\delta(\vec{\omega})$ . Since  $\delta(\vec{\omega}) < 0$ , for each line bundle  $L$ ,  $\tau^n L \in \mathcal{F}$  for  $n \gg 0$  and  $\tau^n L \in \mathcal{T}$  for  $n \ll 0$ . Moreover  $\text{coh}_0 \mathbb{X} \subsetneq \mathcal{T}$  implies that each line bundle lies in  $\mathcal{T}$  or  $\mathcal{F}$  and therefore there must be a line bundle  $\tau^n L \in \mathcal{T}$  with  $\tau^{n+1} L \in \mathcal{F}$ .
- (2) Obviously, the three types of torsion pairs are disjoint. We shall show that if  $(\mathcal{T}, \mathcal{F})$  is not of type (a) then  $(\mathcal{T}, \mathcal{F})$  is of type (b) or (c). Suppose that for any quasi-simple bundle  $E \in \mathcal{T}$ , we have  $\tau E \notin \mathcal{F}$ . Note that we have  $\text{coh}^{\geq \nu_0} \mathbb{X} \subset \mathcal{T} \subset \text{coh}^{>\mu_0} \mathbb{X}$ . Define

$$\begin{aligned} \mu_1 &= \inf\{\mu^-(E) \mid E \in \mathcal{T}\} \\ &= \inf\{\mu^-(E) \mid \text{quasi-simple bundle } E \in \mathcal{T}\}, \end{aligned}$$

which lies in  $\mathbb{R}$ . We have two cases to consider: i) there exists a quasi-simple bundle  $E$  in  $\mathcal{T}$  with  $\mu(E) = \mu_1$ ; 2)  $\mathcal{T}$  contains no quasi-simple bundle with slope  $\mu_1$  but there exists a sequence  $(E_i)$  of quasi-simple bundles in  $\mathcal{T}$  such that  $\mu(E_i) \rightarrow (\mu_1)_+$  as  $i \rightarrow \infty$ .

First we consider case ii). In this case, we have  $\mathcal{T} \subset \text{coh}^{>\mu_1} \mathbb{X}$  and thus  $\text{coh}^{\leq \mu_1} \mathbb{X} \subset \mathcal{F}$ . Moreover, for indecomposable  $F \in \mathcal{F}$ , if  $\mu(F) > \mu_1$  then by Corollary 3.24 there exists

some  $E_i$  with  $\text{Hom}(E_i, F) \neq 0$ , a contradiction to  $\text{Hom}(\mathcal{T}, \mathcal{F}) = 0$ . In particular,  $\mu^+(F) \leq \mu_1$  for  $F \in \mathcal{F}$  since  $\mathcal{F}$  is closed under subobjects. So  $\mathcal{F} \subset \text{coh}^{\leq \mu_1} \mathbb{X}$  and thus

$$(\mathcal{T}, \mathcal{F}) = (\text{coh}^{> \mu_1} \mathbb{X}, \text{coh}^{\leq \mu_1} \mathbb{X}),$$

which is of type (b) or (c).

Now we consider case i). We claim that for any quasi-simple bundle  $E \in \mathcal{T}$  with  $\mu(E) = \mu_1$  we have  $\tau E \in \mathcal{T}$ . Let  $0 \rightarrow A \rightarrow \tau E \rightarrow B \rightarrow 0$  be an exact sequence with  $A \in \mathcal{T}$ ,  $B \in \mathcal{F}$ . Since we are assuming  $\tau E \notin \mathcal{F}$ , we have  $A \neq 0$ . Since  $A \in \mathcal{T}$ , we have  $\mu^-(A) \geq \mu_1$ ; since  $\text{Hom}(A, \tau E) \neq 0$  and since  $\tau E$  is a semistable bundle with  $\mu(\tau E) = \mu_1$ , we have  $\mu^-(A) = \mu_1$ . Let  $C$  be the last semi-stable factor of  $A$ , which lies in  $\mathcal{T}$  and has slope  $\mu_1$ . Then  $\text{Hom}(C, \tau E) \neq 0$ . Since  $\tau E$  is a simple object in  $\text{coh}^{\mu_1} \mathbb{X}$ ,  $E$  is a quotient object of  $C$ . So  $\tau E \in \mathcal{T}$ . Thus our claim is proved. It follows that the  $\tau$ -orbit of any quasi-simple bundle  $E \in \mathcal{T}$  is contained in  $\mathcal{T}$ . Take a quasi-simple bundle  $E$  in  $\mathcal{T}$ . For indecomposable  $F \in \mathcal{F}$ , if  $\mu(F) > \mu_1$  then by Lemma 3.21 there exists some  $\tau^j E$  with  $\text{Hom}(\tau^j E, F) \neq 0$ , a contradiction to  $\text{Hom}(\mathcal{T}, \mathcal{F}) = 0$ . So  $\mathcal{F} \subset \text{coh}^{\leq \mu_1} \mathbb{X}$ . It follows that

$$\text{coh}^{> \mu_1} \mathbb{X} \subset \mathcal{T} \subset \text{coh}^{\geq \mu_1} \mathbb{X}, \quad \text{coh}^{< \mu_1} \mathbb{X} \subset \mathcal{F} \subset \text{coh}^{\leq \mu_1} \mathbb{X}.$$

Let  $\lambda \in \mathbb{P}^1$ . For an indecomposable semistable bundle  $T \in \text{coh}^{\mu_1} \mathbb{X}$ , if  $T$  lies in  $\mathcal{T}$  then the quasi-top of  $T$  as a quotient of  $T$  lies in  $\mathcal{T}$ , which implies  $\text{coh}^{\mu_1} \mathbb{X} \subset \mathcal{T}$ . Hence if  $\text{coh}^{\mu_1} \mathbb{X} \cap \mathcal{T} \neq 0$  then  $\text{coh}^{\mu_1} \mathbb{X} \subset \mathcal{T}$ . Denote

$$P = \{\lambda \in \mathbb{P}^1 \mid \text{coh}^{\mu_1} \mathbb{X} \cap \mathcal{T} \neq 0\}.$$

Then we have

$$(\mathcal{T}, \mathcal{F}) = (\text{add}\{\text{coh}^{> \mu_1} \mathbb{X}, \text{coh}^{\mu_1} \mathbb{X} \mid \lambda \in P\}, \text{add}\{\text{coh}^{\mu_1} \mathbb{X}, \text{coh}^{< \mu_1} \mathbb{X} \mid \lambda \in \mathbb{P}^1 \setminus P\}),$$

which is of type (c). □

We establish bijective correspondences between tilting sheaves, certain bounded t-structures on  $\mathcal{D}^b(\mathbb{X})$  and certain torsion pairs in  $\text{coh} \mathbb{X}$ .

**Proposition 3.33** *Denote  $\mathcal{A} = \text{coh} \mathbb{X}$ . There are bijective correspondences between*

- (1) *torsion pairs  $(\mathcal{T}, \mathcal{F})$  in  $\mathcal{A}$  such that the tilted heart  $\mathcal{F}[1] * \mathcal{T}$  is a length category;*
- (2) *bounded t-structures whose heart is a length category contained in  $\mathcal{A}[1] * \mathcal{A}$ ;*
- (3) *isomorphism classes of basic tilting sheaves in  $\mathcal{A}$ ;*
- (4) *torsion pairs  $(\mathcal{T}, \mathcal{F})$  such that there is  $n = \text{rk} K_0(\mathbb{X})$  pairwise non-isomorphic indecomposable sheaves  $E_1, \dots, E_n$  in  $\mathcal{T}$  with  $\tau E_i \in \mathcal{F}$  for all  $i$ .*

*Moreover, torsion pairs  $(\mathcal{T}, \mathcal{F})$  in (1) with the additional assumption  $\text{coh}_0 \mathbb{X} \subsetneq \mathcal{T} \subsetneq \text{coh} \mathbb{X}$  are in bijection with isoclasses of basic tilting bundles.*

*Proof* The second assertion follows readily from the first one. We show the first assertion. The bijection between (1) and (2) follows from Proposition 2.3. Note that for those  $E_i$ 's in (4), we have  $\text{Hom}(\oplus E_i, \oplus \tau E_i) = 0$ . By Serre duality, we have  $\text{Ext}^1(\oplus E_i, \oplus E_i) = 0$ . Thus  $E_i$ 's can be ordered to be a full exceptional sequence by Proposition 2.20 and Lemma 3.25. So  $\oplus E_i$  is a tilting sheaf. Then the obvious associations between (3) and (4) are evidently inverse to each other.

Now we establish the bijection between (2) and (3). By Theorem 3.6,  $\mathcal{A} = \text{coh} \mathbb{X}$  is derived equivalent to  $\text{mod} \Lambda$  for a canonical algebra  $\Lambda$ . Hence we can apply Theorem 2.22 to conclude that bounded t-structures on  $\mathcal{D}^b(\mathcal{A})$  with length heart are in bijection with equivalence classes of silting objects in  $\mathcal{D}^b(\mathbb{X})$ . Note that if a bounded t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$

has heart  $\mathcal{B} \subset \mathcal{A}[1] * \mathcal{A}$  then  $\mathcal{D}_{\mathcal{A}}^{\leq -1} \subset \mathcal{D}^{\leq 0} \subset \mathcal{D}_{\mathcal{A}}^{\leq 0}$  and thus the Serre functor  $\tau(-)[1]$  of  $\mathcal{D}^b(\mathcal{A})$  is right t-exact with respect to  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ . By Lemma 2.24, in this bijection, a bounded t-structure with length heart  $\mathcal{B} \subset \mathcal{A}[1] * \mathcal{A}$  corresponds to some equivalence class of tilting objects in  $\mathcal{D}^b(\mathbb{X})$ . It remains to show that such a tilting object  $T$  is a sheaf. By Lemma 2.24,  $T, \tau T[1] \in \mathcal{B} \subset \mathcal{A}[1] * \mathcal{A}$ . This forces  $T$  to be a sheaf.  $\square$

*Remark 3.34* Recall that we have a torsion pair  $(\mathcal{T}, \mathcal{F})$  induced by a tilting sheaf  $T$ , where

$$\mathcal{T} = \{E \in \text{coh}\mathbb{X} \mid \text{Ext}^1(T, E) = 0\}, \quad \mathcal{F} = \{E \in \text{coh}\mathbb{X} \mid \text{Hom}(T, E) = 0\}.$$

Since  $T \in \mathcal{T}, \tau T \in \mathcal{F}$ , this torsion pair is just the one corresponding to  $T$ .

*Example 3.35* Consider the torsion pair  $(\mathcal{T}, \mathcal{F}) = (\text{coh}^{\geq \mu}\mathbb{X}, \text{coh}^{< \mu}\mathbb{X})$  for  $\mu \in \mathbb{R}$ . If  $\mathbb{X}$  is of domestic type, similar argument to that in the proof of [30, Theorem 3.5] shows that the direct sum of a complete set of indecomposable bundles with slope in the interval  $[\mu, \mu - \delta(\bar{\omega}))$  is a tilting bundle, whose endomorphism algebra turns out to be a tame hereditary algebra. The induced torsion pair is exactly  $(\text{coh}^{\geq \mu}\mathbb{X}, \text{coh}^{< \mu}\mathbb{X})$ . If  $\mathbb{X}$  is not of domestic type then  $\mathcal{T}$  (resp.  $\mathcal{F}$ ) is closed under  $\tau$  (resp.  $\tau^{-1}$ ) since  $\delta(\bar{\omega}) \geq 0$ . Therefore  $(\text{coh}^{> \mu}\mathbb{X}, \text{coh}^{\leq \mu}\mathbb{X})$  cannot be induced by a tilting sheaf and the tilted heart  $\text{coh}^{\leq \mu}\mathbb{X}[1] * \text{coh}^{> \mu}\mathbb{X}$  is not a length category.

We obtain the following known results as a corollary of Proposition 3.33.

**Corollary 3.36** (1) ([30, Corollary 3.7]). *If  $\mathbb{X}$  is of domestic type then each tilting bundle  $T$  contains at least  $[L(\underline{p}) : \mathbb{Z}\bar{\omega}]$  pairwise nonisomorphic line bundles as its direct summands.*

(2) ([32, Corollary 3.5]). *If  $\mathbb{X}$  is of tubular type then each tilting bundle  $T$  contains a quasi-simple bundle direct summand. For some  $q \in \mathbb{Q}, \Phi_{\infty, q}(T)$  is a tilting sheaf with an exceptional simple sheaf as its direct summand.*

*Proof* Let  $(\mathcal{T}, \mathcal{F})$  be the torsion pair corresponding to  $T$ . Since  $T$  is a bundle,  $\text{coh}_0\mathbb{X} \subsetneq \mathcal{T} \subsetneq \text{coh}\mathbb{X}$ .

- (1) By Lemma 3.32, each  $\tau$ -orbit of a line bundle contains a line bundle  $L \in \mathcal{T}$  with  $\tau L \in \mathcal{F}$ . Each such  $L$  is a direct summand of  $T$ . By Proposition 3.4, we have precisely  $[L(\underline{p}) : \mathbb{Z}\bar{\omega}]$   $\tau$ -orbits of line bundles. So  $T$  contains at least  $[L(\underline{p}) : \mathbb{Z}\bar{\omega}]$  pairwise nonisomorphic line bundles.
- (2) Note that in Lemma 3.32, a torsion pair  $(\mathcal{U}, \mathcal{V})$  in  $\text{coh}\mathbb{X}$  of type 3.32(2b) or 3.32(2c) contains no nonzero sheaf  $F$  with  $F \in \mathcal{U}$  and  $\tau F \in \mathcal{V}$ . So  $(\mathcal{T}, \mathcal{F})$  is of type 3.32(2a), i.e., there exists a quasi-simple bundle  $E$  with  $E \in \mathcal{T}, \tau E \in \mathcal{F}$ .  $E$  is then a direct summand of  $T$ . Let  $q$  be the maximal slope of indecomposable direct summands of  $T$ . Then  $\Phi_{\infty, q}(T)$  is a tilting sheaf with a nonzero torsion direct summand. Since its indecomposable direct summands can be ordered to be a full exceptional sequence, by Lemma 3.26, one of the direct summands is a simple sheaf. This finishes the proof.  $\square$

We end this subsection by determining whether certain torsion pairs yield a noetherian or artinian tilted heart. For  $P \subset \mathbb{P}^1$ , denote by  $(\mathcal{T}_P, \mathcal{F}_P)$  the torsion pair in  $\text{coh}\mathbb{X}$

$$(\text{add}\{\text{coh}_{\lambda}\mathbb{X} \mid \lambda \in P\}, \text{add}\{\text{vect}\mathbb{X}, \text{coh}_{\lambda}\mathbb{X} \mid \lambda \in \mathbb{P}^1 \setminus P\}). \tag{3.7.1}$$

**Lemma 3.37** *Let  $P \subset \mathbb{P}^1$ .*

- (1) *The tilted heart  $\mathcal{B} = \mathcal{F}_P[1] * \mathcal{T}_P$  is noetherian resp. artinian iff  $P = \emptyset$  resp.  $P = \mathbb{P}^1$ .*
- (2) *Suppose  $\mathbb{X}$  is of tubular type. If  $\mu \in \mathbb{R} \setminus \mathbb{Q}$  then the tilted heart  $\mathcal{B} = \text{coh}^{<\mu} \mathbb{X}[1] * \text{coh}^{>\mu} \mathbb{X}$  is neither noetherian nor artinian. If  $\mu \in \mathbb{Q}$ , the tilted heart  $\mathcal{B} = \mathcal{F}[1] * \mathcal{T}$  is noetherian resp. artinian iff  $P = \emptyset$  resp.  $P = \mathbb{P}^1$ , where*

$$(\mathcal{T}, \mathcal{F}) = (\text{add}\{\text{coh}^{>\mu} \mathbb{X}, \text{coh}^{\mu}_{\lambda} \mathbb{X} \mid \lambda \in P\}, \text{add}\{\text{coh}^{\mu}_{\lambda} \mathbb{X}, \text{coh}^{<\mu} \mathbb{X} \mid \lambda \notin P\}).$$

*Proof* (1) If  $P = \emptyset$  then  $\mathcal{B} = \text{coh} \mathbb{X}[1]$ , which is noetherian. If  $P = \mathbb{P}^1$  then  $\mathcal{B} = \text{vect} \mathbb{X}[1] * \text{coh}_0 \mathbb{X} \simeq (\text{coh} \mathbb{X})^{\text{op}}$  is artinian, where the equivalence is induced by the duality functor  $R\mathcal{H}om(-, \mathcal{O})$ . Otherwise,  $\emptyset \neq P \neq \mathbb{P}^1$ . Take  $\lambda \in P, \lambda' \notin P$ . To see that  $\mathcal{B} = \mathcal{F}[1] * \mathcal{T}$  is not artinian, we take an indecomposable torsion sheaf  $F$  supported at  $\lambda'$  such that  $F$  fits into an exact sequence

$$0 \rightarrow \mathcal{O}(n\bar{c}) \rightarrow \mathcal{O}((n+1)\bar{c}) \rightarrow F \rightarrow 0$$

in  $\text{coh} \mathbb{X}$ . Then for each  $n \in \mathbb{Z}$ , we have an exact sequence

$$0 \rightarrow \mathcal{O}(n\bar{c})[1] \rightarrow \mathcal{O}((n+1)\bar{c})[1] \rightarrow F[1] \rightarrow 0$$

in  $\mathcal{B}$ . This implies the existence of a strict infinite descending chain of subobjects of  $\mathcal{O}[1]$  in  $\mathcal{B}$ , which shows  $\mathcal{B}$  is not artinian. Now we show that  $\mathcal{B}$  is not noetherian. There exists a simple sheaf  $S \in \text{coh}_{\lambda} \mathbb{X}$  such that for each  $l \in \mathbb{Z}_{\geq 1}$ , the unique indecomposable sheaf  $S^{[l\mathbf{w}(\lambda)]} \in \text{coh}_{\lambda} \mathbb{X}$  with socle  $S$  and of length  $l\mathbf{w}(\lambda)$ , where  $\mathbf{w}$  is the weight function, fits into an exact sequence

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(l\bar{c}) \rightarrow S^{[l\mathbf{w}(\lambda)]} \rightarrow 0$$

in  $\text{coh} \mathbb{X}$ . In particular,  $S^{[l\mathbf{w}(\lambda)]}$  is a subobject of  $\mathcal{O}[1]$  in  $\mathcal{B}$ . Note also that  $S^{[l\mathbf{w}(\lambda)]}$  is a subobject of  $S^{[(l+1)\mathbf{w}(\lambda)]}$  in  $\mathcal{B}$ . The infinite ascending chain

$$S^{[\mathbf{w}(\lambda)]} \hookrightarrow S^{[2\mathbf{w}(\lambda)]} \hookrightarrow S^{[3\mathbf{w}(\lambda)]} \hookrightarrow \dots$$

of subobjects of  $\mathcal{O}[1]$  shows that  $\mathcal{B}$  is not noetherian. Hence  $\mathcal{B}$  is neither noetherian nor artinian in this case.

- (2) The assertion for  $\mu \in \mathbb{Q}$  is reduced to (1) by using the telescopic functor  $\Phi_{\infty, \mu}$ . So we consider  $\mu \in \mathbb{R} \setminus \mathbb{Q}$ . By applying the duality functor  $R\mathcal{H}om(-, \mathcal{O})$ , we know that

$$\mathcal{B} = \text{coh}^{<\mu} \mathbb{X}[1] * \text{coh}^{>\mu} \mathbb{X} \simeq (\text{coh}^{<-\mu} \mathbb{X}[1] * \text{coh}^{>-\mu} \mathbb{X})^{\text{op}}.$$

To show that  $\mathcal{B}$  is neither noetherian nor artinian, it suffices to show that  $\mathcal{B}$  is not artinian, which in turn follows readily from our claim that each nonzero bundle  $F \in \text{coh}^{>\mu} \mathbb{X}$  fits into an exact sequence  $0 \rightarrow E \rightarrow G \rightarrow F \rightarrow 0$ , where  $E \in \text{coh}^{<\mu} \mathbb{X}$  and  $G \in \text{coh}^{>\mu} \mathbb{X}$ .

Let us show our claim. Let  $F \in \text{coh}^{>\mu} \mathbb{X}$  be a nonzero bundle. By [45, Corollary 1.9], there exists a pair of coprime integers  $(h, k)$  such that

$$0 < \frac{h}{k} - \mu < \frac{1}{k^2}.$$

We can assume further

$$k > \text{rk}(F), \quad \frac{h}{k} + \frac{1}{k^2} < \mu^-(F)$$

since  $k$  can actually be taken to be arbitrarily large. By Lemma 3.14, there is a quasi-simple bundle  $A \in \text{coh}^{\frac{h}{k}} \mathbb{X}$  with coprime rank and degree. In particular, we have  $\text{rk}(A) =$

$k, \text{deg}(A) = h$ . Now let  $F_1, \dots, F_m$  be the semistable factors of  $F$ . Denote  $F_0 = F$ . Let  $F_l^j$  be an indecomposable direct summand of  $F_l$ . Consider the evaluation map

$$\text{ev}: \bigoplus_{i=0}^{p_A-1} \text{Hom}(\tau^i A, F_l^j) \otimes \tau^i A \longrightarrow F_l^j,$$

where  $p_A$  is the  $\tau$ -period of  $A$ . By [37, Theorem 5.1.3],  $\text{ev}$  is either a monomorphism or an epimorphism. Since

$$\text{rk} \left( \bigoplus_{i=0}^{p_A-1} \text{Hom}(\tau^i A, F_l^j) \otimes \tau^i A \right) > \text{rk} F \geq \text{rk} F_l^j,$$

$\text{ev}$  is an epimorphism. In particular, the evaluation map

$$\text{ev}: \bigoplus_{i=0}^{p_A-1} \text{Hom}(\tau^i A, F_l) \otimes \tau^i A \longrightarrow F_l$$

is an epimorphism, whose kernel is denoted by  $E_l$ . For  $1 \leq l \leq m$ ,  $E_l$  is a semistable bundle whose slope  $\mu(E_l)$  satisfies

$$\begin{aligned} \mu(E_l) &= \frac{\bar{\chi}(A, F_l)\text{deg}(A) - \text{deg}(F_l)}{\bar{\chi}(A, F_l)\text{rk}(A) - \text{rk}(F_l)} \\ &= \frac{(\mu(F_l) - \mu(A))\mu(A) - \frac{1}{\text{rk}(A)^2}\mu(F_l)}{(\mu(F_l) - \mu(A)) - \frac{1}{\text{rk}(A)^2}} \quad (\text{by Riemann-Roch theorem}) \\ &< \mu. \end{aligned}$$

Let  $T_{\tau \bullet A}$  be the tubular mutation functor with respect to the  $\tau$ -orbit of  $A$  as recalled in the proof of Lemma 3.13. Note that  $E_l \cong T_{\tau \bullet A}(F_l)[-1]$  for  $0 \leq l \leq m$ . Since  $F_0 = F$  has a filtration with factors  $F_1, \dots, F_m$ ,  $E := E_0$  has a filtration with factors  $E_1, \dots, E_m$ . In particular,  $\mu^+(E) < \mu$ . Hence

$$0 \longrightarrow E \longrightarrow \bigoplus_{i=0}^{p_A-1} \text{Hom}(\tau^i A, F) \otimes \tau^i A \longrightarrow F \longrightarrow 0$$

is the desired exact sequence. We are done. □

### 4 Bounded t-Structures on $\mathcal{D}^b(\mathbb{X})$

Throughout this section,  $\mathbb{X}$  will denote a weighted projective line,  $\mathcal{A} = \text{coh}\mathbb{X}$  the category of coherent sheaves over  $\mathbb{X}$  and  $\mathcal{D} = \mathcal{D}^b(\mathbb{X})$  the bounded derived category of  $\text{coh}\mathbb{X}$ . Moreover,  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  will denote a bounded t-structure on  $\mathcal{D}$  and its heart will be denoted by  $\mathcal{B}$ . The standard t-structure on  $\mathcal{D}^b(\mathbb{X})$  is denoted by  $(\mathcal{D}_{\mathcal{A}}^{\leq 0}, \mathcal{D}_{\mathcal{A}}^{\geq 0})$ .

**Lemma 4.1** *Each bounded t-structure on  $\mathcal{D}^b(\mathbb{X})$  is width-bounded with respect to the standard t-structure. In particular,  $\mathcal{B} \subset \mathcal{D}_{\mathcal{A}}^{[m,n]}$  for some  $m, n \in \mathbb{Z}$ .*

*Proof* Recall that for each  $\mathbb{X}$ , there is a canonical algebra  $\Lambda$  such that  $\mathcal{D}^b(\mathbb{X}) \simeq \mathcal{D}^b(\Lambda)$ . Henceforth we have a bounded t-structure on  $\mathcal{D}^b(\mathbb{X})$  with heart equivalent to  $\text{mod}\Lambda$ . So bounded t-structures are width-bounded with respect to each other (see Example 2.2). □

### 4.1 Bounded t-Structures which Restrict to t-Structures on $\mathcal{D}^b(\text{coh}_0\mathbb{X})$

In this subsection, we characterize when a bounded t-structure on  $\mathcal{D}^b(\mathbb{X})$  restricts to a t-structure on  $\mathcal{D}^b(\text{coh}_0\mathbb{X})$  and then describe this class of t-structures.

The following fact is very useful in analyzing direct summands of truncations of an object.

**Lemma 4.2** *Let  $\mathcal{T}$  be a triangulated category. Assume that  $A \xrightarrow{f} B \xrightarrow{g} C \rightsquigarrow$  is a triangle in  $\mathcal{T}$  with  $\text{Hom}^{-1}(A, C) = 0$ . If  $A = A_1 \oplus A_2$  and correspondingly  $f = (f_1, f_2)$  then  $f_1 = 0$  implies  $A_1 = 0$ . If  $C = C_1 \oplus C_2$  and  $g = (g_1, g_2)^t$  then  $g_1 = 0$  implies  $C_1 = 0$ .*

*Proof*  $f_1 = 0$  implies  $C \cong \text{cone}(f_2) \oplus A_1[1]$  and then  $\text{Hom}(A_1, A_1) \subset \text{Hom}^{-1}(A, C) = 0$  thus  $A_1 = 0$ . Similarly one shows the second assertion. □

**Lemma 4.3** *If  $\mathcal{D}^{[m,n]}$  contains a nonzero bundle then for some  $m \leq l \leq n$ ,  $\mathcal{B}[-l]$  contains a nonzero bundle.*

*Proof* We use induction on  $n - m$ . If  $n = m$  then there is nothing to prove. Assume  $n > m$ . Let  $E$  be a nonzero bundle lying in  $\mathcal{D}^{[m,n]}$ . Consider the triangle  $E_1 \rightarrow E \rightarrow E_2 \rightsquigarrow$ , where  $E_1 = \tau_{\leq n-1} E \in \mathcal{D}^{[m,n-1]}$ ,  $E_2 = \tau_{\geq n} E \in \mathcal{B}[-n]$ . Recall that since  $\text{coh}\mathbb{X}$  is hereditary, each object  $X$  in  $\mathcal{D}^b(\mathbb{X})$  decomposes as  $X \cong \bigoplus \mathcal{H}^i(X)[-i]$ , where  $\mathcal{H}^i(X)$  is the  $i$ -th cohomology of  $X$ . Since  $\text{Hom}^{-1}(E_1, E_2) = 0$ , by Lemma 4.2,  $\mathcal{H}^i(E_1) = 0$  for  $i \neq 0, 1$  and  $\mathcal{H}^j(E_2) = 0$  for  $j \neq 0, -1$ . Hence  $E_1$  decomposes as a direct sum  $A \oplus B[-1]$  and  $E_2$  as a direct sum  $C \oplus D[1]$ , where  $A, B, C, D$  are sheaves. Taking cohomology yields a long exact sequence

$$0 \longrightarrow D \longrightarrow A \longrightarrow E \longrightarrow C \longrightarrow B \longrightarrow 0.$$

If  $A = 0$  then  $D = 0$  and thus  $\text{rk}(C) > 0$ , that is,  $C$  contains a nonzero bundle direct summand. Since  $C \in \mathcal{B}[-n]$ , such a direct summand gives a desired bundle. Since  $\text{Hom}(\text{coh}_0\mathbb{X}, \text{vect}\mathbb{X}) = 0$ , if  $A \neq 0$  then  $A$  cannot be a torsion sheaf by Lemma 4.2. Thus  $A$  contains a nonzero bundle direct summand  $F$ . Now that  $F \in \mathcal{D}^{[m,n-1]}$ , the induction hypothesis assures the existence of the desired bundle. □

Let us make our basic observation on bounded t-structures on  $\mathcal{D}^b(\mathbb{X})$ .

**Lemma 4.4** *The following are equivalent:*

- (1)  $\{i \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} \subset \{j, j + 1\}$  for some  $j \in \mathbb{Z}$ ;
- (2)  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  restricts to a bounded t-structure on  $\mathcal{D}^b(\text{coh}_\lambda\mathbb{X})$  for each  $\lambda \in \mathbb{P}^1$ ;
- (3)  $\mathcal{B}$  contains a shift of some non-exceptional indecomposable torsion sheaf.

*Proof* (2)  $\Rightarrow$  (3) Take an ordinary point  $\lambda$ . The induced bounded t-structure on  $\mathcal{D}^b(\text{coh}_\lambda\mathbb{X})$  has heart  $\mathcal{B}_\lambda = \mathcal{B} \cap \mathcal{D}^b(\text{coh}_\lambda\mathbb{X})$ . Since  $\lambda$  is ordinary, each bounded t-structure on  $\mathcal{D}^b(\text{coh}_\lambda\mathbb{X})$  is a shift of the standard one by Proposition 2.30. Hence a shift of the simple torsion sheaf  $S$  supported at  $\lambda$  lies in  $\mathcal{B}_\lambda \subset \mathcal{B}$ .

(3)  $\Rightarrow$  (1) Suppose  $T$  is a non-exceptional indecomposable torsion sheaf such that  $T[j] \in \mathcal{B}$ . By Lemma 3.20, for each nonzero bundle  $E$ ,  $\text{Ext}^1(T, E) \neq 0$  and  $\text{Hom}(E, T) \neq 0$ . Now that  $T[j] \in \mathcal{B}$ , if  $E[i] \in \mathcal{B}$  then  $i \neq j, j + 1$  will yield a contradiction to  $\text{Hom}^n(\mathcal{B}, \mathcal{B}) = 0$  for  $n < 0$ . Hence  $\{i \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} \subset \{j, j + 1\}$ .

(1)  $\Rightarrow$  (2) We will show that (1) implies that  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  restricts to a bounded t-structure on  $\mathcal{D}^b(\text{coh}_0 \mathbb{X})$ . Then (2) follows since  $\text{coh}_0 \mathbb{X} = \coprod_{\lambda \in \mathbb{P}^1} \text{coh}_\lambda \mathbb{X}$ . Suppose that  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  does not restrict to a t-structure on  $\mathcal{D}^b(\text{coh}_0 \mathbb{X})$ . Then for some torsion sheaf  $T$  and some  $l \in \mathbb{Z}$ ,  $\tau_{\leq l} T \notin \mathcal{D}^b(\text{coh}_0 \mathbb{X})$ . By Lemma 4.2,  $\tau_{\leq l} T$  decomposes as  $A \oplus B[-1]$  with  $A \in \text{coh} \mathbb{X}$ ,  $B \in \text{coh}_0 \mathbb{X}$  and  $\tau_{> l} T$  decomposes as  $C \oplus D[1]$  with  $C \in \text{coh}_0 \mathbb{X}$ ,  $D \in \text{coh} \mathbb{X}$ .  $\tau_{\leq l} T \notin \mathcal{D}^b(\text{coh}_0 \mathbb{X})$  implies that  $A$  contains a nonzero bundle  $E$  as its direct summand. Since  $\text{rk}(A) = \text{rk}(D)$ ,  $D$  also contains such a direct summand  $F$ . Now that  $E \in \mathcal{D}^{\leq l}$ ,  $F \in \mathcal{D}^{\geq l+2}$  and the t-structure is bounded, by Lemma 4.3, both  $\mathcal{B}[-r]$  and  $\mathcal{B}[-s]$  contain nonzero bundles for some  $r \leq l, s \geq l + 2$ . It is then impossible that  $\{i \mid \text{vect} \mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} \subset \{j, j + 1\}$  for some  $j$ .  $\square$

We are going to give a description of bounded t-structures on  $\mathcal{D}^b(\mathbb{X})$  satisfying the conditions in the above lemma. Recall the definition of a proper collection of simple sheaves in Section 3.4. Two such collections are said to be equivalent if they yield the same isoclasses of simple sheaves. Recall also that for  $P \subset \mathbb{P}^1$ , the pair  $(\mathcal{T}_P, \mathcal{F}_P)$  denotes the torsion pair (3.7.1) in  $\text{coh} \mathbb{X}$ . Moreover, we have a split torsion pair  $(\mathcal{S}^{\perp A} \cap \mathcal{T}_P, \mathcal{S}^{\perp A} \cap \mathcal{F}_P)$  in  $\mathcal{S}^{\perp A}$ .

**Proposition 4.5** *Suppose  $\{i \in \mathbb{Z} \mid \text{vect} \mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} = \{j\}$  or  $\{j - 1, j\}$  for some  $j \in \mathbb{Z}$ . Then there is a unique (up to equivalence) proper collection  $\mathcal{S}$  of simple sheaves such that*

- $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the recollement

$$\mathcal{D}^b(\mathcal{S}^{\perp A}) = \mathcal{S}^{\perp \mathcal{D}} \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!} \end{array} \mathcal{D}^b(\mathbb{X}) \begin{array}{c} \xleftarrow{j^!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*} \end{array} \langle \mathcal{S} \rangle_{\mathcal{D}},$$

where  $i_*, j_*$  are the inclusion functors;

- if  $\{i \mid \text{vect} \mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} = \{j\}$  then for a unique  $P \subset \mathbb{P}^1$ , the corresponding t-structure on  $\mathcal{S}^{\perp \mathcal{D}}$  is a shift of the HRS-tilt with respect to the torsion pair  $(\mathcal{S}^{\perp A} \cap \mathcal{T}_P, \mathcal{S}^{\perp A} \cap \mathcal{F}_P)$  in  $\mathcal{S}^{\perp A}$ ;
- if  $\{i \mid \text{vect} \mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} = \{j - 1, j\}$  then the corresponding t-structure on  $\mathcal{S}^{\perp \mathcal{D}}$  is a shift of the HRS-tilt with respect to some torsion pair  $(\mathcal{T}, \mathcal{F})$  in  $\mathcal{S}^{\perp A}$  with  $\mathcal{S}^{\perp A} \cap \text{coh}_0 \mathbb{X} \subsetneq \mathcal{T} \subsetneq \mathcal{S}^{\perp A}$ .

*Proof* By Lemma 4.4,  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  restricts to a bounded t-structure on  $\mathcal{D}^b(\text{coh}_\lambda \mathbb{X})$  for each  $\lambda \in \mathbb{P}^1$ . Denote

$$\begin{aligned} \mathcal{A}_\lambda &= \text{coh}_\lambda \mathbb{X}, & \mathcal{D}_\lambda &= \langle \text{coh}_\lambda \mathbb{X} \rangle_{\mathcal{D}} = \mathcal{D}^b(\text{coh}_\lambda \mathbb{X}), \\ \mathcal{D}_\lambda^{\leq 0} &= \mathcal{D}^{\leq 0} \cap \mathcal{D}_\lambda, & \mathcal{D}_\lambda^{\geq 0} &= \mathcal{D}^{\geq 0} \cap \mathcal{D}_\lambda, & \mathcal{B}_\lambda &= \mathcal{B} \cap \mathcal{D}_\lambda. \end{aligned}$$

Then  $(\mathcal{D}_\lambda^{\leq 0}, \mathcal{D}_\lambda^{\geq 0})$  is a bounded t-structure on  $\mathcal{D}_\lambda$  with heart  $\mathcal{B}_\lambda$ . Observe that each Ext-projective object in  $\mathcal{D}_\lambda^{\leq 0}$  is Ext-projective in  $\mathcal{D}^{\leq 0}$ . Indeed, if  $X \in \mathcal{D}_\lambda^{\leq 0} \subset \mathcal{D}^{\leq 0}$  is  $\mathcal{D}_\lambda^{\leq 0}$ -projective then  $\tau X[1] \in \mathcal{D}_\lambda^{\geq 0} \subset \mathcal{D}^{\geq 0}$ , which implies  $X$  is  $\mathcal{D}^{\leq 0}$ -projective.

For each  $\lambda \in \mathbb{P}^1$ , by Proposition 2.30, there is a unique proper collection  $\mathcal{S}_\lambda$  of simple sheaves supported at  $\lambda$  such that  $(\mathcal{D}_\lambda^{\leq 0}, \mathcal{D}_\lambda^{\geq 0})$  is compatible with

$$\mathcal{S}_\lambda^{\perp \mathcal{D}_\lambda} \begin{array}{c} \xleftarrow{f^*} \\ \xrightarrow{f_*} \\ \xleftarrow{f^!} \end{array} \mathcal{D}_\lambda = \mathcal{D}^b(\text{coh}_\lambda \mathbb{X}) \begin{array}{c} \xleftarrow{g^!} \\ \xrightarrow{g^*} \\ \xleftarrow{g_*} \end{array} \langle \mathcal{S}_\lambda \rangle_{\mathcal{D}_\lambda},$$

where  $f_*, g_*$  are the inclusion functors, and the corresponding t-structure on  $\mathcal{S}_\lambda^{\perp \mathcal{D}_\lambda}$  has heart  $\mathcal{B}_\lambda \cap \mathcal{S}_\lambda^{\perp \mathcal{D}_\lambda} = \mathcal{S}_\lambda^{\perp \mathcal{A}_\lambda}[m_\lambda]$  for some  $m_\lambda$ . If  $\mathcal{S}_\lambda = \emptyset$  (say when  $\lambda$  is an ordinary point), let  $T_\lambda = 0$ . Otherwise,  $\langle \mathcal{S}_\lambda \rangle_{\mathcal{D}_\lambda}$  is triangle equivalent to  $\mathcal{D}^b(\prod_{i=1}^{n_\lambda} \text{mod } k\vec{\mathbb{A}}_{l_{i,\lambda}})$  for some positive integers  $n_\lambda, l_{i,\lambda}$ , where  $k\vec{\mathbb{A}}_l$  is the path algebra of the equioriented  $\mathbb{A}_l$ -quiver. By Theorem 2.22, the t-structure  $(g^* \mathcal{D}_\lambda^{\leq 0}, g^* \mathcal{D}_\lambda^{\geq 0})$  on  $\langle \mathcal{S}_\lambda \rangle_{\mathcal{D}_\lambda}$  corresponds to a basic silting object  $T_\lambda$  in  $\langle \mathcal{S}_\lambda \rangle_{\mathcal{D}_\lambda}$  so that  $\langle T_\lambda \rangle_{\mathcal{D}_\lambda} = \langle \mathcal{S}_\lambda \rangle_{\mathcal{D}_\lambda}$  and  $T_\lambda$  is  $g^* \mathcal{D}_\lambda^{\leq 0}$ -projective. By Lemma 2.18,  $T_\lambda = g_! T_\lambda$  is  $\mathcal{D}_\lambda^{\leq 0}$ -projective and hence  $T_\lambda$  is  $\mathcal{D}^{\leq 0}$ -projective. By Proposition 2.21, the indecomposable direct summands of  $T_\lambda$  can be ordered to form an exceptional sequence. Let  $T = \bigoplus_\lambda T_\lambda, \mathcal{S} = \bigcup_\lambda \mathcal{S}_\lambda$ . We have  $\langle T \rangle_{\mathcal{D}} = \langle \mathcal{S} \rangle_{\mathcal{D}}$  and the indecomposable direct summands of  $T$  can be ordered to form an exceptional sequence. Then by Lemma 2.17,  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the recollement

$$\mathcal{S}^{\perp \mathcal{D}} = T^{\perp \mathcal{D}} \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!} \end{array} \mathcal{D} \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*} \end{array} \langle T \rangle_{\mathcal{D}} = \langle \mathcal{S} \rangle_{\mathcal{D}},$$

where  $i_*, j_!$  are the inclusion functors.

Now let us show that the corresponding t-structure on  $\mathcal{S}^{\perp \mathcal{D}}$  takes the asserted form. Let  $\mathcal{B}_1 = \mathcal{B} \cap \mathcal{S}^{\perp \mathcal{D}}$  be its heart. We have  $\mathcal{S}_\lambda^{\perp \mathcal{A}_\lambda}[m_\lambda] = \mathcal{B}_1 \cap \mathcal{D}_\lambda \subset \mathcal{B}_1$ . Hence for each  $\lambda \in \mathbb{P}^1$ , there is a nonexceptional indecomposable torsion sheaf  $F_\lambda$  such that  $F_\lambda[m_\lambda] \in \mathcal{B}$ . Up to a shift of  $\mathcal{B}$ , we can suppose  $\{i | \text{vect} \mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} = \{1\}$  or  $\{0, 1\}$ . If  $\{i | \text{vect} \mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} = \{1\}$ , let  $E$  be a nonzero bundle such that  $E[1] \in \mathcal{B}$ .  $\text{Hom}(E, F_\lambda) \neq 0$  and  $\text{Ext}^1(F_\lambda, E) \neq 0$  imply that  $m_\lambda \in \{0, 1\}$ . If  $\{i | \text{vect} \mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} = \{0, 1\}$  then we have nonzero bundles  $E_1, E_2$  with  $E_1, E_2[1] \in \mathcal{B}$ .  $\text{Hom}(E_i, F_\lambda) \neq 0$  and  $\text{Ext}^1(F_\lambda, E_i) \neq 0$  ( $i = 1, 2$ ) imply  $m_\lambda = 0$ . Consequently, in either case, we have  $\mathcal{B}_1 \subset \mathcal{S}^{\perp \mathcal{A}}[1] * \mathcal{S}^{\perp \mathcal{A}}$  and thus  $\mathcal{B}_1 = \mathcal{F}[1] * \mathcal{T}$  for some torsion pair  $(\mathcal{T}, \mathcal{F})$  in  $\mathcal{S}^{\perp \mathcal{A}}$ . Moreover, if  $\{i | \text{vect} \mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} = \{1\}$  then  $\mathcal{T} = \text{add}\{\mathcal{S}_\lambda^{\perp \mathcal{A}_\lambda} | \lambda \in P\} = \mathcal{S}^{\perp \mathcal{A}} \cap \mathcal{T}_P$ , where  $P = \{\lambda \in \mathbb{P}^1 | m_\lambda = 0\}$ ; if  $\{i | \text{vect} \mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} = \{0, 1\}$  then  $\mathcal{S}^{\perp \mathcal{A}} \cap \text{coh}_0 \mathbb{X} \subsetneq \mathcal{T} \subsetneq \mathcal{S}^{\perp \mathcal{A}}$ .

Finally, the uniqueness of  $\mathcal{S}$  follows from the uniqueness of  $\mathcal{S}_\lambda$ ; the uniqueness of  $P$  follows from Lemma 2.7. □

*Remark 4.6* Actually, for each bounded t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  on  $\mathcal{D}$ , there exists a unique maximal proper collection  $\mathcal{S}$  of simple sheaves such that  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the admissible subcategory  $\mathcal{S}^{\perp \mathcal{D}}$ . The crucial point to show this is that  $\langle T \rangle_{\mathcal{D}} = \langle \mathcal{S} \rangle_{\mathcal{D}}$ , where  $T$  is the direct sum of a complete set of indecomposable  $\mathcal{D}^{\leq 0}$ -projectives of the form  $E[n]$  with  $E$  a torsion sheaf.

*Remark 4.7* Recall from Theorem 3.15 that we have an equivalence  $\mathcal{S}^{\perp \mathcal{A}} \simeq \text{coh} \mathbb{X}'$  for some weighted projective line  $\mathbb{X}'$ . Via such an equivalence, the torsion pair  $(\mathcal{S}^{\perp \mathcal{A}} \cap \mathcal{T}_P, \mathcal{S}^{\perp \mathcal{A}} \cap \mathcal{F}_P)$  in  $\mathcal{S}^{\perp \mathcal{A}}$  corresponds to the torsion pair  $(\mathcal{T}'_P, \mathcal{F}'_P)$  in  $\text{coh} \mathbb{X}'$ ; a torsion pair  $(\mathcal{T}, \mathcal{F})$  in  $\mathcal{S}^{\perp \mathcal{A}}$  with  $\mathcal{S}^{\perp \mathcal{A}} \cap \text{coh}_0 \mathbb{X} \subsetneq \mathcal{T} \subsetneq \mathcal{S}^{\perp \mathcal{A}}$  corresponds to a torsion pair  $(\mathcal{T}', \mathcal{F}')$  in  $\text{coh} \mathbb{X}'$  with  $\text{coh}_0 \mathbb{X}' \subsetneq \mathcal{T}' \subsetneq \text{coh} \mathbb{X}'$ .

Here we characterize when the heart of a bounded t-structure just described is noetherian, artinian or of finite length.



**Corollary 4.8** *With the notation in Proposition 4.5, in the case  $\{i \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} = \{j\}$ , the heart  $\mathcal{B}$  is not of finite length and  $\mathcal{B}$  is noetherian resp. artinian iff  $P = \emptyset$  resp.  $P = \mathbb{P}^1$ ; in the case  $\{i \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} = \{j - 1, j\}$ , the heart  $\mathcal{B}$  is noetherian (artianian or of finite length) iff so is the tilted heart  $\mathcal{F}[1] * \mathcal{T}$ .*

*Proof* Recall that there exist integers  $n, l_1, \dots, l_n$  such that  $\langle \mathcal{S} \rangle_{\mathcal{A}} \simeq \coprod_{i=1}^n \text{mod}k\tilde{\mathbb{A}}_{l_i}$ . By Lemma 2.23, each bounded t-structure on  $\langle \mathcal{S} \rangle_{\mathcal{D}} = \mathcal{D}^b(\langle \mathcal{S} \rangle_{\mathcal{A}}) \simeq \mathcal{D}^b(\coprod_{i=1}^n \text{mod}k\tilde{\mathbb{A}}_{l_i})$  has a length heart. So the assertion for the case  $\{i \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} = \{j - 1, j\}$  follows from Lemma 2.12. For the case  $\{i \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} = \{j\}$ , by virtue of the equivalence  $\mathcal{S}^{\perp \mathcal{A}} \simeq \text{coh}\mathbb{X}'$  in Theorem 3.15, the assertion follows from Lemma 3.37(1) and Lemma 2.12.  $\square$

**4.2 Bounded t-Structures which do not Restrict to t-Structures on  $\mathcal{D}^b(\text{coh}_0\mathbb{X})$  Even up to Action of  $\text{Aut}\mathcal{D}^b(\mathbb{X})$**

Now we deal with bounded t-structures on  $\mathcal{D}^b(\mathbb{X})$  which does not satisfy the condition considered above even up to the action of  $\text{Aut}\mathcal{D}^b(\mathbb{X})$ . We only have results for the domestic and tubular cases and we rely heavily on the telescopic functors in the tubular case.

The key feature of this class of t-structures is given in the following lemma.

**Lemma 4.9** (1) *If  $\mathbb{X}$  is of domestic type then each indecomposable object in  $\mathcal{B}$  is exceptional iff  $\{i \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} \not\subseteq \{j, j + 1\}$  for any  $j \in \mathbb{Z}$ .*  
 (2)  *$\mathbb{X}$  is of tubular type then each indecomposable object in  $\mathcal{B}$  is exceptional iff  $\{i \mid \text{vect}\mathbb{X}[i] \cap \Phi_{\infty,q}(\mathcal{B}) \neq \emptyset\} \not\subseteq \{j, j + 1\}$  for any  $q \in \tilde{\mathbb{Q}}$  and  $j \in \mathbb{Z}$ .*

*Proof* Each indecomposable object in  $\mathcal{B}$  is of the form  $E[n]$  for some  $n \in \mathbb{Z}$  and some indecomposable bundle or some indecomposable torsion sheaf  $E$ .

- (1) By Theorem 3.9, if  $\mathbb{X}$  is of domestic type then each indecomposable bundle is exceptional. So  $\mathcal{B}$  contains a non-exceptional indecomposable object iff  $\mathcal{B}$  contains a shift of a non-exceptional torsion sheaf, which is equivalent to  $\{i \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} \subseteq \{j, j + 1\}$  for some  $j \in \mathbb{Z}$  by Lemma 4.4. So our assertion holds.
- (2) By Theorem 3.12, if  $\mathbb{X}$  is of tubular type then each indecomposable sheaf is semistable and thus lies in  $\text{coh}^{\mu}\mathbb{X}$  for some  $\mu \in \tilde{\mathbb{Q}}$ .  $\mathcal{B}$  contains a non-exceptional indecomposable object  $E[n]$ , where  $E$  is a sheaf with slope  $q$ , iff the heart  $\Phi_{\infty,q}(\mathcal{B})[-n]$  contains the non-exceptional torsion sheaf  $\Phi_{\infty,q}(E)$ . Thus our assertion follows from Lemma 4.4.  $\square$

We show that  $\mathcal{B}$  contains no cycle if each indecomposable object in  $\mathcal{B}$  is exceptional.

**Lemma 4.10** *Suppose  $\mathbb{X}$  is of domestic or tubular type. If each indecomposable object in  $\mathcal{B}$  is exceptional then a complete set of pairwise non-isomorphic indecomposable objects in  $\mathcal{B}$  can be totally ordered as  $\{X_i\}_{i \in I}$  such that  $\text{Hom}(X_i, X_j) = 0$  if  $i < j$ .*

*Proof* Each indecomposable object in  $\mathcal{B}$  is of the form  $E[n]$  for some indecomposable sheaf  $E$ . Since  $\text{Hom}(E[n], F[m]) = 0$  for  $E, F \in \mathcal{A}$  and  $n > m$ , it suffices to order indecomposables in  $\mathcal{B} \cap \mathcal{A}[n]$ , or rather, indecomposables in  $\mathcal{B}[-n] \cap \mathcal{A}$ . For  $\mathbb{X}$  of domestic or tubular type, each indecomposable sheaf is semistable and  $\text{Hom}(E, F) = 0$  for indecomposable

sheaves  $E, F$  with  $\mu(E) > \mu(F)$ . Thus we only need to consider indecomposable sheaves with the same slope, i.e., those in  $\mathcal{B}[-n] \cap \text{coh}^\mu \mathbb{X}$ . We have assumed these indecomposables to be exceptional.

We consider  $\mu = \infty$  at first. If indecomposables in  $\mathcal{B}[-n] \cap \text{coh}^\infty \mathbb{X} = \mathcal{B}[-n] \cap \text{coh}_0 \mathbb{X}$  cannot be totally ordered as desired then  $\mathcal{B}[-n] \cap \text{coh}_0 \mathbb{X}$  will contain a cycle of indecomposables in some  $\text{coh}_\lambda \mathbb{X}$ . By Lemma 2.28,  $\mathcal{B}$  contains a non-exceptional object, a contradiction. Hence indecomposables in  $\mathcal{B}[-n] \cap \text{coh}^\infty \mathbb{X}$  can be totally ordered as desired. Now we consider  $\mu \in \mathbb{Q}$ . If  $\mathbb{X}$  is of domestic type then indecomposable bundles in  $\text{coh}^\mu \mathbb{X}$  are stable and thus the morphism spaces between each other vanish, whence any order is satisfying. If  $\mathbb{X}$  is of tubular type then using the telescopic functor  $\Phi_{\infty, \mu}$ , we know from the conclusion for  $\mu = \infty$  that the desired ordering also exists. □

Recall the definition of  $\mu(\mathcal{B})$  from Eq. 3.5.1. Observe that each limit point in  $\mu(\mathcal{B})$  is a limit point of  $\mu(\mathcal{B}[l] \cap \mathcal{A})$  for some  $l$  since  $\mathcal{A}$  is hereditary and since  $\mathcal{B} \subset \mathcal{D}_{\mathcal{A}}^{[m,n]}$  for some  $m, n \in \mathbb{Z}$  by Lemma 4.1.

**Lemma 4.11**  $\infty$  is a limit point of  $\mu(\mathcal{B})$  iff  $\{i \mid \text{vect} \mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} \subset \{j, j + 1\}$  for some  $j \in \mathbb{Z}$ .

*Proof* ( $\Rightarrow$ ) If  $\infty$  is a limit point of  $\mu(\mathcal{B})$  then there is a sequence  $(E_i)_{i=1}^\infty$  of objects in some  $\mathcal{B}[l]$ , where  $E_i$ 's are indecomposable bundles, such that

$$\mu(E_i) \rightarrow +\infty \text{ or } \mu(E_i) \rightarrow -\infty \text{ as } i \rightarrow +\infty.$$

If  $\mu(E_i) \rightarrow +\infty$  then by Theorem 3.22, for each nonzero bundle  $F$ ,  $\text{Hom}(F, E_i) \neq 0$  and  $\text{Ext}^1(E_i, F) \neq 0$  for  $i \gg 1$ . Consequently,  $F[k] \in \mathcal{B}$  implies  $k \in \{l, l + 1\}$ . Similar arguments apply to the case  $\mu(E_i) \rightarrow -\infty$ .

( $\Leftarrow$ ) Suppose  $\{i \mid \text{vect} \mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} = \{j\}$  or  $\{j, j + 1\}$ . By Proposition 4.5, we can take a line bundle  $L$  such that  $L \in \mathcal{B}[-j]$ . Moreover, for a simple sheaf  $S$  supported at an ordinary point, i)  $S \in \mathcal{B}[-j]$  or ii)  $S[1] \in \mathcal{B}[-j]$ . If case i) happens,  $L(n\vec{c}) \in \mathcal{B}[-j]$  for  $n \geq 0$ ; if case ii) happens,  $L(n\vec{c}) \in \mathcal{B}[-j]$  for  $n \leq 0$ . In either case,  $\infty$  is a limit point of  $\mu(\mathcal{B})$ . □

The following lemma allows us to apply a telescopic functor in the next proposition.

**Lemma 4.12** If  $\mathbb{X}$  is of tubular type and  $\mu(\mathcal{B})$  has an irrational number as its limit point then for some  $q \in \mathbb{Q}$ ,  $\Phi_{\infty, q}(\mathcal{B})$  coincides with a shift of the tilted heart with respect to some torsion pair in  $\mathcal{A}$ .

*Proof* Suppose that for some  $l \in \mathbb{Z}$ ,  $\mu(\mathcal{B} \cap \mathcal{A}[l])$  has an irrational number  $r$  as its limit point. Then there is a sequence  $(E_i)_{i=1}^\infty$  of indecomposable bundles such that  $E_i \in \mathcal{B}[-l]$  and  $\mu(E_i)$  converges to  $r$ . Let  $E$  be an indecomposable sheaf with  $\mu(E) < r$ . By Corollary 3.24, there are some  $E_i$  with  $\text{Hom}(E, E_i) \neq 0$  and some  $E_j$  with  $\text{Hom}(\tau^{-1}E, E_j) \neq 0$ , which implies  $\text{Ext}^1(E_j, E) \neq 0$ . Thus for  $h \in \mathbb{Z}$ ,  $E[h] \in \mathcal{B}$  implies  $h \in \{l, l + 1\}$ . Similarly, if  $F$  is an indecomposable sheaf with  $\mu(F) > r$ , then for some  $E_i, E_j$ ,  $\text{Hom}(E_i, F) \neq 0$ ,  $\text{Ext}^1(F, E_j) \neq 0$ . For  $h \in \mathbb{Z}$ ,  $F[h] \in \mathcal{B}$  implies  $h \in \{l, l - 1\}$ . Consequently, if  $\mu(\mathcal{B} \cap \mathcal{A}[l])$  has an irrational limit point  $r$  then

$$\{k \in \mathbb{Z} \mid \mathcal{B} \cap \mathcal{A}[k] \neq \emptyset\} \subset \{l - 1, l, l + 1\}$$

and an indecomposable sheaf in  $\mathcal{B}[-1 - l]$  (resp.  $\mathcal{B}[1 - l]$ ) has slope  $< r$  (resp.  $> r$ ).

If  $\mu(\mathcal{B} \cap \mathcal{A}[l + 1])$  also has an irrational number as its limit point then similar arguments as before show that  $\{k \mid \mathcal{B} \cap \mathcal{A}[k] \neq \emptyset\} \subset \{l, l + 1\}$ , that is,  $\mathcal{B} \subset \mathcal{A}[l + 1] * \mathcal{A}[l]$ . Thus  $\mathcal{B}$  is a shift of the tilted heart with respect to some torsion pair in  $\mathcal{A}$ . Consider the case that the set of limit points of  $\mu(\mathcal{B} \cap \mathcal{A}[l + 1])$  is contained in  $\mathbb{Q}$ . Since each indecomposable sheaf in  $\mathcal{B}[-l - 1] \cap \mathcal{A}$  has slope less than  $r$ , there is some rational number  $q < r$  such that each indecomposable sheaf  $E \in \mathcal{B}[-l - 1] \cap \mathcal{A}$  has slope  $\mu(E) \leq q$ . Then

$$\Phi_{\infty,q}(\mathcal{B} \cap \mathcal{A}[l + 1]) \subset \mathcal{A}[l + 1].$$

Since an indecomposable object  $E \in \mathcal{B}[1 - l] \cap \mathcal{A}$  has slope  $\mu(E) > r > q$ , we have

$$\Phi_{\infty,q}(\mathcal{B} \cap \mathcal{A}[l - 1]) \subset \mathcal{A}[l].$$

It follows that

$$\Phi_{\infty,q}(\mathcal{B}) = \Phi_{\infty,q}(\text{add}\{\mathcal{B} \cap \mathcal{A}[l - 1], \mathcal{B} \cap \mathcal{A}[l], \mathcal{B} \cap \mathcal{A}[l + 1]\}) \subset \mathcal{A}[l + 1] * \mathcal{A}[l],$$

as desired. □

The class of bounded t-structures on  $\mathcal{D}^b(\mathbb{X})$  under consideration is reminiscent of bounded t-structures on  $\mathcal{D}^b(\Lambda)$ , where  $\Lambda$  is a representation-finite finite dimensional hereditary algebra, as the following proposition indicates.

**Proposition 4.13** *If one of the following cases occurs:*

- $\mathbb{X}$  is of domestic type and  $\{i \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} \not\subset \{j, j + 1\}$  for any  $j \in \mathbb{Z}$ ,
- $\mathbb{X}$  is of tubular type and  $\{i \mid \text{vect}\mathbb{X}[i] \cap \Phi_{\infty,q}(\mathcal{B}) \neq \emptyset\} \not\subset \{j, j + 1\}$  for any  $q \in \bar{\mathbb{Q}}$  and  $j \in \mathbb{Z}$ ,

then  $\mathcal{B}$  is a length category with finitely many (isoclasses of) indecomposables and each indecomposable object in  $\mathcal{B}$  is exceptional.

*Proof* It has been shown in Lemma 4.9 that each indecomposable object in  $\mathcal{B}$  is exceptional under the given condition. We show that  $\mathcal{B}$  contains finitely many indecomposables. If  $\mathcal{B}$  contains infinitely many indecomposables then for some  $n$ ,  $\mathcal{B}[n] \cap \mathcal{A}$  contains infinitely many indecomposables. But for each  $\mu \in \bar{\mathbb{Q}}$ ,  $\text{coh}^\mu \mathbb{X}$  contains finitely many exceptional indecomposables. Thus  $\mu(\mathcal{B}[n] \cap \mathcal{A})$  has a limit point in  $\bar{\mathbb{R}}$ . Note that an indecomposable object in  $\mathcal{A}$  is either a torsion sheaf or a vector bundle. For  $\mathbb{X}$  of domestic type, since rank on indecomposables is bounded,  $\infty$  is the unique limit point of  $\mu(\mathcal{B}[n] \cap \mathcal{A})$ . By Lemma 4.11,  $\{i \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} \subset \{j, j + 1\}$  for some  $j$ , a contradiction. For  $\mathbb{X}$  of tubular type, under the given assumption, by Lemma 4.12,  $\mu(\mathcal{B})$  contains at most limit points in  $\bar{\mathbb{Q}}$ . If  $q \in \bar{\mathbb{Q}}$  is a limit point of  $\mu(\mathcal{B})$ ,  $\infty$  is a limit point of  $\mu(\Phi_{\infty,q}(\mathcal{B}))$ , whereby yielding a contradiction to our assumption by Lemma 4.11. Thus in either case,  $\mathcal{B}$  contains only finitely many indecomposables. It remains to show that  $\mathcal{B}$  is of finite length. Let  $\{X_1, \dots, X_n\}$  be a complete set of indecomposable objects in  $\mathcal{B}$ . We have  $\text{End}(X_i) = k$ . Moreover, by Lemma 4.10, we can suppose  $\text{Hom}(X_i, X_j) = 0$  for  $i < j$ . Then one sees that if  $\bigoplus_{i=1}^n X_i^{\oplus s_i}$  is a proper sub-object of  $\bigoplus_{i=1}^n X_i^{\oplus t_i}$  then  $(s_1, \dots, s_n) < (t_1, \dots, t_n)$ , where  $<$  refers to the lexicographic order. It follows immediately that  $\mathcal{B}$  must be of finite length. This finishes the proof. □

As a corollary, we obtain a characterization of when a bounded t-structure on  $\mathcal{D}^b(\mathbb{X})$ , where  $\mathbb{X}$  is of domestic type, has a length heart.

**Corollary 4.14** *If  $\mathbb{X}$  is of domestic type then  $\mathcal{B}$  is of finite length iff  $\#\{i \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} > 1$ .*

*Proof* Proposition 4.13 tells us that if  $\{i \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq 0\} \not\subseteq \{j, j + 1\}$  for any  $j$  then  $\mathcal{B}$  is of finite length. So consider those  $\mathcal{B}$  with  $\{i \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq 0\} = \{j\}$  or  $\{j - 1, j\}$  for some  $j$ . By Corollary 4.8, if  $\{i \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq 0\} = \{j\}$  then  $\mathcal{B}$  is not of finite length. So consider the case  $\{i \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq 0\} = \{j - 1, j\}$ . We shall apply Proposition 4.4 and keep the notation there. By Theorem 3.15, we have an equivalence  $\mathcal{S}^{\perp A} \simeq \text{coh}\mathbb{X}'$ , where  $\mathbb{X}'$  is also a weighted projective line of domestic type. By Remark 4.7, the corresponding t-structure on  $\mathcal{D}^b(\mathbb{X}') \simeq \mathcal{D}^b(\mathcal{S}^{\perp A})$  has up to shift the tilted heart  $\mathcal{F}'[1] * \mathcal{T}'$  for some torsion pair  $(\mathcal{T}', \mathcal{F}')$  in  $\text{coh}\mathbb{X}'$  with  $\text{coh}_0\mathbb{X}' \subsetneq \mathcal{T}' \subsetneq \text{coh}\mathbb{X}'$ . By Lemma 3.32(1), we have a line bundle  $L \in \mathcal{T}'$  with  $\tau L \in \mathcal{F}'$ . Let  $(\mathcal{D}_1^{\leq 0}, \mathcal{D}_1^{\geq 0})$  be the bounded t-structure on  $\mathcal{D}_1 := \mathcal{D}^b(\mathbb{X}')$  with heart  $\mathcal{F}'[1] * \mathcal{T}'$ . By Lemma 2.14,  $L$  is  $\mathcal{D}_1^{\leq 0}$ -projective. By Lemma 2.15,  $(\mathcal{D}_1^{\leq 0}, \mathcal{D}_1^{\geq 0})$  is compatible with the admissible subcategory  $L^{\perp \mathcal{D}_1} = \mathcal{D}^b(L^{\perp \text{coh}\mathbb{X}'})$  of  $\mathcal{D}_1 = \mathcal{D}^b(\mathbb{X}')$ . We know from Lemma 3.18(1) that  $L^{\perp \text{coh}\mathbb{X}'} \simeq \text{mod}\Lambda$  for a representation-finite finite dimensional hereditary algebra. Then by Lemma 2.23, each bounded t-structure of  $L^{\perp \mathcal{D}_1} = \mathcal{D}^b(L^{\perp \text{coh}\mathbb{X}'})$  has a length heart. Moreover,  ${}^{\perp \mathcal{D}_1}(L^{\perp \mathcal{D}_1}) = \langle L \rangle_{\mathcal{D}_1} \simeq \mathcal{D}^b(k)$ . Thus the tilted heart  $\mathcal{F}'[1] * \mathcal{T}'$  is of finite length by Lemma 2.12. So is  $\mathcal{B}$ . In conclusion,  $\mathcal{B}$  is of finite length iff  $\sharp\{i \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq 0\} > 1$ .  $\square$

### 4.3 Some Properties of Silting Objects

Recall König-Yang correspondence (see Theorem 2.22) that equivalent classes of silting objects in  $\mathcal{D}^b(\mathbb{X})$  are in bijective correspondence with bounded t-structures on  $\mathcal{D}^b(\mathbb{X})$  with length heart. So we continue to describe some properties of silting objects in  $\mathcal{D}^b(\mathbb{X})$ , which in turn give information on bounded t-structures with length heart.

We obtain the following information on direct summands of  $T$  from our previous conclusions on full exceptional sequences. This holds particularly for a tilting object in  $\mathcal{D}^b(\mathbb{X})$ .

**Proposition 4.15** *Let  $T$  be a silting object in  $\mathcal{D}^b(\mathbb{X})$ .*

- (1) *If  $T$  contains a shift of a torsion sheaf as its direct summand then  $T$  contains a shift of an exceptional simple sheaf as its direct summand.*
- (2) *If  $\mathbb{X}$  is of domestic type then  $T$  contains a shift of some line bundle as its direct summand.*
- (3) *If  $\mathbb{X}$  is of tubular type then for a suitable  $q \in \bar{\mathbb{Q}}$ ,  $\Phi_{\infty, q}(T)$  contains a shift of some exceptional simple torsion sheaf and a shift of a line bundle as its direct summands.*

*Proof* By Proposition 2.20, if  $\mathcal{A} = \bigoplus_{i=1}^l \mathcal{A}_i[n_i]$  is a basic partial silting object in  $\mathcal{D}^b(\mathbb{X})$ , where  $\mathcal{A}_i \in \mathcal{A}$ , then the sequence  $(\mathcal{A}_1[n_1], \dots, \mathcal{A}_l[n_l])$  can be reordered to form an exceptional sequence. So can the sequence  $(\mathcal{A}_1, \dots, \mathcal{A}_l)$ . Moreover, the obtained exceptional sequence is full. Indeed, since  $\text{mod End}(\mathcal{A})$  is equivalent to the heart  $\mathcal{B}$  of the corresponding bounded t-structure, we have  $l = \text{rk } K_0(\text{mod End}(\mathcal{A})) = \text{rk } K_0(\mathcal{B}) = \text{rk } \mathcal{D}^b(\mathbb{X})$ . We know from Lemma 3.25 that the obtained exceptional sequence is full.

Thus (1) follows immediately from Lemma 3.26, (2) from Proposition 3.27 and (3) from Corollary 3.28.  $\square$

A silting object  $T$  in  $\mathcal{D}^b(\mathbb{X})$  is called *concentrated* if  $T$  contains nonzero direct summands in  $\text{vect}\mathbb{X}[m]$  for a unique  $m$ . This is a generalization of the notion of a concentrated tilting complex [37, Definition 9.3.3].

**Lemma 4.16** *A silting object  $T$  in  $\mathcal{D}^b(\mathbb{X})$  is concentrated iff the corresponding bounded t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  satisfies the property  $\{i \in \mathbb{Z} \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} \subset \{j, j + 1\}$  for some  $j \in \mathbb{Z}$ .*

*Proof* Recall that in König-Yang correspondence, the t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  corresponding to  $T$  has heart

$$\mathcal{B} = \{X \in \mathcal{D}^b(\mathbb{X}) \mid \text{Hom}^{\neq 0}(T, X) = 0\}.$$

Let  $T$  be a concentrated silting object, say  $T = T_1 \oplus T_2$  with  $T_1 \in \text{vect}\mathbb{X}[l]$  and  $T_2 \in \mathcal{D}^b(\text{coh}_0\mathbb{X})$ . By Happel-Ringel Lemma (see Proposition 2.19), the indecomposable direct summands of  $T_2$  are exceptional. Hence  $T_2$  is supported at exceptional points. For a simple sheaf  $S$  supported at an ordinary point, we have  $\text{Hom}^{\neq 0}(T_1, S[l]) = 0$  and  $\text{Hom}^k(T_2, S[l]) = 0$  for any  $k \in \mathbb{Z}$  and thus  $S[l]$  lies in  $\mathcal{B}$ . It follows from Lemma 4.4 that  $\{i \in \mathbb{Z} \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} \subset \{j, j + 1\}$  for some  $j$ .

Conversely, suppose  $\{i \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} \subset \{j, j + 1\}$  for some  $j$ . By Proposition 4.5 and Remark 4.7, there is a proper collection  $\mathcal{S}$  of simple sheaves such that the t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the admissible subcategory  $\mathcal{D}^b(\mathcal{S}^{\perp \mathcal{A}})$  of  $\mathcal{D}^b(\mathbb{X})$  and up to shift the corresponding t-structure  $(\mathcal{D}_1^{\leq 0}, \mathcal{D}_1^{\geq 0})$  on  $\mathcal{D}^b(\mathbb{X}') \simeq \mathcal{D}^b(\mathcal{S}^{\perp \mathcal{A}})$  has heart  $\mathcal{F}'[1] * \mathcal{T}'$  for some torsion pair  $(\mathcal{T}', \mathcal{F}')$  in  $\text{coh}\mathbb{X}'$  with  $\text{coh}_0\mathbb{X}' \subsetneq \mathcal{T}' \subsetneq \text{coh}\mathbb{X}'$ . Since  $\mathcal{B}$  is of finite length, so is  $\mathcal{F}'[1] * \mathcal{T}'$  by Lemma 2.12 and thus  $(\mathcal{T}', \mathcal{F}')$  is induced by a tilting bundle in  $\text{coh}\mathbb{X}'$  by Proposition 3.33. Hence indecomposable Ext-projectives in  $\mathcal{D}_1^{\leq 0}$  are bundles. If  $X[n]$  is an indecomposable direct summand of  $T$  with  $X$  a bundle then  $X[n]$  is  $\mathcal{D}^{\leq 0}$ -projective and thus  $i^*X[n]$  is nonzero  $\mathcal{D}_1^{\leq 0}$ -projective by Lemma 2.18, where  $i^*$  is the left adjoint of the composition  $\mathcal{D}^b(\mathbb{X}') \xrightarrow{\sim} \mathcal{D}^b(\mathcal{S}^{\perp \mathcal{A}}) \hookrightarrow \mathcal{D}^b(\mathbb{X})$ . This implies that  $i^*X[n]$  is a nonzero bundle. By Theorem 3.15(2),  $i^*$  is t-exact with respect to the standard t-structures. So we have  $n = 0$ . Hence  $T$  is concentrated.  $\square$

We now give some properties of the endomorphism algebra of a silting object in  $\mathcal{D}^b(\mathbb{X})$ . This generalizes parts of [37, Theorem 9.4.1, 9.5.3].

**Proposition 4.17** *Let  $T$  be a silting object in  $\mathcal{D}^b(\mathbb{X})$  and  $\Gamma = \text{End}(T)$ .*

- (1) *The quiver of  $\Gamma$  has no oriented cycle. In particular,  $\Gamma$  has finite global dimension.*
- (2) *If  $\mathbb{X}$  is of domestic or tubular type then  $\Gamma$  is either representation infinite or representation directed.*
- (3) *For  $\mathbb{X}$  of domestic type,  $\Gamma$  is representation infinite iff  $T$  is concentrated.*
- (4) *For  $\mathbb{X}$  of tubular type,  $\Gamma$  is representation infinite iff  $\Phi_{\infty, q}(T)$  is concentrated for some  $q \in \mathbb{Q}$ .*

*Proof* Let  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  be the bounded t-structure corresponding to  $T$  in König-Yang correspondence. Its heart  $\mathcal{B}$  is equivalent to  $\text{mod } \Gamma$ .

- (1) We can assume  $T$  is basic. Then by Proposition 2.20, indecomposable direct summands of  $T$  can be ordered to form an exceptional sequence. Hence the quiver of  $\Gamma = \text{End}(T)$  has no oriented cycle.
- (2) If  $\Gamma$  is not representation infinite then  $\mathcal{B} \simeq \text{mod } \Gamma$  contains finitely many indecomposables. Thus  $\mathcal{B}$  contains no non-exceptional object by Lemma 4.4 (for the tubular case,

- we may need an additional application of a telescopic functor to apply Lemma 4.4.). By Lemma 4.10, each object in  $\text{mod } \Gamma \simeq \mathcal{B}$  is directed. So  $\Gamma$  is representation directed.
- (3) Suppose  $T$  is concentrated. By Lemma 4.16, we have  $\{i \in \mathbb{Z} \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} \subset \{j, j + 1\}$  for some  $j \in \mathbb{Z}$ . By Lemma 4.11,  $\mathcal{B}$  contains infinitely many indecomposables. Since  $\text{mod } \Gamma \simeq \mathcal{B}$ ,  $\Gamma$  is representation infinite. Conversely, suppose  $\Gamma$  is representation infinite, then  $\mathcal{B}$  contains infinitely many indecomposables. By Proposition 4.13, we have  $\{i \in \mathbb{Z} \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} \subset \{j, j + 1\}$  for some  $j \in \mathbb{Z}$ . Then Lemma 4.16 implies that  $T$  is concentrated.
  - (4) The argument is similar to that for (3), except that we need to take into account the action of a suitable telescopic functor  $\Phi_{\infty,q}$ . We remark that  $\Phi_{\infty,q}(T)$  corresponds to the bounded t-structure with heart  $\Phi_{\infty,q}(\mathcal{B})$ . □

### 4.4 Description of Bounded t-Structures on $\mathcal{D}^b(\mathbb{X})$

We are in a position to formulate our description of bounded t-structures on  $\mathcal{D}^b(\mathbb{X})$  using HRS-tilt and recollement. Recall once again that for  $P \subset \mathbb{P}^1$ ,  $(\mathcal{T}_P, \mathcal{F}_P)$  denotes the torsion pair (3.7.1) in  $\text{coh}\mathbb{X}$ .

We begin with the domestic case.

**Theorem 4.18** *Let  $\mathbb{X}$  be a weighted projective line of domestic type. Suppose  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is a bounded t-structure on  $\mathcal{D}^b(\mathbb{X})$  with heart  $\mathcal{B}$ . Then exactly one of the following holds:*

- (1) *up to the action of  $\text{Pic}\mathbb{X}$ ,  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the recollement*

$$\mathcal{O}^{\perp_{\mathcal{D}}} \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{i_*} \\ \xleftarrow{\quad} \end{array} \mathcal{D} = \mathcal{D}^b(\mathbb{X}) \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \langle \mathcal{O} \rangle_{\mathcal{D}},$$

where  $i_*, j_!$  are the inclusion functors, in which case  $\mathcal{B}$  is of finite length;

- (2) *for a unique (up to equivalence) proper collection  $\mathcal{S}$  of simple sheaves and a unique  $P \subset \mathbb{P}^1$ ,  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the recollement*

$$\mathcal{D}^b(\mathcal{S}^{\perp_{\mathcal{A}}}) = \mathcal{S}^{\perp_{\mathcal{D}}} \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{i_*} \\ \xleftarrow{\quad} \end{array} \mathcal{D} = \mathcal{D}^b(\mathbb{X}) \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \langle \mathcal{S} \rangle_{\mathcal{D}},$$

where  $i_*, j_!$  are the inclusion functors, such that the corresponding t-structure on  $\mathcal{S}^{\perp_{\mathcal{D}}}$  is a shift of the HRS-tilt with respect to the torsion pair  $(\mathcal{S}^{\perp_{\mathcal{A}}} \cap \mathcal{T}_P, \mathcal{S}^{\perp_{\mathcal{A}}} \cap \mathcal{F}_P)$  in  $\mathcal{S}^{\perp_{\mathcal{A}}}$ , in which case  $\mathcal{B}$  is not of finite length and  $\mathcal{B}$  is noetherian resp. artinian iff  $P = \emptyset$  resp.  $P = \mathbb{P}^1$ .

*Proof* If  $\mathcal{B}$  is of finite length then the corresponding basic siltling object is the direct sum of a complete set of indecomposable Ext-projectives in  $\mathcal{D}^{\leq 0}$ . By Lemma 4.15,  $\mathcal{D}^{\leq 0}$  has an Ext-projective object which is a shift of a line bundle and thus up to the action of  $\text{Pic}\mathbb{X}$ ,  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the recollement given in (1). Conversely, if  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the recollement in (1), then  $\mathcal{B}$  is of finite length since bounded t-structures on  $\mathcal{O}^{\perp_{\mathcal{D}}}$  and  $\langle \mathcal{O} \rangle_{\mathcal{D}}$  have length hearts. If  $\mathcal{B}$  is not of finite length then by Proposition 4.13,  $\mathcal{B}$  satisfies the assumption of Proposition 4.5. By Corollary 4.14,  $\mathcal{B}$  is not of finite length iff  $\{i \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} = \{j\}$  for some  $j \in \mathbb{Z}$  and thus  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  fits into type (2) by Proposition 4.5. The assertion on the noetherianness or artinianness of  $\mathcal{B}$  in this case is shown in Corollary 4.8. □

For the tubular case, we need one more lemma characterizing when the heart  $\mathcal{B}$  is of finite length.

**Lemma 4.19** *Suppose  $\mathbb{X}$  is of tubular type. Then  $\mathcal{B}$  is of finite length iff there are two indecomposable sheaves  $E, F$  with  $\mu(E) \neq \mu(F)$  for which  $E[m], F[n]$  are  $\mathcal{D}^{\leq 0}$ -projectives for some  $m, n$ .*

*Proof* ( $\Rightarrow$ ) Let  $T$  be a corresponding silting object. Then by Proposition 4.15, for some  $q \in \mathbb{Q}$ ,  $\Phi_{\infty, q}(T)$  contains a shift of some simple sheaf and a shift of some line bundle as its direct summands. The assertion follows immediately.

( $\Leftarrow$ ) By Proposition 2.21, either  $(E, F)$  or  $(F, E)$  is an exceptional pair. We only consider the case that  $(F, E)$  is an exceptional pair since the other case is similar. By Corollary 2.17,  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the admissible filtration

$$\mathcal{D}^b(\{E, F\}^{\perp A}) = \{E, F\}^{\perp \mathcal{D}} \subset E^{\perp \mathcal{D}} \subset \mathcal{D}.$$

If  $\mu(E) \neq \mu(F)$  then by Lemma 3.18(2),  $\{E, F\}^{\perp A} \simeq \text{mod } \Lambda$  for some representation-finite finite dimensional hereditary algebra  $\Lambda$ . It follows from Corollary 2.13 and Lemma 2.23 that  $\mathcal{B}$  is of finite length. □

Here comes our description of bounded t-structures in the tubular case.

**Theorem 4.20** *Let  $\mathbb{X}$  be a weighted projective line of tubular type. Suppose  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is a bounded t-structure on  $\mathcal{D}^b(\mathbb{X})$  with heart  $\mathcal{B}$ . Then exactly one of the following holds:*

- (1) *for a unique  $\mu \in \mathbb{R} \setminus \mathbb{Q}$ ,  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is a shift of the HRS-tilt with respect to the torsion pair  $(\text{coh}^{>\mu} \mathbb{X}, \text{coh}^{<\mu} \mathbb{X})$  in  $\text{coh} \mathbb{X}$ , in which case  $\mathcal{B}$  is neither noetherian nor artinian;*
- (2) *for a unique  $\mu \in \mathbb{Q}$  and a unique  $P \subset \mathbb{P}^1$ ,  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is a shift of the HRS-tilt with respect to the torsion pair*

$$(\text{add}\{\text{coh}^{>\mu} \mathbb{X}, \text{coh}^{\mu} \mathbb{X} \mid \lambda \in P\}, \text{add}\{\text{coh}^{\mu} \mathbb{X}, \text{coh}^{<\mu} \mathbb{X} \mid \lambda \in \mathbb{P}^1 \setminus P\})$$

*in  $\text{coh} \mathbb{X}$ , in which case  $\mathcal{B}$  is not of finite length and  $\mathcal{B}$  is noetherian resp. artinian iff  $P = \emptyset$  resp.  $P = \mathbb{P}^1$ ;*

- (3) *for a unique  $q \in \mathbb{Q}$ , a unique (up to equivalence) nonempty proper collection  $\mathcal{S}$  of simple sheaves and a unique  $P \subset \mathbb{P}^1$ ,  $\Phi_{\infty, q}((\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}))$  is compatible with the recollement*

$$\mathcal{D}^b(\mathcal{S}^{\perp A}) = \mathcal{S}^{\perp \mathcal{D}} \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{i_*} \\ \xleftarrow{\quad} \end{array} \mathcal{D} = \mathcal{D}^b(\mathbb{X}) \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \langle \mathcal{S} \rangle_{\mathcal{D}},$$

*where  $i_*, j_!$  are the inclusion functors, such that the corresponding t-structure on  $\mathcal{D}^b(\mathcal{S}^{\perp A})$  is a shift of the HRS-tilt with respect to the torsion pair  $(\mathcal{S}^{\perp A} \cap \mathcal{T}_P, \mathcal{S}^{\perp A} \cap \mathcal{F}_P)$  in  $\mathcal{S}^{\perp A}$ , in which case  $\mathcal{B}$  is not of finite length and  $\mathcal{B}$  is noetherian resp. artinian iff  $P = \emptyset$  resp.  $P = \mathbb{P}^1$ ;*

- (4) *for some  $q \in \mathbb{Q}$  and some exceptional simple sheaf  $S$ ,  $\Phi_{\infty, q}((\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}))$  is compatible with the recollement*

$$\mathcal{D}^b(\mathcal{S}^{\perp A}) = \mathcal{S}^{\perp \mathcal{D}} \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{i_*} \\ \xleftarrow{\quad} \end{array} \mathcal{D} = \mathcal{D}^b(\mathbb{X}) \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \langle S \rangle_{\mathcal{D}},$$

*where  $i_*, j_!$  are the inclusion functors, such that the corresponding t-structure on  $\mathcal{D}^b(\mathcal{S}^{\perp A})$  has a length heart, in which case  $\mathcal{B}$  is of finite length.*



*Proof* If  $\mathcal{B}$  is of finite length then by Proposition 4.15, for some  $q \in \bar{\mathbb{Q}}$ , there is some exceptional simple sheaf  $S$  which is  $\Phi_{\infty,q}(\mathcal{D}^{\leq l})$ -projective for some  $l$ . Hence  $\Phi_{\infty,q}((\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}))$  is compatible with the recollement of the form in (4). The corresponding t-structure on  $\mathcal{D}^b(S^{\perp \mathcal{A}})$  has a length heart by Lemma 2.12. Suppose  $\mathcal{B}$  is not of finite length. By Proposition 4.13, for some  $q \in \bar{\mathbb{Q}}$  and some  $j \in \mathbb{Z}$ ,  $\{i \mid \text{vect}\mathbb{X}[i] \cap \Phi_{\infty,q}(\mathcal{B}) \neq 0\} \subset \{j, j + 1\}$ . Thus Proposition 4.5 applies. Moreover, by Lemma 4.10, either (I)  $\Phi_{\infty,q}(\mathcal{D}^{\leq 0})$  contains no nonzero Ext-projective or (II) all indecomposable  $\Phi_{\infty,q}(\mathcal{D}^{\leq 0})$ -projectives has the same slope.

First consider the case (I):  $\Phi_{\infty,q}(\mathcal{D}^{\leq 0})$  contains no nonzero Ext-projective. Then the asserted collection  $\mathcal{S}$  of simple sheaves in Proposition 4.5 is empty by Lemma 2.18. Hence up to shift we have two cases: 1)  $\Phi_{\infty,q}(\mathcal{B}) = \mathcal{F}_P[1] * \mathcal{T}_P$  for some  $P \subset \mathbb{P}^1$ , or 2)  $(\mathcal{T}, \mathcal{F})$  is a torsion pair in  $\text{coh}\mathbb{X}$  with  $\text{coh}_0\mathbb{X} \subsetneq \mathcal{T} \subsetneq \text{coh}\mathbb{X}$ . Moreover, for case 2), there exists no nonzero sheaf  $E \in \mathcal{T}$  with  $\tau E \in \mathcal{F}$  since  $\Phi_{\infty,q}(\mathcal{D}^{\leq 0})$  contains no nonzero Ext-projective. By Lemma 3.32, we have either 2.1)  $(\mathcal{T}, \mathcal{F}) = (\text{coh}^{>\mu}\mathbb{X}, \text{coh}^{\leq\mu}\mathbb{X})$  for some  $\mu \in \mathbb{R} \setminus \mathbb{Q}$ , or 2.2) for some  $\mu \in \mathbb{Q}$  and some  $P \subset \mathbb{P}^1$ ,

$$(\mathcal{T}, \mathcal{F}) = (\text{add}\{\text{coh}^{>\mu}\mathbb{X}, \text{coh}^{\mu}_{\lambda}\mathbb{X} \mid \lambda \in P\}, \text{add}\{\text{coh}^{\mu}_{\lambda}\mathbb{X}, \text{coh}^{<\mu}\mathbb{X} \mid \lambda \notin P\}).$$

If case 2.1) occurs then  $\Phi_{\infty,q}((\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}))$  is of type (1); if 1) or 2.2) occurs,  $\Phi_{\infty,q}((\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}))$  is of type (2). Observe that the class of t-structures of type (1) or (2) is closed under the action of the telescopic functor  $\Phi_{q,\infty} = \Phi_{\infty,q}^{-1}$ . Hence  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is of type (1) or (2). It is evident that types (1) and (2) are disjoint and the assertion on uniqueness is also obvious. The assertion on noetherianness or artinianness is proved in Lemma 3.37.

Now consider the case (II): all indecomposable  $\Phi_{\infty,q}(\mathcal{D}^{\leq 0})$ -projectives has the same slope, which we denote by  $\mu$ . By Lemma 2.18, the compatibility of  $\Phi_{\infty,q}((\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}))$  with the recollement in Proposition 4.5 implies that there is a torsion sheaf which is Ext-projective in some  $\Phi_{\infty,q}(\mathcal{D}^{\leq l})$ . Thus  $\mu = \infty$ . It follows that if an indecomposable sheaf  $E$  is Ext-projective in some  $\mathcal{D}^{\leq l}$  then  $\mu(E) = q$ . This enforces the uniqueness of  $q$ . The uniqueness of  $\mathcal{S}$  and  $P$  is then asserted in Proposition 4.5. To show that  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is of type (3), we will show that it is impossible that  $\{i \mid \text{vect}\mathbb{X}[i] \cap \Phi_{\infty,q}(\mathcal{B}) \neq 0\} = \{j, j + 1\}$ . It sufficies to show that the corresponding t-structure on  $\mathcal{D}^b(\mathbb{X}) \simeq \mathcal{D}^b(\mathcal{S}^{\perp \mathcal{A}})$  is not a shift of HRS-tilt with respect to any torsion pair  $(\mathcal{T}', \mathcal{F}')$  in  $\text{coh}\mathbb{X}'$  with  $\text{coh}_0\mathbb{X}' \subsetneq \mathcal{T}' \subsetneq \text{coh}\mathbb{X}'$  (see Remark 4.7). Assume for a contradiction that it was. Since  $\mathbb{X}'$  is a weighted projective line of domestic type, by Corollary 4.23,  $\mathcal{F}'[1] * \mathcal{T}'$  would be of finite length. Then so would  $\Phi_{\infty,q}(\mathcal{B})$ , a contradiction. This finishes the proof.  $\square$

In light of Lemma 2.7, we can already see certain bijective correspondence from our theorems for bounded t-structures whose heart is not of finite length. In the following corollary, we identify  $\mathbb{Z}$  as the group of autoequivalences of  $\mathcal{D}^b(\mathbb{X})$  generated by the translation functor, which acts freely on the set of bounded t-structures on  $\mathcal{D}^b(\mathbb{X})$ .

**Corollary 4.21** (1) *If  $\mathbb{X}$  is of domestic type then there is a bijection*

$$\{\text{bounded t-structures on } \mathcal{D}^b(\mathbb{X}) \text{ whose heart is not of finite length}\} / \mathbb{Z} \longleftrightarrow \bigsqcup_S \left( \{P \mid P \subset \mathbb{P}^1\} \times \{\text{bounded t-structures on } \langle S \rangle_{\mathcal{D}}\} \right), \quad (4.4.1)$$

where  $\mathcal{S}$  runs through all equivalence classes of proper collections of simple sheaves.



(2) If  $\mathbb{X}$  is of tubular type then there is a bijection

$$\{ \text{bounded t-structures on } \mathcal{D}^b(\mathbb{X}) \text{ whose heart is not of finite length} \} / \mathbb{Z} \longleftrightarrow \mathbb{R} \setminus \mathbb{Q} \bigsqcup \left( \bar{\mathbb{Q}} \times \bigsqcup_S \left( \{ P \mid P \subset \mathbb{P}^1 \} \times \{ \text{bounded t-structures on } \langle \mathcal{S} \rangle_{\mathcal{D}} \} \right) \right), \tag{4.4.2}$$

where  $S$  runs through all equivalence classes of proper collections of simple sheaves.

Suppose  $\mathbb{X}$  is of domestic or tubular type. Corollary 4.21 reduces the classification of bounded t-structures on  $\mathcal{D}^b(\mathbb{X})$  whose heart is not of finite length to the classification of bounded t-structures on  $\langle \mathcal{S} \rangle_{\mathcal{D}} = \mathcal{D}^b(\langle \mathcal{S} \rangle_{\mathcal{A}})$ . Recall that if  $S \neq \emptyset$  then there are positive integers  $m, k_1, \dots, k_m$  such that  $\langle \mathcal{S} \rangle_{\mathcal{A}} \simeq \bigsqcup_{i=1}^m \text{mod} k \vec{\mathbb{A}}_{k_i}$ . By Lemma 2.23, each bounded t-structure on  $\mathcal{D}^b(\text{mod} k \vec{\mathbb{A}}_l)$  has a length heart. So we can achieve the latter classification by calculating silting objects or simple-minded collections in  $\mathcal{D}^b(\text{mod} k \vec{\mathbb{A}}_{k_i})$  by virtue of König-Yang correspondences.

For bounded t-structures on  $\mathcal{D}^b(\mathbb{X})$  with length heart, there is no obvious bijective correspondence from the recollement in Theorem 4.18(1) or Theorem 4.20(4). Recall that  $\mathcal{D}^b(\mathbb{X})$  is triangle equivalent to  $\mathcal{D}^b(\Lambda)$  for a canonical algebra  $\Lambda$ , whose global dimension is at most 2. So the powerful König-Yang correspondences are still applicable. We can try to compute the collections of simple objects in the heart from the recollements using Proposition 2.11. Instead, we can try to compute silting objects in  $\mathcal{D}^b(\mathbb{X})$  from these recollements using [36, Corollary 3.4].

Anyway, for  $\mathbb{X}$  of tubular type, since  $S^{\perp \mathcal{A}} \simeq \text{coh} \mathbb{X}'$  for some weighted projective line of domestic type, Theorem 4.20(4) reduces the combinatorics in classification of bounded t-structures on  $\mathcal{D}^b(\mathbb{X})$  with length heart to that in the classification of bounded t-structures on  $\mathcal{D}^b(\mathbb{X}')$  with length heart; for  $\mathbb{X}$  of domestic type with weight sequence  $(p_1, p_2, p_3)$ , Theorem 4.18(1) reduces the combinatorics in the classification of bounded t-structures on  $\mathcal{D}^b(\mathbb{X})$  with length heart to that in the classification of bounded t-structures on  $\mathcal{O}^{\perp \mathcal{D}} = \mathcal{D}^b(\mathcal{O}^{\perp \mathcal{A}}) \simeq \mathcal{D}^b(k[p_1, p_2, p_3])$  (by Theorem 3.17(2)), where  $k[p_1, p_2, p_3]$  is the path algebra of the equioriented star quiver  $[p_1, p_2, p_3]$  (a Dynkin quiver here).

All in all, for  $\mathbb{X}$  of domestic or tubular type, the combinatorics in the classification of bounded t-structures on  $\mathcal{D}^b(\mathbb{X})$  can be reduced to that in the classification of bounded t-structures on  $\mathcal{D}^b(\Lambda)$  for representation-finite finite dimensional hereditary algebras  $\Lambda$ .

The following example recovers the description of bounded t-structures on  $\mathcal{D}^b(\mathbb{P}^1)$  in [19, §6.10].

*Example 4.22* Let  $\mathbb{X}$  be of trivial weight type  $(p_1, \dots, p_l)$ , that is, each  $p_i = 1$ , and thus  $\text{coh} \mathbb{X} \simeq \text{coh} \mathbb{P}^1$ . Then each indecomposable object in  $\mathcal{A} = \text{coh} \mathbb{X}$  is isomorphic to either a torsion sheaf  $S^{[m]}$  supported at some point  $\lambda \in \mathbb{P}^1$  for some  $m \in \mathbb{Z}_{\geq 1}$ , or a line bundle  $\mathcal{O}(n\vec{c})$  for some  $n \in \mathbb{Z}$ . By Theorem 4.18, a bounded t-structure whose heart is not a length category is up to shift of the form  $(\mathcal{D}_{\mathcal{A}}^{\leq -1} * \mathcal{T}_P, \mathcal{F}_P[1] * \mathcal{D}_{\mathcal{A}}^{\geq 0})$  for some  $P \subset \mathbb{P}^1$ , where

$$(\mathcal{T}_P, \mathcal{F}_P) = (\text{add}\{\text{coh}_{\lambda} \mathbb{X} \mid \lambda \in P\}, \text{add}\{\mathcal{O}(n\vec{c}), \text{coh}_{\lambda} \mathbb{X} \mid n \in \mathbb{Z}, \lambda \notin P\}).$$

To obtain bounded t-structures with length heart, we can compute silting objects directly. Each basic silting object is up to shift of the form  $\mathcal{O}(n\vec{c}) \oplus \mathcal{O}((n + 1)\vec{c})[l]$  for some

$n \in \mathbb{Z}, l \geq 0$ . Such an object is a tilting object iff  $l = 0$ . The t-structure corresponding to the silting object  $\mathcal{O}(n\bar{c}) \oplus \mathcal{O}((n + 1)\bar{c})[l]$  has heart

$$\begin{cases} \text{add}\{\mathcal{O}(n\bar{c}), \mathcal{O}((n - 1)\bar{c})[l + 1]\} \simeq \text{modk} \coprod \text{modk} & \text{if } l > 0, \\ \text{add}\{\text{coh}_0\mathbb{X} \cup \{\mathcal{O}(q\bar{c})[1], \mathcal{O}(m\bar{c}) \mid q < n, m \geq n\}\} \simeq \text{modk}(\bullet \rightrightarrows \bullet) & \text{if } l = 0. \end{cases}$$

### 4.5 Torsion Pairs in $\text{coh}\mathbb{X}$ Revisited

We can now give a more clear description of torsion pairs in  $\text{coh}\mathbb{X}$  since torsion pairs are in bijective correspondence with certain t-structures.

**Proposition 4.23** *Suppose  $\mathbb{X}$  is of domestic type. Each torsion pair  $(\mathcal{T}, \mathcal{F})$  in  $\text{coh}\mathbb{X}$  fits into exactly one of the following types:*

- (1)  $(\mathcal{T}, \mathcal{F})$  is induced by some tilting sheaf, that is, there is a tilting sheaf  $T$  such that  $\mathcal{T} = \{E \in \text{coh}(\mathbb{X}) \mid \text{Ext}^1(T, E) = 0\}$ ,  $\mathcal{F} = \{E \in \text{coh}(\mathbb{X}) \mid \text{Hom}(T, E) = 0\}$ .
- (2) either  $\mathcal{T} \subset \text{coh}_0\mathbb{X}$  or  $\mathcal{F} \subset \text{coh}_0\mathbb{X}$ , and thus  $(\mathcal{T}, \mathcal{F})$  is of the form given in Lemma 3.29.

*Proof* Note that  $\mathcal{T} \not\subset \text{coh}_0\mathbb{X}$  and  $\mathcal{F} \not\subset \text{coh}_0\mathbb{X}$  iff both  $\mathcal{T}$  and  $\mathcal{F}$  contain nonzero bundles. So in this case the tilted heart  $\mathcal{B} = \mathcal{F}[1] * \mathcal{T}$  satisfies  $\{i \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq \emptyset\} = \{0, 1\}$ . By Corollary 4.14,  $\mathcal{B}$  is of finite length. Then by Proposition 3.33,  $(\mathcal{T}, \mathcal{F})$  corresponds to a tilting sheaf  $T$ , which is exactly the one induced by  $T$ . □

**Proposition 4.24** *Suppose  $\mathbb{X}$  is of tubular type. Each torsion pair  $(\mathcal{T}, \mathcal{F})$  in  $\text{coh}\mathbb{X}$  fits into exactly one of the following types:*

- (1)  $(\mathcal{T}, \mathcal{F})$  is induced by a tilting sheaf, that is, there is a tilting sheaf  $T$  such that  $\mathcal{T} = \{E \in \text{coh}(\mathbb{X}) \mid \text{Ext}^1(T, E) = 0\}$ ,  $\mathcal{F} = \{E \in \text{coh}(\mathbb{X}) \mid \text{Hom}(T, E) = 0\}$ .
- (2) for some  $\mu \in \mathbb{R} \setminus \mathbb{Q}$ ,  $(\mathcal{T}, \mathcal{F}) = (\text{coh}^{>\mu}\mathbb{X}, \text{coh}^{<\mu}\mathbb{X})$ ;
- (3) for some  $\mu \in \mathbb{Q}$ , there exists a torsion pair  $(\mathcal{T}_\lambda, \mathcal{F}_\lambda)$  in  $\text{coh}_\lambda^\mu\mathbb{X}$  for each  $\lambda \in \mathbb{P}^1$  such that

$$\mathcal{T} = \text{add}\{\text{coh}^{>\mu}\mathbb{X}, \mathcal{T}_\lambda \mid \lambda \in \mathbb{P}^1\}, \quad \mathcal{F} = \text{add}\{\mathcal{F}_\lambda, \text{coh}^{<\mu}\mathbb{X} \mid \lambda \in \mathbb{P}^1\};$$

- (4)  $\mathcal{F} \subset \text{coh}_0\mathbb{X}$  and thus  $(\mathcal{T}, \mathcal{F})$  is of the form given in Lemma 3.29(2).

*Proof* Consider the HRS-tilt  $(\mathcal{D}_B^{\leq 0}, \mathcal{D}_B^{\geq 0})$  with heart  $\mathcal{B} = \mathcal{F}[1] * \mathcal{T}$ . Obviously types (2), (3) and (4) form disjoint classes. If  $(\mathcal{T}, \mathcal{F})$  is a torsion pair of type (2) or (3) or (4) then either there is no nonzero  $\mathcal{D}_B^{\leq 0}$ -projective or all indecomposable  $\mathcal{D}_B^{\leq 0}$ -projectives have the same slope and hence  $\mathcal{F}[1] * \mathcal{T}$  is not of finite length by Lemma 4.10. Thus types (2), (3) and (4) are disjoint from type (1) by Proposition 3.33. Conversely, suppose that  $(\mathcal{T}, \mathcal{F})$  is a torsion pair in  $\text{coh}\mathbb{X}$  such that  $\mathcal{B}$  is not of finite length. We want to show that  $(\mathcal{T}, \mathcal{F})$  is of type (2), (3) or (4).

We apply Theorem 4.20. If  $(\mathcal{D}_B^{\leq 0}, \mathcal{D}_B^{\geq 0})$  is a t-structure of type Theorem 4.20(1) resp. Theorem 4.20(2) then obviously  $(\mathcal{T}, \mathcal{F})$  is of type (2) resp. (3). Otherwise,  $(\mathcal{D}_B^{\leq 0}, \mathcal{D}_B^{\geq 0})$  is of type Theorem 4.20(3). Denote

$$\tilde{\mathcal{B}} = \Phi_{\infty, q}(\mathcal{B}) = \Phi_{\infty, q}(\mathcal{F}[1] * \mathcal{T}),$$

where  $q$  is the unique element in  $\mathbb{Q}$  asserted in Theorem 4.20(3). From the proof of Theorem 4.20, we see that  $\{i \mid \text{vect}\mathbb{X}[i] \cap \tilde{\mathcal{B}} \neq \emptyset\} = \{j\}$  for some  $j$ . If  $\mathcal{F} \subset \text{coh}_0\mathbb{X}$  then  $(\mathcal{T}, \mathcal{F})$

is of type (4). Suppose  $\mathcal{F}$  contains nonzero bundles. Then by Lemma 3.31,  $\text{coh}^\mu \mathbb{X} \subset \mathcal{F}$  for  $\mu \ll q$ . Now that  $\text{coh}^\mu \mathbb{X}[1] \subset \mathcal{F}[1] \subset \mathcal{B}$ , we have  $\text{vect} \mathbb{X}[1] \cap \tilde{\mathcal{B}} \neq 0$  by Lemma 3.13(2). Hence  $j = 1$ . Moreover, an indecomposable sheaf  $E$  such that  $\Phi_{\infty,q}(E) \in \mathcal{D}^b(\text{coh}_0 \mathbb{X})$  has slope  $\mu(E) = q$ . It follows that  $\tilde{\mathcal{B}} \subset \mathcal{A}[1] * \text{coh}_0 \mathbb{X} \subset \mathcal{A}[1] * \mathcal{A}$ . Thus  $\tilde{\mathcal{B}} = \tilde{\mathcal{F}}[1] * \tilde{\mathcal{T}}$ , where  $(\tilde{\mathcal{T}}, \tilde{\mathcal{F}})$  is the torsion pair

$$(\text{add}\{\tilde{\mathcal{T}}_\lambda \mid \lambda \in \mathbb{P}^1\}, \text{add}\{\text{vect} \mathbb{X}, \tilde{\mathcal{F}}_\lambda \mid \lambda \in \mathbb{P}^1\})$$

for some torsion pair  $(\tilde{\mathcal{T}}_\lambda, \tilde{\mathcal{F}}_\lambda)$  in  $\text{coh}_\lambda \mathbb{X}$ . Let

$$(\mathcal{T}_\lambda, \mathcal{F}_\lambda) = (\Phi_{q,\infty}(\tilde{\mathcal{T}}_\lambda), \Phi_{q,\infty}(\tilde{\mathcal{F}}_\lambda)),$$

which is a torsion pair in  $\text{coh}_\lambda^q \mathbb{X}$ . Then we have

$$(\mathcal{T}, \mathcal{F}) = (\text{add}\{\text{coh}^{>q} \mathbb{X}, \mathcal{T}_\lambda \mid \lambda \in \mathbb{P}^1\}, \text{add}\{\mathcal{F}_\lambda, \text{coh}^{<q} \mathbb{X} \mid \lambda \in \mathbb{P}^1\}),$$

which is of type (3). We are done. □

## 5 Derived Equivalence

### 5.1 Serre Functor and Derived Equivalence

The main theorem of [44] states that given a finite dimensional hereditary algebra  $\Lambda$  and a bounded t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  with heart  $\mathcal{B}$  on  $\mathcal{D}^b(\Lambda)$ , the inclusion  $\mathcal{B} \hookrightarrow \mathcal{D}^b(\Lambda)$  extends to a derived equivalence  $\mathcal{D}^b(\mathcal{B}) \simeq \mathcal{D}^b(\Lambda)$  iff the Serre functor of  $\mathcal{D}^b(\Lambda)$  is right t-exact with respect to the t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ . This motivates us to consider the following

**Assertion 5.1** *For a Hom-finite  $k$ -linear triangulated category  $\mathcal{D}$  with a Serre functor and a bounded t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  on  $\mathcal{D}$  with heart  $\mathcal{B}$ , the inclusion of  $\mathcal{B}$  into  $\mathcal{D}$  extends to an exact equivalence  $\mathcal{D}^b(\mathcal{B}) \simeq \mathcal{D}$  iff the Serre functor is right t-exact.*

The necessity of Assertion 5.1 always holds by [44, Corollary 4.13] whereas [44, Example 9.4, Example 9.5] show that the sufficiency does not hold in general. We put it in the form only to stress the role of the Serre functor. Hopefully there would exist more classes of triangulated categories such that Assertion 5.1 hold. Observe that if  $\mathcal{T}$  is a  $k$ -linear triangulated category that is triangle equivalent to  $\mathcal{D}$  then Assertion 5.1 holds for  $\mathcal{T}$  iff it holds for  $\mathcal{D}$ .

To give an application of our results on bounded t-structures on the bounded derived category  $\mathcal{D}^b(\mathbb{X})$  of coherent sheaves over a weighted projective line  $\mathbb{X}$ , we will prove the following

**Theorem 5.2** *If  $\mathbb{X}$  is of domestic or tubular type then Assertion 5.1 holds for  $\mathcal{D} = \mathcal{D}^b(\mathbb{X})$ .*

Since the result of [44] embraces the wild case, it is tempting to make the following

**Conjecture 5.3** *Given an arbitrary weighted projective line  $\mathbb{X}$ , Assertion 5.1 holds for  $\mathcal{D} = \mathcal{D}^b(\mathbb{X})$ .*

We will see in Lemma 5.13 that this does hold for a certain class of t-structures on  $\mathcal{D}^b(\mathbb{X})$ .

Recall that for  $\mathbb{X}$  of domestic type,  $\text{coh} \mathbb{X}$  is derived equivalent to  $\text{mod } \Gamma$  for a tame hereditary algebra  $\Gamma$ . Thus the conclusion for this case is already covered by [44]. The new

part of Theorem 5.2 is for the tubular case. Recall that a tubular algebra  $\Lambda$ , introduced by Ringel in [43], can be realized as the endomorphism algebra of a tilting sheaf over a weighted projective line of tubular type. In particular,  $\mathcal{D}^b(\Lambda)$  is triangle equivalent to  $\mathcal{D}^b(\mathbb{X})$  for some weighted projective line  $\mathbb{X}$  of tubular type. So Theorem 5.2 yields the following

**Corollary 5.4** *Assume that  $k$  is an algebraically closed field. Assertion 5.1 holds for  $\mathcal{D} = \mathcal{D}^b(\Lambda)$  where  $\Lambda$  is a tubular algebra over  $k$ .*

Here let us review some necessary background. Let  $\mathcal{D}$  be a triangulated category equipped with a bounded t-structure whose heart is denoted by  $\mathcal{B}$ . An exact functor  $F : \mathcal{D}^b(\mathcal{B}) \rightarrow \mathcal{D}$  is called a realization functor if  $F$  is t-exact and the restriction  $F|_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{B}$  is isomorphic to the identity functor of  $\mathcal{B}$ . This is a reasonable functor but the existence of such a functor is a problem. By virtue of the filtered derived category, [7, §3.1] constructed a realization functor for arbitrary bounded t-structure on a triangulated subcategory of  $\mathcal{D}^+(\mathcal{A})$ , where  $\mathcal{A}$  is an abelian category with enough injectives. Beilinson [6] abstracted this theme and introduced the notion of a filtered triangulated category. Given a triangulated category  $\mathcal{D}$  with a filtered triangulated category over it (see [6, Appendix] for the precise definition), [6, Appendix] constructed a realization functor for arbitrary bounded t-structure on  $\mathcal{D}$ . Recently, [16, §3] showed that an algebraic triangulated category indeed admits a filtered triangulated category over it and so generally we have

**Proposition 5.5** [16] *A realization functor exists for any bounded t-structure on an algebraic triangulated category.*

A realization functor is not necessarily an equivalence. For example, Example 4.22 tells us that there is a bounded t-structure on  $\mathcal{D}^b(\mathbb{P}^1)$  with heart equivalent to  $\text{mod}k \amalg \text{mod}k$  but definitely  $\text{mod}k \amalg \text{mod}k$  is not derived equivalent to  $\text{coh}\mathbb{P}^1$ . The following lemma helps us determine when a realization functor is an equivalence.

**Lemma 5.6** [6, Lemma 1.4] *Let  $\mathcal{D}_1, \mathcal{D}_2$  be two triangulated categories with bounded t-structures. Suppose  $\mathcal{A}_1, \mathcal{A}_2$  are the hearts respectively. Let  $F : \mathcal{D}_1 \rightarrow \mathcal{D}_2$  be an exact functor such that  $F$  is t-exact and  $F|_{\mathcal{A}_1} : \mathcal{A}_1 \rightarrow \mathcal{A}_2$  is an equivalence. The following are equivalent:*

- (1)  $F : \mathcal{D}_1 \rightarrow \mathcal{D}_2$  is an equivalence;
- (2) For each  $A, B \in \mathcal{A}_1$ , the map  $F : \text{Hom}_{\mathcal{D}_1}^n(A, B) \rightarrow \text{Hom}_{\mathcal{D}_2}^n(F(A), F(B))$  is an isomorphism.  
 If  $\mathcal{D}_1 = \mathcal{D}^b(\mathcal{A}_1)$  then there is an additional equivalent condition:
- (3) For any  $A, B \in \mathcal{A}_1$ ,  $n > 0$  and  $f \in \text{Hom}_{\mathcal{D}_2}^n(F(A), F(B))$ , there exists a monomorphism  $B \hookrightarrow B'$  in  $\mathcal{A}_1$  effacing  $f$ .

As remarked in [7, Remarque 3.1.17], we have always

$$\text{Hom}_{\mathcal{D}^b(\mathcal{A}_1)}^n(A, B) \xrightarrow{\sim} \text{Hom}_{\mathcal{D}_2}^n(F(A), F(B))$$

for  $A, B \in \mathcal{A}_1$  and  $n \leq 1$ .

Although we don't know the uniqueness of a realization functor, if some realization functor  $F_1 : \mathcal{D}^b(\mathcal{B}) \rightarrow \mathcal{D}$  is an equivalence then any realization functor  $F_2 : \mathcal{D}^b(\mathcal{B}) \rightarrow \mathcal{D}$  is an equivalence by Lemma 5.6. So it makes sense to say that the inclusion  $\mathcal{B} \hookrightarrow \mathcal{D}$  extends

to an exact equivalence  $\mathcal{D}^b(\mathcal{B}) \simeq \mathcal{D}$  if some realization functor  $F : \mathcal{D}^b(\mathcal{B}) \rightarrow \mathcal{D}$  is an equivalence.

If there exists an exact equivalence  $H : \mathcal{D}^b(\mathcal{B}) \xrightarrow{\sim} \mathcal{D}$  which is moreover t-exact then any realization functor  $F : \mathcal{D}^b(\mathcal{B}) \rightarrow \mathcal{D}$  is an equivalence; given an exact autoequivalence  $\Phi$  of  $\mathcal{D}$ , there exists a realization functor  $F : \mathcal{D}^b(\mathcal{B}) \rightarrow \mathcal{D}$  iff there exists a realization functor  $G : \mathcal{D}^b(\Phi(\mathcal{B})) \rightarrow \mathcal{D}$  and  $F$  is an equivalence iff so is  $G$ . We will use these trivial facts implicitly.

A remarkable instance of a realization functor being an equivalence is that for a tilted heart with respect to a (co-)tilting torsion theory introduced in [22].

**Proposition 5.7** *Suppose that  $\mathcal{A}$  is an abelian category and  $(\mathcal{T}, \mathcal{F})$  a torsion pair in  $\mathcal{A}$ . If  $\mathcal{T}$  is a tilting torsion class or  $\mathcal{F}$  is a co-tilting torsion-free class then the inclusion of the tilted heart  $\mathcal{F}[1] * \mathcal{T}$  into  $\mathcal{D}^b(\mathcal{A})$  extends to an exact equivalence  $\mathcal{D}^b(\mathcal{F}[1] * \mathcal{T}) \xrightarrow{\sim} \mathcal{D}^b(\mathcal{A})$ .*

- Remark 5.8* (1) Proposition 5.7 is proved originally in [22] requiring enough projectives or enough injectives in  $\mathcal{A}$  (see [22, Theorem 3.3]). The additional condition is removed in [11] using the derived category of an exact category (see [11, Proposition 5.4.3]). See also [15] for a short proof via an explicit construction of the equivalence functor.
- (2) Generalizing Proposition 5.7, [13] contains a characterization of when the inclusion of the tilted heart  $\mathcal{F}[1] * \mathcal{T}$  into  $\mathcal{D}^b(\mathcal{A})$  extends to an exact equivalence for a torsion pair  $(\mathcal{T}, \mathcal{F})$  in  $\mathcal{A}$ .

### 5.2 Reduction via Ext-Projectives

In [44], one step of the proof of the main theorem (i.e., Assertion 5.1 holds for  $\mathcal{D}^b(\Lambda)$  for a finite dimensional hereditary algebra  $\Lambda$ ) is reduction via Ext-projectives (more precisely, the simple top of an Ext-projective). The reduction relies on [44, Proposition 8.6], but we don't know whether there is a corresponding version in our setup.

Nevertheless, we have a similar but not exactly the same version, i.e., Proposition 5.9 to perform reduction. Although Proposition 5.9 works for a more general class of triangulated categories, we have additional assumption on our Ext-projectives to do reduction and so we have to make efforts to assure the existence of such an Ext-projective object.

Let  $\mathcal{D}$  be a  $k$ -linear algebraic triangulated category of finite type admitting a Serre functor  $\mathbb{S}$ . Let  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  be a bounded t-structure on  $\mathcal{D}$  with heart  $\mathcal{B}$ . These hypothesis will be retained through this subsection.

Let  $X \in \mathcal{D}^{\leq 0}$  be an exceptional object such that  $\mathbb{S}X \in \mathcal{D}^{\geq 0}$ . By Lemma 2.14,  $X$  is  $\mathcal{D}^{\leq 0}$ -projective. Denote  $\mathcal{D}_1 := X^{\perp \mathcal{D}} = {}^{\perp \mathcal{D}}\mathbb{S}X$ . By Lemma 2.15,  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the recollement

$$\begin{array}{ccc}
 \mathcal{D}_1 & \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!} \end{array} & \mathcal{D} & \begin{array}{c} \xleftarrow{j^!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*} \end{array} & \langle X \rangle_{\mathcal{D}}
 \end{array} \tag{5.2.1}$$

where  $i_*, j_!$  are the inclusion functors. We have  $j_*X = \mathbb{S}X$ ; for  $Y \in \mathcal{D}$ , we have  $j^*Y = \text{Hom}^\bullet(X, Y) \otimes X$ . There are triangles

$$\text{Hom}^\bullet(X, Y) \otimes X \xrightarrow{\text{ev}} Y \rightarrow i_*i^*Y \rightsquigarrow, \quad i_*i^!Y \rightarrow Y \xrightarrow{\text{co-ev}} D\text{Hom}^\bullet(Y, \mathbb{S}X) \otimes \mathbb{S}X \rightsquigarrow .$$

$\mathcal{D}_1$  has a Serre functor  $\mathbb{S}_1 = i^!Si_*$  by Proposition 2.4. Moreover, we have an induced t-structure

$$(\mathcal{D}_1^{\leq 0}, \mathcal{D}_1^{\geq 0}) = (\mathcal{D}_1 \cap \mathcal{D}^{\leq 0}, \mathcal{D}_1 \cap \mathcal{D}^{\geq 0})$$

on  $\mathcal{D}_1$  with heart  $\mathcal{B}_1 = \mathcal{D}_1 \cap \mathcal{B}$ . We keep these notation in the following proposition.

**Proposition 5.9** *Let  $X \in \mathcal{D}^{\leq 0}$  be an exceptional object. Suppose that  $X$  and  $\mathbb{S}X$  lie in  $\mathcal{B}$  and that either  $X$  or  $\mathbb{S}X$  is simple in  $\mathcal{B}$ . Then*

- (1)  $\mathbb{S}$  is right t-exact with respect to  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  iff so is  $\mathbb{S}_1$  with respect to the t-structure  $(\mathcal{D}_1^{\leq 0}, \mathcal{D}_1^{\geq 0})$  on  $\mathcal{D}_1$ ;
- (2) the inclusion  $\mathcal{B} \hookrightarrow \mathcal{D}$  extends to an exact equivalence  $\mathcal{D}^b(\mathcal{B}) \simeq \mathcal{D}$  iff the inclusion  $\mathcal{B}_1 \hookrightarrow \mathcal{D}_1$  extends to an exact equivalence  $\mathcal{D}^b(\mathcal{B}_1) \simeq \mathcal{D}_1$ .

*Proof* Since  $X$  is  $\mathcal{D}^{\leq 0}$ -projective, we have  $\text{Ext}_{\mathcal{B}}^1(X, Y) \cong \text{Hom}_{\mathcal{D}}^1(X, Y) = 0$  for all  $Y \in \mathcal{B}$ , and thus  $X$  is a projective object in  $\mathcal{B}$ . Similarly, since  $\mathbb{S}X$  is Ext-injective in  $\mathcal{D}^{\geq 0}$ ,  $\mathbb{S}X$  is an injective object in  $\mathcal{B}$ .

- (1) First we show that the right t-exactness of  $\mathbb{S}_1$  implies that of  $\mathbb{S}$ . Let  $Y \in \mathcal{D}^{\leq 0}$ . Then

$$i^*Y \in \mathcal{D}_1^{\leq 0}, \quad i_*i^*Y \in \mathcal{D}^{\leq 0}$$

and we have a triangle

$$i_*i^! \mathbb{S}i_*i^*Y \rightarrow \mathbb{S}i_*i^*Y \rightarrow D\text{Hom}^\bullet(\mathbb{S}i_*i^*Y, \mathbb{S}X) \otimes \mathbb{S}X \rightsquigarrow .$$

Note that for  $n < 0$ , we have  $X[n] \in \mathcal{D}^{\geq 1}$  and

$$\text{Hom}^n(\mathbb{S}i_*i^*Y, \mathbb{S}X) = \text{Hom}^n(i_*i^*Y, X) = 0.$$

Thus

$$D\text{Hom}^\bullet(\mathbb{S}i_*i^*Y, \mathbb{S}X) \otimes \mathbb{S}X = \bigoplus_{n \geq 0} D\text{Hom}^n(\mathbb{S}i_*i^*Y, \mathbb{S}X) \otimes \mathbb{S}X[n] \in \mathcal{D}^{\leq 0}.$$

If  $\mathbb{S}_1$  is right t-exact then

$$i_*i^! \mathbb{S}i_*i^*Y = i_*\mathbb{S}_1i^*Y \in \mathcal{D}^{\leq 0}.$$

Hence  $\mathbb{S}i_*i^*Y \in \mathcal{D}^{\leq 0}$ . Since  $X$  is  $\mathcal{D}^{\leq 0}$ -projective, we have

$$\begin{aligned} \mathbb{S}(\text{Hom}^\bullet(X, Y) \otimes X) &= \mathbb{S}(\bigoplus_{n \leq 0} \text{Hom}^n(X, Y) \otimes X[-n]) \\ &= \bigoplus_{n \leq 0} \text{Hom}^n(X, Y) \otimes \mathbb{S}X[-n] \\ &\in \mathcal{D}^{\leq 0}. \end{aligned}$$

Then using the triangle

$$\mathbb{S}(\text{Hom}^\bullet(X, Y) \otimes X) \rightarrow Y \rightarrow i_*i^*Y \rightsquigarrow ,$$

one knows that  $\mathbb{S}Y \in \mathcal{D}^{\leq 0}$ . This shows that  $\mathbb{S}$  is right t-exact.

Now we suppose  $\mathbb{S}$  is right t-exact and deduce the equivalence between the right t-exactness of  $\mathbb{S}_1$  and the condition that for each  $Y \in \mathcal{B}_1$ , the co-evaluation map

$$H^0(\mathbb{S}_*Y) \longrightarrow D\text{Hom}(H^0(\mathbb{S}_*Y), \mathbb{S}X) \otimes \mathbb{S}X$$

is an epimorphism in  $\mathcal{B}$ . This equivalence will yield the desired implication, as we will see. For  $Y \in \mathcal{B}_1$ ,  $\mathbb{S}_1Y = i^! \mathbb{S}_*Y$  fits into the triangle

$$i_*\mathbb{S}_1Y \rightarrow \mathbb{S}_*Y \rightarrow D\text{Hom}^\bullet(\mathbb{S}_*Y, \mathbb{S}X) \otimes \mathbb{S}X \rightsquigarrow .$$

Since  $i_*Y, X \in \mathcal{B}$ , we have

$$D\text{Hom}^\bullet(\mathbb{S}_*Y, \mathbb{S}X) \otimes \mathbb{S}X = \bigoplus_{m \geq 0} D\text{Hom}(i_*Y, X[m]) \otimes \mathbb{S}X[m] \in \mathcal{D}^{\leq 0};$$

since  $\mathbb{S}$  is right t-exact, we have  $\mathbb{S}i_*Y \in \mathcal{D}^{\leq 0}$ . Consider the commutative diagram

$$\begin{array}{ccccc}
 Z_1 & \longrightarrow & \tau_{\leq -1}\mathbb{S}i_*Y & \longrightarrow & \bigoplus_{m>0}\text{Hom}(\mathbb{S}i_*Y, \mathbb{S}X[m]) \otimes \mathbb{S}X[m] \\
 \downarrow & & \downarrow & & \downarrow \\
 i_*\mathbb{S}_1Y & \longrightarrow & \mathbb{S}i_*Y & \xrightarrow{\text{co-ev}} & D\text{Hom}^\bullet(\mathbb{S}i_*Y, \mathbb{S}X) \otimes \mathbb{S}X \\
 \downarrow & & \downarrow & & \downarrow \\
 Z_2 & \longrightarrow & H^0(\mathbb{S}i_*Y) & \xrightarrow{\text{co-ev}} & D\text{Hom}(H^0(\mathbb{S}i_*Y), \mathbb{S}X) \otimes \mathbb{S}X,
 \end{array}$$

where rows and columns are distinguished triangles. Then  $Z_1 \in \mathcal{D}^{\leq 0}$  and hence  $i_*\mathbb{S}_1Y \in \mathcal{D}^{\leq 0}$  iff  $Z_2 \in \mathcal{D}^{\leq 0}$ . By the triangle

$$Z_2 \rightarrow H^0(\mathbb{S}i_*Y) \rightarrow D\text{Hom}(H^0(\mathbb{S}i_*Y), \mathbb{S}X) \rightsquigarrow,$$

we have  $Z_2 \in \mathcal{D}^{[0,1]}$ . Taking cohomology tells us that  $Z_2 \in \mathcal{B}$  iff the morphism  $H^0(\mathbb{S}i_*Y) \rightarrow D\text{Hom}(H^0(\mathbb{S}i_*Y), \mathbb{S}X) \otimes \mathbb{S}X$  is epic in  $\mathcal{B}$ . Hence we have the claimed equivalence that  $\mathbb{S}_1$  is right t-exact iff for each  $Y \in \mathcal{B}_1$ , the co-evaluation map

$$H^0(\mathbb{S}i_*Y) \rightarrow D\text{Hom}(H^0(\mathbb{S}i_*Y), \mathbb{S}X) \otimes \mathbb{S}X$$

is epic in  $\mathcal{B}$ .

If  $\mathbb{S}X$  is simple in  $\mathcal{B}$  then clearly the co-evaluation map is an epimorphism. If  $X$  is simple in  $\mathcal{B}$  then  $X$  is a simple projective. Hence for  $Y \in \mathcal{B}_1$ ,

$$\text{Hom}(H^0(\mathbb{S}i_*Y), \mathbb{S}X) \cong \text{Hom}(\mathbb{S}i_*Y, \mathbb{S}X) \cong \text{Hom}(i_*Y, X) = 0$$

and so the co-evaluation map is also an epimorphism.

(2) If  $X$  is simple in  $\mathcal{B}$  then for  $Y \in \mathcal{B}$ , the evaluation map  $\text{Hom}(X, Y) \otimes X \rightarrow Y$  is a monomorphism in  $\mathcal{B}$ . Therefore

$$i_*i^*Y = \text{cone}(\text{Hom}^\bullet(X, Y) \otimes X \rightarrow Y) = \text{cone}(\text{Hom}(X, Y) \otimes X \rightarrow Y)$$

coincides with the cokernel of the evaluation map

$$\text{Hom}(X, Y) \otimes X \rightarrow Y$$

in  $\mathcal{B}$ , whence  $i^*Y \in \mathcal{B}_1$ . It follows that  $i^*$  is t-exact and restricts to an exact functor  $i^*|_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{B}_1$ , which is left adjoint to the inclusion  $\iota = i_{*\mathcal{B}_1} : \mathcal{B}_1 \hookrightarrow \mathcal{B}$ . This implies that the inclusion  $\iota : \mathcal{B}_1 \hookrightarrow \mathcal{B}$  extends to a fully faithful exact functor  $\mathcal{D}^b(\iota) : \mathcal{D}^b(\mathcal{B}_1) \hookrightarrow \mathcal{D}^b(\mathcal{B})$ . Similarly, if  $\mathbb{S}X$  is simple in  $\mathcal{B}$  then  $i^!$  is t-exact and restricts to an exact functor  $i^!|_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{B}_1$ . This also implies that the inclusion  $\iota = i_{*\mathcal{B}_1} : \mathcal{B}_1 \hookrightarrow \mathcal{B}$  extends to a fully faithful embedding  $\mathcal{D}^b(\iota) : \mathcal{D}^b(\mathcal{B}_1) \hookrightarrow \mathcal{D}^b(\mathcal{B})$ . In either case, we have a fully faithful functor  $\mathcal{D}^b(\iota) : \mathcal{D}^b(\mathcal{B}_1) \hookrightarrow \mathcal{D}^b(\mathcal{B})$ .

Let  $F : \mathcal{D}^b(\mathcal{B}) \rightarrow \mathcal{D}$  be a realization functor. Note that  $F$  maps the essential image of  $\mathcal{D}^b(\mathcal{B}_1)$  in  $\mathcal{D}^b(\mathcal{B})$  into  $\mathcal{D}_1$  and  $F_1 := F \circ \mathcal{D}^b(\iota) : \mathcal{D}^b(\mathcal{B}_1) \rightarrow \mathcal{D}_1$  is a realization functor. We now show our assertion.

( $\Rightarrow$ ) If  $F$  is an equivalence then for any  $Y_1, Y_2 \in \mathcal{B}_1$ , we have

$$\text{Hom}_{\mathcal{D}^b(\mathcal{B}_1)}^n(Y_1, Y_2) \xrightarrow{\sim} \text{Hom}_{\mathcal{D}^b(\mathcal{B})}^n(Y_1, Y_2) \xrightarrow{\sim} \text{Hom}_{\mathcal{D}}^n(Y_1, Y_2) = \text{Hom}_{\mathcal{D}_1}^n(Y_1, Y_2).$$

Hence  $F_1$  is an equivalence.

( $\Leftarrow$ ) Assume that  $F_1 : \mathcal{D}^b(\mathcal{B}_1) \rightarrow \mathcal{D}_1$  is an equivalence. Since both  $\mathcal{D}^b(\mathcal{B})$  and  $\mathcal{D}$  are generated by  $\{X\} \cup \mathcal{B}_1$  and also by  $\{\mathbb{S}X\} \cup \mathcal{B}_1$ , to show that  $F$  is an equivalence, it suffices to show that  $F$  induces an isomorphism

$$(*) \quad \text{Hom}_{\mathcal{D}^b(\mathcal{B})}^n(Y_1, Y_2) \xrightarrow{\sim} \text{Hom}_{\mathcal{D}}^n(Y_1, Y_2)$$

for each  $Y_1 \in \{X\} \cup \mathcal{B}_1, Y_2 \in \{\mathbb{S}X\} \cup \mathcal{B}_1$ .  $(*)$  always holds for  $n \leq 1$  and so we need to show that  $(*)$  holds for  $n \geq 2$ . Since  $F_1 : \mathcal{D}^b(\mathcal{B}_1) \rightarrow \mathcal{D}_1$  is an equivalence,  $(*)$  holds for  $Y_1, Y_2 \in \mathcal{B}_1$ . Since  $X$  is Ext-projective in  $\mathcal{D}^{\leq 0}$  and projective in  $\mathcal{B}$ ,

$$\text{Hom}_{\mathcal{D}}^n(X, Y_2) = 0 = \text{Hom}_{\mathcal{D}^b(\mathcal{B})}^n(X, Y_2)$$

for  $Y_2 \in \{\mathbb{S}X\} \cup \mathcal{B}_1$  and  $n \geq 1$ ; since  $\mathbb{S}X$  is Ext-injective in  $\mathcal{D}^{\geq 0}$  and injective in  $\mathcal{B}$ , we have

$$\text{Hom}_{\mathcal{D}}^n(Y_1, \mathbb{S}X) = 0 = \text{Hom}_{\mathcal{D}^b(\mathcal{B})}^n(Y_1, \mathbb{S}X)$$

for  $Y_1 \in \{X\} \cup \mathcal{B}_1$  and  $n \geq 1$ . This finishes the proof. □

We use the following fact to find an object satisfying the assumption of Proposition 5.9. For an exceptional object  $X \in \mathcal{D}$ , denote  $M_X = \text{co-cone}(X \xrightarrow{\eta} \mathbb{S}X)$ , where  $\eta$  is a nonzero morphism. Since  $\text{Hom}(X, \mathbb{S}X) \cong D\text{Hom}(X, X) = k$ ,  $M_X$  is up to isomorphism independent of the choice of  $\eta$ .

**Lemma 5.10** *Let  $X$  be an exceptional Ext-projective object in  $\mathcal{D}^{\leq 0}$ . With the above notation, if  $M_X \in \mathcal{D}^{\leq 0}$  then  $\mathbb{S}X$  is a simple object in  $\mathcal{B}$ ; if  $M_X \in \mathcal{D}^{\geq 1}$  then  $X$  is a simple object in  $\mathcal{B}$ . In particular, if  $M_X[l] \in \mathcal{B}$  for some  $l$  then either  $X$  or  $\mathbb{S}X$  is simple in  $\mathcal{B}$ .*

*Proof* We will use the recollement (5.2.1), with which the t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible. Denote by  $(\mathcal{D}_2^{\leq 0}, \mathcal{D}_2^{\geq 0})$  the corresponding t-structure on  $\langle X \rangle_{\mathcal{D}} \simeq \mathcal{D}^b(k)$ . Since  $j^*X = X \in \mathcal{D}_2^{\leq 0}$  and  $j^*\mathbb{S}X = X \in \mathcal{D}_2^{\geq 0}$ , we know that the heart of  $(\mathcal{D}_2^{\leq 0}, \mathcal{D}_2^{\geq 0})$  is  $\text{add } X$ . Then by Proposition 2.11,  $j_{!*}X$  is simple in  $\mathcal{B}$  and  $j_{!*}X$  fits into the two triangles

$$i_*\tau_{\leq 0}i^!j_!X \rightarrow j_!X \rightarrow j_{!*}X \rightsquigarrow, \quad j_{!*}X \rightarrow j_*X \rightarrow i_*\tau_{\geq 0}i^*j_*X \rightsquigarrow.$$

If  $M_X \in \mathcal{D}^{\geq 1}$  then

$$M_X = i^!M_X \in \mathcal{D}_1^{\geq 1}$$

thus

$$i_*\tau_{\leq 0}i^!j_!X = i_*\tau_{\leq 0}M_X = 0, \quad j_{!*}X \cong j_!X = X;$$

if  $M_X \in \mathcal{D}^{\leq 0}$  then

$$M_X = i^*M_X \in \mathcal{D}_1^{\leq 0}$$

thus

$$i_*\tau_{\geq 0}i^*j_*X = i_*\tau_{\geq 0}(M_X[1]) = 0, \quad j_{!*}X = \mathbb{S}X.$$

These show our first assertion and the second assertion follows easily. □

*Remark 5.11* If  $X, \mathbb{S}X$  lie in  $\mathcal{B}$  then by the definition of  $j_{!*}$ , we have  $j_{!*}(X) = \text{im}(\eta : X \rightarrow \mathbb{S}X)$ , which is the simple top (resp. socle) of  $X$  (resp.  $\mathbb{S}X$ ).

### 5.3 Proof of Theorem 5.2

We prove Theorem 5.2 in this subsection. At first, we consider again the category  $\mathcal{A}_t$  of finite dimensional nilpotent  $k$ -representations of the cyclic quiver  $\hat{\mathbb{A}}_{t-1}$  with  $t$  vertices. The following lemma refines Lemma 2.29 and makes feasible our induction process.



**Lemma 5.12** *For a bounded t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  on  $\mathcal{D}^b(\mathcal{A}_t)$  with heart  $\mathcal{B}$ , which is not a shift of the standard one, there exists a simple object  $X \in \mathcal{A}_t$  such that for some  $n \in \mathbb{Z}$ ,  $X[n]$  is  $\mathcal{D}^{\leq 0}$ -projective and either  $X[n]$  or  $\mathbb{S}X[n]$  is a simple object in  $\mathcal{B}$ , where  $\mathbb{S}$  is the Serre functor of  $\mathcal{D}^b(\mathcal{A}_t)$ .*

*Proof* We will use freely the notation introduced at the start of Section 2.9. Let  $\mathcal{S}$  be the proper collection of simple objects in  $\mathcal{A}_t$  asserted in Proposition 2.30. Then for some  $S \in \mathcal{S}$ ,  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the recollement

$$S^{\perp_{\mathcal{D}}} \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!} \end{array} \mathcal{D} = \mathcal{D}^b(\mathcal{A}_t) \begin{array}{c} \xleftarrow{j^!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*} \end{array} \langle S \rangle_{\mathcal{D}},$$

where  $i_*$ ,  $j_!$  are the inclusion functors. Denote

$$\mathcal{D}_1 = S^{\perp_{\mathcal{D}}}, \mathcal{D}_1^{\leq 0} = \mathcal{D}_1 \cap \mathcal{D}^{\leq 0}, \mathcal{D}_1^{\geq 0} = \mathcal{D}_1 \cap \mathcal{D}^{\geq 0}, \mathcal{B}_1 = \mathcal{D}_1 \cap \mathcal{B}.$$

Then  $(\mathcal{D}_1^{\leq 0}, \mathcal{D}_1^{\geq 0})$  is a bounded t-structure on  $\mathcal{D}_1$  with heart  $\mathcal{B}_1$ .

We will use induction on the pair  $(t, \sharp \mathcal{S})$  to prove our assertion. As the first step of induction, we consider arbitrary  $t$  and  $\sharp \mathcal{S} = 1$ . Then  $\mathcal{S} = \{S\}$  and, up to a shift of  $\mathcal{B}$ , the corresponding t-structure on  $S^{\perp_{\mathcal{D}}}$  has heart  $S^{\perp_{\mathcal{A}_t}}$ . In particular,  $\tau S^{[2]} \in \mathcal{B}$ . Since we have a triangle  $\tau S^{[2]} \rightarrow S \rightarrow \tau S^{[1]} \rightsquigarrow$ ,  $S$  is the desired object by Lemma 5.10. Now suppose  $\sharp \mathcal{S} > 1$ . In particular,  $t > 2$ . By the induction hypothesis, there exist some simple  $S' \in S^{\perp_{\mathcal{A}_t}}$  and some  $l \in \mathbb{Z}$  such that  $S'[l]$  is simple in  $\mathcal{B}_1$  and is moreover  $\mathcal{D}_1^{\leq 0}$ -projective or  $\mathcal{D}_1^{\geq 0}$ -injective. Note that a simple object in  $S^{\perp_{\mathcal{A}_t}}$  is isomorphic to  $\tau S^{[2]}$  or to some simple object in  $\mathcal{A}_t$  nonisomorphic to  $\tau S$ ,  $S$ . If  $S' \cong \tau S^{[2]}$  then we have  $\tau S^{[2][l]} \in \mathcal{B}$  and  $S$  is the desired object by Lemma 5.10. It remains to consider the case when  $S'$  is a simple object in  $\mathcal{A}_t$  nonisomorphic to  $\tau S$  or  $S$ . Up to a shift of  $\mathcal{B}$ , we can suppose  $l = 0$ . Then  $S'$  is either  $\mathcal{D}_1^{\leq 0}$ -projective or  $\mathcal{D}_1^{\geq 0}$ -injective.

If  $S'$  is  $\mathcal{D}_1^{\leq 0}$ -projective then  $\mathbb{S}_1 S' \in \mathcal{D}_1^{\geq 0} \subset \mathcal{D}^{\geq 0}$ , where  $\mathbb{S}_1 = i^! \mathbb{S} i_*$  is the Serre functor of  $\mathcal{D}_1 = S^{\perp_{\mathcal{D}}}$ . Easy computation shows that

$$\mathbb{S}_1 S' = \begin{cases} \tau S'[1] & \text{if } S' \not\cong \tau^{-1} S; \\ \tau S^{[2][1]} & \text{if } S' \cong \tau^{-1} S. \end{cases}$$

If  $S' \not\cong \tau^{-1} S$  then  $\tau S'[1] \in \mathcal{D}^{\geq 0}$  and thus  $S'$  is  $\mathcal{D}^{\leq 0}$ -projective. Moreover  $S'$  is simple in  $\mathcal{B}_1$  thus simple in  $\mathcal{B}$ , whence  $S'$  is the desired object. If  $S' \cong \tau^{-1} S$  then  $\tau S^{[2]} \in \mathcal{D}^{\geq 1}$ . Suppose  $j^* \mathcal{B} = \text{add } S[n]$ . Then  $S \in \mathcal{D}^{\leq n}$ ,  $\tau S^{[1]} \in \mathcal{D}^{\geq n}$ . If  $n \geq 1$  then using the triangle  $\tau S^{[2]} \rightarrow S \rightarrow \tau S^{[1]} \rightsquigarrow$ ,  $\tau S^{[1]} \in \mathcal{D}^{\geq n}$  and  $\tau S^{[2]} \in \mathcal{D}^{\geq 1}$  imply  $S \in \mathcal{D}^{\geq 1}$ . Then  $S' \cong \tau^{-1} S$  is  $\mathcal{D}^{\leq 0}$ -projective. Now that  $\tau^{-1} S$  is simple in  $\mathcal{B}$ ,  $\tau^{-1} S$  is the desired. If  $n \leq 0$  then  $\tau S^{[2]} \in \mathcal{D}^{\geq 1}$  and  $\tau S^{[1]} \in \mathcal{D}^{\geq n}$  imply  $S[n] \in \mathcal{D}^{\geq n}$ , whereby yielding  $S[n] \in \mathcal{B}$  since we already have  $S[n] \in \mathcal{D}^{\leq 0}$ . Now that  $S[n] \in \mathcal{B}$  and  $\tau S[n+1] \in \mathcal{D}^{\geq 0}$ ,  $S[n]$  is  $\mathcal{D}^{\leq 0}$ -projective. Moreover, we have  $\tau S^{[2][n]} \in \mathcal{D}^{\geq 1}$  and thus  $S[n]$  is simple in  $\mathcal{B}$  by Lemma 5.10. Therefore  $S$  is the desired.

Similar arguments apply to the case when  $S'$  is  $\mathcal{D}_1^{\geq 0}$ -injective. The following are some sketchy arguments. Since  $t > 2$ ,  $\tau^2 S \not\cong S$ . We have

$$\mathbb{S}_1^{-1} S' = i^* \mathbb{S}^{-1} i_* S' = \begin{cases} \tau^{-1} S'[-1] & \text{if } S' \not\cong \tau^2 S; \\ \tau S^{[2]}[-1] & \text{if } S' \cong \tau^2 S. \end{cases}$$

Suppose  $j^*\mathcal{B} = \text{add } S[n]$ . If  $S' \not\cong \tau^2 S$  then  $\tau^{-1} S'$  is the desired. If  $S' \cong \tau^2 S$  then  $\tau S$  is the desired when  $n \leq -2$  and  $S$  is the desired when  $n > -2$ . We are done.  $\square$

We show that Assertion 5.1 holds for a class of bounded t-structures on  $\mathcal{D}^b(\mathbb{X})$ , where  $\mathbb{X}$  is a weighted projective line of arbitrary type.

**Lemma 5.13** *Let  $\mathbb{X} = \mathbb{X}(\underline{p}, \underline{\lambda})$  be a weighted projective line. Let  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  be a bounded t-structure on  $\mathcal{D} = \mathcal{D}^b(\mathbb{X})$  whose heart  $\mathcal{B}$  satisfies  $\{i \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq 0\} \subset \{j, j + 1\}$ . Then Assertion 5.1 holds under these additional assumptions.*

*Proof* We have only to show the sufficiency. Let  $\mathcal{S}$  be the proper collection of simple sheaves asserted in Proposition 4.5. If  $\mathcal{S} = \emptyset$  then up to a shift of  $\mathcal{B}$  we have  $\mathcal{B} = \mathcal{F}[1] * \mathcal{T}$  for some torsion pair  $(\mathcal{T}, \mathcal{F})$  in  $\text{coh}\mathbb{X}$ . By Lemma 3.31, either  $\mathcal{T}$  is a tilting torsion class or  $\mathcal{F}$  is a cotilting torsion-free class. Then it follows from Proposition 5.7 that the inclusion  $\mathcal{B} \hookrightarrow \mathcal{D}^b(\mathbb{X})$  extends to an exact equivalence  $\mathcal{D}^b(\mathcal{B}) \xrightarrow{\sim} \mathcal{D}^b(\mathbb{X})$ . In particular, if the weight sequence  $\underline{p}$  is trivial then there is no exceptional simple sheaves and  $\mathcal{S} = \emptyset$  and so the assertion also holds in this case. Now we use induction on the weight sequence  $\underline{p}$  and consider a nontrivial weight sequence  $\underline{p} = (p_1, \dots, p_n)$ . We suppose  $\mathcal{S} \neq \emptyset$ .

Take  $\lambda \in \mathbb{P}^1$  such that  $\mathcal{S}_\lambda = \mathcal{S} \cap \text{coh}_\lambda \mathbb{X} \neq \emptyset$ . By Lemma 4.4,  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  restricts to a bounded t-structure  $(\mathcal{D}_\lambda^{\leq 0}, \mathcal{D}_\lambda^{\geq 0})$  on  $\mathcal{D}^b(\text{coh}_\lambda \mathbb{X})$ . Let  $\mathcal{B}_\lambda = \mathcal{D}^b(\text{coh}_\lambda \mathbb{X}) \cap \mathcal{B}$  be its heart. Recall that  $\text{coh}_\lambda \mathbb{X} \simeq \mathcal{A}_{p_\lambda}$ . By Lemma 5.12, for some exceptional simple sheaf  $S \in \mathcal{S}_\lambda$  and some  $n \in \mathbb{Z}$ ,  $S[n]$  is  $\mathcal{D}_\lambda^{\leq 0}$ -projective and either  $S[n]$  or  $\tau S[n + 1]$  is simple in  $\mathcal{B}_\lambda$ .  $S[n] \in \mathcal{D}^{\leq 0}, \tau S[n + 1] \in \mathcal{D}^{\geq 0}$  imply that  $S[n]$  is  $\mathcal{D}^{\leq 0}$ -projective. Then  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  is compatible with the recollement

$$\mathcal{D}^b(\mathbb{X}') \simeq \mathcal{D}^b(S^{\perp_{\mathcal{A}}}) \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!} \end{array} \mathcal{D} = \mathcal{D}^b(\mathbb{X}) \begin{array}{c} \xleftarrow{j^!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*} \end{array} \langle S \rangle_{\mathcal{D}}, \tag{5.3.1}$$

where  $i_*, j_!$  are the inclusion functors,  $\mathbb{X}' = \mathbb{X}(\underline{p}', \underline{\lambda})$  is a weighted projective line with weight sequence

$$\underline{p}' = (p_1, \dots, p_{i-1}, p_i - 1, p_{i+1}, \dots, p_n)$$

and the exact equivalence  $\mathcal{D}^b(\mathbb{X}') \simeq \mathcal{D}^b(S^{\perp_{\mathcal{A}}})$  is induced by the equivalence  $S^{\perp_{\mathcal{A}}} \simeq \text{coh}\mathbb{X}'$  (see Theorem 3.15). If the Serre functor  $\mathbb{S} = \tau(-)[1]$  is right t-exact then  $S[n], \tau S[n + 1] \in \mathcal{B}$ . One easily shows

$$j_{!*}(S[n]) = \text{im}(\eta : S[n] \rightarrow \tau S[n + 1]) = \begin{cases} S[n] & \text{if } S[n] \text{ is simple in } \mathcal{B}_\lambda \\ \tau S[n + 1] & \text{if } \tau S[n + 1] \text{ is simple in } \mathcal{B}_\lambda \end{cases},$$

where  $\eta : S[n] \rightarrow \tau S[n + 1]$  is any nonzero morphism. Hence either  $S[n]$  or  $\tau S[n + 1]$  is simple in  $\mathcal{B}$ . Then by Proposition 5.9(1), the right t-exactness of the Serre functor  $\mathbb{S}$  of  $\mathcal{D}^b(\mathbb{X})$  implies the right t-exactness of the Serre functor  $\mathbb{S}_1$  of  $\mathcal{D}^b(\mathbb{X}')$ .

Let  $\mathcal{B}_1$  be the heart of the corresponding t-structure on  $\mathcal{D}^b(\mathbb{X}')$ . Since the essential image of  $\text{vect}\mathbb{X}'[i] \cap \mathcal{B}_1$  under the sequence of functors  $\mathcal{D}^b(\mathbb{X}') \simeq \mathcal{D}^b(S^{\perp_{\mathcal{A}}}) \hookrightarrow \mathcal{D}^b(\mathbb{X})$  is contained in  $\text{vect}\mathbb{X}[i] \cap \mathcal{B}$ ,

$$\{i \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq 0\} \subset \{j, j + 1\} \text{ implies } \{i \mid \text{vect}\mathbb{X}'[i] \cap \mathcal{B}_1 \neq 0\} \subset \{j, j + 1\}.$$

By the induction hypothesis, the right t-exactness of  $\mathbb{S}_1$  implies that the inclusion of  $\mathcal{B}_1$  into  $\mathcal{D}^b(\mathbb{X}')$  extends to a derived equivalence  $\mathcal{D}^b(\mathcal{B}_1) \simeq \mathcal{D}^b(\mathbb{X}')$ . Then by Proposition 5.9(2), the inclusion  $\mathcal{B} \hookrightarrow \mathcal{D}^b(\mathbb{X})$  extends to an exact equivalence  $\mathcal{D}^b(\mathcal{B}) \simeq \mathcal{D}^b(\mathbb{X})$ .  $\square$

We eventually arrive at our proof of Assertion 5.1 for  $\mathcal{D} = \mathcal{D}^b(\mathbb{X})$ , where  $\mathbb{X}$  is of domestic or tubular type.

*Proof of Theorem 5.2* We show the sufficiency. Assume that the Serre functor  $\mathbb{S}$  is right t-exact. We have shown in Lemma 5.13 that if  $\{i \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq 0\} \subset \{j, j + 1\}$  then Assertion 5.1 holds. If  $\mathbb{X}$  is of domestic or tubular type and  $\mathcal{B}$  does not satisfy the condition even up to the action of  $\text{Aut}\mathcal{D}^b(\mathbb{X})$  then  $\mathcal{B}$  is of finite length by Proposition 4.13. The remaining argument goes as in [44, §4]. By Theorem 2.22,  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  corresponds to a silting object  $T$  in  $\mathcal{D}^b(\mathbb{X})$ . In particular, we have an equivalence  $F : \mathcal{B} \xrightarrow{\sim} \text{mod End}(T)$ . If  $\mathbb{S}$  is right t-exact then  $T$  is a tilting object by Lemma 2.24, whose endomorphism algebra has finite global dimension by Proposition 4.17. The composition

$$\mathcal{D}^b(\mathcal{B}) \xrightarrow{\mathcal{D}^b(F)} \mathcal{D}^b(\text{End}(T)) \xrightarrow{-\otimes^L T} \mathcal{D}^b(\mathbb{X})$$

is an exact equivalence which maps  $\mathcal{B}$  into  $\mathcal{B}$ . Thus the inclusion  $\mathcal{B} \hookrightarrow \mathcal{D}^b(\mathbb{X})$  extends to an exact equivalence  $\mathcal{D}^b(\mathcal{B}) \simeq \mathcal{D}^b(\mathbb{X})$ . □

*Remark 5.14* We make a final remark on a potential approach to Conjecture 5.3, based on the validity of the following

**Conjecture 5.15** *Let  $\mathbb{X}$  be a weighted projective line of arbitrary type. For any bounded t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  on  $\mathcal{D}^b(\mathbb{X})$ ,  $\mathcal{D}^{\leq 0}$  contains no nonzero Ext-projective iff it is a shift of the HRS-tilt with respect to some torsion pair  $(\mathcal{T}, \mathcal{F})$  in  $\text{coh}\mathbb{X}$  such that there is no nonzero sheaf  $E \in \mathcal{T}$  with  $\tau E \in \mathcal{F}$ .*

The sufficiency obviously holds. The necessity holds in the domestic and tubular case by our description of bounded t-structures.

The aforementioned potential approach is as follows. Let  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  be a bounded t-structure on  $\mathcal{D}^b(\mathbb{X})$  with heart  $\mathcal{B}$ . We can first try to show that Assertion 5.1 holds when  $\mathcal{D}^{\leq 0}$  contains no nonzero Ext-projective. For example, if Conjecture 5.15 holds, then Assertion 5.1 holds by Lemma 3.31 and Proposition 5.7. Then we consider the case when  $\mathcal{D}^{\leq 0}$  contains a nonzero Ext-projective. Suppose all indecomposable Ext-projectives are torsion sheaves and suppose Conjecture 5.15 is true. Then the heart  $\mathcal{B}$  satisfies  $\{i \mid \text{vect}\mathbb{X}[i] \cap \mathcal{B} \neq 0\} \subset \{j, j + 1\}$  for some  $j \in \mathbb{Z}$  and Assertion 5.1 holds by Lemma 5.13. It remains to consider the case when some indecomposable bundle  $E$  is  $\mathcal{D}^{\leq 0}$ -projective (up to a shift of  $\mathcal{B}$ ). On one hand, it’s possible that our previous approach still works, i.e., we can still apply Proposition 5.9 in some way. On the other hand, since  $E$  is exceptional, by Proposition 3.17,  $E^{\perp_{\text{coh}\mathbb{X}}} \simeq \text{mod}H$  for some hereditary algebra  $H$ . Stanley and van Roosmalen’s result [44] may apply here.

**Acknowledgements** The question of this article originated from a seminar on Bridgeland’s stability conditions organized by Prof. Xiao-Wu Chen, Prof. Mao Sheng and Prof. Bin Xu. I thank these organizers who gave me the opportunity to report. I am grateful to the participants for their patience and critical questions. Thanks are once again due to Prof. Xiao-Wu Chen, my supervisor, for his guidance and kindness.

I thank Peng-Jie Jiao for discussion, thank Prof. Helmut Lenzing for carefully answering my question on stable bundles over a tubular weighted projective line, thank Prof. Zeng-Qiang Lin for communication on realization functors, thank Prof. Hagen Meltzer for his lectures on weighted projective lines, thank Prof. Dong Yang for explaining the results in [28] and for stimulating conversations, and thank Prof. Pu Zhang for a series of lectures on triangulated categories based on his newly-written book titled “triangulated categories and derived categories” (in Chinese).

Moreover, I would like to express my deep gratitude to two anonymous referees. The referee of the first version of this article gave me a long list of suggestions, which pointed out many mistakes and inaccuracies

and helped in improving the exposition and in reshaping some parts of this article. Another referee of the second version of this article also pointed out several mistakes, many typos and grammar problems.

This work is supported by the National Science Foundation of China (No. 11522113 and No. 115771329) and also by the Fundamental Research Funds for the Central Universities.

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