Algebras and Representation Theory (2020) 23:1977–1982
https://doi.org/10.1007/s10468-019-09920-5

A Note on the Rational Non-linear Characters of Groups

M. Norooz-Abadian[1] · H. Sharifi[1]

Received: 1 February 2019 / Accepted: 1 August 2019 / Published online: 27 August 2019

Abstract
We study finite groups with rational valued non-linear characters and obtain the structure of these types of groups with derived subgroup of prime order. In particular, in the case of the size of derived subgroup of a group is two we conclude that this group is rational if and only if the group is a direct product of an extraspecial 2-group with an elementary abelian 2-group.

Keywords Rational valued non-linear characters · Frobenius group

Mathematics Subject Classification (2010) 20C15

1 Introduction

Let us call a finite group G rational group or $\mathbb{Q}$-group if every character $\chi \in \mathrm{Irr}(G)$ is rational valued. We also call it is $\mathbb{Q}_1$-group whenever every non-linear irreducible character is rational-valued.

Elementary results on $\mathbb{Q}_1$-groups are collected in the paper [3]. In particular, they characterized $\mathbb{Q}_1$-groups by their vanishing-off subgroup, as introduced in [6]. In [8] authors investigated the structure of Frobenius $\mathbb{Q}_1$-groups. Basmaji studied metabelian $\mathbb{Q}_1$-groups [1]. Especially, he achieved some facts about $\mathbb{Q}_1$-groups with cyclic derived subgroup. Motivated by [1, 7–9], we classify $\mathbb{Q}_1$-groups with derived subgroup of prime order.

Throughout the paper we consider finite groups, and we employ the following notation and terminology:

The semi-direct product of group K with group H is denoted by $K : H$. $A * B$ is the central product of groups A and B, i.e., $A * B = AB$ with $[A, B] = 1$, where $[A, B]$ is the commutator of A and B. The symbol $\mathbb{Z}_n$ denotes a cyclic group of order n. For a prime p and a non-negative integer n, the symbol $E(p^n)$ denotes the elementary abelian p-group of

Presented by: Alistair Savage

✉ H. Sharifi
hsharifi@shahed.ac.ir

1 Department of Mathematics, Faculty of Science, Shahed University, Tehran, Iran

order p^n, p a prime number; Q_8 and D_8 are employed to denote the quaternion and dihedral group of order 8, respectively.

The main results of this work are the following

Theorem 1.1 *Suppose that G is a $\mathbb{Q}_1$-group and its derived subgroup G' is of order 2. Then $G \cong (A_1 * A_2 * \ldots * A_n) \times E(2^m)$, where A_i, $1 \leq i \leq n$, is isomorphic to one of the following*

(i) $\langle a, b \mid a^4 = b^4 = 1, a^b = a^3 \rangle$;
(ii) $\langle a, b, c \mid a^4 = b^4 = c^2 = 1, [a, c] = [b, c] = 1, [a, b] = c \rangle$;
(iii) $\langle a, b, c \mid a^4 = b^2 = c^2 = 1, [a, c] = [b, c] = 1, [a, b] = c \rangle$;
(iv) Q_8;
(v) D_8.

Theorem 1.2 *Suppose that G is a $\mathbb{Q}_1$-group such that $|G'| = p$, where p is an odd prime. Then G is a $\mathbb{Q}_1$-group if and only if one the following occurs*

(i) $G \cong (\mathbb{Z}_p : \mathbb{Z}_{p-1}) \times E(2^n)$
(ii) $G \cong (\mathbb{Z}_p : \mathbb{Z}_{2(p-1)}) \times E(2^n)$

2 A Review and Preliminary Results

We review some facts about rational groups. Let G be a finite group. Let $\mathrm{nl}(G)$ denote the set of non-linear irreducible characters of G.

An element $x \in G$ is called rational if $\chi(x) \in \mathbb{Q}$ for every $\chi \in \mathrm{Irr}(G)$. Also, $\chi \in \mathrm{Irr}(G)$ is called a rational character if $\chi(x) \in \mathbb{Q}$ for every $x \in G$.

Lemma 2.1 ([5, p.11] *and* [4, p.31]) *A finite group G is a $\mathbb{Q}$-group if and only if for every $x \in G$ of order n the elements x and x^m are conjugate in G, whenever $(m, n) = 1$. Equivalently, $N_G(\langle x \rangle)/C_G(\langle x \rangle) \cong \mathrm{Aut}(\langle x \rangle)$ for each $x \in G$.*

Theorems 2.2 and 2.5 can be found in [3].

Theorem 2.2 *Let G be a non-abelian $\mathbb{Q}_1$-group. Then the following are true:*

(1) $|G|$ *is even.*
(2) *A quotient of G is a $\mathbb{Q}_1$-group.*
(3) $Z(G)$ *is an elementary abelian 2-group.*

Definition 2.3 The vanishing-off subgroup of a non-abelian finite group G is defined as follows:

$$V(G) = \langle g \in G \mid \exists \chi \in \mathrm{nl}(G) : \chi(g) \neq 0 \rangle.$$

We need to the following facts about $V(G)$.

Lemma 2.4 *([6]) If G is a non-abelian finite group, then G' and $Z(G)$ are subgroups of $V(G)$.*

Theorem 2.5 *Let G be a non-abelian finite group. Then G is a $\mathbb{Q}_1$-group if and only if every element of $V(G)$ is a rational element.*

The following result about metabelian $\mathbb{Q}_1$-groups which is proved in [1], is the base of this article.

Proposition 2.6 *Let G be a finite metabelian $\mathbb{Q}_1$-group and assume that G' is a cyclic subgroup. Then, $|G'|$ is a prime or $|G'|$ divides 12. Furthermore, If $|G'| = p$, where p is an odd prime, then $|G| = 2^t p(p-1)$; if $|G'| \in \{2, 4\}$, then $|G| = 2^t$, and if $|G'| \in \{6, 12\}$ then $|G| = 2^t \cdot 3$.*

Let us recall the following fact from [2].

Lemma 2.7 *Let G be a minimal non-abelian p-group. Then $|G'| = p$ and G is one the following groups*

(i) $G \cong \langle a, b \mid a^{p^m} = b^{p^n} = 1, a^b = a^{1+p^{m-1}} \rangle$, *where* $2 \leq m$ *and* $1 \leq n$;
(ii) $G \cong \langle a, b, c \mid a^{p^m} = b^{p^n} = c^2 = 1, [a, c] = [b, c] = 1, [a, b] = c \rangle$ *and if* $p = 2$, *then* $2 < m + n$;
(iii) $G \cong Q_8$.

Proposition 2.8 *Let G be a minimal non-abelian p-group. If G is a $\mathbb{Q}_1$-group, then one of the following holds*

(1) $G \cong \langle a, b \mid a^4 = b^4 = 1, a^b = a^3 \rangle$;
(2) $G \cong \langle a, b, c \mid a^4 = b^4 = c^2 = 1, [a, c] = [b, c] = 1, [a, b] = c \rangle$;
(3) $G \cong \langle a, b, c \mid a^4 = b^2 = c^2 = 1, [a, c] = [b, c] = 1, [a, b] = c \rangle$;
(4) $G \cong Q_8$;
(5) $G \cong D_8$.

Proof Theorem 2.2 forces $p = 2$. Suppose that G is isomorphic to case (i) in Lemma 2.7. Then $(a^2)^b = (a^b)^2 = a^{2+2^m} = a^2$. This implies $a^2 \in Z(G)$. Since $Z(G)$ is an elementary abelian subgroup, a^2 must be an involution or trivial. Therefore a is an involution or is of order 4. On the other hand, $a^{b^2} = (a^b)^b = a^9 = a$. Thus $b^2 \in Z(G)$. Therefore, a, b are also an involution or of order 4. But $2 \leq m$, thus $o(a) = 4$. If $o(b) = 4$, then we have the case (1) and if b is an involution, then we have the case (5).

Now, we consider the case (i) in Lemma 2.7. We have $(a^2)^b = (a^b)^2 = (ac)^2 = a^2c^2 = a^2$. Similarly in the previous case, $o(a) = 2$ or 4. On the other hand, $a^{b^2} = (a^b)^b = (ac)^b = a^b c^b = ac^2 = a$. Therefore, a, b are involutions or $o(b) = 4$. With condition $2 < m + n$, either $m = n = 2$ which is case (2) or $m = 2$ and $n = 1$ which is case (3). □

3 Proof of Theorems

Let's start with a reminder of a lemma.

Lemma 3.1 [6, Lemma 2.1] *Let g be an element of a group G. Then the following are equivalent*

(1) $\chi(g) = 0$ *for all* $\chi \in \mathrm{nl}(G)$.
(2) $|C_G(g)| = |G : G'|$.

We say $g \in G$ is an anti-central element of G if g satisfies one of the equivalent conditions of Lemma 3.1.

Proposition 3.2 *Let $G = A * B$, where A and B are subgroups of G such that $A' \subseteq B$ and $|A'| = |B'| = 2$. If A and B are $\mathbb{Q}_1$-groups, then G is also a $\mathbb{Q}_1$-group with $|G'| = 2$.*

Proof Simply from $A' \subseteq B$, we conclude that $|G'| = 2$. Now, we prove that G is a $\mathbb{Q}_1$-group.

First, we show that if $g = ab \in V(G)$, where $a \in A$ and $b \in B$, then $a \in V(A)$ and $b \in V(B)$. We consider two cases:

case(i) g is not an anti-central element. So, $\chi_0(g) \neq 0$ for some $\chi_0 \in \mathrm{Irr}(G)$. By Lemma 3.1, $|C_G(g)| \neq |G : G'| = |G|/2$. Since $|G|/2 = |G : G'| \leq |C_G(g)|$, $g \in Z(G)$. Now, let $c \in A$. Thus $cab = cg = gc = abc = acb$. Therefore, $a \in Z(A)$. By [6, Lemma 2.4], there exist $\chi_1 \in \mathrm{Irr}(A)$ such that $\chi_1(a) \neq 0$. Similarly, there exist a $\chi_2 \in \mathrm{Irr}(B)$ such that $\chi_2(b) \neq 0$. Consequently, a, b are generators of $V(A)$ and $V(B)$, respectively.

case(ii) g is an anti-central element. Thus $g = g_1 g_2 \ldots g_n$, where g_i for $1 \leq i \leq n$, are generators of $V(G)$. Suppose that for $1 \leq i \leq n$, $g_i = a_i b_i$. Then $g = a_1 \ldots a_n b_1 \ldots b_n$. Therefore $a = a_1 \ldots a_n$ and $b = b_1 \ldots b_n$. By case(i), $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are generators of $V(A)$ and $V(B)$, respectively. Thus $a \in V(A)$ and $b \in V(B)$.

Now, let $g = ab \in G$ and $g \in V(G)$. Then, by Theorem 2.5 and above cases a and b are rational in G. Suppose $o(g) = n$ and m be a positive integers such that $(m, n) = 1$. Thus m is an odd integer. Therefore m is prime to $o(a)$ and $o(b)$. Therefore, by Lemma 2.1, there exists $h, k \in G$ such that $a^h = a^m$ and $b^k = b^m$. Hence, by the properties of central product, we have $g^{hk} = (ab)^{hk} = a^h b^k = a^m b^m = g^m$. This implies that g is a rational element in G and hence G is a $\mathbb{Q}_1$-group. □

We have the following lemma about p-groups with derived subgroup of order p.

Proposition 3.3 [2, Lemma 4.2] *Let G be a p-group with $|G'| = p$. Then $G = (A_1 * A_2 * \ldots * A_n)Z(G)$, where $A_1, A_2, \ldots, A_n$ are minimal non-abelian.*

We are now ready to prove the first main theorem.

Proof of Theorem 1.1. Let G be a $\mathbb{Q}_1$-group with $|G'| = 2$. Then, it follows from Proposition 3.3 that $G = (A_1 * A_2 * \ldots * A_n)Z(G)$, where each $A_i, 1 \leq i \leq n$, is isomorphic to one of the groups which are introduced in Proposition 2.8. Thus, if we set $H = (A_1 * A_2 * \ldots * A_n) \cap Z(G)$, then $G = (A_1 * A_2 * \ldots * A_n) \times K$, where $K \cong \frac{Z(G)}{H}$. Since $Z(G)$ is elementary abelian, K is also elementary abelian.

The converse follows obviously from Proposition 3.2. Therefore, we get the proof of Theorem 1.1. □

The following is an immediate consequence of Theorem 1.1.

Corollary 3.4 *Suppose G is a group with $|G'| = 2$. Then G is a $\mathbb{Q}$-group if and only if G is a direct product of an extraspecial 2-group with an elementary abelian 2-group.*

Now we are going to prove Theorem 1.2. The proof of this theorem was pointed to us by I.M. Isaacs and is independent of [1].

The following is essential.

Lemma 3.5 *Let $K \lhd G$ and G/K is abelian. Suppose $\theta \in \mathrm{Irr}(K)$ such that every character in $\mathrm{Irr}(G|\theta)$ is rational. If θ is extendible to its stabilizer T in G, then T/K is an elementary abelian 2-group.*

Proof Let $\chi \in \mathrm{Irr}(G)$ lie over θ. By [4, Theorem 6.11], $\chi = \psi^G$ for some character $\psi \in \mathrm{Irr}(T)$ lying over θ. Also by a famous theorem of Gallagher, as ν runs over $\mathrm{Irr}(T/K)$, all of the characters $\nu\psi$ are different and induce different irreducible characters of G, for more details we refer the reader to [4, Corollary 6.17].

We argue now that T/K is an elementary abelian 2-group. To see this, let $\nu \in \mathrm{Irr}(T/K)$ and we take σ to be a field automorphism in Galois group of $\mathrm{Gal}(\mathbb{Q}(T)/\mathbb{Q})$, where $\mathbb{Q}(T)$ is the field which generated by the values of irreducible characters of T. It suffices to show that $\nu^\sigma = \nu$. Let μ be an extension of ν. Then

$$(\nu\psi)^G = \mu\chi = (\mu\chi)^\sigma = \mu^\sigma\chi = (\nu^\sigma\psi)^G.$$

We give the proof only for the second equality which is less trivial as follow. Since μ is an extension of ν, the restriction of $\mu\chi$ to K is $\nu_K\chi_K$, and since K is included in the kernel of ν, so $(\mu\chi)_K$ is equal to χ_K and therefore θ is a constituent of $(\mu\chi)_K$. Therefore, according to the assumption of lemma, $\mu\chi$ is rational and consequently, $(\mu\chi)^\sigma$ is equal to $\mu\chi$.

Finally, it follows that $\nu^\sigma = \nu$. The proof is complete. □

Proof of Theorem 1.2. Suppose that G is a $\mathbb{Q}_1$-group and $|G'| = p > 2$. Let α be a faithful character of G', and T be the stabilizer of α in G. It is easy to see that $T = \mathrm{C}_G(G')$. Also, since the irreducible characters of G that lie over α are rational, it follows that all of the $p-1$ faithful linear characters of G' are conjugate in G, and thus $|G : T| = p - 1$ and G/T is cyclic.

On the other hand, since every rational element of odd order belongs to G', so by [9, Proposition 2.3], G' is a Hall subgroup of G. So that, applying Problem 6.18 of [4], α extends to T. Therefore, by Lemma 3.5, T/G' is an elementary abelian 2-group. Thus, if E is a Sylow 2-subgroup of T, then E is elementary abelian and $T = G'E$. Since $T = \mathrm{C}_G(G')$, we see that the E will centralize G', and this implies that E is normal in T and hence characteristic in T, and thus, E is normal in G.

Now, T/E has a complement C/E in G/E, and C/E is cyclic of order $p-1$. Then, $G = G'E$ and G' meets C trivially. Also, let c generates C modulo E. If c has order $p-1$, then $C = E \times \langle c\rangle$. Otherwise, $\langle c\rangle$ is cyclic of order $2(p-1)$ and the result follows. □

Corollary 3.6 *Suppose G is a finite group with $|G'| = p$, where p is an odd prime. Then G is a $\mathbb{Q}$-group if and only if G is a direct product of $\mathbb{S}_3$ with an elementary abelian 2-group, where $\mathbb{S}_3$ is the symmetric group on three letters.*

Acknowledgments The authors express their deep gratitude to the referee for her/his extensive comments leading to a much clearer presentation.

References

1. Basmaji, B.G.: Rational non linear characters of metabelian groups. Proc. Amer. Math. Soc **85**(2), 175–180 (1982)

2. Ya, G., Janko, Z.: Berkovich Groups of Prime Power Order, vol. 1. Walter de Gruyter, Berlin (2008)
3. Darafsheh, M.R., Iranmanesh, A., Moosavi, S.A.: Groups whose non-linear irreducible characters are rational valued. Arch. Math. (Basel) **94**(5), 411–418 (2010)
4. Isaacs, I.M.: Character Theory of Finite Groups. Academic press, Cambridge (1976)
5. Kletzing, D.: Structure and Representations of $\mathbb{Q}$-group, Lecture Notes in Math, vol. 1084. Springer, Berlin (1984)
6. Lewis, M.L.: The vanishing-off subgroup. J. Algebra **321**, 1313–1325 (2009)
7. Norooz-Abadian, M., Sharifi, H.: A note on the rationality of Sylow 2-subgroups of rational groups . Math. Notes **101**, 1066–1067 (2017)
8. Norooz-Abadian, M., Sharifi, H.: Frobenius $\mathbb{Q}_1$-groups. Arch. Math. (Basel) **105**, 509–517 (2015)
9. Norooz-Abadian, M., Sharifi, H.: Sylow 2-subgroup of solvable $\mathbb{Q}_1$-groups. C. R. Acad. Sci. Paris, Ser. I **355**, 20–23 (2016)