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The Grassmann Algebra and its Differential Identities

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Abstract
Let G be the infinite dimensional Grassmann algebra over an infinite field F of characteristic different from two. In this paper we study the differential identities of G with respect to the action of a finite dimensional Lie algebra L of inner derivations. We explicitly determine a set of generators of the ideal of differential identities of G. Also in case F is of characteristic zero, we study the space of multilinear differential identities in n variables as a module for the symmetric group S_n and we compute the decomposition of the corresponding character into irreducibles. Finally, we prove that unlike the ordinary case the variety of differential algebras with L action generated by G has no almost polynomial growth.

Keywords Polynomial identity · Differential identity · Codimension · Cocharacter

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1 Introduction

Let A be an associative algebra over an infinite field F of characteristic $p \neq 2$. A well-established method of studying the growth of the polynomial identities of A is that of determining some numerical invariants allowing to give a quantitative description (e.g. [6, 10]). In particular a lot of information for the polynomial identities is carried by the codimension sequence, $c_n(A)$, $n \geq 1$, and in case F is of characteristic zero, by the S_n-cocharacter sequence, $\chi_n(A)$, $n \geq 1$, of the algebra A. Similar sequences are defined when studying the polynomial identities of algebras with an additional structure such as group-graded algebras, algebras with an action of a group by automorphism and anti-automorphism, algebras with an action of a Lie algebra by derivations (e.g. [3, 4, 8, 9, 15]).

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One of the most interesting questions in this context is to compute the growth of the codimension sequence of an algebra. Regev in [19] proved that any associative algebra A satisfying a non trivial polynomial identity has codimensions exponentially bounded. Later in [14] Kemer showed that such codimensions are either polynomially bounded or grow exponentially. One of the mostly notable algebras with exponential codimension growth is the infinite dimensional Grassmann algebra G; in case F is of characteristic zero, Latyshev in [17] determined a basis of the ideal of its polynomial identities . Later, its codimension sequence and its cocharacter sequence were determined in [16, 18], respectively. Moreover, in 2001 Giambruno and Koshlukov (see [7]) exhibited a basis of the polynomial identities satisfied by the Grassmann algebra over a field of positive characteristic.

In this context it is often convenient to use the language of varieties of algebras. Given a variety of algebras $\mathcal{V}$, the growth of $\mathcal{V}$ is defined as the growth of the sequence of codimensions of any algebra A generating $\mathcal{V}$, i.e., $\mathcal{V} = \mathrm{var}(A)$. In [13] Kemer proved that $\mathrm{var}(G)$ has almost polynomial growth, i.e., $\mathrm{var}(G)$ has exponential growth but every proper subvariety has polynomial growth.

In light of the above, it seems interesting to study the structure of the polynomial identities of the infinite dimensional Grassmann algebra with an additional structure. In this perspective, in [1] Anisimov computed the graded codimension growth of G when a cyclic group of prime order q by automorphism and anti-automorphism acts on G. In particular in [2] he found the exact values of the codimensions with involution of the Grassmann algebra for two concrete involutions. Later in [5] Di Vincenzo, Kolshlukov and da Silva determined the $\mathbb{Z}_q$-graded codimensions and cocharacters of G.

The purpose of this paper is to study the growth of the differential identities of the Grassmann algebra G. More precisely, we consider G with the action of a finite dimensional Lie algebra L of its inner derivations. Since this action on G can be naturally extended to the action of its universal enveloping algebra $U(L)$, it is natural to define the differential identities of G, i.e., the polynomials in non-commutative variables $x^h = h(x)$, $h \in U(L)$, vanishing on G (see [11, 15]).

In this paper we explicitly construct a set of generators for the ideal of differential identities of the Grassmann algebra over an infinite field F of characteristic $p \neq 2$ and also we compute its differential codimensions. As a consequence it turns out that the growth of the differential identities of G is exponential, as in the ordinary case. However, we prove that unlike the ordinary case G with the action of a finite dimensional Lie algebra of inner derivations does not generate a variety of almost polynomial growth; in fact we exhibit a subvariety of almost polynomial growth. Furthermore, in case F is of characteristic zero we determine the decomposition of the differential cocharacter of G in its irreducible components by computing all the corresponding multiplicities.

2 Preliminaries

Throughout this paper F will denote an infinite field of characteristic $p \neq 2$. Let A be an associative algebra. Recall that a derivation of A is a linear map $\partial : A \to A$ such that

$$\partial(xy) = \partial(x)y + x\partial(y), \qquad \forall x, y \in A.$$

In particular an inner derivation induced by $x \in A$ is the derivation $\mathrm{ad}\, x : A \to A$ of A define by $(\mathrm{ad}\, x)(y) = [x, y]$, for all $y \in A$. The set of all derivations of A is a Lie algebra denoted by $\mathrm{Der}(A)$, and the set $\mathrm{ad}(A)$ of all inner derivations of A is a Lie subalgebra of $\mathrm{Der}(A)$.

Let L be a Lie algebra of derivations of A. If $U(L)$ is its universal enveloping algebra, then the L-action on A can be naturally extended to a $U(L)$-action.

Given a basis $B = \{h_i | i \in I\}$ of $U(L)$, we let $F\langle X|L\rangle$ be the free associative algebra over F with free formal generators $x_j^{h_i}$, $i \in I$, $j \in \mathbb{N}$. If $h = \sum_{i \in I} \alpha_i h_i$, $\alpha_i \in F$, where only a finitely many of α_i are nonzero, then we put $x^h := \sum_{i \in I} \alpha_i x^{h_i}$. We also write $x_i = x_i^1$, $1 \in U(L)$, and then we set $X = \{x_1, x_2, \dots\}$. We let $U(L)$ act on $F\langle X|L\rangle$ by the following

$$h(x_{j_1}^{h_{i_1}} x_{j_2}^{h_{i_2}} \dots x_{j_n}^{h_{i_n}}) = x_{j_1}^{hh_{i_1}} x_{j_2}^{h_{i_2}} \dots x_{j_n}^{h_{i_n}} + \dots + x_{j_1}^{h_{i_1}} x_{j_2}^{h_{i_2}} \dots x_{j_n}^{hh_{i_n}},$$

where $h, h_{i_1}, h_{i_2}, \dots, h_{i_n} \in B$. $F\langle X|L\rangle$ is called the free associative algebra with derivations on the countable set X and its elements are called differential polynomials (see [9, 11, 15]). Notice that if $L \subseteq \mathrm{ad}(A)$, i.e., L is a Lie algebra of inner derivations of A, then $F\langle X|L\rangle$ is a free associative algebra with action of inner derivations.

A polynomial $f(x_1, \dots, x_n) \in F\langle X|L\rangle$ is a polynomial identity with derivation of A or differential identity of A if $f(a_1, \dots, a_n) = 0$ for any $a_i \in A$, and we write $f \equiv 0$. We denote by

$$\mathrm{Id}^L(A) = \{f \in F\langle X|L\rangle | f \equiv 0 \text{ on } A\}$$

the T_L-ideal of differential identities of A, i.e., $\mathrm{Id}^L(A)$ is an ideal of $F\langle X|L\rangle$ invariant under the $U(L)$-action. We also denote by

$$P_n^L = \mathrm{span}\{x_{\sigma(1)}^{h_1} \dots x_{\sigma(n)}^{h_n} | \sigma \in S_n, h_i \in B\}$$

the space of multilinear differential polynomials in $x_1, \dots, x_n$, $n \geq 1$. The non-negative integer

$$c_n^L(A) = \dim \frac{P_n^L}{P_n^L \cap \mathrm{Id}^L(A)}$$

is called the nth differential codimension of A.

If S_n is the symmetric group of degree n, then the space P_n^L has a natural structure of left S_n-module induced by defining for $\sigma \in S_n$, $\sigma(x_i^h) = x_{\sigma(i)}^h$. Since $P_n^L \cap \mathrm{Id}^L(A)$ is stable under the S_n-action the space

$$P_n^L(A) = \frac{P_n^L}{P_n^L \cap \mathrm{Id}^L(A)}$$

is a left S_n-module. If F is of characteristic zero, the character of $P_n^L(A)$, $\chi_n^L(A)$, is called nth differential cocharacter of A, and it can be decompose as

$$\chi_n^L(A) = \sum_{\lambda \vdash n} m_\lambda^L \chi_\lambda,$$

where λ is a partition of n, χ_λ is the irreducible S_n-character associated to λ, and $m_\lambda^L \geq 0$ is the corresponding multiplicity.

We denote by P_n the space of multilinear ordinary polynomials in $x_1, \dots, x_n$ and by $\mathrm{Id}(A)$ the T-ideal of the free algebra $F\langle X\rangle$ of polynomial identities of A. We also write $c_n(A)$ for the nth codimension of A and $\chi_n(A)$ for the nth cocharacter of A. Since $U(L)$ is an algebra with unit, we can identify in a natural way P_n with a subspace of P_n^L. Hence we have $P_n \subseteq P_n^L$ and $P_n \cap \mathrm{Id}(A) = P_n \cap \mathrm{Id}^L(A)$. Thus it follows that $c_n(A) \leq c_n^L(A)$, for all $n \geq 1$.

Next we focus on the Grassmann algebra G.

Let V be a vector space with a countable basis $\{e_1, e_2, \dots\}$ over F. The Grassmann algebra G of V is the associative algebra with the following basis over F

$$\{1, e_{i_1} \dots e_{i_k} \mid 1 \leq i_1 < \dots < i_k\}$$

and multiplication induced by $e_i e_j = -e_j e_i$, for all $i, j \geq 1$. Notice that G can be decomposed in a natural way as the direct sum of the subspaces

$$G_0 = \text{span}_F\{e_{i_1} \dots e_{i_{2k}} \mid i_1 < \dots < i_{2k}, k \geq 0\}$$

and

$$G_1 = \text{span}_F\{e_{i_1} \dots e_{i_{2k+1}} \mid i_1 < \dots < i_{2k+1}, k \geq 0\},$$

i.e. $G = G_0 \oplus G_1$. Note that G_0 is the center of G, and that the elements of G_1 anticommute.

Recall that if $g = e_{i_1} \dots e_{i_n} \in G$, the set $\text{Supp}\{g\} = \{e_{i_1}, \dots, e_{i_n}\}$ is called the support of g. Let now $g_1, \dots, g_t \in G_1$ be such that $\text{Supp}\{g_i\} \cap \text{Supp}\{g_j\} = \emptyset$, for all $i, j \in \{1, \dots, t\}$. Since char $F \neq 2$, we set

$$\delta_{i=2}^{-1} \,\text{ad}\, g_i, \quad i = 1, \dots, t.$$

Then for all $g \in G$, we have

$$\delta_i(g) = \begin{cases} 0, & \text{if } g \in G_0 \\ g_i g, & \text{if } g \in G_1 \end{cases}, \quad i = 1, \dots, t.$$

We shall consider $L = \text{span}_F\{\delta_1, \dots, \delta_t\} \subset \text{ad}(G)$. Since for all $g \in G$, $[\delta_i, \delta_j](g) = 0$, $i, j \in \{1, \dots, t\}$, L is a t-dimensional abelian Lie algebra of inner derivations of G. We shall denote by $\widetilde{G}$ the algebra G with this L-action. Throughout this paper $F\langle X|L\rangle$ will be the free associative algebra with inner derivations on X.

3 Differential Codimensions

We start by describing the differential identities of $\widetilde{G}$. Recall that in the ordinary case we have the following result (see[7, 16]).

Theorem 1 *Let G be the infinite dimensional Grassmann algebra over an infinite field F of characteristic $p \neq 2$. Then*

(1) $\text{Id}(G) = \langle [x_1, x_2, x_3] \rangle_T$.
(2) $\{x_{i_1} \dots x_{i_m}[x_{j_1}, x_{j_2}] \dots [x_{j_{2q-1}}, x_{2q}] : i_1 < \dots < i_m, j_1 < \dots < j_{2q}, 2q + m = n\}$ *is a basis of P_n modulo* $(P_n \cap \text{Id}(G))$.
(3) $c_n(G) = 2^{n-1}$.

Remark 2 It can be checked that

$$[x_1, x_2][x_1, x_2] \equiv 0 \tag{1}$$

is a consequence of $[x_1, x_2, x_3] \equiv 0$ in G (see for example [10]). Since $[x_1, x_2, x_3] \equiv 0$ is also a differential identity on $\widetilde{G}$, then the linearization of Eq. 1 leads to the identity $[x_1, x_2][x_3, x_4] \equiv -[x_3, x_2][x_1, x_4]$ on $\widetilde{G}$. Notice that the linearization is harmless because char $F \neq 2$ and the degree of x_1 is equal to 2.

Next we prove the main result of this section. For a real number x we denote by $\lfloor x \rfloor$ its integer part.

Theorem 3 *Let F be an infinite field of characteristic $p \neq 2$ and $\widetilde{G}$ be the infinite dimensional Grassmann algebra over F with $L = \mathrm{span}_F\{\delta_1, \dots, \delta_t\}$-action. Then*

(1) $\mathrm{Id}^L(\widetilde{G}) = \langle [x_1, x_2, x_3] \rangle_{T_L}$.
(2) $c_n^L(\widetilde{G}) = 2^t 2^{n-1} - \sum_{j=1}^{\lfloor t/2 \rfloor} \sum_{i=2j}^{t} \binom{t}{i}\binom{n}{i-2j}$.

Proof Let $Q = \langle [x_1, x_2, x_3] \rangle_{T_L}$. It is readily checked that $Q \subseteq \mathrm{Id}^L(\widetilde{G})$. Let $f \in F\langle X|L \rangle$ be a differential polynomial in $x_1, \dots, x_n$. Since $1 \in \widetilde{G}$, f can be written as a linear combination of products of the type

$$x_{i_1}^{\alpha_1} \dots x_{i_k}^{\alpha_k} w_1 \dots w_m \tag{2}$$

where $\alpha_i \in U(L)$, $\alpha_i \neq 1$, for $1 \leq i \leq k$, and $w_1 \dots, w_m$ are left normed commutators in the $x_j^{\beta_h}$s, $\beta_h \in U(L)$. Notice that $[x_1^{\gamma_1}, x_2^{\gamma_2}, x_3^{\gamma_3}] \equiv 0$ and $[x_1^{\gamma_1}, x_2] \equiv 0$ with $\gamma_i \in U(L)$, for $1 \leq i \leq 3$, are consequence of $[x_1, x_2, x_3] \equiv 0$. Then, modulo Q, in Eq. 2 we have $w_j = [x_{j_h}, x_{j_k}]$, for all $j = 1, \dots, m$, and they are central. Also since $x^{\delta_i \delta_j} \in Q$, for all $i, j \in \{1, \dots, t\}$, it follows that in Eq. 2 $\alpha_i \in \{\delta_1, \dots, \delta_t\}$ modulo Q. Moreover it is clear that $x^{\delta_i} x^{\delta_j} \equiv 0$ is a consequence of $[x_1, x_2][x_1, x_3] \equiv 0$ and by Remark 2 $[x_1, x_2][x_3, x_4] + [x_3, x_2][x_1, x_4] \in Q$. Then we may assume that f is multilinear. Now observe that $x_1^{\delta_i} x_2^{\delta_j} \equiv -x_2^{\delta_i} x_1^{\delta_j}$ and $x_1^{\delta_i}[x_2, x_3] \equiv -x_3^{\delta_i}[x_2, x_1]$ are consequences of $[x_1, x_2][x_3, x_4] \equiv -[x_3, x_2][x_1, x_4]$. Then f can be written, modulo Q, as a linear combination of elements of the type

$$x_1^{\delta_{h_1}} \dots x_k^{\delta_{h_k}} [x_{k+1}, x_{k+2}] \dots [x_{k+2q-1}, x_{k+2q}], \tag{3}$$

with

$$h_1 < \dots < h_k, \quad k + 2q = n, \quad 0 \leq k \leq t. \tag{4}$$

Next we prove that these elements are linearly independent modulo $\mathrm{Id}^L(\widetilde{G})$.

For any $0 \leq k \leq t$, consider $\Delta_k = \{\delta_{h_1}, \dots, \delta_{h_k}\} \subseteq \{\delta_1, \dots, \delta_t\}$, set $X_{\Delta_k} = x_1^{\delta_{h_1}} \dots x_k^{\delta_{h_k}} [x_{k+1}, x_{k+2}] \dots [x_{k+2q-1}, x_{k+2q}]$ and suppose that $f = \sum_{\Delta_k} \alpha_{\Delta_k} X_{\Delta_k} \in \mathrm{Id}^L(\widetilde{G})$. In order to show that all coefficients α_{Δ_k} are zero we consider the following evaluations: for any $\Delta_k = \{\delta_{h_1}, \dots, \delta_{h_k}\}$ we choose $x_1 = g_1^{'}, \dots, x_{k+2q} = g_{k+2q}^{'}$ where $g_i^{'} \in G_1$, $1 \leq i \leq k + 2q$, and for all $r \in \{1, \dots, t\} \setminus \{h_1, \dots, h_k\}$, there exists $s \in \{1, \dots, k + 2q\}$ such that $\mathrm{Supp}\{g_s^{'}\} \cap \mathrm{Supp}\{g_r\} \neq \emptyset$. If we make these evaluations for increasing value of k $(0 \leq k \leq t)$, by the properties of the polynomial in Eq. 3, it follows that $\alpha_{\Delta_k} = 0$ for any Δ_k. Thus the elements (3) are linearly independent modulo $\mathrm{Id}^L(\widetilde{G})$, and this proves that $\mathrm{Id}^L(\widetilde{G}) = Q$.

Notice that if we consider the multilinear differential polynomials, then the elements

$$x_{i_1} \dots x_{i_m} x_{j_1}^{\delta_{h_1}} \dots x_{j_k}^{\delta_{h_k}} [x_{j_{k+1}}, x_{j_{k+2}}] \dots [x_{j_{k+2q-1}}, x_{j_{k+2q}}], \tag{5}$$

with

$$i_1 < \dots < i_m, \; j_1 < \dots < j_{k+2q}, \; h_1 < \dots < h_k, \; m + k + 2q = n, \; 0 \leq k \leq t, \tag{6}$$

are a basis of P_n^L modulo $P_n^L \cap \mathrm{Id}^L(\widetilde{G})$. Thus we count for any fixed n, the total number of elements in Eq. 5 subject to the conditions (6), i.e. the nth differential codimension $c_n^L(G)$. If $0 \leq k \leq t$, then this number is equal to

$$s_k = \binom{t}{k} \sum_{q=0}^{\lfloor (n-k)/2 \rfloor} \binom{n}{k+2q}.$$

Notice that $s_0 = 2^{n-1}$ and $s_1 = \binom{t}{1}2^{n-1}$. Moreover, if $k = 2l$ with $l \geq 1$,

$$s_{2l} = \binom{t}{2l}\left(\sum_{r=0}^{\lfloor n/2\rfloor}\binom{n}{2r} - \sum_{p=0}^{l-1}\binom{n}{2p}\right) = \binom{t}{2l}\left(2^{n-1} - \sum_{p=0}^{l-1}\binom{n}{2p}\right).$$

Finally, in case $k = 2l + 1$ with $l \geq 1$,

$$\begin{aligned} s_{2l+1} &= \binom{t}{2l+1}\left(\sum_{r=0}^{\lfloor (n-1)/2\rfloor}\binom{n}{2r+1} - \sum_{p=0}^{l-1}\binom{n}{2p+1}\right) \\ &= \binom{t}{2l+1}\left(2^{n-1} - \sum_{p=0}^{l-1}\binom{n}{2p+1}\right). \end{aligned}$$

Thus

$$\begin{aligned} c_n^L(\widetilde{G}) &= \sum_{k=0}^{t} s_k = 2^{n-1} + \binom{t}{1}2^{n-1} + \sum_{l=1}^{\lfloor t/2\rfloor}\binom{t}{2l}\left(2^{n-1} - \sum_{p=0}^{l-1}\binom{n}{2p}\right) \\ &\quad + \sum_{l=1}^{\lfloor (t-1)/2\rfloor}\binom{t}{2l+1}\left(2^{n-1} - \sum_{p=0}^{l-1}\binom{n}{2p+1}\right) \\ &= 2^t2^{n-1} - \sum_{l=1}^{\lfloor t/2\rfloor}\binom{t}{2l}\sum_{p=0}^{l-1}\binom{n}{2p} - \sum_{l=1}^{\lfloor (t-1)/2\rfloor}\binom{t}{2l+1}\sum_{p=0}^{l-1}\binom{n}{2p+1} \\ &= 2^t2^{n-1} - \sum_{i=2}^{t}\binom{t}{i}\binom{n}{i-2} - \sum_{l=2}^{\lfloor t/2\rfloor}\binom{t}{2l}\sum_{p=0}^{l-2}\binom{n}{2p} \\ &\quad - \sum_{l=2}^{\lfloor (t-1)/2\rfloor}\binom{t}{2l+1}\sum_{p=0}^{l-2}\binom{n}{2p+1} = \dots \\ &= 2^t2^{n-1} - \sum_{j=1}^{\lfloor t/2\rfloor}\sum_{i=2j}^{t}\binom{t}{i}\binom{n}{i-2j}. \end{aligned}$$

□

Recall that two functions $\varphi_1(n)$ and $\varphi_2(n)$ are asymptotically equal and we write $\varphi_1(n) \approx \varphi_2(n)$ if $\lim_{n\to\infty}\varphi_1(n)/\varphi_2(n) = 1$. Then the following corollary is an obvious consequence of the previous theorem.

Corollary 4 $c_n^L(\widetilde{G}) \approx 2^t2^{n-1}$.

The proof of Theorem 3 suggests a convenient decomposition of $P_n^L(\widetilde{G})$. For any $n \geq 1$ and for all $\gamma_1, \dots, \gamma_k \in L$ distinct, we set

$$\Phi_{\gamma_1,\dots,\gamma_k} = \{\gamma_1, \dots, \gamma_k, \underbrace{1, \dots, 1}_{n-k}\}.$$

We define

$$P_n^{\Phi_{\gamma_1,\dots,\gamma_k}} = \mathrm{span}_F\{x_{\sigma(1)}^{\varepsilon_1}\cdots x_{\sigma(n)}^{\varepsilon_n} \mid \sigma \in S_n,\ \varepsilon_i \in \Phi_{\gamma_1,\dots,\gamma_k}\},$$

a S_n-submodule of P_n^L. Since for all $\gamma_1, \dots, \gamma_k, \beta_1, \dots, \beta_k \in L$, $P_n^{\Phi_{\gamma_1,\dots,\gamma_k}}$ and $P_n^{\Phi_{\beta_1,\dots,\beta_k}}$ are isomorphic as S_n-modules, we introduce the notation

$$P_{n,k}^L = P_n^{\Phi_{\delta_1,\dots,\delta_k}}.$$

In particular, for $k = 0$ we have $P_{n,0}^L = P_n$. Hence for any $0 \le k \le t$, we set

$$P_{n,k}^L(\widetilde{G}) = \frac{P_{n,k}^L}{P_{n,k}^L \cap \mathrm{Id}^L(\widetilde{G})}$$

and

$$c_{n,k}^L(\widetilde{G}) = \dim_F P_{n,k}^L(\widetilde{G}).$$

As consequence of proof of the Theorem 3 we have the following.

Corollary 5 $c_n^L(\widetilde{G}) = \sum_{k=0}^t \binom{t}{k} c_{n,k}^L(\widetilde{G})$, *where*

$$c_{n,k}^L(\widetilde{G}) = \begin{cases} 2^{n-1}, & \text{if } k = 0, 1 \\ 2^{n-1} - \sum_{j=0}^{\lfloor k/2 \rfloor - 1} \binom{n}{2j}, & \text{if } k \ge 2 \text{ is even} \\ 2^{n-1} - \sum_{j=0}^{\lfloor k/2 \rfloor - 1} \binom{n}{2j+1}, & \text{if } k \ge 3 \text{ is odd} \end{cases}.$$

Next we shall be concerned with the growth of the differential codimension of $\widetilde{G}$.

Recall that if $\mathcal{V} = \mathrm{var}^L(A)$ is a variety of algebras with derivation generated by an algebra A, i.e. the Lie algebra L acts on A as derivations, then the growth of $\mathcal{V}$ is the growth of the sequence $c_n^L(\mathcal{V}) = c_n^L(A)$, $n \ge 1$. We say that $\mathcal{V}$ has polynomial growth if $c_n^L(\mathcal{V})$ is polynomially bounded and $\mathcal{V}$ has almost polynomial growth if $c_n^L(\mathcal{V})$ is not polynomially bounded but every proper subvariety of $\mathcal{V}$ has polynomial growth.

Notice that by Corollary 4 $\mathrm{var}^L(\widetilde{G})$ has exponential growth, nevertheless it has no almost polynomial growth. In fact, the Grassmann algebra G (ordinary case) is an algebra with L-action where δ_i, $i = 1, \dots, t$, acts trivially on G, i.e., $x^{\delta_i} \equiv 0$, $i = 1, \dots, t$, are differential identities of G. Then it follows that $G \in \mathrm{var}^L(\widetilde{G})$, but by Theorem 1 $c_n(G) = 2^{n-1}$. Thus we have the following result.

Theorem 6 $\mathrm{var}^L(\widetilde{G})$ *has no almost polynomial growth.*

4 Differential Cocharacter

Throughout this section F will be a field of characteristic zero.

We start by recalling a notation. For integers $d, l \ge 0$, we define a hook shaped part of the plane of arm d and leg l as

$$H(d, l) = \{\lambda = (\lambda_1, \lambda_2, \dots) \vdash n \ge 1 \mid \lambda_{d+1} \le l\}.$$

In particular, if λ is a partition of $n \ge 1$, then $\lambda \subset H(1, 1)$ if

$$\lambda = (p, 1, \dots, 1) = (p, 1^{n-p}), \quad p \ge 1.$$

Let $\chi_n(G) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$ be the nth (ordinary) cocharacter of G where $m_\lambda \ge 0$ is the multiplicity corresponding to the irreducible character χ_λ. Then in the ordinary case we have the following theorem (see [18]).

Theorem 7 *If G is the infinite dimensional Grassmann algebra over a field F of characteristic zero, then* $\chi_n(G) = \sum_{\substack{\lambda\vdash n\\ \lambda\subset H(1,1)}} \chi_\lambda$.

Let now $\chi^L_{n,k}(\widetilde{G})$ be the character of the S_n-module $P^L_{n,k}(\widetilde{G})$. Then we can write

$$\chi^L_{n,k}(\widetilde{G}) = \sum_{\lambda\vdash n} m^L_{\lambda,k}\chi_\lambda, \tag{7}$$

where $m^L_{\lambda,k} \geq 0$ is the multiplicity corresponding to the irreducible character χ_λ.

Next we shall compute the multiplicities $m^L_{\lambda,k}$ in Eq. 7.

For any partition $\lambda \vdash n$ let T_λ be a Young tableau of shape λ and e_{T_λ} the corresponding minimal essential idempotent of the group algebra FS_n. Recall that $e_{T_\lambda} = \sum_{\substack{\sigma\in R_{T_\lambda}\\ \tau\in C_{T_\lambda}}} (\operatorname{sgn}\tau)\sigma\tau$ where R_{T_λ} and C_{T_λ} are the subgroups of row and column permutations of T_λ, respectively.

Lemma 8 *If* $\chi^L_{n,k}(\widetilde{G}) = \sum_{\lambda\vdash n} m^L_{\lambda,k}\chi_\lambda$ *is the character of* $P^L_{n,k}(\widetilde{G})$*, then we have:*

(1) $m^L_{\lambda,k} = 1$, *if* $\lambda = (n-r+1, 1^{r-1})$ *and* $r \geq k$, $r \neq 0$;

(2) $m^L_{\lambda,k} = 0$ *in all other cases.*

Proof If $k = 0$, we have $P^L_{n,0} = P_n$ and $\chi^L_{n,0}(\widetilde{G}) = \chi_n(G)$. Then by Theorem 7 the theorem is proved in case $k = 0$.

Suppose that $k \geq 1$. Assume that $\delta_1, \ldots, \delta_k$, act on $P^L_{n,k}(\widetilde{G})$. If $\lambda = (n-r+1, 1^{r-1})$ and $r \geq k$, we define T_λ to be the tableau

$$\begin{array}{|c|c|c|c|}\hline 1 & r+1 & \ldots & n \\ \hline 2 \\ \cline{1-1} \vdots \\ \cline{1-1} r \\ \cline{1-1}\end{array}\,.$$

Then $R_{T_\lambda} = S_{n-r+1}\{1, r+1, \ldots, n\}$ and $C_{T_\lambda} = S_r$, where $S_{n-r+1}\{1, r+1, \ldots, n\}$ denotes the symmetric group acting on the set $\{1, r+1, \ldots, n\}$. We associate to T_λ the polynomial

$$\begin{aligned} w_r^{\delta_1\ldots\delta_k} &= e_{T_\lambda}(x_1^{\delta_1}\ldots x_k^{\delta_k} x_{k+1}\ldots x_n) \\ &= \left(\sum_{\sigma\in S_{n-r+1}\{1,r+1,\ldots,n\}} \sigma\right)\left(\sum_{\tau\in S_r} (\operatorname{sgn}\tau) x^{\delta_1}_{\tau(1)}\ldots x^{\delta_k}_{\tau(k)} x_{\tau(k+1)}\ldots x_{\tau(r)}\right) x_{r+1}\ldots x_n. \end{aligned}$$

We claim that $w_r^{\delta_1\ldots\delta_k}$, $r \geq k$, is not an identity of $\widetilde{G}$. In fact, we consider the evaluation $\varphi : F\langle X|L\rangle \to G$ such that

$$\varphi(x_i) = e_i, \quad 1 \leq i \leq r,$$

and

$$\varphi(x_{r+1}) = e_{r+1}e_{r+2}, \ldots, \varphi(x_n) = e_{2n-r-1}e_{2n-r}$$

such that for all $i \in \{1,\ldots,2n-r\}$, $e_i \notin \mathrm{Supp}\{g_j\}$, for all $j \in \{1,\ldots,k\}$. Then, since for all $i \in \{1,\ldots,r\}$ and $j \in \{1,\ldots,k\}$, $\varphi(x_i^{\delta_j}) = g_j\varphi(x_i)$, we obtain

$$\begin{aligned}
&\varphi(\sum_{\tau\in S_r}(\operatorname{sgn}\tau)x_{\tau(1)}^{\delta_1}\ldots x_{\tau(k)}^{\delta_k}x_{\tau(k+1)}\ldots x_{\tau(r)})\\
&=\sum_{\tau\in S_r}(\operatorname{sgn}\tau)g_1\varphi(x_{\tau(1)})\ldots g_k\varphi(x_{\tau(k)})\varphi(x_{\tau(k+1)})\ldots\varphi(x_{\tau(r)})\\
&=(\pm g_1\ldots g_k)\sum_{\tau\in S_r}(\operatorname{sgn}\tau)e_{\tau(1)}\ldots e_{\tau(r)}=\pm(r!)g_1\ldots g_k e_1\ldots e_r\neq 0.
\end{aligned}$$

Thus, since $\varphi(x_{r+1}),\ldots,\varphi(x_n)$ are central in G,

$$\varphi(w_r^{\delta_1\ldots\delta_k}) = \pm(r!)(n-r+1)!g_1\ldots g_k e_1\ldots e_{2n-r}\neq 0.$$

We have proved that $w_r^{\delta_1\ldots\delta_k}$ is not an identity of $\widetilde{G}$. Hence this implies that $m_{\lambda,k}^L \geq 1$, if $\lambda = (n-r+1, 1^{r-1})$ and $r \geq k$. Then, since $c_{n,k}^L(\widetilde{G}) = \sum_{\lambda\vdash n} m_{\lambda,k}^L\chi_\lambda(1)$, we have

$$\sum_{r=k}^{n}\chi_{(n-r+1,1^{r-1})}(1) \leq c_{n,k}^L(\widetilde{G}). \tag{8}$$

By the hook formula $\chi_{(n-r+1,1^{r-1})}(1) = \binom{n-1}{r-1}$ (see [12]), then, if $k=1$, we have

$$\sum_{r=1}^{n}\chi_{(n-r+1,1^{r-1})}(1) = \sum_{r=1}^{n}\binom{n-1}{r-1} = 2^{n-1}.$$

On the other hand, by Corollary 5, $c_{n,1}^L(\widetilde{G}) = 2^{n-1}$. Then, if $k=1$ we get the equality in Eq. 8, and in this case the theorem is proved. Suppose then $k \geq 2$,

$$\sum_{r=k}^{n}\chi_{(n-r+1,1^{r-1})}(1) = \sum_{r=1}^{n}\binom{n-1}{r-1} - \sum_{r=1}^{k-1}\binom{n-1}{r-1} = 2^{n-1} - \sum_{r=1}^{k-1}\binom{n-1}{r-1}.$$

Hence in order to get the equality in Eq. 8 we need to prove that

$$2^{n-1} - \sum_{r=1}^{k-1}\binom{n-1}{r-1} \geq c_{n,k}^L(\widetilde{G}).$$

Thus, if $k=2l$ with $l \geq 1$, by Corollary 5 we need to check that

$$2^{n-1} - \sum_{r=1}^{2l-1}\binom{n-1}{r-1} \geq 2^{n-1} - \sum_{j=0}^{l-1}\binom{n}{2j}.$$

But by induction on $l \geq 1$, it is easy to verify that $\sum_{r=1}^{2l-1}\binom{n-1}{r-1} = \sum_{j=0}^{l-1}\binom{n}{2j}$ and also in this case the theorem is proved. Suppose finally that $k = 2l+1$ with $l \geq 1$. Since $\sum_{r=1}^{2l-1}\binom{n-1}{r-1} = \sum_{j=0}^{l-1}\binom{n}{2j+1}$, by Corollary 5 we get the equality in Eq. 8 and the theorem is proved. □

Theorem 9 *Let F be a field of characteristic zero and $\widetilde{G}$ be the infinite dimensional Grassmann algebra over F with $L = \mathrm{span}_F\{\delta_1,\ldots,\delta_t\}$-action. If $\chi_n^L(\widetilde{G}) = \sum_{\lambda\vdash n} m_\lambda^L\chi_\lambda$ is the nth differential cocharacter of $\widetilde{G}$, then we have:*

(1) $m_\lambda^L = \begin{cases}\sum_{i=0}^{r}\binom{t}{i}, & r < t\\ 2^t, & r \geq t\end{cases}$, *if* $\lambda = (n-r+1, 1^{r-1})$;

(2) $m_\lambda^L = 0$ *in all other cases.*

Proof By Corollary 5, $m_\lambda^L = \sum_{k=0}^{t} \binom{t}{k} m_{\lambda,k}^L$. Then by using Lemma 8 we get the proof of the theorem. □

References

1. Anisimov, N.: $\mathbb{Z}_p$-codimension of $\mathbb{Z}_p$-identities of Grassmann algebra. Commun. Algebra **29**, 4211–4230 (2001)
2. Anisimov, N.: Codimensions of identities with the Grassmann algebra involution, (Russian) Vestnik Moskov. Univ. Ser. I Mat. Mekh. **77**(3), 25–29 (2001). Translation in Moscow Univ. Math. Bull. **56**(3), 25–29 (2001)
3. Bahturin, Yu.A., Giambruno, A., Zaicev, M.V.: G-identities on associative algebras. Proc. Am. Math. Soc. **127**, 63–69 (1999)
4. Bahturin, Yu.A., Zaicev, M.V.: Identities of graded algebras and codimension growth. Trans. Amer. Math. Soc. **356**, 3939–3950 (2004)
5. Di Vincenzo, O.M., Koshlukov, P., da Silva, V.R.T.: On $\mathbb{Z}_p$-graded identities and cocharacters of the Grassmann algebra. Commun. Algebra **45**(1), 343–356 (2017)
6. Drensky, V.: Free Algebras and PI-algebras, Graduate Course in Algebra. Springer, Singapore (2000)
7. Giambruno, A., Koshlukov, P.: On the identities of the Grassmann algebras in characteristic $p > 0$. Isr. J. Math. **122**, 305–316 (2001)
8. Giambruno, A., Mishchenko, S., Zaicev, M.: Group actions and asymptotic behaviour of graded polynomial identities. J. Lond. Math. Soc. **66**(2), 295–312 (2002)
9. Giambruno, A., Rizzo, C.: Differential identities, 2×2 upper triangular matrices and varieties of almost polynomial growth. J. Pure Appl. Algebra **223**(4), 1710–1727 (2019)
10. Giambruno, A., Zaicev, M.: Polynomial identities and asymptotic methods. Math. Surv. Monogr., AMS, Providence, RI, 122 (2005)
11. Gordienko, A.S., Kochetov, M.V.: Derivations, gradings, actions of algebraic groups, and codimension growth of polynomial identities. Algebr. Represent. Theory **17**(2), 539–563 (2014)
12. James, G., Kerber, A.: The Representation Theory of the Symmetric Group. Addison-Wesley, London (1981)
13. Kemer, A.R.: Varieties of finite rank. Proc. 15th All the Union Algebraic Conf., Krasnoyarsk. **2**, 73 (1979). (in Russian)
14. Kemer, A.R.: Ideals of Identities of Associative Algebras, AMS Translations of Mathematical Monograph, p. 87. AMS, Providence (1988)
15. Kharchenko, V.K.: Differential identities of semiprime rings. Algebra Logic **18**, 86–119 (1979)
16. Krakowski, D., Regev, A.: The polynomial identities of the Grassmann algebra. Trans. Am. Math. Soc. **181**, 429–438 (1973)
17. Latyshev, V.N.: Algebras with identical relations. (Russian) Dokl. Akad. Nauk SSSR **146**, 1003–1006 (1962)
18. Olsonn, J.B., Regev, A.: Colength sequence of some T-ideals. J. Algebra **38**, 100–111 (1976)
19. Regev, A.: Existence of identities in $A \otimes B$. Israel J. Math. **11**, 131–152 (1972)