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Generalized Oscillator Representations of the Twisted Heisenberg-Virasoro Algebra

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Abstract
In this paper, we first obtain a general result on sufficient conditions for tensor product modules to be simple over an arbitrary Lie algebra. We classify simple smooth modules over the infinite-dimensional Heisenberg algebra $\mathfrak{H}$, and then obtain a lot of simple modules over the twisted Heisenberg-Virasoro algebra $\widetilde{\mathcal{V}}$ from generalized oscillator representations of $\widetilde{\mathcal{V}}$ by extending these $\mathfrak{H}$-modules. Using generalized oscillator representations we give the necessary and sufficient conditions for Whittaker modules over $\widetilde{\mathcal{V}}$ (in the more general setting) to be simple. We use the "shifting technique" to determine the necessary and sufficient conditions for the tensor products of highest weight modules and modules of intermediate series over $\widetilde{\mathcal{V}}$ to be simple. At last we establish the "embedding trick" to obtain a lot more simple $\widetilde{\mathcal{V}}$-modules.

Keywords Heisenberg algebra · Virasoro algebra · Twisted Heisenberg-Virasoro algebra · Whittaker module · Weight module · Simple module

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1 Introduction

We denote by $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$, and $\mathbb{C}$ the sets of all integers, nonnegative integers, positive integers, and complex numbers, respectively. For a Lie algebra L we denote by $U(L)$ the universal enveloping algebra of L.

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The twisted Heisenberg-Virasoro algebra $\widetilde{\mathcal{V}}$ is the universal central extension of the Lie algebra $\{f(t)\frac{d}{dt}+g(t)|f,g\in\mathbb{C}[t,t^{-1}]\}$ of differential operators of order at most one on the Laurent polynomial algebra $\mathbb{C}[t,t^{-1}]$, see [1]. More precisely, the twisted Heisenbeg-Virasoro algebra $\widetilde{\mathcal{V}}$ is a Lie algebra over $\mathbb{C}$ with the basis

$$\{t^{n+1}\frac{d}{dt}, t^n, z_1, z_2, z_3|n\in\mathbb{Z}\}$$

and subject to the Lie brackets given by

$$[t^{n+1}\frac{d}{dt}, t^{m+1}\frac{d}{dt}] = (m-n)t^{m+n+1}\frac{d}{dt} + \delta_{n,-m}\frac{n^3-n}{12}z_1, \tag{1.1}$$

$$[t^{n+1}\frac{d}{dt}, t^m] = mt^{m+n} + \delta_{n,-m}(n^2+n)z_2, \tag{1.2}$$

$$[t^n, t^m] = n\delta_{n,-m}z_3, \tag{1.3}$$

$$[\widetilde{\mathcal{V}}, z_1] = [\widetilde{\mathcal{V}}, z_2] = [\widetilde{\mathcal{V}}, z_3] = 0. \tag{1.4}$$

The center of $\widetilde{\mathcal{V}}$ is spanned by z_1, z_2, z_3, and t^0. We define $d_n = t^{n+1}\frac{d}{dt}$, $I_i = t^i$, and both symbols will be used according to contexts. The Lie algebra $\widetilde{\mathcal{V}}$ has a natural $\mathbb{Z}$-gradation with respect to $\mathrm{ad}(d_0)$:

$$\widetilde{\mathcal{V}}_n = \mathbb{C}d_n + \mathbb{C}I_n, \forall n \in \mathbb{Z}\setminus\{0\}, \tag{1.5}$$

$$\widetilde{\mathcal{V}}_0 = \mathbb{C}d_0 + \mathbb{C}I_0 + \mathbb{C}z_1 + \mathbb{C}z_2 + \mathbb{C}z_3. \tag{1.6}$$

The Lie algebra $\widetilde{\mathcal{V}}$ has a Witt subalgebra $\mathcal{W} = \mathbb{C}[t]\frac{d}{dt}$, a Virasoro subalgebra $\mathcal{V}$ with basis $\{d_i, z_1|i\in\mathbb{Z}\}$, and a Heisenberg subalgebra $\mathfrak{H}$ with basis $\{I_i, z_3|i\in\mathbb{Z}\}$.

The twisted Heisenberg-Virasoro algebra $\widetilde{\mathcal{V}}$ has been studied by Arbarello, De Concini, Kac, and Procesi in [1], where a connection is established between the second cohomology of certain moduli spaces of curves and the second cohomology of the Lie algebra of differential operators of order at most one. They also proved that when the central element of the Heisenberg subalgebra acts in a non-zero way, a simple highest weight module for $\widetilde{\mathcal{V}}$ is isomorphic to the tensor product of a simple module for the Virasoro algebra and a simple module for the infinite-dimensional Heisenberg algebra. For a more general result, see Theorem 12.

To introduce our results in this paper, we will first recall and define some concepts.

For $\lambda\in\mathbb{C}$, $s, r\in\mathbb{Z}_+$, we define the following subalgebras and quotient algebras of $\widetilde{\mathcal{V}}$:

$$\mathcal{W} = \mathrm{Der}(\mathbb{C}[t]) = \mathrm{span}\{d_i|i\geq -1\}, \quad \widetilde{\mathcal{W}} = \mathbb{C}[t]\frac{d}{dt} + \mathbb{C}[t], \tag{1.7}$$

$$\mathfrak{a} = \mathrm{span}\{d_i|i\geq 0\}, \quad \widetilde{\mathfrak{a}} = \mathrm{span}\{d_i, I_i|i\geq 0\}, \tag{1.8}$$

$$\mathcal{V}^{(r)} = \mathrm{span}\{d_i|i\geq r\}, \quad \widetilde{\mathcal{V}}^{(r,s)} = \mathrm{span}\{t^{r+i}, d_{s+i}|i\geq 0\}, \tag{1.9}$$

$$\mathfrak{a}_r = \mathfrak{a}/\mathcal{V}^{(r+1)}, \quad \widetilde{\mathfrak{a}}_{r,s} = \widetilde{\mathfrak{a}}/\widetilde{\mathcal{V}}^{(r,s)}, \tag{1.10}$$

$$\mathcal{V}[\lambda] = \mathbb{C}[t,t^{-1}](t-\lambda)\frac{d}{dt} + \mathbb{C}z_1, \tag{1.11}$$

$$\widetilde{\mathcal{V}}[\lambda] = \mathrm{span}\{t^i, t^i(t-\lambda)\frac{d}{dt}, z_1, z_2, z_3|i\in\mathbb{Z}\}. \tag{1.12}$$

For any $\mathbb{Z}$-graded Lie algebra $L = \oplus L_i$, denote by $\mathcal{O}_L$ the category of all L-modules V satisfying

Condition A For any $v\in V$, there exists a positive integer n (depending on v) such that $L_i v = 0$ for all $i\geq n$.

Modules in $\mathcal{O}_L$ for affine Kac-Moody algebras L are called *smooth modules* by D. Kazhdan and G. Lusztig in [14, 15].

Let V be a module over a Lie algebra L. We say that V is *trivial* if $LV = 0$. Denote by $\mathrm{Soc}_L(V)$ the *socle* of the L-module V, i.e., $\mathrm{Soc}_L(V)$ is the sum of the minimal nonzero submodules of V.

Recall that a module V over a Lie algebra L is called *locally finite* provided that any $v \in V$ belongs to a finite dimensional L-submodule. The module is called *locally nilpotent* provided that for any $v \in V$ there exists an $n \in \mathbb{N}$ such that $a_1 a_2 \cdots a_n(v) = 0$ for all $a_1, a_2, \ldots, a_n \in L$.

When V is a simple module over L, where L is one of $\widetilde{\mathcal{V}}, \mathcal{V}, \widetilde{\mathcal{W}}, \mathcal{W}$, the following lemma gives some equivalent conditions for Condition A.

Lemma 1 *Let L be one of $\widetilde{\mathcal{V}}, \mathcal{V}, \widetilde{\mathcal{W}}, \mathcal{W}$, and V be a simple L-module. Denote $L^{(k)} = \sum_{i \geq k} L_i$. Then the following conditions are equivalent:*

(a) $V \in \mathcal{O}_L$.
(b) *There exists some* $0 \neq v \in V$ *and* $s \in \mathbb{N}$ *such that* $L^{(s)} v = 0$.
(c) *There exists* $k \in \mathbb{N}$ *such that* V *is a locally finite* $L^{(k)}$*-module.*
(d) *There exists* $m \in \mathbb{N}$ *such that* V *is a locally nilpotent* $L^{(m)}$*-module.*

Proof $(a) \Rightarrow (b)$ is trivial. Using PBW Theorem, we can easily deduce $(b) \Rightarrow (a), (c), (d)$.

$(c) \Rightarrow (b)$. Let $0 \neq v_0 \in V$. Then $W = U(L^{(k)})v_0$ is a finite dimensional $L^{(k)}$ module. Hence $L^{(k)} / \mathrm{ann}_{L^{(k)}}(W)$ is finite dimensional. By similar arguments as in Section 3.3 in [21], we obtain $d_{k_1} \in \mathrm{ann}_{L^{(k)}}(W)$ for some $k_1 > k$. So $L^{(s)} \subset \mathrm{ann}_{L^{(k)}}(W)$ for some $s > k + k_1$. Thus $L^{(s)} v_0 = 0$.

$(d) \Rightarrow (b)$. Let $0 \neq v \in V$. There exists some $n \in \mathbb{N}$ with $a_1 a_2 \cdots a_n v = 0$ for all $a_1, a_2, \ldots, a_n \in L^{(m)}$. It is straightforward to verify that there exists some $s \in \mathbb{N}$ such that

$$L^{(s)} \subset \mathrm{span}\{a_1 a_2 \cdots a_n | a_i \in L^{(m)}\} \subset U(L^{(m)}),$$

which completes the proof. □

The simple modules in $\mathcal{O}_{\mathcal{W}}$ are studied in [20].

Lemma 2 [20, Lemma 2]

(a). Suppose that $A \in \mathcal{O}_{\mathfrak{a}}$ is simple and nontrivial. Then there exists some $r \in \mathbb{Z}_+$ such that $d_i A = 0$ for all $i > r$ and d_r acts bijectively on A. Consequently, A is a simple $\mathfrak{a}_r$-module for some $r \in \mathbb{N}$.
(b). Let $A \in \mathcal{O}_{\mathfrak{a}}$, $W, W_1 \in \mathcal{O}_{\mathcal{W}}$ be all nontrivial simple modules.

(1) *The $\mathcal{W}$-module $\mathrm{Ind}_{\mathfrak{a}}^{\mathcal{W}}(A)$ is simple in $\mathcal{O}_{\mathcal{W}}$;*
(2) *The $\mathfrak{a}$-module $\mathrm{Soc}_{\mathfrak{a}}(W) \in \mathcal{O}_{\mathfrak{a}}$ is simple, and an essential $\mathfrak{a}$-submodule of W, i.e. the intersection of all nonzero $\mathfrak{a}$-submodules of V;*
(3) *We have $W \cong \mathrm{Ind}_{\mathfrak{a}}^{\mathcal{W}} \mathrm{Soc}_{\mathfrak{a}}(W)$ and $A = \mathrm{Soc}_{\mathfrak{a}}(\mathrm{Ind}_{\mathfrak{a}}^{\mathcal{W}} A)$;*
(4) *We have $W \cong W_1$ if and only if $\mathrm{Soc}_{\mathfrak{a}}(W) \cong \mathrm{Soc}_{\mathfrak{a}}(W_1)$.*

Consequently, W is the induced module from a simple $\mathfrak{a}_r$-module for some $r \in \mathbb{N}$.

Let us recall some results for Whittaker modules over $\mathcal{V}$ studied in [22] and [17].

For any nonnegative integer $m \in \mathbb{Z}_+$, let $\psi_m : \mathcal{V}^{(m)} + \mathbb{C}z_1 \to \mathbb{C}$ be a Lie algebra homomorphism. Then we have the one dimensional module $\mathbb{C}_{\psi_m} = \mathbb{C}w_{\psi_m}$ over $\mathcal{V}^{(m)} + \mathbb{C}z_1$ with $x \cdot w_{\psi_m} = \psi_m(x)w_{\psi_m}, \forall x \in \mathcal{V}^{(m)} + \mathbb{C}z_1$. The induced $\mathcal{V}$-module

$$W_{\psi_m} = \mathrm{Ind}_{\mathcal{V}^{(m)}+\mathbb{C}z_1}^{\mathcal{V}} \mathbb{C}_{\psi_m} \tag{1.13}$$

is called the universal Whittaker module with respect to ψ_m.

Lemma 3 [17, Theorem 7] *For any $m \geq 1$, W_{ψ_m} is simple if and only if $(\psi(d_{2m}), \psi(d_{2m-1})) \neq (0, 0)$.*

Let us recall a result on Whittaker modules over $\mathfrak{H}$ from [9]. Suppose that $\theta : \mathbb{C}[t] + \mathbb{C}z_3 \to \mathbb{C}$ is a linear map. Then $\mathbb{C}w_\theta$ becomes a one dimensional $\mathbb{C}[t]+\mathbb{C}z_3$ module defined by $xw_\theta = \theta(x)w_\theta$ for all $x \in \mathbb{C}[t] + \mathbb{C}z_3$. The induced $\mathfrak{H}$-module $W_\theta = \mathrm{Ind}_{\mathbb{C}[t]+\mathbb{C}z_3}^{\mathfrak{H}} \mathbb{C}w_\theta$ is called a Whittaker module with respect to θ.

Lemma 4 [9, Proposition 6] *The $\mathfrak{H}$-module W_θ is simple if and only if $\theta(z_3) \neq 0$.*

The simple modules in $\mathcal{O}_{\mathcal{V}}$ are studied in [21]. From Lemma 1 and Theorem 2 in [21], we have

Lemma 5 *Let $V \in \mathcal{O}_{\mathcal{V}}$ be simple.*

(1) *The $\mathfrak{a}$-module $\mathrm{Soc}_{\mathfrak{a}}(V)$ is simple. Furthermore it is an essential $\mathfrak{a}$-submodule of V, i.e. the intersection of all nonzero $\mathfrak{a}$-submodule of V;*
(2) *If V is not a highest weight module, then $V \cong \mathrm{Ind}_{\mathfrak{a}+\mathbb{C}z_1}^{\mathcal{V}} \mathrm{Soc}_{\mathfrak{a}}(V)$, where the action of z_1 is a scalar;*
(3) *Suppose $V, V_1 \in \mathcal{O}_{\mathcal{V}}$. Then $V \cong V_1$ if and only if $\mathrm{Soc}_{\mathfrak{a}}(V) \cong \mathrm{Soc}_{\mathfrak{a}}(V_1)$ and z_1 acts on V, V_1 as the same scalar.*

This paper is organized as follows. In Section 2, we obtain a general result on sufficient conditions for tensor product modules to be simple over an arbitrary Lie algebra (Theorem 7). This result can be applied to many known cases and several cases in the present paper. In Section 3, we first classify simple modules in $\mathcal{O}_{\mathfrak{H}}$ (Proposition 9), and then construct generalized oscillator representations of $\widetilde{\mathcal{V}}$ by extending the the $\mathfrak{H}$-module structure on simple modules in $\mathcal{O}_{\mathfrak{H}}$ (Theorem 10). Many simple modules in $\mathcal{O}_{\widetilde{\mathcal{V}}}$ with nonzero action of z_3 are proved to be decomposed into a tensor product of an oscillator representation of $\widetilde{\mathcal{V}}$ and a simple $\mathcal{V}$-module (Theorem 12). Then we apply this theory and generalized oscillator representations to completely determine conditions for Whittaker modules W_{φ_m} over $\widetilde{\mathcal{V}}$ (in the more general setting as in [2]) to be simple, see Theorem 15. Quite surprisingly, the conditions are a couple of degree two equations involving only the values of $d_{2m}, d_{2m-1}, I_m, I_{m-1}, z_2$ and z_3 under the Whittaker function φ_m. In Section 4, we use the "shifting technique" to determine the necessary and sufficient conditions for the tensor products of highest weight modules and modules of intermediate series over $\widetilde{\mathcal{V}}$ to be simple (Theorem 26). The case Theorem 26 (1) is an anomaly, unlike the other cases or the Virasoro algebra case [8], it has nothing to do with singular vectors. In Section 5, we first classify simple modules in $\mathcal{O}_{\widetilde{\mathcal{W}}}$ (Lemma 29), and then establish the "embedding trick" to make these simple $W \in \mathcal{O}_{\widetilde{\mathcal{W}}}$ into simple $\widetilde{\mathcal{V}}$-modules $W[\lambda]$ for any $\lambda \in \mathbb{C}^*$ (Proposition 32). By taking tensor product, we obtain more simple $\widetilde{\mathcal{V}}$-modules (Theorem 34). Their isomorphism classes are determined in Theorem 35.

2 Simplicity of Tensor Product Modules

In this section, we will prove some general results for tensor product modules, which will be frequently used later.

Let V be a module over a Lie algebra L. For any $v \in V$, the *annihilator* of v is defined as $\mathrm{ann}_L(v) = \{g \in L | gv = 0\}$. For any $S \subset V$, define

$$\mathrm{ann}_L(S) = \cap_{v \in S} \mathrm{ann}_L(v).$$

Lemma 6 *Let L be a Lie algebra over $\mathbb{C}$ with a countable basis, and V be a simple L-module. For any $n \in \mathbb{Z}_+$ and any linearly independent subset $\{v_1, v_2, \dots v_n\} \subset V$, and any subset $\{v_1', \dots, v_n'\} \subset V$, there exists some $u \in U(L)$, such that*

$$uv_i = v_i', \forall i = 1, 2, \dots, n.$$

Proof Denote $R = U(L)/\mathrm{ann}_{U(L)}(V)$. Then R is an associative algebra with countable basis. It is well known that any endomorphism of a simple module over a countably generated associative $\mathbb{C}$-algebra is a scalar (Proposition 2.6.5 in [6]). Thus $\mathrm{Hom}_R(V, V) \cong \mathbb{C}$. Note that V is a faithful and simple R-module. From the Jacobson Density Theorem (Page 197, [12]), we know that R is isomorphic to a dense ring of endomorphisms of the $\mathbb{C}$-vector space V. The Lemma follows. □

Now we can give some useful sufficient conditions for a tensor product module to be simple.

Theorem 7 *Let L be a Lie algebra over $\mathbb{C}$ with a countable basis, and V_1, V_2 be L-modules. Suppose that one of the following conditions holds:*

(1) *The module V_1 is simple and $\mathrm{ann}_L(v) + \mathrm{ann}_L(S) = L$ for all $v \in V_1$ and all finite subsets $S \subset V_2$;*
(2) *For any finite subset $S \subset V_2$, V_1 is a simple $\mathrm{ann}_L(S)$-module.*

Then

1. *Any submodule of $V_1 \otimes V_2$ is of the form $V_1 \otimes V_2'$, for a submodule V_2' of V_2;*
2. *If V_1, V_2 are simple, then $V_1 \otimes V_2$ is simple.*

Proof (1) $\Rightarrow$ (2). Suppose that Condition 1 holds. For any nonzero $v \in V_1$ and a finite subset S in V_2, we have

$$V_1 = U(L)v = U(\mathrm{ann}_L(v) + \mathrm{ann}_L(S))v$$

$$= U(\mathrm{ann}_L(S))U(\mathrm{ann}_L(v))v = (U(\mathrm{ann}_L(S))v,$$

So V_1 is simple as $\mathrm{ann}_L(S)$-module. Thus Condition 2 holds.

Now we suppose that Condition 2 holds.

(a) Let M be a nonzero submodule of $V_1 \otimes V_2$. For any nonzero $X \in M$, write $X = \sum_{i=1}^{s} v_{1,i} \otimes v_{2,i} \in M$ with minimal s. Then $\{v_{1,1}, \dots, v_{1,s}\}$ and $\{v_{2,1}, \dots, v_{2,s}\}$ are linearly independent sets.

Denote $L_2 = \mathrm{ann}_L(\{v_{2,j} | j = 1, 2 \ldots, s\})$. Then V_1 is a simple L_2-module. From Lemma 6, there exists some $u_0 \in U(L_2)$ such that $u_0 v_{1,1} = v_{1,1}$, and $u_0 v_{1,i} = 0$ for $i = 2, \ldots, s$. So

$$u_0 X = u_0(\sum_{i=1}^{s} v_{1,i} \otimes v_{2,i}) = \sum_{i=1}^{s}(u_0 v_{1,i} \otimes v_{2,i}) = v_{1,1} \otimes v_{2,1} \in M.$$

For any $u \in U(L_2)$, we have $(uv_{1,1}) \otimes v_{2,1} = u(v_{1,1} \otimes v_{2,i}) \in M$. Thus $V_1 \otimes v_{2,1} = (U(L_2)v_{1,1}) \otimes v_{2,1} \subset M$. Similarly we have

$$V_1 \otimes v_{2,i} \subset M, \forall i = 1, 2, \ldots, s. \tag{2.1}$$

We have proved that

$$M = V_1 \otimes V_2', \tag{2.2}$$

where $V_2' = \{v_2 \in V_2 | V_1 \otimes v_2 \subseteq M\}$. For any $v_1 \in V_1$, $v_2 \in V_2'$, and for all $g \in L$, we have $v_1 \otimes gv_2 = g(v_1 \otimes v_2) - gv_1 \otimes v_2 \in M$. So V_2' is an L-submodule of V_2.

Part (b) follows from (a). □

Example 1 Theorem 3.3 in [11] can follow from our Theorem 7. For other applications, see Theorems 11, 13, 33.

Lemma 8 *Let W be a module over a Lie algebra L, $\mathfrak{g}$ be a subalgebra of L, and B be a $\mathfrak{g}$-module. Then the L-module homomorphism $\tau : \mathrm{Ind}_{\mathfrak{g}}^{L}(W \otimes B) \to W \otimes \mathrm{Ind}_{\mathfrak{g}}^{L}(B)$ induced from the inclusion map $W \otimes B \to W \otimes \mathrm{Ind}_{\mathfrak{g}}^{L}(B)$ is an L-module isomorphism.*

Proof Take a subspace $V \subset L$ such that $L = \mathfrak{g} \oplus V$. Let $\{l_i | i \in I\}$, $\{b_j | j \in J_1\}$, $\{w_j | j \in J_2\}$ be bases of V, B and W, respectively. Denote $S = \{l_{i_1}^{k_1} l_{i_2}^{k_2} \cdots l_{i_m}^{k_m} | i_1 \prec i_2 \prec \cdots \prec k_m\}$, where $\prec$ is a total order on I, and $U_n = \mathrm{span}\{l_{i_1}^{k_1} l_{i_2}^{k_2} \cdots l_{i_m}^{k_m} \in S \mid k_1 + \ldots + k_m = n\}$. Then

$$T_1 = \{x(w_s \otimes b_t) \mid x \in S, s \in J_1, t \in J_2\},$$

$$T_2 = \{w_s \otimes (xb_t) \mid x \in S, s \in J_1, t \in J_2\}$$

are bases of $\mathrm{Ind}_{\mathfrak{g}}^{L}(W \otimes B)$ and $W \otimes \mathrm{Ind}_{\mathfrak{g}}^{L}(B)$, respectively.

For any $x = l_{i_1}^{k_1} l_{i_2}^{k_2} \cdots l_{i_m}^{k_m} \in S$, by induction on $n = \sum_{i=1}^{m} k_i$ we can prove that

$$\tau(x(w_s \otimes b_t)) \in w_s \otimes (xb_t) + W \otimes \sum_{i<n}(U_i B), \tag{2.3}$$

from which one can deduce that τ is bijective. □

3 Generalized Oscillator Representations of $\widetilde{\mathcal{V}}$

In this section we will first classify simple modules in $\mathcal{O}_{\mathfrak{H}}$, and then construct generalized oscillator representations of $\widetilde{\mathcal{V}}$ by extending the the $\mathfrak{H}$-module structure on simple modules in $\mathcal{O}_{\mathfrak{H}}$. Since the representation space is in general not the Fock space, we call the resulting representations as generalized oscillator representations of $\widetilde{\mathcal{V}}$. When the representation space is indeed the Fock space, we actually obtain the usual oscillator representations of $\widetilde{\mathcal{V}}$. This generalizes the construction in Section 2.3 of [13].

3.1 Simple Modules in $\mathcal{O}_{\mathfrak{H}}$

For any $m \in \mathbb{N}$, we define

$$\mathfrak{T}_m = \text{span}\{I_i, z_3 | i = -m, -m+1, \dots, m-1, m\},$$

$$\mathfrak{H}_m = \text{span}\{I_i, z_3 | i \geq -m, i \in \mathbb{Z}\}.$$

Proposition 9 *Let B, B' be simple modules over $\mathfrak{T}_m$ for some $m \in \mathbb{N}$ with nonzero action of z_3.*

(a). The $\mathfrak{H}$-module $\text{Ind}_{\mathfrak{H}_m}^{\mathfrak{H}} B$ is simple, where B is regarded as $\mathfrak{H}_m$-module by $I_i B = 0$, ll all $i > m$. Moreover, all nontrivial simple modules in $\mathcal{O}_{\mathfrak{H}}$ can be obtained in this way.
(b). As $\mathfrak{H}$-modules, $\text{Ind}_{\mathfrak{H}_m}^{\mathfrak{H}} B \cong \text{Ind}_{\mathfrak{H}_m}^{\mathfrak{H}} B'$ if and only if $B \cong B'$ as $\mathfrak{T}_m$-modules.

Proof Let $K_m = \text{span}\{I_{-m-j} | j \in \mathbb{N}\}$.

(a). Using PBW Theorem and nonzero action of z_3, we can easily prove that the $\mathfrak{H}$-module $\text{Ind}_{\mathfrak{H}_m}^{\mathfrak{H}} B = U(K_m)B$ is simple.

Now suppose that $V \in \mathcal{O}_{\mathfrak{H}}$ is simple and nontrivial. Take a nonzero $v \in V$ such that $m \in \mathbb{N}$ is minimal with $I_{m+j}v = 0$ for all $j \in \mathbb{Z}_+$.

Claim 1 $U(\mathfrak{H}_m)v$ is a simple $\mathfrak{H}_m$-module.

Since $V = U(K_m)U(\mathfrak{H}_m)v = U(K_m)U(\mathfrak{T}_m)v$ is nontrivial, we see that $U(\mathfrak{T}_m)v \not\subseteq \mathbb{C}v$, hence the action of z_3 is nonzero. We can deduce that V is a free $U(K_m)$-module on $U(\mathfrak{H}_m)v$. Since V is a simple $\mathfrak{H}$-module, we deduce that $B = U(\mathfrak{H}_m)v$ is a simple $\mathfrak{H}_m$-module.

Thus $V = \text{Ind}_{\mathfrak{H}_m}^{\mathfrak{H}} B$.

(b). Noting that B and B' are the socles of $\mathfrak{H}_m$-modules $\text{Ind}_{\mathfrak{H}_m}^{\mathfrak{H}} B$ and $\text{Ind}_{\mathfrak{H}_m}^{\mathfrak{H}} B'$ respectively, we see that, if $\text{Ind}_{\mathfrak{H}_m}^{\mathfrak{H}} B \cong \text{Ind}_{\mathfrak{H}_m}^{\mathfrak{H}} B'$ as $\mathfrak{H}_m$-modules, hence $B \cong B'$ as $\mathfrak{T}_m$-modules. The converse is trivial.

□

From the above theorem, it follows that classifying all simple modules in $\mathcal{O}_{\mathfrak{H}}$ is equivalent to classifying all simple modules over the finite-dimensional Heisenberg algebras $\mathfrak{T}_m$ for all $m \in \mathbb{N}$, which in tern is equivalent to classifying all simple modules over the rank m Weyl algebras A_m. Such a classification is only known for $m = 1$, see [5].

Example 2 Take $m = 1$, make $\mathbb{C}v$ into a module over $\mathfrak{b} = \mathbb{C}I_0 + \mathbb{C}I_{-1} + \mathbb{C}z_3$ by $I_0 v = \dot{I}_0 v$, $I_{-1}v = 0$, $z_3 v = \dot{z}_3 v$ for $\dot{I}_0, \dot{z}_3 \in \mathbb{C}$ with $\dot{z}_3 \neq 0$. We have the simple weight $\mathfrak{T}_1$-module $B = \text{Ind}_{\mathfrak{b}}^{\mathfrak{T}_1} \mathbb{C}v$. Then we obtain the simple weight $\mathfrak{H}$-module $\text{Ind}_{\mathfrak{H}_1}^{\mathfrak{H}} B$. In the same way as in Section 3.1 in [3], one can construct a lot of simple weight modules in $\mathcal{O}_{\mathfrak{H}}$.

Example 3 Let $B = \mathbb{C}[t, t^{-1}]$ be a simple $\mathfrak{T}_1$-module defined by $I_{-1}t^n = t^{n+1}$, $I_1 t^n = t^{n-1}(\alpha + n)$, $I_0 t^n = at^n$, $z_3 t^n = t^n$ for $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ and $a \in \mathbb{C}$. Then we obtain the simple $\mathfrak{H}$-module $\text{Ind}_{\mathfrak{H}_1}^{\mathfrak{H}} B$. In this manner using simple modules over the Weyl algebra as in [5], one can construct a lot of simple modules in $\mathcal{O}_{\mathfrak{H}}$.

3.2 Generalized Oscillator Representations of $\widetilde{\mathcal{V}}$

In this subsection, we prove that for any $\dot{z_2} \in \mathbb{C}$, a simple representation $H \in \mathcal{O}_{\mathfrak{H}}$ with nonzero action of z_3 can always be extended to a representation of $\widetilde{\mathcal{V}}$ with z_2 acting as scalar $\dot{z_2}$. Here we will use oscillator-like representations on H.

Let σ be an endomorphism of some Lie algebra L, and V be any module of L. We can make V into another L-module, by defining the new action of L on V as

$$x \circ v = \sigma(x)v, \forall x \in L, v \in V. \tag{3.1}$$

We will call the new module *the twisted module of* V *by* σ, and denote it by V^σ. Two L-modules V and W are said to be equivalent if there exists some automorphism σ of L such that $V \simeq W^\sigma$.

For any

$$\alpha = \sum_{i\in\mathbb{Z}} a_i t^i \in \mathbb{C}[t, t^{-1}], b \in \mathbb{C}, \tag{3.2}$$

we have the $\sigma = \sigma_{\alpha,b} \in \mathrm{Aut}(\widetilde{\mathcal{V}})$ defined as

$$\sigma(d_n) = d_n + t^n(\alpha + nb) - (n+1)a_{-n}z_2 - \left(\frac{\sum_i a_i a_{-n-i}}{2} + a_{-n}nb\right)z_3 + \delta_{n,0}b\left(z_2 + \frac{b}{2}z_3\right) \tag{3.3}$$

$$\sigma(t^n) = t^n + \delta_{n,0}bz_3 - a_{-n}z_3, \sigma(z_1) = z_1 - 24bz_2 - 12b^2z_3, \tag{3.4}$$

$$\sigma(z_2) = z_2 + bz_3, \sigma(z_3) = z_3. \tag{3.5}$$

For more details, see Page 712 in [17].

Theorem 10 *Let $\dot{z_2}, \dot{z_3} \in \mathbb{C}$ with $\dot{z_3} \neq 0$. Let $H \in \mathcal{O}_{\mathfrak{H}}$ be a simple module with z_3 acting as the scalar $\dot{z_3}$. Then the $\mathfrak{H}$-module structure on H can be extended to an $\widetilde{\mathcal{V}}$-module by*

$$z_1 = 1 - \frac{12\dot{z_2}^2}{\dot{z_3}}, z_2 = \dot{z_2}, z_3 = \dot{z_3}, \tag{3.6}$$

$$d_k = -\frac{1}{2\dot{z_3}} \sum_{i\in\mathbb{Z}} : I_{-i} I_{i+k} : + \frac{(k+1)\dot{z_2}}{\dot{z_3}} I_k, \forall k \in \mathbb{Z}. \tag{3.7}$$

where, for all $i, j \in \mathbb{Z}$, the normal order is defined as

$$: I_i I_j := \begin{cases} I_i I_j, & \text{if } i < j, \\ I_j I_i, & \text{otherwise.} \end{cases}$$

The resulting $\widetilde{\mathcal{V}}$-module structure on H will be denoted by $H(\dot{z_2})$.

Proof Let $b = \frac{\dot{z_2}}{\dot{z_3}}$. Note that the righthand side in Eq. 3.7 is well defined as an operator on H, i.e., $-\frac{1}{2\dot{z_3}} \sum_{i\in\mathbb{Z}} : I_{-i} I_{i+k} : v + b(k+1)I_k v$ makes sense for any $v \in H$ since there are only finitely many nonzero terms. It is straightforward to check that $H(\dot{z_2}) = H(0)^{\sigma_{b,b}}$.

So we only need to verify that $H(0)$ is a $\widetilde{\mathcal{V}}$ module. The proof is similar to the arguments in Section 2.3 of [13]. The verifications are tedious but straightforward. We omit the details. □

We see that (3.7) is the oscillator-like actions on H. So we call the $\widetilde{\mathcal{V}}$-module $H(\dot{z}_2)$ a *generalized oscillator representation* of $\widetilde{\mathcal{V}}$.

Note that for any $v \in H \in \mathcal{O}_{\mathfrak{H}}$ with $z_3 \neq 0$ and $\dot{z_2} \in \mathbb{C}$, if $I_i v = 0, \forall i \geq n$ for some $n \in \mathbb{N}$, then in $H(\dot{z_2})$, we have

$$d_i v = 0, \forall i \geq 2n. \tag{3.8}$$

We see that as an $\mathcal{V}$-module, $H(\dot{z_2}) \in \mathcal{O}_{\mathcal{V}}$ which does not give new simple $\mathcal{V}$-modules.

If the $\mathfrak{H}$-module H is a weight module, then $H(\lambda)$ is a weight module. When H is a highest weight module over $\mathfrak{H}$, the $\mathcal{V}$-module $H(\dot{z_2})$ was constructed in Section 3.4 of [13]. For any $\lambda \in \mathbb{C}$, the $\widetilde{\mathcal{V}}$-module $H(\lambda)$ is simple if and only if H is simple as $\mathfrak{H}$-module.

Note that the action of z_1 on $H(\dot{z}_2)$ is determined by z_2 and z_3. In the next subsection we will give a method so that the action z_1 can be arbitrary.

Example 4 Let H be the simple $\mathfrak{H}$-module $\mathrm{Ind}_{\mathfrak{H}_1}^{\mathfrak{H}} B$ constructed in Examples 2 and 3. Using Theorem 10 we can obtain simple $\widetilde{\mathcal{V}}$-modules $H(\dot{z}_2)$. From the simple weight modules H defined in Example 2, we obtain simple weight modules over $\widetilde{\mathcal{V}}$ whose nonzero weight spaces are infinite-dimensional.

Example 5 Let $\dot{z}_2, \dot{z}_3, \dot{I}_0, a \in \mathbb{C}$. Define $\theta(z_3) = \dot{z}_3 \neq 0, \theta(I_0) = \dot{I}_0, \theta(I_1) = a, \theta(I_i) = 0$ for $i > 1$. Then we have the simple $\mathfrak{H}$-module $H = W_\theta$ in Lemma 4. From Theorem 10 we have the simple $\widetilde{\mathcal{V}}$-module $H(\dot{z}_2)$ which is a Whittaker module defined in [16]. Unlike Corollary 4.5 in [16], the $\widetilde{\mathcal{V}}$-module $H(\dot{z}_2)$ is not free over $U(\widetilde{\mathcal{V}}^-)$.

3.3 Simple Modules from Tensor Product

Let V be a $\mathcal{V}$-module. Then we can regard V as $\widetilde{\mathcal{V}}$-module by defining $(\mathbb{C}z_2 + \mathfrak{H})V = 0$. The resulting $\widetilde{\mathcal{V}}$-module will be denoted by $V^{\widetilde{\mathcal{V}}}$.

Theorem 11 *Let $\lambda, \mu \in \mathbb{C}$, let V, W be $\mathcal{V}$-modules, and let $H, K \in \mathcal{O}_{\mathfrak{H}}$ be simple with nonzero action of z_3.*

(1) *Any $\widetilde{\mathcal{V}}$-submodule of $V^{\widetilde{\mathcal{V}}} \otimes H(\lambda)$ is of the form $(V')^{\widetilde{\mathcal{V}}} \otimes H(\lambda)$ for a $\mathcal{V}$-submodule V' of V. Thus any $\widetilde{\mathcal{V}}$-quotient of $V^{\widetilde{\mathcal{V}}} \otimes H(\lambda)$ is of the form $(V^{\widetilde{\mathcal{V}}} \otimes H(\lambda))/((V')^{\widetilde{\mathcal{V}}} \otimes H(\lambda)) \cong (V/V')^{\widetilde{\mathcal{V}}} \otimes H(\lambda)$. In particular, $V^{\widetilde{\mathcal{V}}} \otimes H(\lambda)$ is simple as $\widetilde{\mathcal{V}}$-module if and only if V is a simple $\mathcal{V}$-module.*

(2) *$V^{\widetilde{\mathcal{V}}} \otimes H(\lambda) \cong W^{\widetilde{\mathcal{V}}} \otimes K(\mu)$ if and only if $\lambda = \mu$, $V \cong W$, and $H \cong K$.*

Proof (1) Note that $\mathfrak{H} \subset \mathrm{ann}_{\widetilde{\mathcal{V}}}(V)$, and $H(\lambda)$ is a simple $\mathfrak{H}$-module. Thus the statement follows from Theorem 7.

(2) The sufficiency is trivial. Now suppose that $\psi : V^{\widetilde{\mathcal{V}}} \otimes H(\lambda) \to W^{\widetilde{\mathcal{V}}} \otimes K(\mu)$ is a $\widetilde{\mathcal{V}}$-module isomorphism. By comparing the action of z_2, we have $\lambda = \mu$.

Let us fix a nonzero $h_0 \in H$ and arbitrary $v \in V$, Write $\psi(v \otimes h_0) = \sum_{i=1}^{s} w_i \otimes k_i$ with minimal s. By Lemma 6, there exists some $u_0 \in U(\mathfrak{H})$ such that $u_0 k_i = \delta_{1,i} k_i, i = 1, 2, \ldots, s$. Therefore

$$\begin{aligned}\varphi(v \otimes uu_0h_0) = \ & \varphi(uu_0(v \otimes h_0)) = uu_0(\sum_{i=1}^{s} w_i \otimes k_i) \\ & = \sum_{i=1}^{s} w_i \otimes uu_0k_i = w_1 \otimes uk_1, \forall u \in U(\mathfrak{H}).\end{aligned}$$

Let $h_1 = u_0h_0$, so we have proved that

$$\psi(v \otimes uh_1) = w_1 \otimes uk_1, \forall u \in U(\mathfrak{H}). \tag{3.9}$$

From the simplicity of H and K, we have $U(\mathfrak{H})h_1 = H$ and $U(\mathfrak{H})k_1 = K$. From Eq. 3.9, we have $\tau_v : H \to K$ defined by $\tau_v(uh_1) = uk_1, \forall u \in U(\mathfrak{H})$ is a well-defined $U(\mathfrak{H})$ module isomorphism.

Without lose of generality, we may assume that $H = K$. Then ψ becomes an $\widetilde{\mathcal{V}}$-module isomorphism from $V^{\widetilde{\mathcal{V}}} \otimes H(\lambda)$ to $W^{\widetilde{\mathcal{V}}} \otimes H(\lambda)$. Recall that any endomorphism of a simple module over a countably generated associative $\mathbb{C}$-algebra is a scalar. So τ_v is a scalar and from Eq. 3.9 we may define the map $\nu : V \to W$ such that $\psi(v \otimes h) = \nu(v) \otimes h, \forall v \in V, h \in H$. Now from $d_n\varphi(v \otimes h) = \varphi(d_n(v \otimes h))$, we have $d_n\nu(v) \otimes h = \nu(d_n v) \otimes h, \forall v \in V, h \in H$. Hence $d_n\nu(v) = \nu(d_n(v))$. So $\nu : V \to W$ is a $\mathcal{V}$-module isomorphism. □

Theorem 12 *Let $V \in \mathcal{O}_{\widetilde{\mathcal{V}}}$ be simple with nonzero action of z_3. If V contains a simple $\mathfrak{H}$-submodule H, then $V \cong H(\dot{z_2}) \otimes U^{\widetilde{\mathcal{V}}}$ as $\widetilde{\mathcal{V}}$-modules for some $\dot{z_2} \in \mathbb{C}$ and some simple module $U \in \mathcal{O}_{\mathcal{V}}$.*

Proof Suppose that I_0, z_1, z_2, z_3 act on V as scalars $\dot{I_0}, \dot{z_1}, \dot{z_2}, \dot{z_3}$, where we have assumed that $\dot{z_3} \neq 0$. Let $0 \neq v \in H$ and $n_1, n_2 \in \mathbb{N}$ with $d_i v = 0$, for all $i \geq n_1$ and $I_j v = 0$, for all $j \geq n_2$. Denote $n = \max\{n_1, 2n_2\}$. It is not hard to show that H is a simple module over $L = \mathrm{span}\{d_i, I_j, z_1, z_2, z_3 | i \geq n, j \in \mathbb{Z}\}$. Define the one dimensional L-module $\mathbb{C}v_0$ by d_i, I_j, z_2, z_3 acting as zero for all $i \geq n, j \in \mathbb{Z}$, and z_1 acting as $\dot{z_1} + \frac{12\dot{z_2}^2}{\dot{z_3}} - 1$. It is clear that $H \cong H(\dot{z_2}) \otimes \mathbb{C}v_0$ as L-module. Now from Lemma 8, we have

$$\mathrm{Ind}_L^{\widetilde{\mathcal{V}}}(H) \cong \mathrm{Ind}_L(H(\dot{z_2}) \otimes \mathbb{C}v_0) \cong H(\dot{z_2}) \otimes \mathrm{Ind}_L^{\widetilde{\mathcal{V}}} \mathbb{C}v_0.$$

Note that $\mathrm{Ind}_L^{\widetilde{\mathcal{V}}} \mathbb{C}v_0 \in \mathcal{O}_{\mathcal{V}}$, and that V is a simple quotient module over $\mathrm{Ind}_L^{\widetilde{\mathcal{V}}}(H)$. Now the theorem follows from Theorem 11. □

Open Problem It will be interesting to classify all simple modules in $\mathcal{O}_{\widetilde{\mathcal{V}}}$.

3.4 Whittaker Modules Over $\widetilde{\mathcal{V}}$

Now we will focus on the so called Whittaker modules over $\widetilde{\mathcal{V}}$.

For any $m \in \mathbb{Z}_+$, recall $\widetilde{\mathcal{V}}^{(0,m)} = \mathrm{span}\{d_{m+i}, t^i | i \in \mathbb{Z}_+\}$. Let $\varphi_m : \widetilde{\mathcal{V}}^{(0,m)} + \sum_{i=1}^3 \mathbb{C}z_i \to \mathbb{C}$ be a Lie algebra homomorphism. Then we have the one dimensional module $\mathbb{C}_{\varphi_m} = \mathbb{C}w_{\varphi_m}$ over $\widetilde{\mathcal{V}}^{(0,m)} + \sum_{i=1}^3 \mathbb{C}z_i$ with $x \cdot w_{\varphi_m} = \varphi_m(x)w_{\varphi_m}, \forall x \in \widetilde{\mathcal{V}}^{(0,m)} + \sum_{i=1}^3 \mathbb{C}z_i$. The induced $\widetilde{\mathcal{V}}$-module

$$\widetilde{W}_{\varphi_m} = \mathrm{Ind}^{\widetilde{\mathcal{V}}}_{\widetilde{\mathcal{V}}^{(0,m)} + \sum_{i=1}^3 \mathbb{C}z_i} \mathbb{C}_{\varphi_m} \tag{3.10}$$

will be called the universal Whittaker module with respect to φ_m. And any nonzero quotient of $\widetilde{W}_{\varphi_m}$ will be called a Whittaker module with respect to φ_m.

Note that $\varphi_m([\widetilde{\mathcal{V}}^{(0,m)}, \widetilde{\mathcal{V}}^{(0,m)}]) = 0$. Then we have

$$\varphi_m(d_{2m+j}) = \varphi_m(I_{m+j}) = 0, \forall j \in \mathbb{N}.$$

Remark that if $m = 0$, $\widetilde{W}_{\varphi_m}$ will be a highest weight module.

For the above φ_m with $\varphi_m(z_3) \neq 0$, we define a new Lie algebra homomorphism $\varphi'_m : \mathcal{V}^{(m)} + \mathbb{C}z_1 \to \mathbb{C}$ as follows

$$\varphi'_m(z_1) = \varphi_m(z_1) - 1 + 12\frac{\varphi_m(z_2)^2}{\varphi_m(z_3)},$$

$$\varphi'_m(d_k) = 0, \forall k \geq 2m + 1,$$

$$\varphi_m'(d_k) = \varphi_m(d_k) + \frac{(\sum_{i=0}^{k}\varphi_m(I_i)\varphi_m(I_{k-i})) - 2(m+1)\varphi_m(I_m)\varphi_m(z_2)}{2\varphi_m(z_3)},$$

for all $k = m, m+1, \ldots, 2m$. Then we have the universal whittaker $\mathcal{V}$-module $W_{\varphi_m'}$:

$$W_{\varphi_m'} = \mathrm{Ind}_{\mathcal{V}^{(m)}+\mathbb{C}z_1}^{\mathcal{V}} \mathbb{C}w_{\varphi_m'} \tag{3.11}$$

where $x \cdot w_{\varphi_m'} = \varphi_m'(x)w_{\varphi_m'}, \forall x \in \mathcal{V}^{(m)} + \mathbb{C}z_1$.

Theorem 13 *Suppose that $m \in \mathbb{Z}_+$, and φ_m and φ_m' are given above with $\varphi_m(z_3) \neq 0$. Let $H = U(\mathfrak{H})w_{\varphi_m}$ in $\widetilde{W}_{\varphi_m}$.*

(1) *We have $\widetilde{W}_{\varphi_m} \cong H(\varphi_m(z_2)) \otimes W_{\varphi_m'}^{\widetilde{\mathcal{V}}}$. Consequently, each simple whittaker module with respect to φ_m is isomorphic to $H(\varphi_m(z_2)) \otimes T^{\widetilde{\mathcal{V}}}$ for a simple quotient T of $W_{\varphi_m'}$.*

(2) *The $\widetilde{\mathcal{V}}$-module $\widetilde{W}_{\varphi_m}$ is simple if and only if $W_{\varphi_m'}$ is a simple $\mathcal{V}$-module. Consequently, for $m \in \mathbb{N}$, the module $\widetilde{W}_{\varphi_m}$ is simple if and only if $(\varphi_m'(d_{2m-1}), \varphi_m'(d_{2m})) \neq (0,0)$, i.e.,*

$$2\varphi_m(d_{2m})\varphi_m(z_3) + \varphi_m(I_m)^2 - 2(m+1)\varphi_m(I_m)\varphi_m(z_2) \neq 0, \ \text{or}$$
$$\varphi_m(d_{2m-1})\varphi_m(z_3) + \varphi_m(I_m)\varphi_m(I_{m-1}) - (m+1)\varphi_m(I_m)\varphi_m(z_2) \neq 0.$$

(3) *Let T_1, T_2 be simple quotients of $W_{\varphi_m'}$. Then $H(\varphi_m(z_2)) \otimes T_1^{\widetilde{\mathcal{V}}} \cong H(\varphi_m(z_2)) \otimes T_2^{\widetilde{\mathcal{V}}}$ if and only if $T_1 \cong T_2$.*

Proof From Lemma 4, we know that H is a simple $\mathfrak{H}$-module.

(1) Define $L = \mathrm{span}\{d_{m+i}, I_j, z_1, z_2, z_3 | i \in \mathbb{Z}_+, j \in \mathbb{Z}\}$. From simple computations we see that

$$H \cong \mathrm{Ind}_{\widetilde{\mathcal{V}}^{(0,m)}+\sum_{i=1}^{3}\mathbb{C}z_i}^{L} \mathbb{C}w_{\varphi_m} \cong H(\dot{z_2}) \otimes \mathbb{C}w_{\varphi_m'}$$

as L-modules, where the action of L on $\mathbb{C}w_{\varphi_m'}$ is given by $(\mathfrak{H}+\mathbb{C}z_2+\mathbb{C}z_3)w_{\varphi_m'} = 0, xw_{\varphi_m'} = \varphi_m'(x)w_{\varphi_m'}$ for all $x \in \mathcal{V}^{(m)} + \mathbb{C}z_1$. Therefore from Lemma 8, we have

$$\widetilde{W}_{\varphi_m} \cong \mathrm{Ind}_L^{\widetilde{\mathcal{V}}}(\mathrm{Ind}_{\widetilde{\mathcal{V}}^{(0,m)}+\sum_{i=1}^{3}\mathbb{C}z_i}^{L} \mathbb{C}w_{\varphi_m}) \cong \mathrm{Ind}_L^{\widetilde{\mathcal{V}}}(H(\dot{z_2}) \otimes \mathbb{C}w_{\varphi_m'})$$
$$\cong H(\dot{z_2}) \otimes \mathrm{Ind}_L^{\widetilde{\mathcal{V}}} \mathbb{C}w_{\varphi_m'}) \cong H(\dot{z_2}) \otimes W_{\varphi_m'}^{\widetilde{\mathcal{V}}}.$$

Parts (2) and (3) follows from Lemma 3, Theorem 11 and some easy computations. □

Next we consider $\widetilde{W}_{\varphi_m}$ for $\varphi(z_3) = 0$.

Theorem 14 *Suppose that $m \geq 1$ and the Lie algebra homomorphism $\varphi_m : \widetilde{\mathcal{V}}^{(0,m)} + \sum_{i=1}^{3}\mathbb{C}z_i \to \mathbb{C}$ is with $\varphi_m(z_3) = 0$. Then the universal Whittaker module $\widetilde{W}_{\varphi_m}$ is simple if and only if $\varphi_m(I_m) \neq 0$.*

Proof **Case 1** $\varphi_m(I_m) \neq 0$.

Since $\varphi_m(d_{2m+j}) = \varphi_m(I_{m+j}) = 0$ for all $j \in \mathbb{N}$, we may choose some $\alpha = \sum_{i=-m}^{-1} a_i t^i \in \mathbb{C}[t, t^{-1}]$ such that

$$0 = \varphi_m(d_n) + \varphi_m(t^n\alpha), \forall n = m+1, m+2, \ldots, 2m.$$

Then $\widetilde{W}_{\varphi_m}^{\sigma_{\alpha,0}}$ becomes a new Whittaker module with the new action $d_k \circ w_{\varphi_m} = 0$ for all $k > m$. Since $\varphi(z_3) = 0$, the action of $\mathfrak{H}$ on $\widetilde{W}_{\varphi_m}$ is unchanged. Without lose of generality, we may assume that

$$\varphi_m(d_k) = 0, \forall k > m. \tag{3.12}$$

Denote $B = \mathrm{Ind}_{\mathcal{V}^{(0,m)}}^{\widetilde{\mathcal{V}}^{(0,0)}} \mathbb{C}w_{\varphi_m}$. Then $\{d_{m-1}^{i_{m-1}} \cdots d_0^{i_0} w_{\varphi_m} | (i_{m-1}, \ldots, i_0) \in \mathbb{Z}_+^m\}$ is a basis of B. It is straightforward to check that I_m acts injectively on B, and B is simple as $\widetilde{\mathcal{V}}^{(0,0)}$-module. Now from Theorem 1 in [7] (for $d = 0$ there), we see that $\widetilde{W}_{\varphi_m} \cong \mathrm{Ind}_{\widetilde{\mathcal{V}}^{(0,0)}+\sum_{i=1}^3 \mathbb{C}z_i}^{\widetilde{\mathcal{V}}} B$ is simple.

Case 2 $\varphi_m(I_m) = 0$.

Let

$$w = (\sum_{i=0}^{m-1} a_i d_i + \sum_{i=1}^{m+1} b_i I_{-i} + cI_{-1}I_{-1}) w_{\varphi_m} \in \widetilde{W}_{\varphi_m},$$

where $a_0, \ldots, a_{m-1}, b_1, \ldots, b_{m+1}, c \in \mathbb{C}$ will be determined. We will make w into a Whittaker vector. From $d_{m+i}w = \varphi_m(d_{m+i})w$ and $I_{1+i}w = \varphi_m(I_{1+i})w$ for $i = 0, 1, ..., m$, we obtain the following homogenous linear system with $2m + 2$ variables $a_0, \ldots, a_{m-1}, b_1, \ldots, b_{m+1}, c$ and $2m + 1$ equations:

$$\sum_{i=0}^{m-1} a_i \varphi_m([I_k, d_i]) = 0, \quad k = 1, \ldots, m-1;$$

$$(-m-1)b_{m+1} + 2c\varphi_m([d_m, I_{-1}]) = 0;$$

$$\sum_{i=0}^{m-1} a_i \varphi_m([d_m, d_i]) + \sum_{i=1}^{m} b_i \varphi_m([d_m, I_{-i}]) = 0;$$

$$\sum_{i=0}^{m-1} a_i \varphi_m([d_{m+l}, d_i]) + \sum_{i=1}^{m+1} b_i \varphi_m([d_{m+l}, I_{-i}]) = 0, \quad l = 1, 2 \ldots, m.$$

It is clear that the homogenous linear system has a nonzero solution since the number of equations is less than the number of variables.

Thus w generates a nonzero proper submodule of $\widetilde{W}_{\varphi_m}$. So we have proved that $\widetilde{W}_{\varphi_m}$ is not simple in this case. □

Now we summarize the established results of this subsection into the following main theorem.

Theorem 15 *Let $m \in \mathbb{N}$ and $\varphi_m : \widetilde{\mathcal{V}}^{(0,m)} + \sum_{i=1}^3 \mathbb{C}z_i \to \mathbb{C}$ be a Lie algebra homomorphism.*

(1) *Suppose $\varphi_m(z_3) \neq 0$. Then the Whittaker module $\widetilde{W}_{\varphi_m}$ is simple if and only if*

$$2\varphi_m(d_{2m})\varphi_m(z_3) + \varphi_m(I_m)^2 - 2(m+1)\varphi_m(I_m)\varphi_m(z_2) \neq 0, \ \ or$$

$$\varphi_m(d_{2m-1})\varphi_m(z_3) + \varphi_m(I_m)\varphi_m(I_{m-1}) - (m+1)\varphi_m(I_m)\varphi_m(z_2) \neq 0.$$

(2) *Suppose $\varphi_m(z_3) = 0$. Then the Whittaker module $\widetilde{W}_{\varphi_m}$ is simple if and only if $\varphi_m(I_m) \neq 0$.*

We remark that if $m = 1$, then the modules $\widetilde{W}_{\varphi_m}$ are the usual Whittaker module defined in [16] and [22].

4 Tensor Products of Highest Weight Modules and Modules of Intermediate Series

In this section, we will determine the necessary and sufficient conditions for the tensor products of highest weight modules and modules of intermediate series over $\widetilde{\mathcal{V}}$ to be simple.

Let $(\dot{I_0}, \dot{d_0}, \dot{z_1}, \dot{z_2}, \dot{z_3}) \in \mathbb{C}^5$, $\mathcal{I}$ be the left ideal in $U(\widetilde{\mathcal{V}})$ generated by

$$\{d_n, I_n, d_0 - \dot{d_0}, I_0 - \dot{I_0}, z_1 - \dot{z_1}, z_2 - \dot{z_2}, z_2 - \dot{z_3} | n \in \mathbb{N}\}.$$

Then $M(\dot{I_0}, \dot{d_0}, \dot{z_1}, \dot{z_2}, \dot{z_3}) = U(\widetilde{\mathcal{V}})/\mathcal{I}$ is a Verma module which is a free $U(\widetilde{\mathcal{V}}^-)$ module generated by the highest weight vector $w = 1 + \mathcal{I}$. It has a unique maximal submodule $J(\dot{I_0}, \dot{d_0}, \dot{z_1}, \dot{z_2}, \dot{z_3})$, and the quotient module

$$V(\dot{I_0}, \dot{d_0}, \dot{z_1}, \dot{z_2}, \dot{z_3}) = M(\dot{I_0}, \dot{d_0}, \dot{z_1}, \dot{z_2}, \dot{z_3})/J(\dot{I_0}, \dot{d_0}, \dot{z_1}, \dot{z_2}, \dot{z_3})$$

is simple. We will denote the image of w by $\bar{w}$. These modules are studied in [1, 4].

Now we recall the modules of intermediate series from [18]. For any $a, b, F \in \mathbb{C}$, $A(a, b; F) = \mathbb{C}[x, x^{-1}]$ is a $\widetilde{\mathcal{V}}$-module with the action

$$z_1 = z_2 = z_3 = 0, \tag{4.1}$$

$$d_n x^m = (a + m + nb)x^{m+n}, \tag{4.2}$$

$$I_n x^m = F x^{m+n}, \forall m, n \in \mathbb{Z}. \tag{4.3}$$

It is well known that $A(a, b; F)$ is reducible if and only if $a \in \mathbb{Z}$, $b \in \{0, 1\}$ and $F = 0$. Denote by $A'(a, b; F)$ the unique nontrivial simple sub-quotient of $A(a, b; F)$. Recall that $A'(a, b, F) \cong A'(0, 0, 0)$ if $A(a, b; F)$ is not simple.

It has been shown in [18] that an simple $\widetilde{\mathcal{V}}$-module with finite-dimensional weight spaces is either a highest (or lowest) weight module, or isomorphic to some $A'(a, b; F)$.

As usual, we use $M(\dot{d_0}, \dot{z_1})$ and $V(\dot{d_0}, \dot{z_1})$ to denote the Verma module and irreducible highest weight module over $\mathcal{V}$, respectively, and use $A(a, b)$ and $A'(a, b)$ to denote the intermediate series $\mathcal{V}$-modules.

Lemma 16 *(1) Suppose* $\dot{z_3} \neq 0$. *Then* $V(\dot{I_0}, \dot{d_0}, \dot{z_1}, \dot{z_2}, \dot{z_3}) \otimes A'(a, b; 0)$ *is simple if and only if* $V(\dot{d_0} + \frac{1}{2\dot{z_3}}\dot{I_0}^2 - \frac{\dot{z_2}}{\dot{z_3}}\dot{I_0}, \dot{z_1} - 1 + \frac{12\dot{z_2}^2}{\dot{z_3}}) \otimes A'(a, b)$ *is a simple* $\mathcal{V}$*-module (the later part is completely determined in* [8]*).*

(2) If $F \neq 0$*, then* $A(a, b; F) \otimes V(\dot{d_0}, \dot{z_1})^{\widetilde{\mathcal{V}}}$ *is simple.*

Proof (1). Denote by H the highest weight $\mathfrak{H}$-module with $I_0 = \dot{I_0}$ and $z_3 = \dot{z_3}$. From Theorem 13 or Section 6 in [1] we know that

$$V(\dot{I_0}, \dot{d_0}, \dot{z_1}, \dot{z_2}, \dot{z_3}) \cong H(\dot{z_2}) \otimes V(\dot{d_0} + \frac{1}{2\dot{z_3}}\dot{I_0}^2 - \frac{\dot{z_2}}{\dot{z_3}}\dot{I_0}, \dot{z_1} - 1 + \frac{12\dot{z_2}^2}{\dot{z_3}})^{\widetilde{\mathcal{V}}},$$

$$V(\dot{I_0}, \dot{d_0}, \dot{z_1}, \dot{z_2}, \dot{z_3}) \otimes A'(a, b; 0)$$
$$\cong H(\dot{z_2}) \otimes (V(\dot{d_0} + \frac{1}{2\dot{z_3}}\dot{I_0}^2 - \frac{\dot{z_2}}{\dot{z_3}}\dot{I_0}, \dot{z_1} - 1 + \frac{12\dot{z_2}^2}{\dot{z_3}}) \otimes A'(a, b))^{\widetilde{\mathcal{V}}}.$$

Therefore the result follows from Theorem 11.

(2) Note that for any finite subset $S \subset V(\dot{d_0}, \dot{z_1})$, there exists some r such that $\mathcal{V}^{(r)} + \mathfrak{H} + \mathbb{C}z_2 \subset \mathrm{ann}_{\widetilde{\mathcal{V}}}(S)$. However $A(a, b; F)$ is simple as $\mathcal{V}^{(r)} + \mathfrak{H} + \mathbb{C}z_2$ module. Thus the result follows from Theorem 7.

□

Lemma 17 *The module $M(\dot{I}_0, \dot{d}_0, \dot{z}_1, \dot{z}_2, \dot{z}_3) \otimes A'(a, b; F)$ is not simple.*

Proof Let w be the highest weight vector of $M(\dot{I}_0, \dot{d}_0, \dot{z}_1, \dot{z}_2, \dot{z}_3)$. Suppose that $x^k, x^{k+1} \neq 0$ in $A'(a, b; F)$. The lemma is clear from the following claim.

Claim We have $w \otimes x^k \notin U(\widetilde{\mathcal{V}})(w \otimes x^{k+1})$.

Note that

$$U(\widetilde{\mathcal{V}})(w \otimes x^{k+1}) = U(\widetilde{\mathcal{V}}^-)U(\widetilde{\mathcal{V}}^+)(w \otimes x^{k+1}) \subset \sum_{i\in\mathbb{Z}_+} U(\widetilde{\mathcal{V}}^-)(w \otimes x^{k+1+i}).$$

Using the PBW Basis of $U(\widetilde{\mathcal{V}}^-)$, it is easy to verify that $w \otimes x^k$ can not be written as $\sum_{i=1}^r u_{-i}(w \otimes x^{k+i})$, where $u_{-i} \in U(\widetilde{\mathcal{V}}^-)_{-i}$ with $u_{-r} \neq 0$, since the term $u_{-r}w \otimes x^{k+r}$ in the expression of $\sum_{i=1}^r u_{-i}(w \otimes x^{k+i})$ cannot be canceled. □

Corollary 18 *If $\dot{I}_0 \neq 0$, then $V(\dot{I}_0, \dot{d}_0, \dot{z}_1, 0, 0) \otimes A'(a, b; F)$ is not simple.*

Proof From Theorem 2 in section 6 of [1] or Theorem 1 in [7], we have

$$V(\dot{I}_0, \dot{d}_0, \dot{z}_1, 0, 0) = M(\dot{I}_0, \dot{d}_0, \dot{z}_1, 0, 0)$$

if $\dot{I}_0 \neq 0$. The result follows from Lemma 17. □

Lemma 19 *Suppose that $F \neq 0$. Then any nonzero submodule M of $V(\dot{I}_0, \dot{d}_0, \dot{z}_1, \dot{z}_2, \dot{z}_3) \otimes A(a, b; F)$ contains $\bar{w} \otimes x^k$ for some k.*

Proof Take a nonzero $\beta = \sum_{i=0}^s v_{-i} \otimes x^{k+i} \in M$, where $v_{-i} \in U(\widetilde{\mathcal{V}}^-)_{-i}\bar{w}$. Replacing β with $u\beta$ for some $u \in U(\widetilde{\mathcal{V}}^+)$ if necessary, we may assume that $v_0 = \bar{w}$. Choose n such that $d_j v_{-i} = I_j v_{-i} = 0, \forall j \geq n, i = 1, 2, \ldots, s$. Note that $A(a, b; F)$ is simple as $\mathcal{V}^{(n)} + \mathfrak{H} + \mathbb{C}z_2$ module. Therefore from Lemma 6, we may choose some $u \in U(\mathcal{V}^{(n)} + \mathfrak{H} + \mathbb{C}z_2)$ with $ux^{k+i} = \delta_{0,i}x^0$ for all $j = 1, 2, \ldots, s$. Rewrite $u = \sum_i u_i u_i'$ with $u_i \in U(\mathfrak{H})$ and $u_i' \in U(\mathcal{V}^{(n)})$. Note that

$$I_j I_i X = F I_{i+j} X, \ \forall i, j \in \mathbb{Z}, X \in A(a, b; F).$$

For sufficient large l, replacing $I_j I_i$ with $F I_{i+j}$ in $I_l u$, we obtain $u' \in U(\widetilde{\mathcal{V}}^{(n,n)})$ with $u'x^{k+i} = t^l ux^{k+i} = F\delta_{0,i}x^l, \forall i = 0, 1, \ldots, s$. Now $0 \neq u'\beta = F\bar{w} \otimes x^l \in M$. □

Now we use the "shifting technique". We will denote $v_k \otimes x^i$ as $v_k \otimes y^{i+k}$ for all $i, k \in \mathbb{Z}$, where $v_k \in V(\dot{I}_0, \dot{d}_0, \dot{z}_1, \dot{z}_2, \dot{z}_3)$ with $d_0 v_k = (\dot{d}_0 + k)v_k$. Then

$$M(\dot{I}_0, \dot{d}_0, \dot{z}_1, \dot{z}_2, \dot{z}_3) \otimes A(a, b; F) = M(\dot{I}_0, \dot{d}_0, \dot{z}_1, \dot{z}_2, \dot{z}_3) \otimes \mathbb{C}[y, y^{-1}]$$

with the actions

$$d_n(u_k w \otimes y^i) = ((d_n - k + a + i + nb)u_k w) \otimes y^{n+i}, \tag{4.4}$$

$$I_n(u_k w \otimes y^i) = ((I_n + F)u_k w) \otimes y^{i+n}, \tag{4.5}$$

for all $u_k \in U(\widetilde{\mathcal{V}})_k = \{u \in U(\widetilde{\mathcal{V}}) | [d_0, u] = ku\}$. For simplicity we define

$$W^{(k)} = \sum_{i\in\mathbb{Z}_+} U(\widetilde{\mathcal{V}})(w \otimes y^{k+i}) \subset M(\dot{I}_0, \dot{d}_0, \dot{z}_1, \dot{z}_2, \dot{z}_3) \otimes A(a, b; F),$$

$$W_n^{(k)} = W^{(k)} \cap (M(\dot{I}_0, \dot{d}_0, \dot{z}_1, \dot{z}_2, \dot{z}_3) \otimes y^n), \forall n \in \mathbb{Z}.$$

Lemma 20 (1) $W^{(k)} = \sum_{i\in\mathbb{Z}_+} U(\widetilde{\mathcal{V}}^-)(w\otimes y^{k+i})$.

(2) $W^{(k)} \supset \oplus_{i\geq k} M(\dot{I}_0, \dot{d}_0, \dot{z_1}, \dot{z_2}, \dot{z_3})\otimes y^i$.

(3) $M(\dot{I}_0, \dot{d}_0, \dot{z_1}, \dot{z_2}, \dot{z_3})\otimes y^{k-1} = W^{(k)}_{k-1}\oplus \mathbb{C}(w\otimes y^{k-1})$.

(4) *Suppose that* P *is a weight vector in* $U(\widetilde{\mathcal{V}}^-)$ *such that*

$$Pw\otimes y^{k-1}\in W^{(k)}_{k-1}\subset M(\dot{I}_0, \dot{d}_0, \dot{z_1}, \dot{z_2}, \dot{z_3})\otimes A(a,b;F),$$

then $(U(\widetilde{\mathcal{V}}^-)Pw)\otimes y^{k-1}\subset W^{(k)}_{k-1}$.

Proof (1). It follows from $U(\widetilde{\mathcal{V}})(w\otimes y^i) = U(\widetilde{\mathcal{V}}^-)U(\widetilde{\mathcal{V}}^+ + \widetilde{\mathcal{V}}^0)(w\otimes y^i)\subset \sum_{j\in\mathbb{Z}_+} U(\widetilde{\mathcal{V}}^-)(w\otimes y^{i+j})$.

(2). Using (1), Eqs. 4.4 and 4.5, by induction on $s+m$ it is straightforward to prove that $I_{-j_1}\ldots I_{-j_s}d_{-l_1}\cdots d_{-l_m}w\otimes y^i\in W^{(k)}$ for all $i\geq k$ and $j_1,\ldots,j_s,l_1,\ldots,l_m\in\mathbb{N}$.

(3). This follows from (2) and the proof of Lemma 17.

(4). Suppose that $P\in U(\widetilde{\mathcal{V}}^-)_m$. From (2), Eqs. 4.4 and 4.5, we have

$$\begin{aligned}(d_{-i}Pw)\otimes y^{k-1} =& d_{-i}(Pw\otimes y^{k+i-1}))\\ &-(a-m+k-1-ib)(Pw)\otimes y^{k-1}\in W^{(k)}_{k-1},\end{aligned} \tag{4.6}$$

$$\begin{aligned}(I_{-i}Pw)\otimes y^{k-1} =& I_{-i}(Pw\otimes y^{k+i-1}))\\ &-FPw\otimes y^{k-1}\in W^{(k)}_{k-1}, \forall i\in\mathbb{N}.\end{aligned} \tag{4.7}$$

Therefore we may prove (4) by induction on m.

□

For any $n\in\mathbb{N}$, from Lemma 20 (3), similar to φ_n defined in [8] we may define the linear map $\rho_n : U(\widetilde{\mathcal{V}}^-)\to\mathbb{C}$ inductively as follows

$$\rho_n(1) = 1, \tag{4.8}$$
$$\rho_n(I_{-i}u) = -F\rho_n(u), \tag{4.9}$$
$$\rho_n(d_{-i}u) = -(a+k+i+n-ib)\rho_n(u), \forall u\in U(\widetilde{\mathcal{V}}^-)_{-k}. \tag{4.10}$$

Remark 21 It is clear that ρ_n depends only on a, b, F, n.

Lemma 22 *Let* $P\in U(\widetilde{\mathcal{V}}_-)$. *Then*

(1) $Pw\otimes y^n \equiv \rho_n(P)w\otimes y^n \pmod{W^{(n+1)}}$;

(2) $Pw\otimes y^n\in W^{(n+1)}$ *if and only if* $\rho_n(P) = 0$.

Proof The proof for (1) is similar to that of Lemma 8 in [8] but one also has to consider I_{-k}. Part (2) follows from (1). □

Let us recall some following results from Theorem 1 in [4].

Theorem 23 *Let* $(\dot{I}_0, \dot{d}_0, \dot{z_1}, \dot{z_2}, \dot{z_3})\in\mathbb{C}^5$ *with* $\dot{z_3} = 0$.

(1) *If* $\frac{\dot{I}_0}{\dot{z_2}}\notin\mathbb{Z}$ *or* $\frac{\dot{I}_0}{\dot{z_2}} = 1$, *then* $M(\dot{I}_0, \dot{d}_0, \dot{z_1}, \dot{z_2}, 0)$ *is simple.*

(2) *If* $1-\frac{\dot{I}_0}{\dot{z_2}}\in\mathbb{Z}\backslash\{0\}$, *then* $M(\dot{I}_0, \dot{d}_0, \dot{z_1}, \dot{z_2}, 0)$ *possesses a singular weight vector* $v\in M_p$ *where* $p = |1-\frac{\dot{I}_0}{\dot{z_2}}|$, *and the quotient module* $M(\dot{I}_0, \dot{d}_0, \dot{z_1}, \dot{z_2}, 0)/U(\widetilde{\mathcal{V}}^-)w$ *is simple.*

Remark 24 (1) Let $M(\dot{d_0}, \dot{z_1})$ be the Verma module over $\mathcal{V}$. Then it is well-known that (see [10], for example), there exist two weight vectors Q_1w', Q_2w' such that $U(\mathcal{V}^-)Q_1w' + U(\mathcal{V}^-)Q_2w'$ is the maximal proper submodule of $M(\dot{d_0}, \dot{z_1})$, where $Q_1, Q_2 \in U(\mathcal{V}^-)$ and w' is the highest weight vector in $M(\dot{d_0}, \dot{z_1})$.

(2) If $\dot{z_3} = \dot{z_2} = 0$ and $\dot{I_0} \neq 0$, from Theorem 1 in [7] we know that $M(\dot{I_0}, \dot{d_0}, \dot{z_1}, \dot{z_2}, \dot{z_3})$ is simple.

(3) If $\dot{z_3} \neq 0$, denote by H the highest weight $\mathfrak{H}$-module with $I_0 = \dot{I_0}$ and $z_3 = \dot{z_3}$. From Theorem 13 we know that

$$M(\dot{I_0}, \dot{d_0}, \dot{z_1}, \dot{z_2}, \dot{z_3}) \cong H(\dot{z_2}) \otimes M(\dot{d_0} + \frac{1}{2\dot{z_3}}\dot{I_0}^2 - \frac{\dot{z_2}}{\dot{z_3}}\dot{I_0}, \dot{z_1} - 1 + \frac{12\dot{z_2}^2}{\dot{z_3}})^{\widetilde{\mathcal{V}}}.$$

Let $J = U(\mathcal{V}^-)Q_1w_2 + U(\mathcal{V}^-)Q_2w_2$ be the maximal proper submodule of $M(\dot{d_0} + \frac{1}{2\dot{z_3}}\dot{I_0}^2 - \frac{\dot{z_2}}{\dot{z_3}}\dot{I_0}, \dot{z_1} - 1 + \frac{12\dot{z_2}^2}{\dot{z_3}})$ where $Q_1, Q_2 \in U(\mathcal{V}^-)$ and w_2 is a highest weight vector. From Theorem 13 we know that $U(\widetilde{\mathcal{V}}^-)(w_1 \otimes Q_1w_2) + U(\widetilde{\mathcal{V}}^-)(w_1 \otimes Q_2w_2)$ is the unique maximal proper submodule of $H(\dot{z_2}) \otimes M(\dot{d_0} + \frac{1}{2\dot{z_3}}\dot{I_0}^2 - \frac{\dot{z_2}}{\dot{z_3}}\dot{I_0}, \dot{z_1} - 1 + \frac{12\dot{z_2}^2}{\dot{z_3}})^{\widetilde{\mathcal{V}}}$,where w_1, w_2 are highest weight vectors of $H(\dot{z_2})$ and $M(\dot{d_0} + \frac{1}{2\dot{z_3}}\dot{I_0}^2 - \frac{\dot{z_2}}{\dot{z_3}}\dot{I_0}, \dot{z_1} - 1 + \frac{12\dot{z_2}^2}{\dot{z_3}})$ respectively. So we have proved that the maximal proper $\widetilde{\mathcal{V}}$-submodule of $M(\dot{I_0}, \dot{d_0}, \dot{z_1}, \dot{z_2}, \dot{z_3})$ can also be generated by at most two singular weight vectors.

Suppose that $(\dot{I_0}, \dot{z_2}, \dot{z_3}) \neq (0, 0, 0)$, from the remark above, we always have weight vectors (or zero) $\widetilde{Q}_1, \widetilde{Q}_2 \in U(\widetilde{\mathcal{V}}^-)$ such that the maximal proper $\widetilde{\mathcal{V}}$-submodule of $M(\dot{I_0}, \dot{d_0}, \dot{z_1}, \dot{z_2}, \dot{z_3})$ is generated by $\widetilde{Q}_1w, \widetilde{Q}_2w$ as $U(\widetilde{\mathcal{V}}^-)$ modules. Recall that $\rho_n(\widetilde{Q}_1), \rho_n(\widetilde{Q}_2)$ are defined in Eqs. 4.8, 4.9, 4.10.

Lemma 25 *Suppose that* $(\dot{I_0}, \dot{z_2}, \dot{z_3}) \neq (0, 0, 0)$ *and* $V(\dot{I_0}, \dot{d_0}, \dot{z_1}, \dot{z_2}, \dot{z_3}) \otimes A'(a, b; F)$ *satisfies*

Condition B For any nonzero submodule V_1 of $V(\dot{I_0}, \dot{d_0}, \dot{z_1}, \dot{z_2}, \dot{z_3}) \otimes A'(a, b, F)$, there exists some k (depending on V_1), such that $\bar{w} \otimes x^{k+i} \in V_1$ for all $i \in \mathbb{Z}_+$.

1. *Suppose that* $A(a, b; F)$ *is simple. Then* $V(\dot{I_0}, \dot{d_0}, \dot{z_1}, \dot{z_2}, \dot{z_3}) \otimes A(a, b; F)$ *is simple if and only if* $(\rho_n(\widetilde{Q}_1), \rho_n(\widetilde{Q}_2)) \neq (0, 0)$ *for all* $n \in \mathbb{Z}$.
2. *If* $(a, b, F) = (0, 0, 0)$*, then* $V(\dot{I_0}, \dot{d_0}, \dot{z_1}, \dot{z_2}, \dot{z_3}) \otimes A'(0, 0, 0)$ *is simple if and only if* $(\rho_n(\widetilde{Q}_1), \rho_n(\widetilde{Q}_2)) \neq (0, 0)$ *for all* $n \in \mathbb{Z}\backslash\{0\}$.

Proof Denote by $J = U(\widetilde{\mathcal{V}}^-)\widetilde{Q}_1w + U(\widetilde{\mathcal{V}}^-)\widetilde{Q}_2w$ the maximal proper submodule of $M(\dot{I_0}, \dot{d_0}, \dot{z_1}, \dot{z_2}, \dot{z_3})$.

(1) From condition B and Lemmas 20, 22, it is clear that the module $V(\dot{I_0}, \dot{d_0}, \dot{z_1}, \dot{z_2}, \dot{z_3}) \otimes A(a, b; F)$ is simple if and only if $J \otimes y^n + W_n^{(n+1)} = M(\dot{I_0}, \dot{d_0}, \dot{z_1}, \dot{z_2}, \dot{z_3}) \otimes y^n$ for all $n \in \mathbb{Z}$ if and only if

$$(U(\widetilde{\mathcal{V}}^-)\widetilde{Q}_1w + U(\widetilde{\mathcal{V}}^-)\widetilde{Q}_2w) \otimes y^n \not\subset W_n^{(n+1)}, \forall n \in \mathbb{Z},$$

if and only if

$$\{Q_1w \otimes y^n, Q_2w \otimes y^n\} \not\subset W_n^{(n+1)}, \quad \forall\ n \in \mathbb{Z}$$

if and only if $(\rho_n(\widetilde{Q}_1), \rho_n(\widetilde{Q}_2)) \neq (0, 0)$ for all $n \in \mathbb{Z}$.

(2) The proof is similar to that of Part (1). The only difference is that we don't need $(\rho_0(Q_1), \rho_0(Q_2)) \neq (0,0)$ since the image of $w \otimes y^0$ is zero in $V(\dot{I}_0, \dot{d}_0, \dot{z}_1, \dot{z}_2, \dot{z}_3) \otimes A'(0,0,0)$.

□

Now we summarize the established results into the following main result in this section.

Theorem 26 *Let* $a, b, F, \dot{I}_0, \dot{d}_0, \dot{z}_1, \dot{z}_2, \dot{z}_3 \in \mathbb{C}$.

(1) *If* $F \neq 0$, *then* $V(0, \dot{d}_0, \dot{z}_1, 0, 0) \otimes A'(a, b; F)$ *is simple.*
(2) $V(0, \dot{d}_0, \dot{z}_1, 0, 0) \otimes A'(a, b; 0)$ *is simple if and only if* $V(\dot{d}_0, \dot{z}_1) \otimes A'(a, b)$ *is a simple* $\mathcal{V}$*-module.*
(3) *Suppose that* $(\dot{I}_0, \dot{z}_2, \dot{z}_3) \neq (0,0,0)$ *and* $A(a, b; F)$ *is simple. Then* $V(\dot{I}_0, \dot{d}_0, \dot{z}_1, \dot{z}_2, \dot{z}_3) \otimes A(a, b; F)$ *is simple if and only if* $(\rho_n(\widetilde{Q}_1), \rho_n(\widetilde{Q}_2)) \neq (0,0)$ *for all* $n \in \mathbb{Z}$.
(4) *Suppose that* $(\dot{I}_0, \dot{z}_2, \dot{z}_3) \neq (0,0,0)$ *and* $(a, b, F) = (0,0,0)$. *Then* $V(\dot{I}_0, \dot{d}_0, \dot{z}_1, \dot{z}_2, \dot{z}_3) \otimes A'(0,0,0)$ *is simple if and only if the pairs* $(\rho_n(\widetilde{Q}_1), \rho_n(\widetilde{Q}_2)) \neq (0,0)$ *for all* $n \in \mathbb{Z}\backslash\{0\}$.

Proof (1) is from Lemma 16. (2) is trivial. If $\dot{z}_3 = \dot{z}_2 = 0$ and $\dot{I}_0 \neq 0$, we have the theorem from Corollary 18. If $F = 0$ and $\dot{z}_3 \neq 0$, the theorem follows from Lemma 16. For other case, we need to check the Condition B in Lemma 25. If $F \neq 0$, we have Condition B hold from Lemma 19. If $F = \dot{z}_3 = 0$ and $\dot{z}_2 \neq 0$, we have Condition B hold from the proof of Theorem 45 in [23]. □

The simplicity of the tensor product $A'(a, b; 0) \otimes V(\dot{I}_0, \dot{d}_0, \dot{z}_1, \dot{z}_2, 0)$ is obtained in [23]. Let us recover it as an example.

Theorem 27 [23] *Suppose that* $\dot{z}_3 = 0$ *and* $\dot{z}_2 \neq 0$.

(1) *For* $1 - \frac{\dot{I}_0}{\dot{z}_2} = p \in \mathbb{N}$, *the module* $A'(a, b; 0) \otimes V(\dot{I}_0, \dot{d}_0, \dot{z}_1, \dot{z}_2, 0)$ *is simple if and only if* $a - pb \notin \mathbb{Z}$.
(2) *For* $\frac{\dot{I}_0}{\dot{z}_2} - 1 \in \mathbb{N}$, *the module* $A'(a, b; 0) \otimes V(\dot{I}_0, \dot{d}_0, \dot{z}_1, \dot{z}_2, 0)$ *is not simple.*
(3) *The module* $A'(a, b; F) \otimes V(0, 0, \dot{z}_1, \dot{z}_2, 0)$ *is simple if and only if* $a - b \notin \mathbb{Z}$.

Proof Suppose that $F = 0$. Using Lemma 23, from easy computations we see that $\rho_n(Q) = 0$ if $\frac{\dot{I}_0}{\dot{z}_2} - 1 \in \mathbb{N}$, and $\rho_n(Q) = \rho_n(d_{-p}) = -(a + p + n - pb)$ if $1 - \frac{\dot{I}_0}{\dot{z}_2} = p \in \mathbb{N}$, which imply (1) and (2).

Now suppose that $(\dot{I}_0, \dot{d}_0, \dot{z}_1, \dot{z}_2, \dot{z}_3) = (0, 0, \dot{z}_1, \dot{z}_2, 0)$, then it is straightforward to see that $Q = d_{-1}$. And we have $\rho_n(Q) = \rho_n(d_{-1}) = -(a + 1 + n - b)$, which implies (3). □

Example 6 Suppose that $\dot{z}_3 \neq 0$ and $F \neq 0$. Let H be the highest weight module over $\mathfrak{H}$ with $I_0 = \dot{I}_0$ and $z_3 = \dot{z}_3$. Then $M(\dot{I}_0, -\frac{\dot{I}_0^2}{2\dot{z}_3} + \frac{\dot{I}_0\dot{z}_2}{\dot{z}_3}, 2 - \frac{12\dot{z}_2^2}{\dot{z}_3}, \dot{z}_2, \dot{z}_3) \cong H(\dot{z}_2) \otimes M(0,1)^{\widetilde{\mathcal{V}}}$. The maximal submodule of $M(0,1)$ is generated by the weight vector of weight -1. It is straightforward to obtain that $\widetilde{Q}_1 = \widetilde{Q}_2 = d_{-1} + \frac{\dot{I}_0}{\dot{z}_3} I_{-1}$.

We know that $\rho_n(\widetilde{Q}_1) = -(a + 1 + n - b) - \frac{\dot{I}_0}{\dot{z}_3} F$. Therefore

$$V(\dot{I}_0, -\frac{\dot{I}_0^2}{2\dot{z}_3} + \frac{\dot{I}_0\dot{z}_2}{\dot{z}_3}, 2 - \frac{12\dot{z}_2^2}{\dot{z}_3}, \dot{z}_2, \dot{z}_3) \otimes A(a, b, F)$$

is simple if and only if $b-a-\frac{\dot{I_0}}{\dot{z_3}}F \notin \mathbb{Z}$. If $n = b-a-1-\frac{\dot{I_0}}{\dot{z_3}}F \in \mathbb{Z}$, then $U(\widetilde{\mathcal{V}})\cdot(\bar{w}\otimes y^{n+1})$ is the unique minimal submodule of $V(\dot{I_0}, -\frac{\dot{I_0}^2}{2\dot{z_3}} + \frac{\dot{I_0}\dot{z_2}}{\dot{z_3}}, 2 - \frac{12\dot{z_2}^2}{\dot{z_3}}, \dot{z_2}, \dot{z_3}) \otimes A(a, b; F)$, which is simple and the quotient

$$V(\dot{I_0}, -\frac{\dot{I_0}^2}{2\dot{z_3}} + \frac{\dot{I_0}\dot{z_2}}{\dot{z_3}}, 2 - \frac{12\dot{z_2}^2}{\dot{z_3}}, \dot{z_2}, \dot{z_3}) \otimes A(a, b; F)/(U(\widetilde{\mathcal{V}}) \cdot (\bar{w} \otimes y^{n+1}))$$

is a highest weight module with the highest weight vector $\bar{w} \otimes y^n + U(\widetilde{\mathcal{V}}) \cdot (\bar{w} \otimes y^{n+1})$.

5 Simple $\widetilde{\mathcal{V}}$-Modules from $\mathcal{O}_{\widetilde{\mathcal{W}}}$

We will use the algebras defined in Section 1: $\widetilde{\mathcal{W}}$, $\widetilde{\mathfrak{a}}$, $\widetilde{\mathcal{V}}^{(r,s)}$ and $\widetilde{\mathcal{V}}[\lambda]$.

In this section we will first classify all simple modules in $\mathcal{O}_{\widetilde{\mathcal{W}}}$, then use the "embedding trick" to make these simple $\widetilde{\mathcal{W}}$-modules into simple $\widetilde{\mathcal{V}}$-modules.

5.1 Simple Modules in $\mathcal{O}_{\widetilde{\mathcal{W}}}$

For any $B \in \mathcal{O}_{\widetilde{\mathfrak{a}}}$ and $0 \neq v \in B$, define $\mathrm{ord}_{\widetilde{\mathfrak{a}}}(v)$, the order of v, to be the minimal nonnegative integer r with $I_{r+i}v = 0$ for all $i \geq 0$. And $\mathrm{ord}_{\widetilde{\mathfrak{a}}}(B)$, the order of B, is defined to be the maximal order of all its elements or ∞ if it doesn't exist.

Lemma 28 *Suppose that $B \in \mathcal{O}_{\widetilde{\mathfrak{a}}}$ is simple.*

(1) *We have $ord_{\widetilde{\mathfrak{a}}}(B) = ord_{\widetilde{\mathfrak{a}}}(v)$ for all $0 \neq v \in B$. And there exists some $(r, s) \in \mathbb{Z}_+^2$ such that $\widetilde{\mathcal{V}}^{(r,s)}B = 0$, i.e., B can be regarded as a simple module over $\widetilde{\mathfrak{a}}_{r,s}$.*
(2) *If $ord_{\widetilde{\mathfrak{a}}}(B) = 0$, then B is a simple $\mathfrak{a}$-module.*
(3) *If $r = ord_{\widetilde{\mathfrak{a}}}(B) > 0$, then the action of I_{r-1} on B is bijective.*

Proof For any nonzero $v, v' \in B$, since B is simple, there exists some $u \in U(\widetilde{\mathfrak{a}})$, such that $v' = uv$. It is straightforward to check that $I_iv' = I_iuv = 0, \forall i \geq \mathrm{ord}_{\widetilde{\mathfrak{a}}}(v)$. So $\mathrm{ord}_{\widetilde{\mathfrak{a}}}(v) \geq \mathrm{ord}_{\widetilde{\mathfrak{a}}}(v')$. Similarly we have $\mathrm{ord}_{\widetilde{\mathfrak{a}}}(v') \geq \mathrm{ord}_{\widetilde{\mathfrak{a}}}(v)$. Thus $\mathrm{ord}_{\widetilde{\mathfrak{a}}}(v') = \mathrm{ord}_{\widetilde{\mathfrak{a}}}(v) = r$. Now suppose that $d_iv = 0, \forall i \geq k$. Then take $s > \max\{k, r\}$. It is easy to verify that $\widetilde{\mathcal{V}}^{(r,s)}B = 0$. So we have proved (1). Part (2) is trivial.

Now suppose that $r = \mathrm{ord}_{\widetilde{\mathfrak{a}}}(B) > 0$. Consider the subspace $X = \{v \in B | I_{r-1}v = 0\}$ which is a proper subspace of B. Then X and $I_{r-1}B$ are $\widetilde{\mathfrak{a}}$-submodules of B. Since B is simple, $X \neq B$ and $I_{r-1}B \neq 0$, we deduce that $X = 0$ and $I_{r-1}B = B$, i.e., I_{r-1} is bijective on B. Part (3) follows. □

Lemma 29 *Let $B \in \mathcal{O}_{\widetilde{\mathfrak{a}}}$, $W, W_1 \in \mathcal{O}_{\widetilde{\mathcal{W}}}$ be nontrivial simple modules.*

(1) *The module $\mathrm{Ind}_{\widetilde{\mathfrak{a}}}^{\widetilde{\mathcal{W}}}(B)$ is simple in $\mathcal{O}_{\widetilde{\mathcal{W}}}$;*
(2) *The module $\mathrm{Soc}_{\widetilde{\mathfrak{a}}}(W)$ is a simple $\widetilde{\mathfrak{a}}$-module, and an essential $\widetilde{\mathfrak{a}}$-submodule of W;*
(3) *We have $W \cong \mathrm{Ind}_{\widetilde{\mathfrak{a}}}^{\widetilde{\mathcal{W}}} \mathrm{Soc}_{\widetilde{\mathfrak{a}}}(W)$, $B = \mathrm{Soc}_{\widetilde{\mathfrak{a}}}(\mathrm{Ind}_{\widetilde{\mathfrak{a}}}^{\widetilde{\mathcal{W}}} B)$;*
(4) *We have $W \cong W_1$ if and only if $\mathrm{Soc}_{\widetilde{\mathfrak{a}}}(W) \cong \mathrm{Soc}_{\widetilde{\mathfrak{a}}}(W_1)$.*

Proof (1). Let M be any nonzero submodule of $\mathrm{Ind}_{\widetilde{\mathfrak{a}}}^{\widetilde{\mathcal{W}}}(B) = \mathbb{C}[d_{-1}] \otimes B$. Choose $0 \neq v = \sum_{i=0}^{s} d_{-1}^{i} \otimes v_i \in M$ with minimal s, where $v_i \in B$. Denote $r = \mathrm{ord}_{\widetilde{\mathfrak{a}}}(B)$. If $r = 0$, the result follows from Lemma 2. Thus we assume that $r > 0$. Note that in $U(\widetilde{\mathcal{W}})$ we have

$$I_r d_{-1}^{s} = d_{-1}^{s} I_r - rsd_{-1}^{s-1} I_{r-1} + \sum_{i=0}^{s-2} d_{-1}^{i} U(\mathfrak{H}).$$

If $s > 0$, then

$$0 \neq I_r v \in -srd_{-1}^{s-1} \otimes I_{r-1} v_s + \sum_{i=0}^{s-2} d_{-1}^{i} \otimes B \subset M,$$

which contradicts to the minimality of s. So $s = 0$, i.e., $v \in 1 \otimes B$. Therefore $M = \mathrm{Ind}_{\widetilde{\mathfrak{a}}}^{\widetilde{\mathcal{W}}}(B)$, and $\mathrm{Ind}_{\widetilde{\mathfrak{a}}}^{\widetilde{\mathcal{W}}}(B)$ is simple.

(2). Fix some $0 \neq w \in W$ with minimal $\mathrm{ord}_{\widetilde{\mathfrak{a}}} w = r$. Let $M = U(\widetilde{\mathfrak{a}})w$. Then $\mathrm{ord}_{\widetilde{\mathfrak{a}}} M = r$, and $W = \mathbb{C}[d_{-1}]M$. If $r = 0$, the result follows from Lemma 2. Thus we assume that $r > 0$. For any $v = \sum_{i=0}^{s} d_{-1}^{i} w_i \in W$ with $w_s \neq 0$ and $w_i \in M$ for $i = 0, \dots, s$, we have

$$0 \neq I_{r+s-1} v = (-1)^s (r+s-1)(r+s) \cdots r I_{r-1} w_s \in M, \tag{5.1}$$

$$I_{r+i} v = 0, \forall i \geq s. \tag{5.2}$$

Thus $\mathrm{ord}_{\widetilde{\mathfrak{a}}}(v) = r + s$. So $W = \mathbb{C}[d_{-1}] \otimes M$ and $W \cong \mathrm{Ind}_{\widetilde{\mathfrak{a}}}^{\widetilde{\mathcal{W}}} M$. Recall that W is a simple $\widetilde{\mathcal{W}}$ module. Thus M is simple as $\widetilde{\mathfrak{a}}$-module, and it is essential from Eq. 5.1.

Part (3) is an obvious consequence of (1) and (2). Part (4) follows from (3). □

Example 7 Consider some $r \in \mathbb{N}$ and set

$$\mu = (\mu_r, \mu_{r+1}, \dots, \mu_{2r}), \kappa = (\kappa_0, \kappa_1, \kappa_2, \dots, \kappa_r) \in \mathbb{C}^{r+1}.$$

Define the one dimensional $\widetilde{\mathcal{V}}^{(0,r)}$-module $\mathbb{C}v_{\mu,\kappa}$ with the action

$$d_{2r+i} v_{\mu,\kappa} = I_{i+r} v_{\mu,\kappa} = 0, \ \ \forall i \in \mathbb{N},$$

$$I_i v_{\mu,\kappa} = \kappa_i v_{\mu,\kappa}, \ \ d_{r+i} v_{\mu,\kappa} = \mu_{r+i} v_{\mu,\kappa}, \ \ \forall k = 0, 1, \dots, r.$$

Then we have the induced $\widetilde{\mathcal{W}}$-module $W_{\mu,\kappa} = \mathrm{Ind}_{\widetilde{\mathcal{V}}^{(0,r)}}^{\widetilde{\mathcal{W}}} \mathbb{C}v_{\mu,\kappa}$.

Lemma 30 *The $\widetilde{\mathcal{W}}$-module $W_{\mu,\kappa}$ is simple if and only if* $(\mu_{2r}, \mu_{2r-1}, \kappa_r) \neq (0, 0, 0)$.

Proof If $(\mu_{2r}, \mu_{2r-1}) \neq (0, 0)$, then from Lemma 3, and Theorem 2 in [MZ], we know that $W_{\mu,\kappa}$ is a simple $\mathcal{W}$-module, hence a simple $\widetilde{\mathcal{W}}$-module.

Now suppose that $(\mu_{2r}, \mu_{2r-1}) = (0, 0)$ and $\kappa_r \neq 0$. We make $\mathbb{C}v_{\mu,\kappa}$ to be a $\widetilde{\mathcal{V}}^{(0,r)} + \sum_{i=1}^{3} \mathbb{C}z_i$ module by $z_1 = z_2 = 0, z_3 = 1$. Then by Theorem 13(2), $\mathrm{Ind}_{\widetilde{\mathcal{V}}^{(0,r)}+\sum_{i=1}^{3}\mathbb{C}z_i}^{\widetilde{\mathcal{V}}} \mathbb{C}v_{\mu,\kappa}$ is a simple $\widetilde{\mathcal{V}}$-module. Thus $\mathrm{Ind}_{\widetilde{\mathcal{V}}^{(0,r)}+\sum_{i=1}^{3}\mathbb{C}z_i}^{\widetilde{\mathcal{W}}+\sum_{i=1}^{3}\mathbb{C}z_i} \mathbb{C}v_{\mu,\kappa}$ is a simple $\widetilde{\mathcal{W}} + \sum_{i=1}^{3} \mathbb{C}z_i$ module, that is, $W_{\mu,\kappa}$ is a simple $\widetilde{\mathcal{W}}$-module. On the other hand, if $(\mu_{2r}, \mu_{2r-1}, \kappa_r) = (0, 0, 0)$, then it is straightforward to verify that $d_{r-1} v_{\mu,\kappa}$ generates a proper $\widetilde{\mathcal{W}}$ submodule. □

5.2 The "Embedding Trick"

Let $W \in \mathcal{O}_{\widetilde{\mathcal{W}}}$. Then W can be naturally regarded as a module over $\mathbb{C}[[t]]\frac{d}{dt}+\mathbb{C}[[t]]$. Regard $\mathbb{C}[t,(t+\lambda)^{-1}]\frac{d}{dt}+\mathbb{C}[t,(t+\lambda)^{-1}]$ as a subalgebra of $\mathbb{C}[[t]]\frac{d}{dt}+\mathbb{C}[[t]]$. We will use the expression

$$(t+\lambda)^m=\sum_{i=0}^{\infty}\binom{m}{i}\lambda^{m-i}t^i \in \mathbb{C}[[t]],\ \forall m\in\mathbb{Z}, \lambda\in\mathbb{C}^*,$$

where $\binom{m}{i}=\frac{m\cdot(m-1)\cdots(m-i+1)}{i!}$. Let

$$\sigma_\lambda : \widetilde{\mathcal{V}} \to \mathbb{C}[t,(t+\lambda)^{-1}]\frac{d}{dt}+\mathbb{C}[t,(t+\lambda)^{-1}]$$

be the epimorphism of Lie algebras defined by

$$\sigma_\lambda(f(t))=f(t+\lambda),\ \ \sigma_\lambda(f(t)\frac{d}{dt})=f(t+\lambda)\frac{d}{dt}, \forall f(t)\in\mathbb{C}[t,t^{-1}],$$

and $\sigma_\lambda(z_i)=0, i=1,2,3$. So we have the $\widetilde{\mathcal{V}}$-module $W[\lambda]=W$ with the action

$$z_i\circ w=0, i=1,2,3, \tag{5.3}$$

$$(f(t)\frac{d}{dt})\circ w=\sigma_\lambda(f(t)\frac{d}{dt})v=(f(t+\lambda)\frac{d}{dt})v, \tag{5.4}$$

$$f(t)\circ w=\sigma_\lambda(f(t))v=f(t+\lambda)v, \forall\ f(t)\in\mathbb{C}[t,t^{-1}], w\in W. \tag{5.5}$$

We call the above method to make $\widetilde{\mathcal{W}}$-module W into $\widetilde{\mathcal{V}}$-module $W[\lambda]$ the "embedding trick".

Remark 31 Note that $\sigma_\lambda|_{\widetilde{\mathcal{W}}}$ is a Lie algebra automorphism. Hence for any $W\in\mathcal{O}_{\widetilde{\mathcal{W}}}$, $W[\lambda]$ is equivalent to W as $\widetilde{\mathcal{W}}$-modules. It is easy to see that $W[\lambda]\notin\mathcal{O}_{\widetilde{\mathcal{V}}}$ for any $\lambda\in\mathbb{C}^*$ unless W is trivial.

Proposition 32 *Suppose that $W, W'\in\mathcal{O}_{\widetilde{\mathcal{W}}}$ are simple and nontrivial, $\lambda,\lambda'\in\mathbb{C}^*$.*

(a) *The module $W[\lambda]$ is a simple $\widetilde{\mathcal{V}}$-module.*
(b) *We have $W[\lambda]\cong \mathrm{Ind}_{\widetilde{\mathcal{V}}[\lambda]}^{\widetilde{\mathcal{V}}}\mathrm{Soc}_{\tilde{\mathfrak{a}}}(W)=\mathbb{C}[d_0]\otimes\mathrm{Soc}_{\tilde{\mathfrak{a}}}(W)$, where $\mathrm{Soc}_{\tilde{\mathfrak{a}}}(W)$ is considered as a $\widetilde{\mathcal{V}}[\lambda]$-module.*
(c) *We have $W[\lambda]\cong W'[\lambda']$ as $\widetilde{\mathcal{V}}$-modules if and only if $\lambda=\lambda'$ and $W\cong W'$ as $\widetilde{\mathcal{W}}$-modules.*

Proof Part (a) follows from Remark 31.
(b). For any $w\in\mathrm{Soc}_{\tilde{\mathfrak{a}}}(W)$, any $n\in\mathbb{Z}$, we have

$$(t^n)\circ w=(t+\lambda)^n w=\sum_{i=0}^{\infty}\binom{n}{i}\lambda^{m-i}t^i w,$$

$$((t-\lambda)t^n\frac{d}{dt})\circ w=(t(t+\lambda)^n\frac{d}{dt})w=\sum_{i=0}^{\infty}\binom{n}{i}\lambda^{m-i}(t^{i+1}\frac{d}{dt})w.$$

So $\mathrm{Soc}_{\tilde{\mathfrak{a}}}(W)$ is a $\widetilde{\mathcal{V}}[\lambda]$-module. The rest follows from the definition of $W[\lambda]$ and Lemma 29.

The sufficiency of Part (c) is trivial. Now suppose that $\psi : W[\lambda] \to W'[\lambda']$ is a $\widetilde{\mathcal{W}}$-module isomorphism. Suppose that $\lambda \neq \lambda'$. For any $0 \neq v \in W[\lambda]$, there exists some $k \in \mathbb{N}$ such that $(\mathbb{C}[t, t^{-1}](t-\lambda)^k \frac{d}{dt} + \mathbb{C}[t, t^{-1}](t-\lambda)^k + \mathbb{C}z_1 + \mathbb{C}z_2 + \mathbb{C}z_2) \circ v = 0$. Hence

$$(\mathbb{C}[t, t^{-1}](t-\lambda)^k \frac{d}{dt} + \mathbb{C}[t, t^{-1}](t-\lambda)^k + \sum_{i=1}^{3} \mathbb{C}z_i) \circ \psi(v) = 0. \tag{5.6}$$

And also there exists some $k' \in \mathbb{N}$, such that

$$(\mathbb{C}[t, t^{-1}](t-\lambda')^{k'} \frac{d}{dt} + \mathbb{C}[t, t^{-1}](t-\lambda')^{k'} + \sum_{i=1}^{3} \mathbb{C}z_i) \circ \psi(v) = 0. \tag{5.7}$$

From Eqs. 5.6 and 5.7, we have $\widetilde{\mathcal{V}} \circ \psi(v) = 0$, a contradiction. So $\lambda = \lambda'$. For any $f(t)\frac{d}{dt} + g(t) \in \mathcal{W}$ and $v \in W$, we have

$$\psi((f(t)\frac{d}{dt} + g(t))v) = \psi((f(t-\lambda)\frac{d}{dt} + g(t-\lambda)) \circ v)$$

$$= (f(t-\lambda)\frac{d}{dt} + g(t-\lambda)) \circ \psi(v) = (f(t)\frac{d}{dt} + g(t))\psi(v) \in W'.$$

Thus $\psi : W \to W'$ is a $\mathcal{W}$-module isomorphism. □

Example 8 Let $\mathcal{A}(b_1, b_2)$ be the Verma module over $\widetilde{\mathcal{W}}$ with the highest weight vector v_{b_1,b_2} of highest weight $(b_1, b_2) \in \mathbb{C}^2$, i.e., $d_0 v_{b_1,b_2} = b_1 v_{b_1,b_2}$, $I_0 v_{b_1,b_2} = b_2 v_{b_1,b_2}$, and $I_i v_{b_1,b_2} = d_i v_{b_1,b_2} = 0$ for all $i > 0$. Denote $\Omega(\lambda; b_1, b_2) = \mathcal{A}(b_1, b_2)[\lambda]$. From Lemma 29 then $\Omega(\lambda; b_1, b_2)$ is simple if and only if $(b_1, b_2) \neq (0, 0)$.

For any $\lambda \in C^*$, the action of $\widetilde{\mathcal{V}}$ on $\Omega(\lambda; b_1, b_2) = \mathbb{C}[d_0] \circ v_{b_1,b_2}$ is

$$\begin{aligned}
z_1 = z_2 = z_3 = 0,&\\
d_m \circ (d_0^i \circ v_{b_1,b_2}) =&(d_0 - m)^i \circ (d_m \circ v_{b_1,b_2})\\
=&(d_0 - m)^i \circ (((t+\lambda)^{m+1} \frac{d}{dt}) v_{b_1,b_2})\\
=&(d_0 - m)^i \circ (\lambda^{m+1} d_{-1} + (m+1)\lambda^m d_0) v_{b_1,b_2})\\
=&(d_0 - m)^i \circ ((m\lambda^m d_0 + \lambda^m (d_0 + \lambda d_{-1}) v_{b_1,b_2})\\
=&\lambda^m ((d_0 - m)^i (m b_1 + d_0)) \circ v_{b_1,b_2}, \ \forall m \in \mathbb{Z}, i \in \mathbb{Z}_+,
\end{aligned}$$

$$\begin{aligned}
I_m \circ (d_0^i \circ v_{b_1,b_2}) =&(d_0 - m)^i \circ (I_m \circ v_{b_1,b_2})\\
=&(d_0 - m)^i \circ (((t+\lambda)^{m+1}) v_{b_1,b_2})\\
=&(\lambda^{m+1} b_2)(d_0 - m)^i \circ v_{b_1,b_2}, \forall m \in \mathbb{Z}, i \in \mathbb{Z}_+.
\end{aligned}$$

We see that, as $\mathcal{V}$-modules, $\Omega(\lambda; b_1, b_2)$ is isomorphic to $\Omega(\lambda, b_1 + 1)$ defined in Section 4.3 of [19], where $\mathcal{V}$-modules $\Omega(\lambda, b_1 + 1) = \mathbb{C}[\partial]$ with the action:

$$d_n \cdot \partial^k = \lambda^n (\partial + n b_1)(\partial - n)^k, \forall\, k \in \mathbb{Z}_+, n \in \mathbb{Z}. \tag{5.8}$$

Example 9 Let $W_{\mu,\kappa}$ be as defined in Example 7 with $r = 1$. Then $W_{\mu,\kappa}[\lambda] = \mathbb{C}[d_0, d_{-1}] v_{\mu,\kappa}$ is simple if and only if $\mu \neq 0$ or $\kappa_1 \neq 0$. Take

$$\{d_0^i \circ (d_0^j v_{\mu,\kappa}) | i, j \in \mathbb{Z}_+\}$$

as a basis of $W_{\mu,\kappa}[\lambda]$. Then action of $\widetilde{\mathcal{V}}$ on this basis is

$$\begin{aligned}
&z_1 = z_2 = z_3 = 0,\\
&d_m \circ (d_0^i \circ (d_0^j v_{\mu,\kappa}))\\
=&\lambda^m((d_0 - m)^i d_0) \circ (d_0^j v_{\mu,\kappa}) + m\lambda^m(d_0 - m)^i \circ (d_0^{j+1} v_{\mu,\kappa})\\
&+ \frac{m^2+m}{2}\lambda^{m-1}\mu_1(d_0 - m)^i \circ ((d_0 - 1)^j v_{\mu,\kappa})\\
&+ \frac{m^3-m}{6}\lambda^{m-2}\mu_2(d_0 - m)^i \circ (d_0 - 2)^j v_{\mu,\kappa}),\\
&I_m \circ (d_0^i \circ (d_0^j v_{\mu,\kappa}))\\
=&(d_0 - m)^i \circ ((\lambda^m \kappa_0 d_0^j + m\lambda^{m-1}(d_0 - 1)^j \kappa_1)v_{\mu,\kappa}).
\end{aligned}$$

5.3 Simplicity of Tensor Product Modules

In this subsection we will use Theorem 7 to construct more simple $\widetilde{\mathcal{V}}$-modules by taking tensor product of simple modules constructed in this paper.

Lemma 33 *Let $W_1, W_2, \ldots, W_n \in \mathcal{O}_{\widetilde{\mathcal{W}}}$ be simple and nontrivial, and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}^*$ pairwise distinct. Then the $\widetilde{\mathcal{V}}$-module*

$$W_1[\lambda_1] \otimes W_2[\lambda_2] \otimes \cdots \otimes W_n[\lambda_n]$$

is simple.

Proof We will prove the lemma by induction on n. It is obvious for $n = 1$. Now suppose that $n > 1$. Denote $V_1 = W_1[\lambda_1] \otimes W_2[\lambda_2] \otimes \cdots \otimes W_{n-1}[\lambda_{n-1}]$ and $V_2 = W_n[\lambda_n]$.

From the inductive hypothesis, V_1 is simple.

Take $p(t) = (t - \lambda_1)\ldots(t - \lambda_{n-1})$. From the definition of $W[\lambda]$, we see that for any finite subset $v_1 \in V_1$, and $S_2 \subset V_2$, there exists some $k_0 \in \mathbb{N}$ such that

$$(\mathbb{C}[t, t^{-1}]p(t)^{k_0}\frac{d}{dt} + \mathbb{C}[t, t^{-1}]p(t)^{k_0} + \mathbb{C}z_1 + \mathbb{C}z_2 + \mathbb{C}z_2) \circ v_1 = 0,$$

$$(\mathbb{C}[t, t^{-1}](t - \lambda_n)^{k_0}\frac{d}{dt} + \mathbb{C}[t, t^{-1}](t - \lambda_n)^{k_0} + \mathbb{C}z_1 + \mathbb{C}z_2 + \mathbb{C}z_2) \circ S_2 = 0.$$

Note that $\mathbb{C}[t, t^{-1}]p(t)^{k_0} + \mathbb{C}[t, t^{-1}](t - \lambda_n)^{k_0} = \mathbb{C}[t, t^{-1}]$. From Theorem 7, we know that $V_1 \otimes V_2 = W_1[\lambda_1] \otimes W_2[\lambda_2] \otimes \cdots \otimes W_n[\lambda_n]$ is simple. □

Theorem 34 *Let $W_1, W_2, \ldots, W_n \in \mathcal{O}_{\widetilde{\mathcal{W}}}$ be simple and nontrivial, $V \in \mathcal{O}_{\widetilde{\mathcal{V}}}$ be simple, and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}^*$ be pairwise distinct. Then the $\widetilde{\mathcal{V}}$-module*

$$V \otimes W_1[\lambda_1] \otimes W_2[\lambda_2] \otimes \cdots \otimes W_n[\lambda_n]$$

is simple.

Proof Denote $V_1 = V$ and $V_2 = W_1[\lambda_1] \otimes W_2[\lambda_2] \otimes \cdots \otimes W_n[\lambda_n]$. From Lemma 33, V_2 is simple. Take $p(x) = (t - \lambda_1)\ldots(t - \lambda_n)$. Then see that for any $v \in V_1$, and any finite subset $S_2 \subset V_2$, there exists some $k_0 \in \mathbb{N}$ such that

$$(\mathbb{C}[t]t^{k_0}\frac{d}{dt} + \mathbb{C}[t]t^{k_0})v = 0,$$

$$(\mathbb{C}[t,t^{-1}]p(t)^{k_0}\frac{d}{dt}+\mathbb{C}[t,t^{-1}]p(t)^{k_0}+\mathbb{C}z_1+\mathbb{C}z_2+\mathbb{C}z_2)\circ S_2=0.$$

Clearly, $\mathbb{C}[t]t^{k_0}+\mathbb{C}[t,t^{-1}]p(t)^{k_0}=\mathbb{C}[t,t^{-1}]$. In fact, $I=\mathbb{C}[t,t^{-1}]p(t)^{k_0}$ has finite codimension in $\mathbb{C}[t,t^{-1}]$. Say $\mathbb{C}[t,t^{-1}]=T\oplus I$ as vector spaces where T is a finite-dimensional subspace of $\mathbb{C}[t,t^{-1}]$. Then we have $t^iT\subset\mathbb{C}[t]t^{k_0}$ for some sufficient large i since T is finite dimensional. Therefore

$$\mathbb{C}[t,t^{-1}]=t^i\mathbb{C}[t,t^{-1}]=t^i(T\oplus I)=t^iT+t^iI\subseteq\mathbb{C}[t]t^{k_0}+I\subseteq\mathbb{C}[t,t^{-1}].$$

So $\mathbb{C}[t]t^{k_0}+\mathbb{C}[t,t^{-1}]p(t)^{k_0}=\mathbb{C}[t,t^{-1}]$. Now by Theorem 7, we know $V_1\otimes V_2=V\otimes W_1[\lambda_1]\otimes W_2[\lambda_2]\otimes\cdots\otimes W_n[\lambda_n]$ is a simple $\widetilde{\mathcal{V}}$-module. □

Example 10 Let $\lambda_1,\ldots,\lambda_n\in\mathbb{C}^*$ are pairwise distinct, $a_1,\ldots,a_n,b_1,\ldots,b_n\in\mathbb{C}$, and $V\in\mathcal{O}_{\widetilde{\mathcal{V}}}$ is simple. From Example 8 and using Theorem 34, we obtain the simple module $\Omega(\lambda_1;a_1,b_1)\otimes\Omega(\lambda_2;a_1,b_2)\otimes\cdots\otimes\Omega(\lambda_n;a_n,b_n)\otimes V$ if each $(a_i,b_i)\neq(0,0)$. Further, if we take $b_i=0$ for all i, and $V\in\mathcal{O}_{\mathcal{V}}$, then $\Omega(\lambda_1;a_1,0)\otimes\Omega(\lambda_2;a_1,0)\otimes\cdots\otimes\Omega(\lambda_n;a_n,0)\otimes V$ is also a simple $\mathcal{V}$-module since $\mathfrak{H}$ acts trivially. Such simple $\mathcal{V}$-modules were obtained in [24, 25].

5.4 Isomorphism Classes

In this subsection we will determine the isomorphism classes of the simple tensor product $\widetilde{\mathcal{V}}$-modules discussed in Theorem 34.

Theorem 35 *Let $W_1,W_2,\ldots,W_n,W_1',W_2',\ldots,W_m'\in\mathcal{O}_{\widetilde{\mathcal{W}}}$ be simple and nontrivial, $V,V'\in\mathcal{O}_{\widetilde{\mathcal{V}}}$ be simple, and $\lambda_1,\lambda_2,\ldots,\lambda_n,\lambda_1',\lambda_2',\ldots,\lambda_m'\in\mathbb{C}^*$. Then the simple $\widetilde{\mathcal{V}}$-modules $W_1[\lambda_1]\otimes W_2[\lambda_2]\otimes\cdots\otimes W_n[\lambda_n]\otimes V$ and $W_1'[\lambda_1']\otimes W_2'[\lambda_2']\otimes\cdots\otimes W_m'[\lambda_m']\otimes V'$ are isomorphic iff $m=n$, $V\cong V'$, $\lambda_i=\lambda_i'$, $W_i\cong W_i'$ for $i=1,2,\ldots,n$ after re-indexing the modules $W'[\lambda']$.*

Proof The "if" part is obvious. We need only to consider the "only if" part, that is, we suppose that the simple $\widetilde{\mathcal{V}}$-modules $W_1[\lambda_1]\otimes W_2[\lambda_2]\otimes\cdots\otimes W_n[\lambda_n]\otimes V$ and $W_1'[\lambda_1']\otimes W_2'[\lambda_2']\otimes\cdots\otimes W_m'[\lambda_m']\otimes V'$ are isomorphic.

For any $v\in V$ or $v\in V'$, there is an $l\in\mathbb{N}$ such that

$$\left(\mathbb{C}[t]t^l\frac{d}{dt}+\mathbb{C}[t]t^l\right)v=0.$$

Let $p_1(t)=(t-\lambda_1)\ldots(t-\lambda_n)$ and $p_2(t)=(t-\lambda_1')\ldots(t-\lambda_n')$. From the given isomorphism, we know that, for any nonzero

$$X=w_1\otimes\cdots\otimes w_n\otimes v\in W_1[\lambda_1]\otimes W_2[\lambda_2]\otimes\cdots\otimes W_n[\lambda_n]\otimes V,$$

there exists $k_0\in\mathbb{N}$ such that

$$\left(\mathbb{C}[t](p_1(t)t)^{k_0}\frac{d}{dt}+\mathbb{C}[t](p_1(t)t)^{k_0}\right)\circ w_i=0,\ i=1,2,\ldots,n \tag{5.9}$$

$$\left(\mathbb{C}[t](p_1(t)t)^{k_0}\frac{d}{dt}+\mathbb{C}[t](p_1(t)t)^{k_0}\right)v=0,\ \text{and} \tag{5.10}$$

$$\left(\mathbb{C}[t](p_2(t)t)^{k_0}\frac{d}{dt}+\mathbb{C}[t](p_2(t)t)^{k_0}\right)X=0. \tag{5.11}$$

If $p_1(t) \nmid p_2(t)$, without lose of generality, we may assume that $(t-\lambda_1) \nmid p_2(t)$. Let $p(t) = p_1(t)p_2(t)/(t-\lambda_1)$. Then

$$\left(\mathbb{C}[t](p(t)t)^{k_0}\frac{d}{dt} + \mathbb{C}[t](p(t)t)^{k_0}\right)(w_2 \otimes \cdots \otimes w_n \otimes v) = 0.$$

Combining with Eq. 5.11, we know that

$$\left(\mathbb{C}[t](p(t)t)^{k_0}\frac{d}{dt} + \mathbb{C}[t](p(t)t)^{k_0}\right) \circ w_1 = 0. \tag{5.12}$$

Note that there exists some $k \in \mathbb{Z}$ such that

$$\left(\mathbb{C}[t,t^{-1}](t-\lambda_1)^k\frac{d}{dt} + \mathbb{C}[t,t^{-1}](t-\lambda_1)^k + \sum_{i=1}^{3}\mathbb{C}z_i\right) \circ w_1 = 0. \tag{5.13}$$

From Eqs. 5.12 and 5.13, we have $\widetilde{\mathcal{V}} \circ w_1 = 0$, which is a contradiction. Thus $p_1(t)|p_2(t)$, and similarly $p_2(t)|p_1(t)$, to give $p_1(t) = p_2(t)$. So we have $m = n$, and we may assume that $\lambda_i = \lambda_i'$ for $i = 1, 2, \ldots, n$.

Let $\psi : W_1[\lambda_1] \otimes \cdots \otimes W_n[\lambda_n] \otimes V \to W_1'[\lambda_1] \otimes \cdots \otimes W_n'[\lambda_n] \otimes V'$ be a $\widetilde{\mathcal{V}}$-module isomorphism. Fix some nonzero

$$X = w_1 \otimes \ldots w_n \otimes v \in W_1[\lambda_1] \otimes W_2[\lambda_2] \otimes \cdots \otimes W_n[\lambda_n] \otimes V.$$

Write $\psi(X) = \sum_{i=1}^{s} w_{1,i}' \otimes Y_i$ with minimal s, where $w_{1,i}' \in W_1'[\lambda_1]$, $Y_i \in W_2'[\lambda_2] \otimes \cdots \otimes W_n'[\lambda_n] \otimes V'$. We know that Y_i's are linearly independent. Denote $p(t) = t(t-\lambda_2)\ldots(t-\lambda_n)$, then there exists $k_1 \in \mathbb{N}$ such that

$$\left(\mathbb{C}[t]p(t)^{k_1}\frac{d}{dt} + \mathbb{C}[t]p(t)^{k_1}\right) Y_i = 0, \ \forall i = 1, 2, \ldots, s,$$

$$\left(\mathbb{C}[t]p(t)^{k_1}\frac{d}{dt} + \mathbb{C}[t]p(t)^{k_1}\right)(w_2 \otimes \cdots \otimes w_n \otimes v) = 0.$$

Denote $L = \mathbb{C}[t]p(t)^{k_1}\frac{d}{dt} + \mathbb{C}[t]p(t)^{k_1}$. Then for any $w \in W_1[\lambda]$, $w' \in W_1'[\lambda]$, it is straightforward to check that

$$\mathrm{ann}_{\widetilde{\mathcal{V}}}(w) + L = \mathrm{ann}_{\widetilde{\mathcal{V}}}(w') + L = \widetilde{\mathcal{V}}. \tag{5.14}$$

For any $w \in W_1$, we have

$$W_1[\lambda_1] = U(\widetilde{\mathcal{V}}) \circ w = U(\mathrm{ann}_L(w) + L) \circ w$$

$$= U(L)U(\mathrm{ann}_L(w)) \circ w = U(L) \circ w.$$

that is, $W_1[\lambda_1]$ is a simple L-module. Similarly we have $W_1'[\lambda_1]$ is also simple as L-module. Now from Lemma 6, there exists some $u_0 \in U(L)$ such that $u_0 \circ w_{1,i}' = \delta_{i,1}w_{1,1}'$. Then

$$\psi((u_0 \circ w_1) \otimes (w_2 \otimes \cdots w_n \otimes v)) = w_{1,1}' \otimes Y_1.$$

And

$$\psi((uu_0) \circ w_1) \otimes (w_2 \otimes \cdots w_n \otimes v))) = (u \circ w_{1,1}') \otimes Y_1, \forall u \in U(L).$$

Recall that $W_1[\lambda_1]$, $W_1'[\lambda_1]$ are simple $\widetilde{\mathcal{V}}$ -module. Therefore we have the well-defined linear map $\varphi : W_1[\lambda_1] \to W_2[\lambda_2]$ defined by $\varphi(u \circ (u_0 \circ w_1)) = u \circ w_{1,1}'$ for all $u \in U(L)$, which is obvious an L-module isomorphism. Now we only need to show that it is also a $\widetilde{\mathcal{V}}$-module homomorphism. Now for any $w \in W_1[\lambda]$, it is easy to check that $L + (\mathrm{ann}_{\widetilde{\mathcal{V}}}(w) \cap$

$\text{ann}_{\widetilde{\mathcal{V}}}\,\varphi(w)) = \widetilde{\mathcal{V}}$. Hence for any $x \in \widetilde{\mathcal{V}}$, we may write $x = x_1 + x_2$ with $x_1 \in L$ and $x_2 \in \text{ann}_{\widetilde{\mathcal{V}}}(w) \cap \text{ann}_{\widetilde{\mathcal{V}}}\,\varphi(w)$. Now

$$\varphi(x \circ w) = \varphi(x_1 \circ w) = x_1 \circ \varphi(w) = x \circ \varphi(w),$$

i.e., φ is a $\widetilde{\mathcal{V}}$-module isomorphism.

Similarly, we have $W_i[\lambda_i] \cong W_i'[\lambda_i]$ for $i = 2, 3, \ldots, n$, and $V \cong V'$. Now the theorem follows from Proposition 32 (c). □

It is easy to see that the simple modules for $m \geq 1$ in Theorem 35 are not isomorphic to any generalized oscillator representations.

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